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Basics of Statistics and Coherence Theory

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Two simple calculations:

1) $I = \int_{-\infty}^{\infty} e^{-\frac{ax^2}{2}} dx \,, \quad a > 0$

$$I^2 = \int_{-\infty}^{\infty} e^{-\frac{ax^2}{2}} dx \int_{-\infty}^{\infty} e^{-\frac{ay^2}{2}} dy = \iint_{-\infty}^{\infty} e^{-\frac{a(x^2+y^2)}{2}} dx\,dy$$

change to polars

$$= \int_0^{\infty}\int_0^{2\pi} e^{-\frac{ar^2}{2}} r\, d\theta\, dr = 2\pi \int_0^{\infty} e^{-\frac{ar^2}{2}} r\,dr = 2\pi \int_0^{\infty} e^{-au} du$$

$u = \frac{r^2}{2}, \quad du = r\,dr$

$$= 2\pi \left. \frac{e^{-au}}{-a} \right|_0^{\infty} = \frac{2\pi}{a} \qquad \therefore\ I = \int_{-\infty}^{\infty} e^{-\frac{ax^2}{2}} dx = \sqrt{\frac{2\pi}{a}}$$

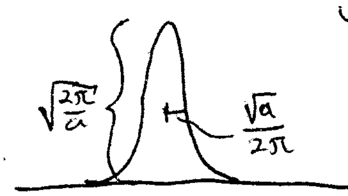
2) $D(\nu) = \int_{-\infty}^{\infty} e^{i2\pi x\nu} dx = \int_{-\infty}^{\infty} \underbrace{\left[\lim_{a\to 0} e^{-\frac{ax^2}{2}}\right]}_{1} e^{i2\pi x\nu} dx$

$$= \lim_{a\to 0} \int_{-\infty}^{\infty} e^{-\frac{ax^2}{2} + i2\pi x\nu} dx = \lim_{a\to 0} \int_{-\infty}^{\infty} e^{-\frac{a}{2}\left[x^2 - 2x\left(\frac{2\pi i}{a}\nu\right)\right]} dx$$

$$= \lim_{a\to 0} \int_{-\infty}^{\infty} e^{-\frac{a}{2}\Big[\underbrace{x - \frac{2\pi i\nu}{a}}_{x'}\Big]^2} dx\; e^{\frac{a}{2}\left(\frac{2\pi i\nu}{a}\right)^2} = \lim_{a\to 0} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{a}{2}x'^2} dx'}_{\sqrt{\frac{2\pi}{a}}}\; e^{-\frac{2\pi^2\nu^2}{a}}$$

$dx' = dx$

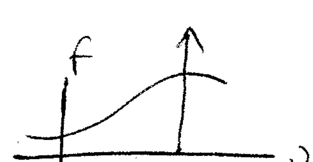
$$= \lim_{a\to 0} \sqrt{\frac{2\pi}{a}}\, e^{-\frac{2\pi^2\nu^2}{a}}$$



$$\int_{-\infty}^{\infty} D(\nu)\, d\nu = \lim_{a\to 0} \sqrt{\frac{2\pi}{a}} \sqrt{\frac{2\pi a}{(2\pi)^2}} = 1$$

so $D(\nu) = \int_{-\infty}^{\infty} e^{i2\pi x\nu} dx = \delta(\nu)$

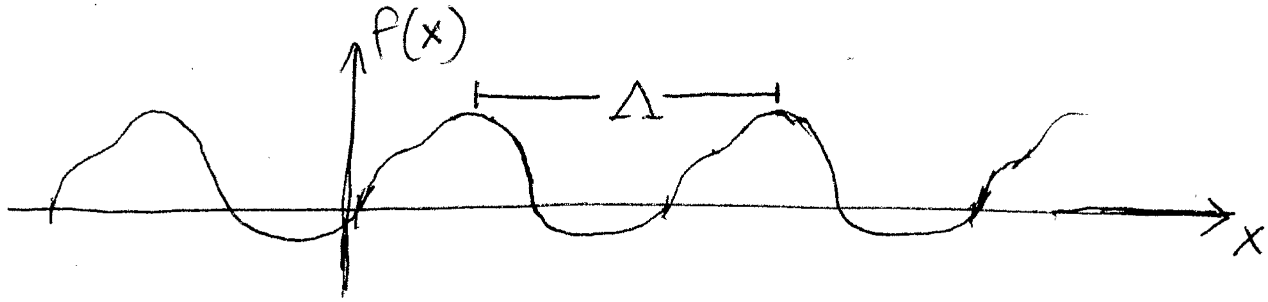
Property: $\int_{-\infty}^{\infty} f(\nu)\, \delta(\nu - \nu')\, d\nu = f(\nu')$



$[\delta(\nu)] = [\frac{1}{\nu}]$, $\quad \delta(a\nu) = \frac{\delta(\nu)}{}$

Fourier Series

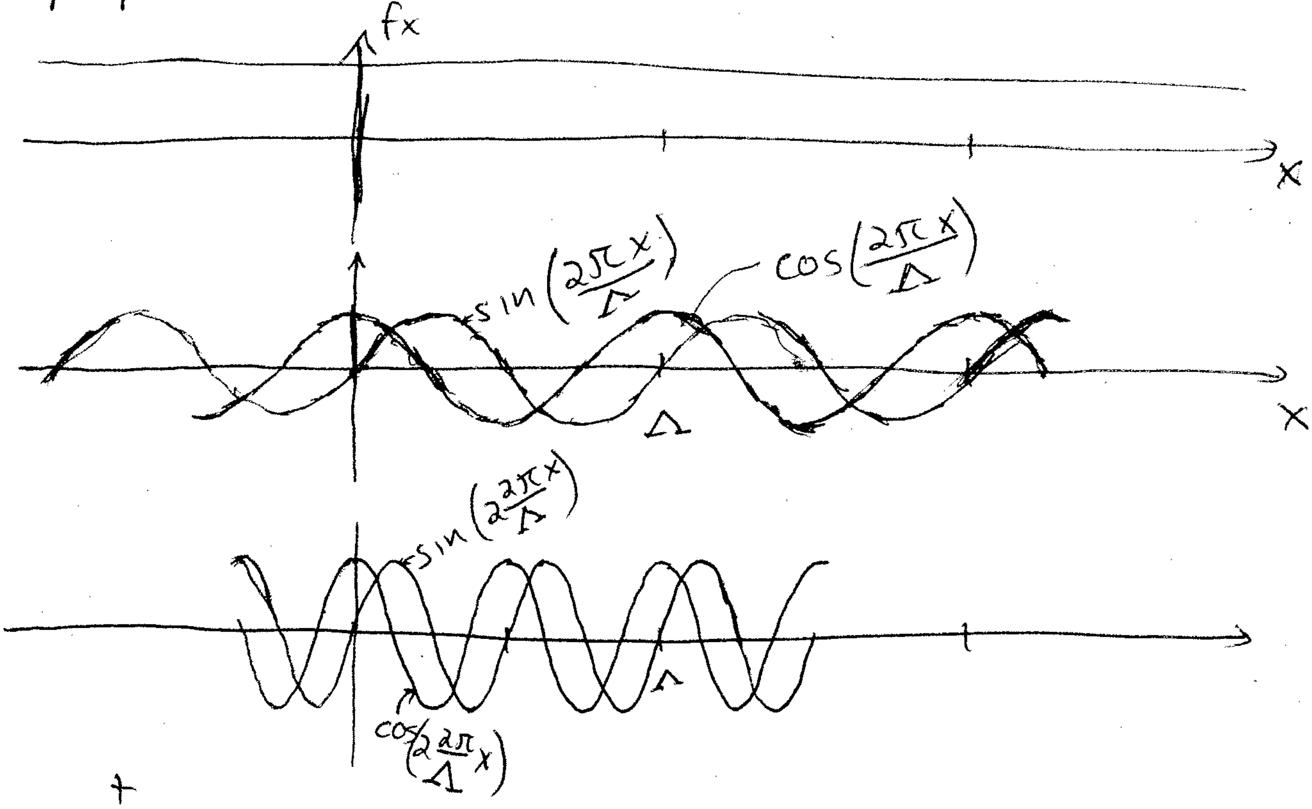
Periodic function:



$$f(x+m\Delta) = f(x) \quad , m = \text{integer}$$

Fourier theorem:

Any periodic function can be expressed as a superposition of sinusoidal functions:



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Propose

$$f(x) = a_0 + \sum_{m=1}^{\infty} a_m \cos\left(\frac{2\pi m x}{\Delta}\right) + \sum_{m=1}^{\infty} b_m \sin\left(\frac{2\pi m x}{\Delta}\right)$$

Recall:

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

so we can rewrite

$$f(x) = \underbrace{a_0}_{C_0} + \sum_{m=1}^{\infty} \underbrace{\left(\frac{a_m}{2} + \frac{b_m}{2i}\right)}_{C_m} e^{i\frac{2\pi m x}{\Lambda}} + \sum_{m=1}^{\infty} \underbrace{\left(\frac{a_m}{2} - \frac{b_m}{2i}\right)}_{C_{-m}} e^{-i\frac{2\pi m x}{\Lambda}}$$

$$\boxed{f(x) = \sum_{m=-\infty}^{\infty} C_m e^{i\frac{2\pi m x}{\Lambda}}}$$

Fourier Series or Fourier Synthesis.

Notice that, if $f(x)$ is real a_m & b_m are real,

so $\underline{C_{-m} = C_m^*}$

Finding the Coefficients

Consider $\displaystyle\int_{x_0-\frac{\Lambda}{2}}^{x_0+\Lambda/2} f(x)\,dx =$

$= \Lambda C_0$

so $C_0 = \dfrac{1}{\Lambda}\displaystyle\int_{x_0-\frac{\Lambda}{2}}^{x_0+\frac{\Lambda}{2}} f(x)\,dx$ = average of the function.

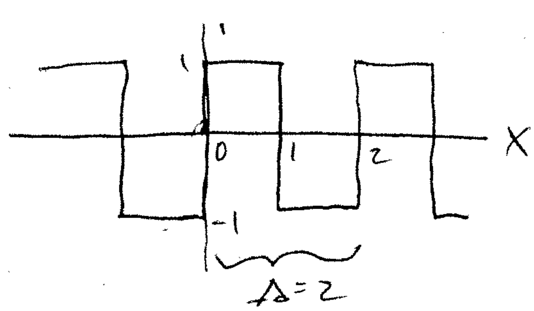
Now try

$$\frac{1}{\Lambda}\int_{x_0-\frac{\Lambda}{2}}^{x_0+\frac{\Lambda}{2}} f(x)\, e^{-i\frac{2\pi m' x}{\Lambda}}\,dx =$$

$= C_{m'}$

so $$\boxed{C_m = \frac{1}{\Lambda}\int_{x_0-\frac{\Lambda}{2}}^{x_0+\frac{\Lambda}{2}} f(x)\, e^{-i\frac{2\pi m x}{\Lambda}}\,dx}$$ Fourier analysis

Example:



$C_0 = 0$

$$C_m = \frac{1}{2}\int_{-1}^{1} f(x)\, e^{-i\frac{\cancel{2}\pi m x}{\cancel{2}}}\,dx$$

$$= \frac{1}{2}\int_{-1}^{0}(-1)\, e^{-i\pi m x}\,dx + \frac{1}{2}\int_{0}^{1}(1)\, e^{-i\pi m x}\,dx$$

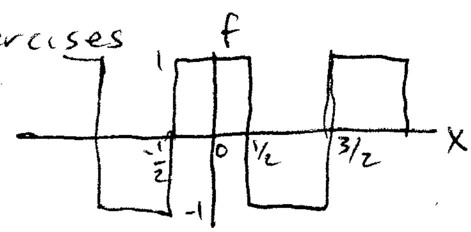
$$= \frac{1}{2}\left[-\left.\frac{e^{-i\pi m x}}{-i\pi m}\right|_{-1}^{0} + \left.\frac{e^{-i\pi m x}}{-i\pi m}\right|_{0}^{1} \right]$$

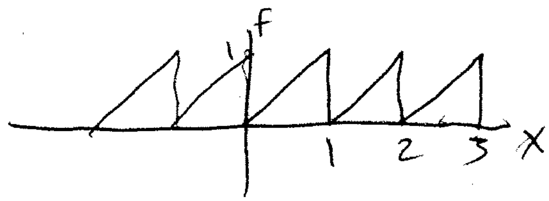
$$= \frac{i}{2\pi m}\left[e^{-i\pi m} - 1 - 1 + e^{i\pi m} \right]$$

$$= \frac{i}{2\pi m}\left[\underbrace{\cos(\pi m)}_{\substack{1 \text{ for even } m \\ -1 \text{ for odd } m}} - 1 \right] = \begin{cases} 0, & m = \text{even} \\ \frac{-i}{\pi m}, & m \text{ odd} \end{cases}$$

Plot series from $-M$ to M. Play.

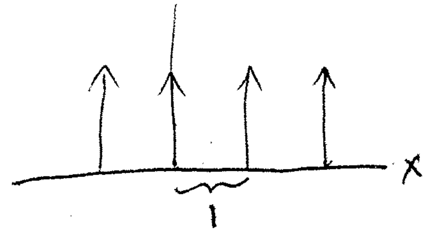
Excercises

a) 

b) 

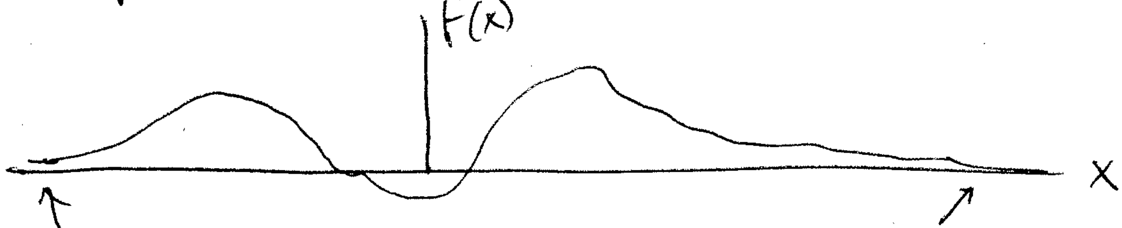
c) $\cos(ax)$, $\sin(ax)$

d) $\cos^2(ax)$

e) Dirac comb 

Fourier transforms

Non periodic function:



must go to zero at infinity, such that

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \text{finite.}$$

Fourier theorem is still valid, but there is no periodicity to constrain the allowed frequencies, so in principle we need all of them.

Propose

$$\boxed{f(x) = \int_{-\infty}^{\infty} \tilde{f}(\nu)\, e^{i 2\pi \nu x}\, d\nu}$$

Fourier Synthesis or Inverse Fourier Transformation

To find $\tilde{f}$, as with series, consider

$$\int_{-\infty}^{\infty} f(x)\, e^{-i2\pi\nu' x}\, dx = \iint_{-\infty}^{\infty} \tilde{f}(\nu)\, e^{i2\pi\nu x}\, d\nu\, e^{-i2\pi\nu' x}\, dx$$

$$= \int_{-\infty}^{\infty} \tilde{f}(\nu) \underbrace{\int_{-\infty}^{\infty} e^{i2\pi(\nu-\nu')x}\, dx}_{\delta(\nu-\nu')}\, d\nu = \tilde{f}(\nu')$$

so

$$\boxed{\tilde{f}(\nu) = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi\nu x}\, dx}$$

Fourier analysis or Fourier transformation

Other conventions:

$$\tilde{f}(p) = \sqrt{\frac{k}{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikxp}\, dx$$

$$f(x) = \sqrt{\frac{k}{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(p)\, e^{ikxp}\, dp$$

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{-i\omega t}\, d\omega$$

Properties 1 Let $\hat{\mathfrak{F}}_{x\to\nu}[f(x)] = \tilde{f}(\nu)$

- Shift $\hat{\mathfrak{F}}_{x\to\nu}[f(x-a)] = \int_{-\infty}^{\infty} f(\underbrace{x-a}_{x'})\, e^{-i2\pi x\nu}\, dx$

$$= \int_{-\infty}^{\infty} f(x')\, e^{-i2\pi(x'+a)\nu}\, dx' = e^{-i2\pi a\nu} \int_{-\infty}^{\infty} f(x')\, e^{-i2\pi x'\nu}\, dx'$$

$$= \tilde{f}(\nu)\, e^{-i2\pi a\nu}$$

- Phase $\hat{\mathfrak{F}}_{x\to\nu}[f(x)\, e^{iax}] = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi x(\nu - \frac{a}{2\pi})}\, dx$

$$= \tilde{f}\left(\nu - \frac{a}{2\pi}\right)$$

- Scale $\hat{\mathfrak{F}}_{x\to\nu}[f(ax)] = \int_{-\infty}^{\infty} f(\underbrace{ax}_{x'})\, e^{-i2\pi x\nu}\, dx$

$$= \int_{-\infty\,\mathrm{sgn}(a)}^{\infty\,\mathrm{sgn}(a)} f(x')\, e^{-i2\pi x'\frac{\nu}{a}}\, \frac{dx'}{a}$$

$$= \frac{\mathrm{sgn}(a)}{a}\, \tilde{f}\left(\frac{\nu}{a}\right) = \frac{\tilde{f}\left(\frac{\nu}{a}\right)}{|a|}$$

• Derivative $\hat{\mathcal{F}}_{x\to\nu}[f'(x)] = \int_{-\infty}^{\infty} f'(x)\, e^{-i2\pi x\nu}\, dx$

integrate by parts

$u = e^{-i2\pi x\nu}$ $\quad dv = f'dx$

$du = -i2\pi\nu e^{-i2\pi x\nu} dx, \quad v = f$

$$= \cancel{f(x)e^{-i2\pi x\nu}\Big|_{-\infty}^{\infty}}\to 0 - (-i2\pi)\nu \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi\nu x} dx$$

$$= 2\pi i\nu\, \tilde{f}(\nu))$$

By using this repeatedly, can show

$$\hat{\mathcal{F}}_{x\to\nu}\left[f^{(n)}(x)\right] = (2\pi i\nu)^n\, \tilde{f}(\nu)$$

• Powers of x: $\hat{\mathcal{F}}_{x\to\nu}[x^n f(x)] = \int_{-\infty}^{\infty} f(x)\, \underbrace{x^n e^{-i2\pi x\nu}}_{\frac{1}{(2\pi i)^n}\frac{d^n}{d\nu^n} e^{-i2\pi x\nu}} dx$

$$= \left(\frac{i}{2\pi}\right)^n \frac{d^n}{d\nu^n}\int_{-\infty}^{\infty} f(x) e^{-i2\pi x\nu} dx = \left(\frac{i}{2\pi}\right)^n \tilde{f}^{(n)}(\nu)$$

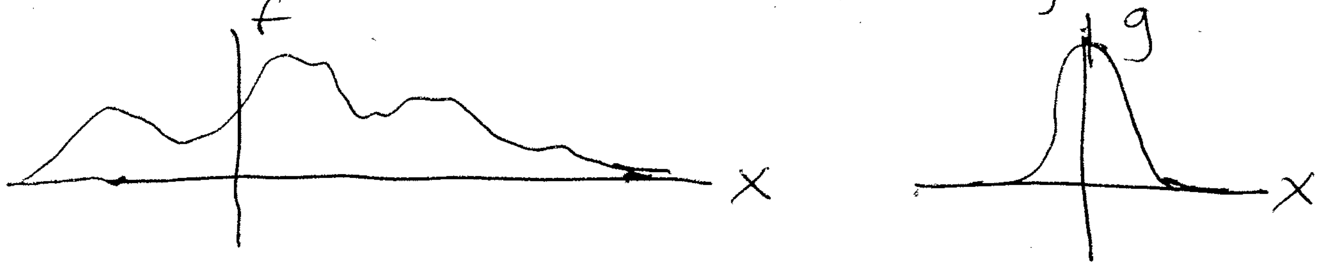
• Real functions: if $f(x) = f^*(x)$, then

$$\tilde{f}^*(\nu) = \left[\int_{-\infty}^{\infty} f(x)\, e^{-i2\pi x\nu} dx\right]^* = \int_{-\infty}^{\infty} \underbrace{f^*(x)}_{f(x)}\, e^{i2\pi x\nu} dx$$

$$= \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi x(-\nu)} dx = \tilde{f}(-\nu)$$

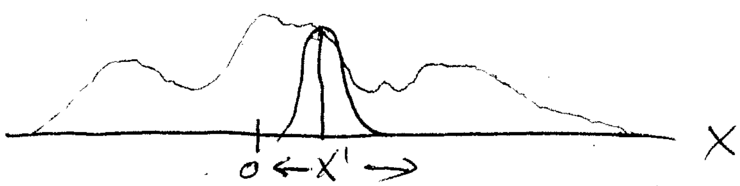
Convolution:

Consider two functions $f(x)$ & $g(x)$.



The convolution of f & g corresponds to a "blurring" of f with g:

$$f * g(x) = \int_{-\infty}^{\infty} f(x')\, g(x-x')\, dx'$$



Notice that this operation is commutative, i.e. that it can also considered as a blurring of g with f:

$$f * g = \int_{-\infty}^{\infty} f(x')\, g(\underbrace{x-x'})\, dx'$$

$$x'' = x - x', \quad dx'' = -dx'$$

$$= -\int_{\infty}^{-\infty} f(x-x'')\, g(x'')\, dx'' = \int_{-\infty}^{\infty} f(x-x'')\, g(x'')\, dx''!$$

Norm: $\|f\|$

the squared norm of f is defined as

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

For optical fields, it is associated with total power.

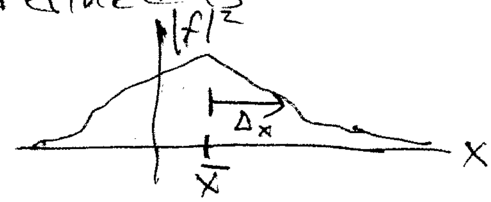
Similarly

$$\|\tilde{f}\|^2 = \int_{-\infty}^{\infty} |\tilde{f}(\nu)|^2 d\nu$$

Centroid:

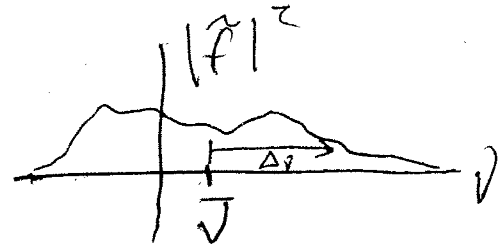
The centroid of a function is defined as

$$\bar{x} = \frac{\int_{-\infty}^{\infty} x |f(x)|^2 dx}{\|f\|^2}$$



similarly

$$\bar{\nu} = \frac{\int_{-\infty}^{\infty} \nu |\tilde{f}(\nu)|^2 d\nu}{\|\tilde{f}\|^2}$$



Standard deviation (measure of spread or width)

$$\Delta_x = \left[\frac{\int_{-\infty}^{\infty} (x-\bar{x})^2 |f(x)|^2 dx}{\|f\|^2} \right]^{1/2}$$

$$\Delta_\nu = \left[\frac{\int_{-\infty}^{\infty} (\nu-\bar{\nu})^2 |\tilde{f}(\nu)|^2 d\nu}{\|\tilde{f}\|^2} \right]^{1/2}$$

Properties 2

- Parseval's theorem

$$\|\tilde{f}\|^2 = \int_{-\infty}^{\infty} \tilde{f}^*(\nu)\, f(\nu)\, d\nu = \int_{-\infty}^{\infty} \tilde{f}^*(\nu) \int_{-\infty}^{\infty} f(x) e^{-i2\pi x\nu} dx\, d\nu$$

$$= \int_{-\infty}^{\infty} f(x) \left[\int_{-\infty}^{\infty} \tilde{f}(\nu)\, e^{i2\pi x\nu} d\nu \right] dx = \int_{-\infty}^{\infty} f(x)\, f^*(x)\, dx$$

$$= \|f\|^2$$

so the norm is the same for $\tilde{f}$ as for f.

- Product: $\hat{\mathcal{F}}_{x\to\nu}\left[f(x)g(x)\right] = \int_{-\infty}^{\infty} f(x)\, g(x)\, e^{-i2\pi x\nu} dx$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \tilde{f}(\nu')\, e^{i2\pi x\nu'} d\nu' \right] g(x)\, e^{-i2\pi x\nu} dx$$

$$= \int_{-\infty}^{\infty} \tilde{f}(\nu') \int_{-\infty}^{\infty} g(x)\, e^{-i2\pi x(\nu-\nu')} dx\, d\nu'$$

$$= \int_{-\infty}^{\infty} \tilde{f}(\nu')\, \tilde{g}(\nu-\nu')\, d\nu' = \tilde{f} * \tilde{g}(\nu)$$

- Convolution: $\hat{\mathcal{F}}_{x\to\nu}\left[f*g\right] = \iint_{-\infty}^{\infty} f(x')\, g(\underbrace{x-x'}_{x''})\, e^{-i2\pi x\nu} dx\, dx'$ $\quad dx = dx''$

$$= \iint_{-\infty}^{\infty} f(x')\, g(x'')\, e^{-i2\pi(x'+x'')\nu} dx'\, dx'' = \int_{-\infty}^{\infty} f(x')\, e^{-i2\pi x'\nu} dx' \int_{-\infty}^{\infty} g(x'')\, e^{-i2\pi x''\nu} dx''$$

$$= \tilde{f}(\nu)\, \tilde{g}(\nu)$$

- Heisenberg uncertainty relation

$$\Delta_x \Delta_p \geqslant \frac{1}{4\pi},$$

where the equality holds only for Gaussian functions

Proof of uncertainty relation

Let, for now, $\bar{x}=0$, $\bar{\nu}=0$, so

$$\Delta_x^2 = \frac{\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx}{\|f\|^2}, \quad \Delta_\nu^2 = \frac{\int_{-\infty}^{\infty} \nu^2 |\tilde{f}(\nu)|^2 d\nu}{\|f\|^2}$$

By parseval's theorem

$$\int_{-\infty}^{\infty} \nu^2 |\tilde{f}|^2 d\nu = \int_{-\infty}^{\infty} |\nu \tilde{f}|^2 d\nu = \int_{-\infty}^{\infty} |\hat{\mathcal{F}}^{-1}(\nu \tilde{f})|^2 d\nu$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} |f'(x)|^2 dx,$$

So

$$\Delta_\nu^2 = \frac{\int_{-\infty}^{\infty} |f'(x)|^2 dx}{(2\pi)^2 \|f\|^2}$$

Now, consider the integral:

$$I = \iint_{-\infty}^{\infty} |x_1 f(x_1) f'(x_2) - x_2 f(x_2) f'(x_1)|^2 dx_1 dx_2.$$

Because this is the integral of a nonnegative quantity, it must be nonnegative:

$$I \geq 0.$$

Now use the fact that $|a|^2 = a^* a$:

$$I = \iint_{-\infty}^{\infty} [x_1 f^*(x_1) f'^*(x_2) - x_2 f^*(x_2) f'^*(x_1)][x_1 f(x_1) f'(x_2) - x_2 f(x_2) f'(x_1)] dx_1 dx_2$$

$$I = \iint_{-\infty}^{\infty} \Big[x_1^2 |f(x_1)|^2 |f'(x_2)|^2 - x_1 x_2 f^*(x_1) f'(x_1) f(x_2) f'^*(x_2)$$

$$- x_1 x_2 f(x_1) f'^*(x_1) f^*(x_2) f'(x_2) + x_2^2 |f(x_2)|^2 |f'(x_1)|^2 \Big] dx_1 dx_2$$

$$= \int_{-\infty}^{\infty} x_1^2 |f(x_1)|^2 dx_1 \int_{-\infty}^{\infty} |f'(x_2)|^2 dx_2 - \int_{-\infty}^{\infty} x_1 f^*(x_1) f'(x_1) dx_1 \int_{-\infty}^{\infty} x_2 f(x_2) f'^*(x_2) dx_2$$

$$- \int_{-\infty}^{\infty} x_1 f(x_1) f'^*(x_1) dx_1 \int_{-\infty}^{\infty} x_2 f^*(x_2) f'(x_2) dx_2 + \int_{-\infty}^{\infty} |f'(x_1)|^2 dx_1 \int_{-\infty}^{\infty} x_2^2 |f(x_2)|^2 dx_2$$

but $\displaystyle\int_{-\infty}^{\infty} x_n^2 |f(x_n)|^2 dx_n = ||f||^2 \Delta_x^2$

$\displaystyle\int_{-\infty}^{\infty} |f'(x_n)|^2 dx_n = ||f||^2 \Delta_\nu^2 (2\pi)^2$

Let us define:

$$A = \int_{-\infty}^{\infty} x_n f^*(x_n) f'(x_n) dx_n$$

then

$$I = ||f||^4 \Delta_x^2 \Delta_\nu^2 - AA^* - A^*A + ||f||^4 \Delta_x^2 \Delta_\nu^2$$

$$= 2||f||^4 \Delta_x^2 \Delta_\nu^2 - 2|A|^2 \geq 0$$

so

$$\Delta_x^2 \Delta_\nu^2 \geq \frac{|A|^2}{(2\pi)^2 ||f||^4}$$

Note that

$$A = \int_{-\infty}^{\infty} x f^*(x) f'(x) dx \quad \text{integrate by parts}$$

$u = x f^*$ $\qquad dv = f' dx$

$du = (f^* + x f'^*) dx$ $\qquad v = f$

$$A = \cancel{x f^*(x) f(x) \Big|_{-\infty}^{\infty}}^{\,0} - \int_{-\infty}^{\infty} \left(f^*(x) + x f'^*(x)\right) f(x)\, dx$$

$$= -\int_{-\infty}^{\infty} |f(x)|^2 dx - \int_{-\infty}^{\infty} x f(x) f'^*(x)\, dx$$

$$= -\|f\|^2 - A^*$$

so $\underbrace{A + A^*}_{2\,\mathrm{Re}\{A\}} = -\|f\|^2$

$$\mathrm{Re}\{A\} = \frac{-\|f\|^2}{2}$$

$$\frac{|A|^2}{\|f\|^4} = \frac{(\mathrm{Re}\{A\})^2 + (\mathrm{Im}\{A\})^2}{\|f\|^4} = \frac{1}{4} + \frac{(\mathrm{Im}\{A\})^2}{\|f\|^4} \geq \frac{1}{4}$$

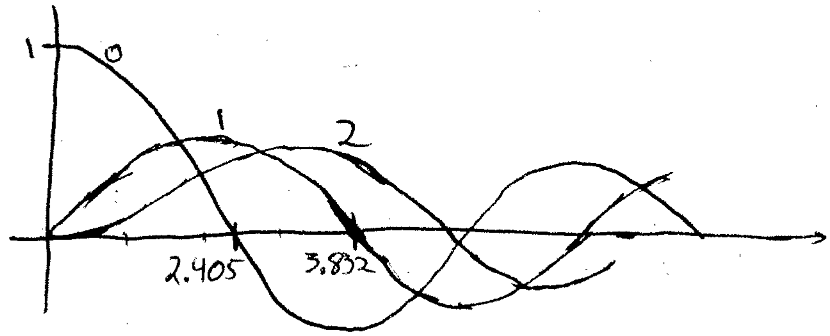
so

$$\Delta_x^2 \Delta_\nu^2 \geq \frac{|A|^2}{(2\pi)^2 \|f\|^4} \geq \frac{1}{4(2\pi)^2}$$

$\therefore$ $\Delta_x^2 \Delta_\nu^2 \geq \frac{1}{4(2\pi)^2}$ and $\boxed{\Delta_x \Delta_\nu > \frac{1}{4\pi}}$

Bessel functions of the first kind

$J_n(x)$



They are solutions of

$$x^2 J_n'' + x J_n' + (x^2 - n^2) J_n = 0$$

- For $x \ll 1$

$$J_n(x) \approx \frac{x^n}{2^n n!}$$

- For $x \gg 1$

$$J_n(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

- Parity:

$$J_{-n}(x) = (-1)^n J_n(x) \quad , \quad J_n(-x) = (-1)^n J_n(x)$$

- Integral form

$$J_n(x) = \frac{1}{2\pi i^n} \int_0^{2\pi} e^{i(x\cos\theta + n\theta)} d\theta$$

- Jacobi-Anger expansion

$$e^{ix\cos\theta} = \sum_{n=-\infty}^{\infty} J_n(x)\, i^n e^{in\theta}$$

- Closure relation

$$\int_0^\infty x\, J_n(ux)\, J_n(vx)\, dx = \frac{1}{u}\,\delta(u-v)$$

- Derivative identity

$$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x)$$

- Integral identity (from previous)

$$\int_0^x x'^n J_{n-1}(x')\, dx' = x^n J_n(x)$$

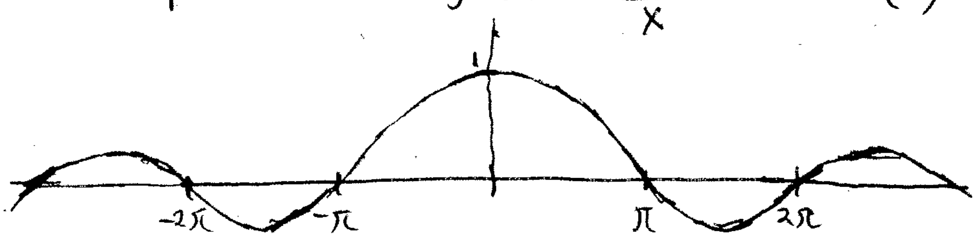
- Recursion relation

$$J_{n+1} + J_{n-1} = \frac{2n\, J_n}{x}$$

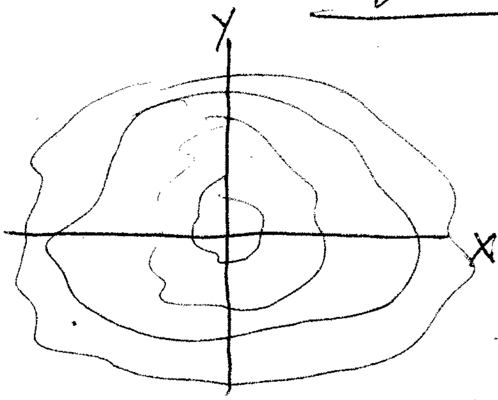
- Spherical Bessel functions

$$j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+\frac{1}{2}}(x)$$

In particular $j_0(x) = \dfrac{\sin x}{x} = \mathrm{sinc}(x)$



2D Fourier transform



$f(x,y)$

let $\underline{x} = (x,y)$

$$\tilde{f}(\underline{\nu}) = \iint_{-\infty}^{\infty} f(\underline{x})\, e^{-i2\pi \underline{x}\cdot\underline{\nu}} \underbrace{dx\,dy}_{d^2x}$$ Fourier Transform

where $\underline{\nu} = (\nu_x, \nu_y)$

$$f(\underline{x}) = \iint_{-\infty}^{\infty} \tilde{f}(\underline{\nu})\, e^{i2\pi \underline{x}\cdot\underline{\nu}} \underbrace{d\nu_x\, d\nu_y}_{d^2\nu}$$ Inverse Fourier Transform

Similar properties:

- Shift $\hat{\mathfrak{f}}_{\underline{x}\to\underline{\nu}}\left[f(\underline{x}-\underline{a})\right] = \tilde{f}(\underline{\nu})\, e^{-i2\pi \underline{a}\cdot\underline{\nu}}$
- Phase: $\hat{\mathfrak{f}}_{\underline{x}\to\underline{\nu}}\left[f(\underline{x})\, e^{i\underline{a}\cdot\underline{x}}\right] = \tilde{f}\left(\underline{\nu} - \frac{\underline{a}}{2\pi}\right)$
- Scale: $\hat{\mathfrak{f}}_{\underline{x}\to\underline{\nu}}\left[f(a\underline{x})\right] = \dfrac{\tilde{f}\left(\frac{1}{a}\underline{\nu}\right)}{a^2}$
- Gradient: $\hat{\mathfrak{f}}_{\underline{x}\to\underline{\nu}}\left[\nabla f(\underline{x})\right] = 2\pi i\, \underline{\nu}\, \tilde{f}(\underline{\nu})$
- Powers of x, y: $\hat{\mathfrak{f}}_{\underline{x}\to\underline{\nu}}\left[x^m y^n f(\underline{x})\right] = \left(\frac{i}{2\pi}\right)^{m+n} \frac{\partial^m}{\partial \nu_x^m} \frac{\partial^n}{\partial \nu_y^n} \tilde{f}(\underline{\nu})$
- Real Functions: if $f(\underline{x}) = f^*(\underline{x})$ then $\tilde{f}^*(\underline{\nu}) = \tilde{f}(-\underline{\nu})$
- Parseval: $\|f\|^2 = \iint_{-\infty}^{\infty} |f(\underline{x})|^2 d^2x = \iint_{-\infty}^{\infty} |\tilde{f}(\underline{\nu})|^2 d^2\nu = \|\tilde{f}\|^2$

- Product: $\hat{\mathcal{F}}_{\underline{x}\to\underline{\nu}}\left[f(\underline{x})\,g(\underline{x})\right] = \iint_{-\infty}^{\infty} \tilde{f}(\underline{\nu}')\,\tilde{g}(\underline{\nu}-\underline{\nu}')\,d^2\nu' = \tilde{f}*\tilde{g}(\underline{\nu})$
- Convolution $\hat{\mathcal{F}}_{\underline{x}\to\underline{\nu}}\left[f*g(\underline{x})\right] = \tilde{f}(\underline{\nu})\,\tilde{g}(\underline{\nu})$
- Uncertainty $\quad \Delta_x \Delta_\nu \geq \dfrac{1}{2\pi}$

where
$$\Delta_x^2 = \frac{\iint_{-\infty}^{\infty} (x^2+y^2)\,|f(\underline{x})|^2\,d^2x}{\|f\|^2}$$
$$\Delta_\nu^2 = \frac{\iint_{-\infty}^{\infty} (\nu_x^2+\nu_y^2)\,|\tilde{f}(\underline{\nu})|^2\,d^2\nu}{\|f\|^2}$$

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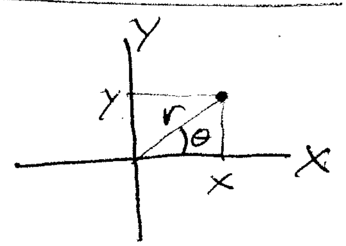
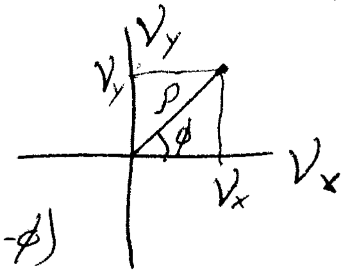
functions with rotational symmetry

$$f(\underline{x}) = f(|\underline{x}|) = f(r)$$



$$\tilde{f}(\underline{\nu}) = \iint_{-\infty}^{\infty} f(|\underline{x}|)\, e^{-i2\pi \underline{x}\cdot\underline{\nu}}\, d^2x$$



change to polars

$d^2x = r\,dr\,d\theta, \quad \underline{x}\cdot\underline{\nu} = r\rho\cos(\theta-\phi)$

$$\tilde{f}(\underline{\nu}) = \int_0^{2\pi}\int_0^{\infty} f(r)\, e^{-i2\pi r\rho\cos(\underbrace{\theta-\phi}_{\theta'})}\, r\,dr\,d\theta$$
$$= \int_0^{\infty} f(r) \int_{-\phi}^{-\phi+2\pi} e^{-i2\pi r\rho\cos\theta'}\,d\theta'\; r\,dr$$
$$= \int_0^{\infty} f(r) \int_0^{2\pi} e^{-i2\pi r\rho\cos\theta'}\,d\theta'\; r\,dr$$

$$\tilde{F}(\underline{\nu}) = \int_0^\infty f(r) \underbrace{\int_0^{2\pi} e^{-i 2\pi r \rho \cos\theta'} d\theta'}_{2\pi J_0(2\pi r \rho)} r\,dr$$

indep. of ϕ

$$\boxed{\tilde{F}(\rho) = 2\pi \int_0^\infty f(r)\, J_0(2\pi r \rho)\, r\,dr}$$

Fourier-Bessel transform or Hankel transform

Can show similarly

$$\boxed{f(r) = 2\pi \int_0^\infty \tilde{F}(\rho)\, J_0(2\pi r \rho)\, \rho\, d\rho}$$

Inverse FB transform or Inverse Hankel transform

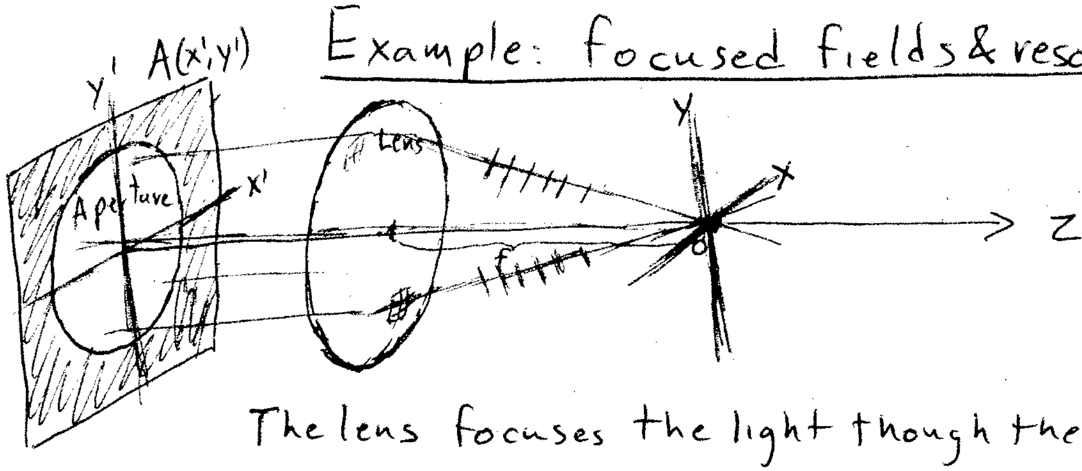
Check:

$$f(r) = 2\pi \int_0^\infty 2\pi \int_0^\infty f(r')\, J_0(2\pi r' \rho)\, r' dr'\, J_0(2\pi r \rho)\, \rho\, d\rho$$

$$= \int_0^\infty f(r') \underbrace{\left[(2\pi)^2 \int_0^\infty J_0(2\pi r' \rho)\, J_0(2\pi r \rho)\, \rho\, d\rho\, r'\right]}_{\delta(r-r')} dr'$$

$$= f(r) \checkmark$$

Example: focused fields & resolution



The lens focuses the light though the aperture. Near the focus, each "ray" behaves like a plane wave. The direction of the plane wave, $\vec{p}=(p_x,p_y,p_z)$, is approximately (for a perfect lens) given by

$$(p_x,p_y) \approx \frac{(x',y')}{f}\ , \quad p_z=\sqrt{1-p_x^2-p_y^2} \approx 1-\frac{p_x^2+p_y^2}{2}$$

Let $\underline{p}=(p_x,p_y)$, $\underline{x}=(x,y)$, $\bar{A}(\underline{p}) = A(x',y') = A(f\underline{p})$.
Then, the focal field is given by the superposition of plane waves:

$$U(\vec{r}) = \frac{U_0}{2\pi}\iint \bar{A}(\underline{p})\,\underbrace{e^{ik\vec{p}\cdot\vec{r}}}_{\text{plane wave}}\,\underbrace{dp_x\,dp_y}_{d^2p} = \frac{U_0}{2\pi}\iint \bar{A}(\underline{p})\,e^{ik\underline{p}\cdot\underline{x}}\,e^{ikp_z z}\,d^2p$$

$$\approx \frac{U_0 e^{ikz}}{2\pi}\iint \bar{A}(\underline{p})\,e^{ik\underline{p}\cdot\underline{x}}\,e^{-ik\frac{|\underline{p}|^2}{2}z}\,d^2p$$

Notice that, at the focal plane $(z=0)$:

$$U(\underline{x},0) \approx \frac{U_0}{2\pi}\iint \bar{A}(\underline{p})\,e^{ik\underline{p}\cdot\underline{x}}\,d^2p \quad \propto \text{ inverse FT of } \bar{A}.$$

On the other hand, along the axis $(\underline{x}=\underline{0})$

$$U(0,0,z) \approx \frac{U_0 e^{ikz}}{2\pi}\iint \bar{A}(\underline{p})\,e^{-ik\frac{|\underline{p}|^2}{2}z}\,d^2p$$

Consider rotationally symmetric apertures:

$\bar{A}(\rho) = A(\rho)$, where $\rho = |\rho|$, then. Numerical Aperture

$$U(r\cos\theta, r\sin\theta, z) \approx \frac{U_0}{2\pi} e^{ikz} \int_0^{NA}\int_0^{2\pi} \bar{A}(\rho)\, e^{ikr\rho\cos(\theta-\phi)} e^{-ik\frac{\rho^2 z}{2}} \rho\, d\phi\, d\rho$$

$$= U_0 e^{ikz} \int_0^{NA} \bar{A}(\rho)\, J_0(kr\rho)\, e^{-ik\frac{\rho^2 z}{2}} \rho\, d\rho$$

independent of θ.

In general, no analytic solution for all r, z

Again, consider focal plane:

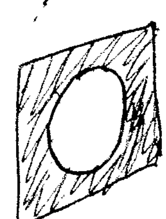
$$\underline{U(r,0) = U_0 \int_0^{NA} \bar{A}(\rho)\, J_0(kr\rho)\, \rho\, d\rho}$$ Inverse Hankel transform

And, along axis

$$U(0,z) = U_0 e^{ikz} \int_0^{NA} \bar{A}(\rho)\, e^{-ik\frac{\rho^2}{2} z} \rho\, d\rho$$

$u = \frac{\rho^2}{2}$, $du = \rho\, d\rho$

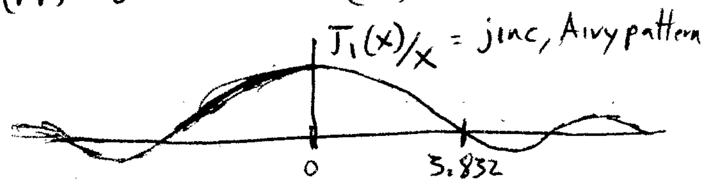
$$\underline{= U_0 e^{ikz} \int_0^{\frac{NA^2}{2}} \bar{A}(\sqrt{2u})\, e^{-ikuz}\, du}$$ Fourier transform.



Example 1: $\bar{A}(\rho) = \begin{cases} 1, & \rho \le NA \\ 0, & \rho > NA \end{cases}$

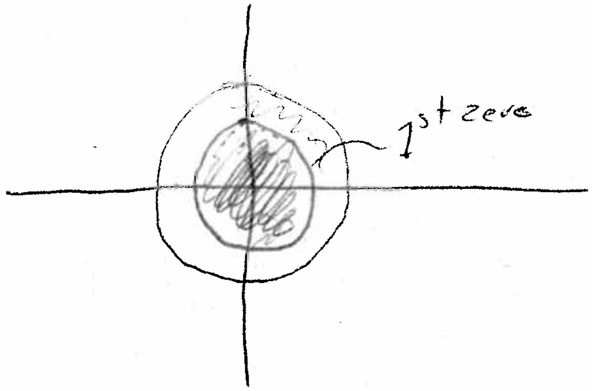
$$U(r,0) = U_0 \int_0^{NA} J_0(\underbrace{kr\rho}_{\tau})\, \rho\, d\rho = \frac{U_0}{(kr)^2} \int_0^{krNA} J_0(\tau)\, \tau\, d\tau = \frac{U_0}{(kr)^2} J_1(\tau)\, \tau \Big|_0^{krNA}$$

$$\underline{= U_0\, NA\, \frac{J_1(krNA)}{kr}}$$

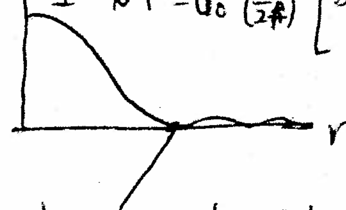


Using $NA \approx \frac{d}{2f}$, where d = aperture diameter, and $k = \frac{2\pi}{\lambda}$

$$U(r,0) \approx U_0 \left(\frac{d}{2f}\right)^2 \frac{J_1\left(\frac{\pi d}{\lambda f} r\right)}{\left(\frac{\pi d}{\lambda f} r\right)}$$



$$I = |U| = U_0^2\left(\frac{d}{2f}\right)\left[\frac{J_1\left(\frac{\pi d}{\lambda f} r\right)}{\left(\frac{\pi d}{\lambda f} r\right)}\right]$$



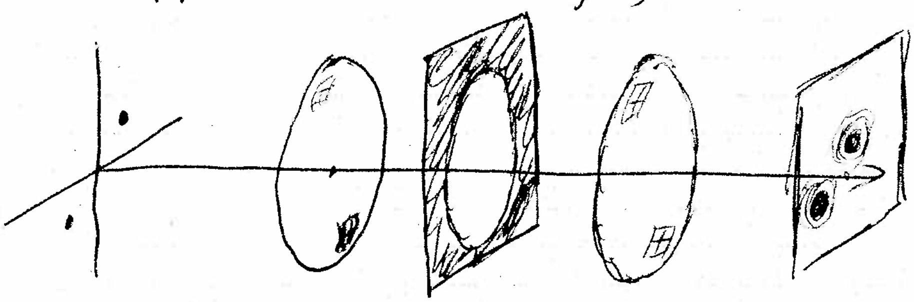
first zero at $\frac{\pi d}{\lambda f} r = 3.832$

$$r = \frac{f\lambda}{d}\,\frac{3.832}{\pi} = 1.22\,\frac{f}{d}\,\lambda$$

$$= 0.609\,\frac{\lambda}{NA}$$

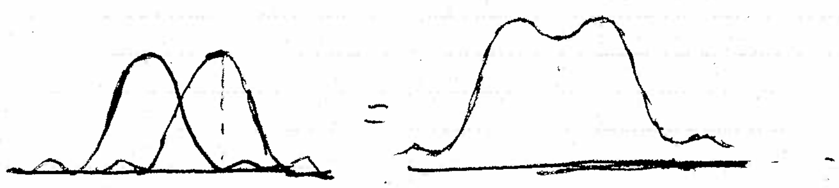
Rayleigh resolution criterion:

Suppose we are imaging two points (using incoherent illumination)



The image of each is an Airy pattern.

Rayleigh's criterion is that we can differenciate the two if the separation is at least the distance from the maximum to the first zero:



$$\Delta r \geq 0.609\,\frac{\lambda}{NA}$$

Longitudinal field:

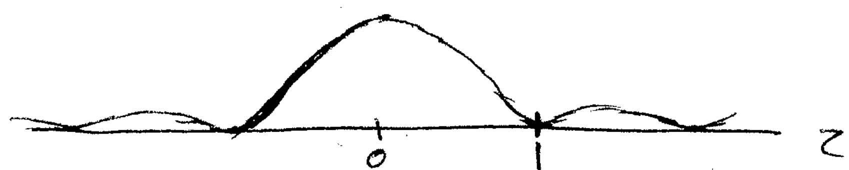
$$U(0,z) = U_0 e^{ikz} \int_0^{\frac{NA^2}{2}} e^{-ikuz}\, du = U_0 e^{ikz} \left. \frac{e^{-ikuz}}{-ikz} \right|_0^{\frac{NA^2}{2}}$$

$$= U_0 e^{ikz} \left[\frac{e^{-ikz\frac{NA^2}{2}} - 1}{-ikz} \right] = \frac{U_0 e^{ikz} e^{-ikz\frac{NA^2}{4}} NA^2}{2}$$

$$\cdot \left[\frac{e^{ikz\frac{NA^2}{4}} - e^{-ikz\frac{NA^2}{4}}}{2i\left(kz\frac{NA^2}{4}\right)} \right] = U_0\, e^{ikz\left(1-\frac{NA^2}{4}\right)} \frac{NA^2}{2} \, \frac{\sin\left(kz\frac{NA^2}{4}\right)}{kz\frac{NA^2}{4}}$$

$$= U_0 \frac{NA^2}{2} e^{ikz\left(1-\frac{NA^2}{4}\right)} \operatorname{sinc}\left(kz\frac{NA^2}{4}\right)$$

$$I(0,z) = U_0^2 \, \frac{NA^4}{4} \operatorname{sinc}^2\left(kz\frac{NA^2}{4}\right)$$



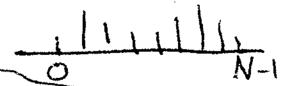
$kz\frac{NA^2}{4} = \pi \Rightarrow z = \frac{2\lambda}{NA^2}$

Other examples:

- $\bar{A} = \delta(p - NA)$, • $\bar{A} = \begin{cases} 1 - \frac{p^2}{NA^2}, & p \le NA, \\ 0, & p > NA. \end{cases}$

Discrete Fourier transform (DFT)

Instead of $f(x)$ we have f_n, $n=0,\ldots,N-1$

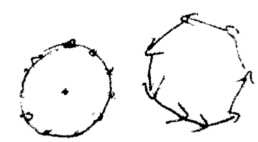


$$\boxed{F_m = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f_n\, e^{-i\frac{2\pi mn}{N}}}$$ Discrete Fourier transform

Inverse: try:

$$f_{n'} = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} F_m\, e^{i\frac{2\pi mn'}{N}}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} f_n \underbrace{\sum_{m=0}^{N-1} e^{i\frac{2\pi(n'-n)m}{N}}}_{N\,\delta_{n'-n}}$$



So:

$$\boxed{f_n = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} F_m\, e^{i\frac{2\pi mn}{N}}}$$ Inverse Discrete Fourier transform

Approximating FT with DFT

(Notice that the sums can be shifted,) if we define

$$f_{n-N} = f_n$$

$$F_m = \frac{1}{\sqrt{N}} \sum_{n=-\lfloor\frac{N-1}{2}\rfloor}^{\lfloor\frac{N}{2}\rfloor} f_n\, e^{-i\frac{2\pi mn}{N}}$$

Let f_n be a sampling of $f(x)$:

$$f_n = f(n\Delta x)$$

$$F_m = \frac{1}{\sqrt{N}} \sum_{n=-\lfloor\frac{N-1}{2}\rfloor}^{\lfloor\frac{N}{2}\rfloor} f(n\Delta x)\, e^{-i\frac{2\pi mn}{N}}$$

For very large N, and small Δx, can approximate the sum as an integral

$$F_m \approx \frac{1}{\sqrt{N}} \int_{-X_1}^{X_2} f(x)\, e^{\frac{-i2\pi m x}{N\Delta x}} \frac{dx}{\Delta x}$$

where $n\Delta x \rightarrow x$

$$X_1 = \left\lfloor \frac{N-1}{2} \right\rfloor \Delta x, \quad X_2 = \left\lfloor \frac{N}{2} \right\rfloor \Delta x$$

Assume $\underset{\text{big}}{N} \underset{\text{small}}{\Delta x} = \text{big} \gg$ width of $f(x)$.

note $X_1 \approx X_2 \approx \frac{N\Delta x}{2} = \text{big}$.

Then

$$F_m \approx \frac{1}{\sqrt{N}\,\Delta x} \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi x \left(\frac{m}{N\Delta x}\right)} dx$$

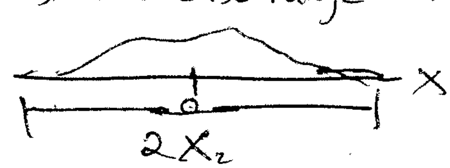
$$= \frac{\tilde{f}\left(\frac{m}{N\Delta x}\right)}{\sqrt{N}\,\Delta x}$$

So the sampling distance in ν is $\frac{1}{N\Delta x} \approx \frac{1}{2X_2}$

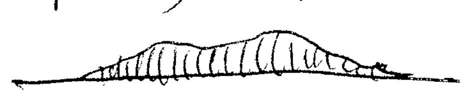
where $2X_2$ is the width over which we're sampling $f(x)$.

Therefore:

- To increase resolution in $\tilde{f}(\nu)$ $\longrightarrow$ must increase range in $f(x)$

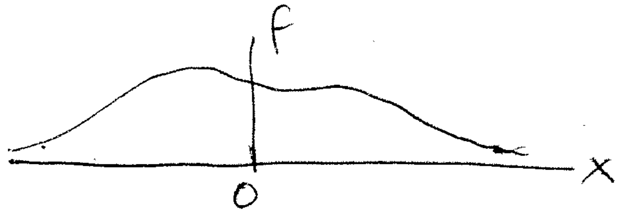


- To increase range in $\tilde{f}(\nu)$ and avoid aliasing $\longrightarrow$ must decrease sampling spacing in $f(x)$

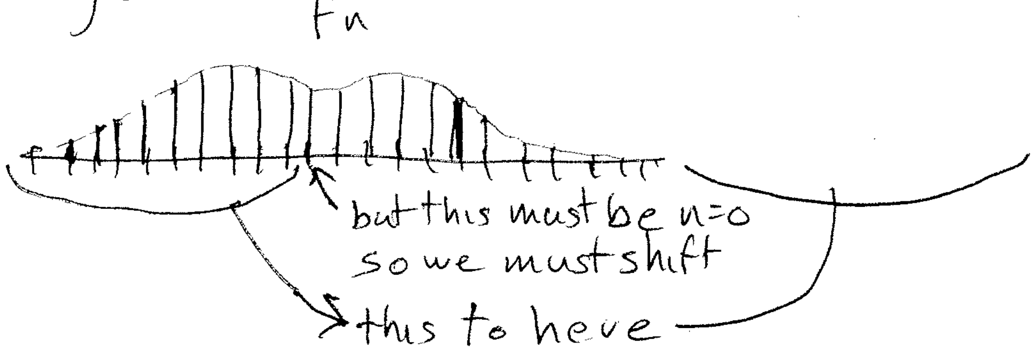


Shifting the functions.

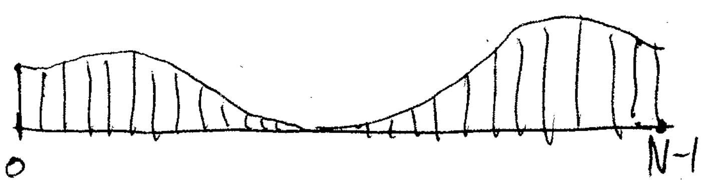
Notice that, if we sample:



we get

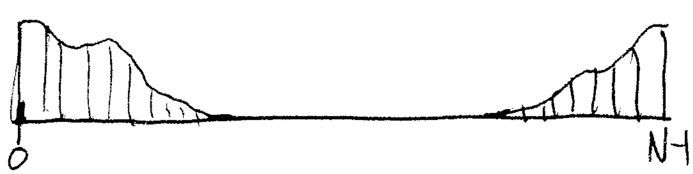


so we get



→ this is f_n

Similarly, once we get F_m, it will look like



To reconstruct $\tilde{f}(\nu)$ we must cut the second half and place it before the first. We also need to multiply by $\sqrt{N}\Delta x$.

Fast Fourier transform (FFT)

Notice that the, for each m, the DFT involves the sum of N terms. Since m runs from 0 to $N-1$, then N^2 must be performed. The time of computation can therefore be expected to be proportional to N^2.

The FFT is an algorithm for performing the DFT whose time of computation is proportional to $N \log N$. While it can work for any N, its simplest form can be understood if $N = 2^M$ (so that $M = \log_2 N$):

$$F_m = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f_n e^{\frac{-i2\pi nm}{N}} = \frac{1}{\underset{\text{write as } 2\frac{N}{2}}{\sqrt{N}}} \left[\underbrace{\sum_{n'=0}^{\frac{N}{2}-1} f_{2n'} e^{\frac{-i2\pi(2n')m}{N}}}_{\text{terms with even } n} + \underbrace{\sum_{n'=0}^{\frac{N}{2}-1} f_{(2n'+1)} e^{\frac{-i2\pi(2n'+1)m}{N}}}_{\text{terms with odd } n} \right]$$

$$= \frac{1}{\sqrt{2}} \left[\underbrace{\frac{1}{\sqrt{\frac{N}{2}}} \sum_{n'=0}^{\frac{N}{2}-1} f_{2n'} e^{\frac{-i2\pi n'm}{(N/2)}}} + e^{-i\frac{2\pi m}{N}} \underbrace{\frac{1}{\sqrt{\frac{N}{2}}} \sum_{n'=0}^{\frac{N}{2}-1} f_{(2n'+1)} e^{\frac{-i2\pi n'm}{(N/2)}}} \right]$$

Each of these two sums is itself a DFT of size $\frac{N}{2}$.

They can be joined:

$$F_m = \frac{1}{\sqrt{N}} \sum_{n'=0}^{\frac{N}{2}-1} \left(f_{2n'} + e^{-\frac{i2\pi m}{N}} f_{(2n'+1)} \right) e^{\frac{-i2\pi n'm}{(N/2)}}$$

The same separation can be done M times.

2D DF

$$\boxed{F_{m_1,m_2} = \frac{1}{N}\sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1} f_{n_1,n_2}\, e^{-i2\pi\frac{(m_1n_1+m_2n_2)}{N}}}$$

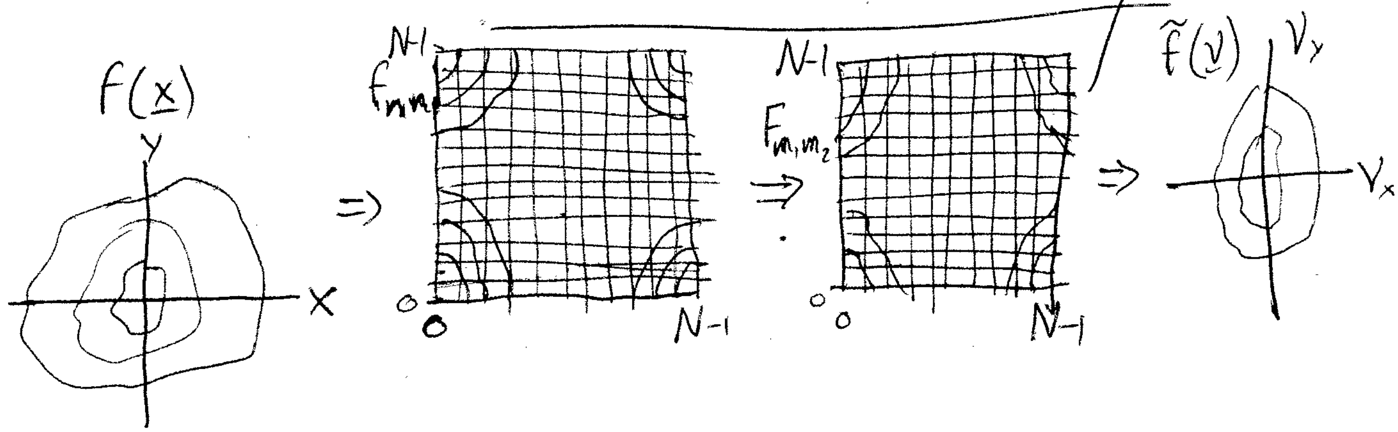
$$\boxed{f_{n_1,n_2} = \frac{1}{N}\sum_{m_1=0}^{N-1}\sum_{m_2=0}^{N-1} F_{m_1,m_2}\, e^{i2\pi\frac{(m_1n_1+m_2n_2)}{N}}}$$

Using 2D DFT to approximate 2D FT:

if $f_{n_1,n_2} = f(n_1\Delta x, n_2\Delta x)$,

and $N\Delta x$ is bigger than width of f, then:

$$F_{m_1,m_2} \approx \frac{1}{N\Delta x^2}\,\tilde{f}\left(\frac{m_1}{N\Delta x}, \frac{m_2}{N\Delta x}\right)$$



Fast Fourier transform: time $\propto N^2 \log N$

Element of the theory of Random Processes

Let $u(t)$ be a real variable that depends on time, and presents random fluctuations. Suppose that we perform N experiments to measure $u(t)$. We call these ${}^n u(t)$. Together, these measurements are called an "ensemble of realizations" of $u(t)$.

In optics, $u(t)$ can be the electric field (or one of its components) at a given point.

The time average of a realization is defined as:

$$\langle {}^n u(t) \rangle_t = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} {}^n u(t)\, dt.$$

We can also define the time average of a function of u:

$$\langle F({}^n u(t)) \rangle_t = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F({}^n(u(t))\, dt.$$

The ensemble average of a realization is defined as:

$$\langle u(t) \rangle_e = \frac{1}{N} \sum_n {}^n u(t).$$

This average is meaningful if N is large. Formally, we can let $N \to \infty$. The result can also be written in terms of a probability density $p_1(u;t)$, so that the probability that, at time t, u is in the interval $[u_0, u_0 + du]$ is $p_1(u_0;t)\,du$. The ensemble average is then given by

$$\langle u(t) \rangle_e = \int u\, p_1(u;t)\, du,$$

and the ensemble average of $F(u(t))$ is

$$\langle F(u(t)) \rangle_e = \int F(u)\, p_1(u;t)\, du.$$

One can also define ensemble averages at different times, eg.

$$\langle u(t_1)\, u(t_2)\rangle_e = \frac{1}{N} \sum_n {}^n u(t_1)\, {}^n u(t_2).$$

In terms of probability densities, one needs a joint probability $p_2(u_1, u_2; t_1, t_2)$, so

$$\langle u(t_1)\, u(t_2)\rangle_e = \iint p_2(u_1, u_2; t_1, t_2)\, u_1 u_2\, du_1\, du_2$$

This is called the "autocorrelation" of u.
Similarly one can calculate:

$$\langle F_1(u(t_1))\, F_2(u(t_2))\rangle_e = \iint F_1(u_1)\, F_2(u_2)\, p_2(u_1, u_2; t_1, t_2)\, du_1\, du_2.$$

Analogous definitions are possible for more times.

<u>Stationarity</u>: a process is stationary if:

$p_1(u;t)$ is independent of t,

$p_2(u_1, u_2; t_1, t_2)$ depends only on $t_2 - t_1$,

$p_3(u_1, u_2, u_3; t_1, t_2, t_3)$ depends only on $t_2 - t_1$, $t_3 - t_1$,

etc.

This implies that:

$\langle F(u(t))\rangle_e$ is independent of time,

$\langle F_1(u(t_1))\, F_2(u(t_2))\rangle_e$ depends only on the time difference $t_2 - t_1$,

etc.

An example is light from the sun or a light bulb.

<u>Ergodicity</u>: a process is ergodic if, for any n and F,

$$\langle F({}^n u(t))\rangle_t = \langle F(u(t))\rangle_e,$$

$$\langle F_1({}^n u(t))\, F_2({}^n u(t+\tau))\rangle_t = \langle F_1(u(t))\, F_2(u(t+\tau))\rangle_e,$$

etc. All these are independent of t.

Complex signal representation

Consider a monochromatic signal (which is not stochastic)

$$u(t) = u_0 \cos(\omega t + \phi_0)$$

It is mathematically easier to work instead with a complex signal

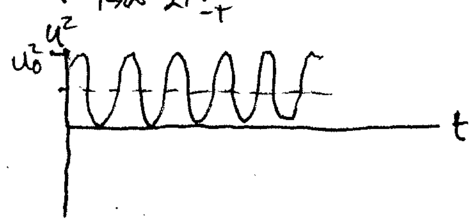
$$U(t) = u_0\, e^{i(\omega t + \phi_0)}$$

so that $\mathrm{Re}\{U(t)\} = u(t)$. Notice that both u and U have zero mean:

$$\langle u(t)\rangle_t = \langle U(t)\rangle_t = 0$$

The second moment of $u(t)$ is

$$\langle u^2(t)\rangle_t = \lim_{T\to\infty} u_0^2 \frac{1}{2T}\int_{-T}^{T} \cos^2(\omega t + \phi_0)\, dt = \frac{u_0^2}{2}$$



However, the second moment of $U(t)$ vanishes:

$$\langle U^2(t)\rangle_t = \lim_{T\to\infty} u_0^2 \frac{1}{2T}\int_{-T}^{T} e^{2i\omega t}\, dt\, e^{2i\phi_0} = 0$$

To get the correct result, that is, one that agrees with the one of the real function, we must conjugate one of the Us and divide by two:

$$\frac{1}{2}\langle U^*(t)\, U(t)\rangle_t = \lim_{T\to\infty} \frac{u_0^2}{2}\frac{1}{2T}\int_{-T}^{T} 1\, dt = \frac{u_0^2}{2}$$

Similarly, for the autocorrelation

$$\langle u(t)\, u(t+\tau)\rangle_t = \lim_{T\to\infty} u_0^2 \frac{1}{2T}\int_{-T}^{T} \cos(\omega t+\phi_0)\cos(\omega t+\phi_0+\omega\tau)\,dt$$

$$= u_0^2 \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\Big[\cos\big(\underbrace{\omega t+\phi_0+\tfrac{\omega\tau}{2}}_{\delta}-\tfrac{\omega\tau}{2}\big)\Big]\Big[\cos\big(\omega t+\phi_0+\tfrac{\omega\tau}{2}+\tfrac{\omega\tau}{2}\big)\Big]dt$$

$$= u_0^2 \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\Big[\cos(\delta)\cos\left(\tfrac{\omega\tau}{2}\right)+\sin(\delta)\sin\left(\tfrac{\omega\tau}{2}\right)\Big]\Big[\cos(\delta)\cos\left(\tfrac{\omega\tau}{2}\right)-\sin(\delta)\sin\left(\tfrac{\omega\tau}{2}\right)\Big]dt$$

$$= u_0^2 \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\Big[\cos^2(\delta)\cos^2\left(\tfrac{\omega\tau}{2}\right)-\sin^2(\delta)\sin^2\left(\tfrac{\omega\tau}{2}\right)\Big]dt$$

$$= u_0^2\Big[\cos^2\left(\tfrac{\omega\tau}{2}\right)\underbrace{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\cos^2\left(\omega t+\phi_0+\tfrac{\omega\tau}{2}\right)dt}_{1/2}$$

$$-\sin^2\left(\tfrac{\omega\tau}{2}\right)\underbrace{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sin^2\left(\omega t+\phi_0+\tfrac{\omega\tau}{2}\right)dt}_{1/2}\Big]$$

$$= \frac{u_0^2}{2}\Big[\cos^2\left(\tfrac{\omega\tau}{2}\right)-\sin^2\left(\tfrac{\omega\tau}{2}\right)\Big] = \frac{u_0^2}{2}\cos(\omega\tau)$$

If we use the complex representation, conjugating the first, we get

$$\frac{\langle U^*(t)\, U(t+\tau)\rangle_t}{2} = \lim_{T\to\infty}\frac{u_0^2}{2}\frac{1}{2T}\int_{-T}^{T} e^{-i(\cancel{\omega t}+\cancel{\phi_0})}\, e^{i(\cancel{\omega t}+\omega\tau+\cancel{\phi_0})}\,dt$$

$$= \frac{u_0^2}{2} e^{i\omega\tau}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}dt = \frac{u_0^2}{2}e^{i\omega\tau}$$

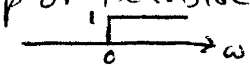
so

$$\mathrm{Re}\left\{\frac{\langle U^*(t)\, U(t+\tau)\rangle_t}{2}\right\} = \langle u(t)\, u(t+\tau)\rangle_t$$

For non-monochromatic real signals $u(t)$, the complex representation is defined as

$$U(t) = \hat{\mathcal{F}}^{-1}_{\omega \to t}\left[2\,\Theta(\omega)\,\hat{\mathcal{F}}_{t\to\omega}\, u(t)\right]$$

step or Heaviside function



$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty \tilde{u}(\omega)\, e^{-i\omega t}\, d\omega.$$

That is, we only use positive frequencies. Notice that, because $u(t)$ is real, $\tilde{u}(-\omega) = \tilde{u}^*(\omega)$, so we do not lose any information. Notice that

$$\mathrm{Re}\{U(t)\} = \mathrm{Re}\left\{\frac{2}{\sqrt{2\pi}} \int_0^\infty \hat{u}(\omega)\, e^{-i\omega t}\, d\omega\right\}$$

$$= \mathrm{Re}\left\{\frac{1}{\sqrt{2\pi}} \int_0^\infty \underbrace{\tilde{u}(\omega)}_{\tilde{u}^*(-\omega)} e^{-i\omega t}\, d\omega\right\} + \mathrm{Re}\left\{\frac{1}{\sqrt{2\pi}} \int_0^\infty \hat{u}(\omega)\, e^{-i\omega t}\, d\omega\right\}$$

$$= \mathrm{Re}\left\{\left[\frac{1}{\sqrt{2\pi}} \int_0^\infty \underbrace{\tilde{u}(-\omega)}_{\omega' = -\omega} e^{i\omega t}\, d\omega\right]^*\right\} + \mathrm{Re}\left\{\frac{1}{\sqrt{2\pi}} \int_0^\infty \tilde{u}(\omega)\, e^{-i\omega t}\, d\omega\right\}$$

$$= \mathrm{Re}\left\{\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \hat{u}(\omega')\, e^{-i\omega' t}\, d\omega'\right]^*\right\} + \mathrm{Re}\left\{\frac{1}{\sqrt{2\pi}} \int_0^\infty \hat{u}(\omega)\, e^{-i\omega t}\, d\omega\right\}$$

$\omega = \omega'$. Also notice that $*$ can be removed due to Real part.

$$= \mathrm{Re}\left\{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \hat{u}(\omega)\, e^{-i\omega t}\, d\omega\right\} + \mathrm{Re}\left\{\frac{1}{\sqrt{2\pi}} \int_0^\infty \hat{u}(\omega)\, e^{-i\omega t}\, d\omega\right\}$$

$$= \mathrm{Re}\left\{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \hat{u}(\omega)\, e^{-i\omega t}\, d\omega\right\} = \underline{\mathrm{Re}\{U(t)\} = u(t)}$$

It can be shown that, if $\langle u(t)\rangle_t = 0$, then $\langle U(t)\rangle_t = 0$ and

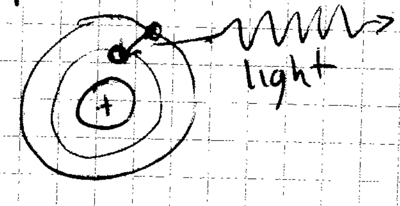
$$\langle u(t)\, u(t+\tau)\rangle_t = \mathrm{Re}\left\{\frac{\langle U^*(t)\, U(t+\tau)\rangle_t}{2}\right\}.$$

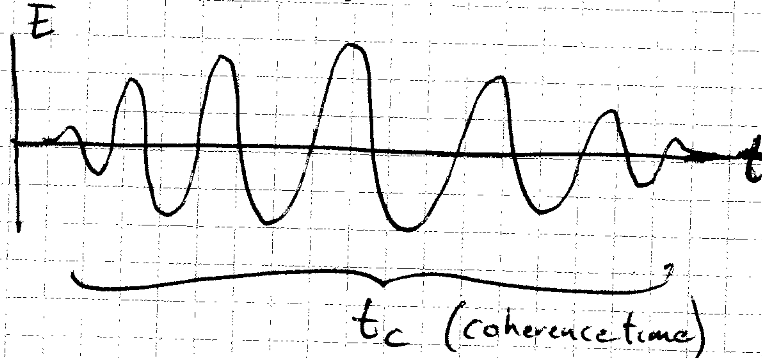
Note $\langle U^*(t+\tau)\, U(t)\rangle_t = \langle U^*(t)\, U(t-\tau)\rangle_t$.

Coherence and partial coherence

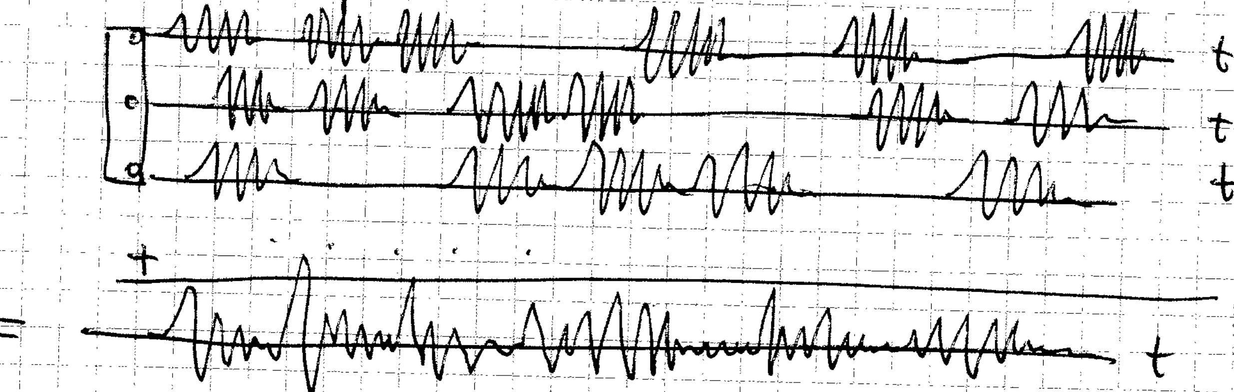
Temporal coherence

Pure monochromatic fields do not exist.
In practice, an atom or molecule emits a "photon" following a transition:



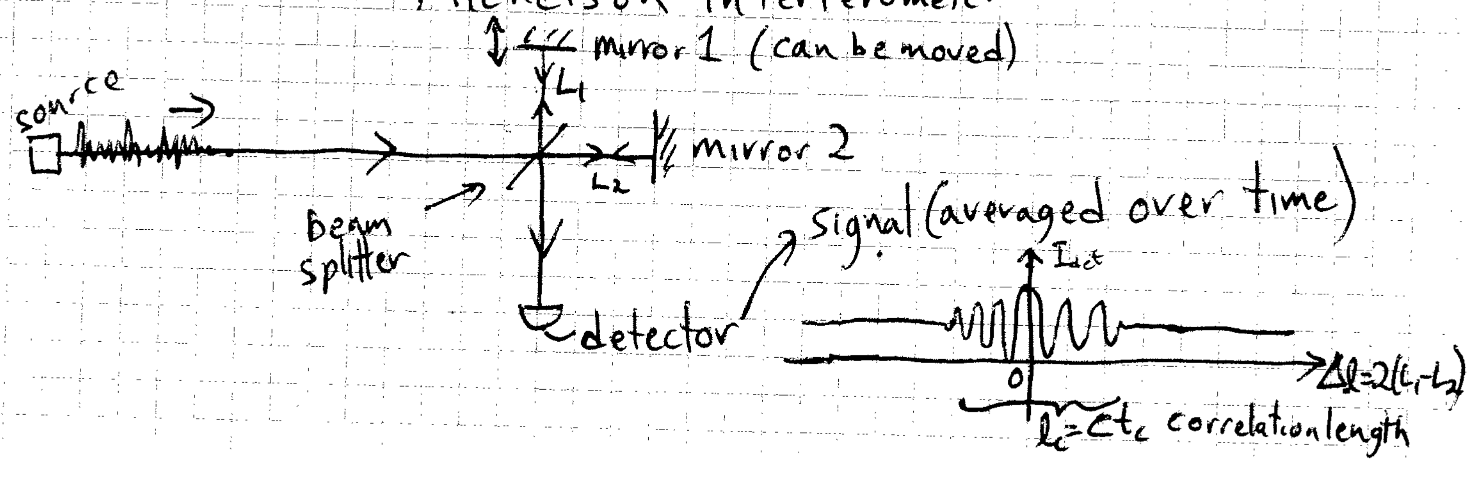


In an incoherent source, each particle emits light pulses randomly, uncorrelated from each other and from those of other particles



The oscillations are only correlated for time differences smaller than t_c.

Michelson interferometer



Where

$$I_{det} = \left\langle \left| \frac{E(t)}{2} + \frac{E(t+\overset{\frac{\Delta l}{c}=\tau}{\tau})}{2} \right|^2 \right\rangle_t = \frac{1}{4}\left\langle \left(E(t)+E(t+\tau)\right)^* \left(E(t)+E(t+\tau)\right) \right\rangle_t$$

$$= \underbrace{\frac{\langle |E(t)|^2\rangle_t}{4}}_{\frac{I_0}{4}} + \underbrace{\frac{\langle |E(t+\tau)|^2\rangle_t}{4}}_{\frac{I_0}{4}} + \frac{2}{4}\,\mathrm{Re}\underbrace{\left\langle E^*(t+\tau)\,E(t)\right\rangle_t}_{\mathcal{C}(\tau)}$$

$$= \frac{I_0 + \mathrm{Re}[\mathcal{C}(\tau)]}{2}$$

Notice: The Correlation $\mathcal{C}(\tau)$ satisfies

$$\mathcal{C}(0) = I_0 \Rightarrow I_{det}(0) = I_0$$

$$\mathcal{C}(\tau \gg t_c) = 0 \Rightarrow I_{det}(\tau \gg t_c) = \frac{I_0}{2}$$

$$\mathcal{C}(-\tau) = \mathcal{C}^*(\tau)$$

Wiener-Khinchin theorem (roughly)

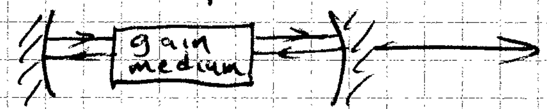
$$\mathcal{C}(\tau) = \langle E^*(t+\tau)\,E(t)\rangle_t \propto \int \underbrace{E^*(t+\tau)}_{\text{write as}\left[\frac{1}{\sqrt{2\pi}}\int \tilde{E}(\omega)e^{-i\omega(t+\tau)}d\omega\right]^*} E(t)\,dt$$

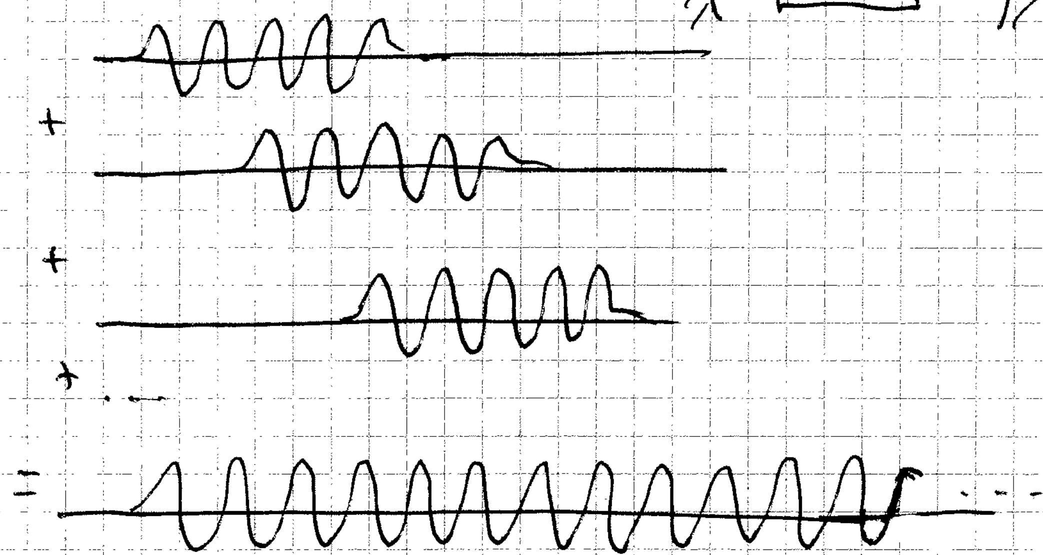
$$\mathcal{C}(\tau) \propto \frac{1}{\sqrt{2\pi}}\iint \tilde{E}^*(\omega)\,e^{+i\omega(t+\tau)}\,E(t)\,dt\,d\omega$$

$$= \int \hat{E}^*(\omega)\underbrace{\frac{1}{\sqrt{2\pi}}\int E(t)\,e^{i\omega t}dt}_{\tilde{E}(\omega)}\;e^{i\omega\tau}d\omega$$

$$= \int \underbrace{|\tilde{E}(\omega)|^2}_{S(\omega)=\text{spectrum}} e^{i\omega\tau}d\omega, \quad \text{so } \mathcal{C}(\tau) = I_0\frac{\hat{\mathcal{F}}^{-1}_{\omega\to\tau}S(\omega)}{\left[\hat{\mathcal{F}}^{-1}_{\omega\to\tau}S(\omega)\right]\Big|_{\tau=0}}$$

In a laser, on the other hand, each photon triggers the coherent emission of another one (stimulated emission), so the pulses are not (entirely) random:

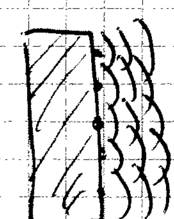




Very long correlation time & length!

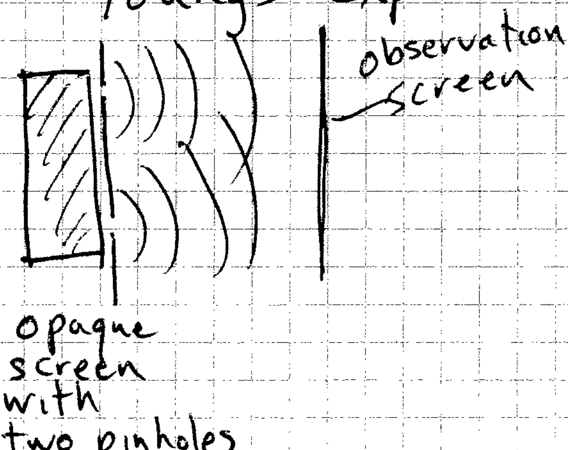
Spatial coherence

extended incoherent source



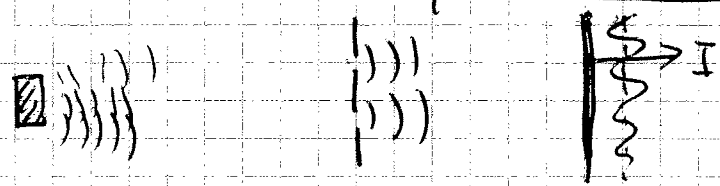
the emissions of each point are statistically uncorrelated

Young's experiment:

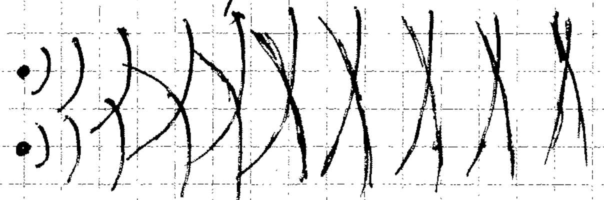


The interference between the light transmissions through the two pinholes washes out statistically over time, so the intensity at the observation screen looks uniform.

Upon propagation from the source, however, the field acquires some spatial coherence:



Explanation: consider a source that consists of only two incoherent emitters



as we move away, their wavefronts look more and more similar.

Cross-correlation function

Consider a component of the electric field, $E(\vec{r},t)$, which is a function of space and time. Assume that this field is statistically stationary. Then the cross-correlation function is defined as:

$$\Gamma(\vec{r}_1,\vec{r}_2;\tau) = \langle E^*(\vec{r}_1,t)\,E(\vec{r}_2,t+\tau)\rangle$$

Notice that: $\underbrace{\Gamma(\vec{r},\vec{r};\tau)}_{\text{same pt.}}$ is the autocorrelation of the field at the point $\vec{r}$, and that $\Gamma(\vec{r},\vec{r};0)=I(\vec{r})$ is the intensity at $\vec{r}$. Also notice that

$$\Gamma(\vec{r}_2,\vec{r}_1;\tau) = \langle E^*(\vec{r}_2,t)E(\vec{r}_1,t+\tau)\rangle = \langle E^*(\vec{r}_1,t+\tau)E(\vec{r}_2,t)\rangle^*$$

$$= \langle E^*(\vec{r}_1,t)\,E(\vec{r}_2,t-\tau)\rangle^* = \Gamma^*(\vec{r}_1,\vec{r}_2;-\tau)$$

The complex degree of coherence is defined as

$$\gamma(\vec{r}_1,\vec{r}_2;\tau) = \frac{\Gamma(\vec{r}_1,\vec{r}_2;\tau)}{\sqrt{I(\vec{r}_1)I(\vec{r}_2)}}$$

It can be shown that

$$0 \le |\gamma(\vec{r}_1,\vec{r}_2;\tau)| \le 1$$

and

$$\gamma(\vec{r},\vec{r};0) = \frac{\Gamma(\vec{r},\vec{r};0)}{I(\vec{r})} = 1$$

Coherence Theory in the frequency domain

We define the cross-spectral density as:

$$W(\vec{r}_1,\vec{r}_2;\omega)=\underset{\text{our convention}}{\frac{1}{\sqrt{2\pi}}}\int\Gamma(\vec{r}_1,\vec{r}_2;\tau)\,e^{i\omega\tau}\,d\tau$$

From Wiener-Khinchin thm:

$$W(\vec{r},\vec{r};\omega)=\frac{1}{\sqrt{2\pi}}\int\underbrace{\Gamma(\vec{r},\vec{r};\tau)}_{\text{auto correlation}}e^{i\omega\tau}d\tau=S(\vec{r};\omega)=\text{spectrum}$$

It is not easy, but it can be shown that

$$W(\vec{r}_1,\vec{r}_2;\omega)=\langle U^*(\vec{r}_1;\omega)\,U(\vec{r}_2;\omega)\rangle_e$$

where $U(\vec{r};\omega)e^{-i\omega t}$ is a monochromatic field of frequency ω, which forms part of an ensemble.

In free space, U satisfies the Helmholtz equation:

$$\nabla^2U+k^2U=0,\quad\text{where } k=\frac{\omega}{c}.$$

Therefore, $W(\vec{r}_1,\vec{r}_2;\omega)$ satisfies

$$(\nabla_1^2+k^2)W=(\nabla_2^2+k^2)W=0.\qquad\left(\begin{array}{l}\text{Wolf equations}\\\text{in the frequency domain}\end{array}\right)$$

The freespace propagation of a coherent field $U(\vec{r};\omega)$ given its knowledge at an initial plane $z=0$ and the fact that it only propagates towards larger z, is given by

$$U(\vec{r};\omega)=\iint U(\vec{r}\,';\omega)\,K(\vec{r}\,',\vec{r};\omega)\,dx'dy',\quad\text{where } \vec{r}\,'=(x',y',0)\text{ and}$$

K is the Rayleigh-Sommerfeld propagator

$$K(\vec{r}\,',\vec{r};\omega)=\left(\frac{1}{i\lambda}+\underbrace{\frac{1}{2\pi|\vec{r}\,'-\vec{r}|}}_{\text{small}}\right)\frac{z}{|\vec{r}-\vec{r}\,'|}\,\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}$$

Therefore

$$W(\vec{r}_1, \vec{r}_2; \omega) = I_0 \iint_{x'^2+y'^2 \le a^2} \frac{e^{-ik \frac{\Delta \underline{x} \cdot \underline{x}'}{R}}}{\lambda^2 R^2} \, dx' dy'$$

change to polars: $\underline{x}' = (\rho' \cos\theta', \rho' \sin\theta')$

$\Delta \underline{x} = (|\Delta \underline{x}| \cos\theta, |\Delta \underline{x}| \sin\theta)$

Then

$$W(\vec{r}_1, \vec{r}_2; \omega) = \frac{I_0}{\lambda^2 R^2} \int_0^{2\pi} \int_0^a e^{-ik \frac{|\Delta \underline{x}| \rho'}{R} \cos(\theta' - \theta)} \rho' d\rho' d\theta'$$

$$= \frac{I_0 2\pi}{\lambda^2 R^2} \int_0^a J_0 \Big[\overbrace{\frac{k |\Delta \underline{x}| \rho'}{R}}^{u = k|\Delta \underline{x}| \rho'/R} \Big] \rho' d\rho' \qquad d\rho' = \frac{R}{k|\Delta \underline{x}|} du$$

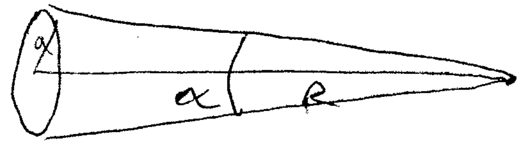
$\uparrow$ Bessel function

$$= \underbrace{\frac{I_0}{\lambda^2 R^2} 2\pi \int_0^{\frac{k|\Delta \underline{x}| a}{R}} u J_0(u) \, du \frac{R^2}{k^2 |\Delta \underline{x}|^2}}_{\text{from properties of Bessel } J.} \qquad \left(\text{use } k = \frac{2\pi}{\lambda}\right)$$

$$= \frac{I_0}{2\pi |\Delta \underline{x}|^2} \frac{k |\Delta \underline{x}| a}{R} J_1\left(\frac{k |\Delta \underline{x}| a}{R} \right)$$

$$= \frac{I_0}{2\pi} \left(\frac{ka}{R}\right)^2 \frac{J_1\left(\frac{ka}{R} |\Delta \underline{x}|\right)}{\frac{ka}{R} |\Delta \underline{x}|}$$

Notice that $\frac{2a}{R} = \alpha$ where α is the angle subtended by the source at the observation points $\vec{r}_1, \vec{r}_2$



In the paraxial approximation, the propagator can be approximated as the Fresnel propagator

$$K(\vec{r}',\vec{r};\omega) \approx \frac{e^{ikz}}{i\lambda z}\, e^{-ik\frac{[(x-x')^2+(y-y')^2]}{\lambda z}} .$$

(*)

The cross-spectral density then propagates as

$$\boxed{W(\vec{r}_1,\vec{r}_2;\omega) = \iiiint W(\vec{r}_1{}',\vec{r}_2{}';\omega)\, K^*(\vec{r}_1{}',\vec{r}_1;\omega)\, K(\vec{r}_2{}',\vec{r}_2;\omega)\, dx_1'dy_1'dx_2'dy_2'}$$

Let us assume that the initial field is almost spatially incoherent.
We can then use

$$W(\vec{r}_1{}',\vec{r}_2{}';\omega) \approx I(\vec{r}_1{}')\, \delta(\vec{r}_2{}'-\vec{r}_1{}')$$

Substituting this in the propagation formula gives:

$$W(\vec{r}_1,\vec{r}_2;\omega) \approx \iiiint I(\vec{r}_1{}')\,\delta(\vec{r}_2{}'-\vec{r}_1{}')\, K^*(\vec{r}_1{}',\vec{r}_1;\omega)\, K(\vec{r}_2{}',\vec{r}_2;\omega)\, dx_1'dy_1'dx_2'dy_2'$$

$$= \underline{\iint I(\vec{r}')\, K^*(\vec{r}',\vec{r}_1;\omega)\, K(\vec{r}',\vec{r}_2;\omega)\, dx'dy'}$$

This corresponds to the Van Cittert-Zernike theorem (in the frequency domain), which explains why coherence increases under propagation from the source.

(*) Can also use the Far-field or Fraunhofer propagator

$$K(\vec{r}',\vec{r};\omega) \approx \frac{z\, e^{ik|\vec{r}|}}{i\lambda |\vec{r}|^2}\, e^{-ik\frac{\vec{r}'\cdot\vec{r}}{|\vec{r}|}}$$

when $|\vec{r}| \gg \sqrt{\lambda a}$, where a is the size of the source.

Example: distant uncorrelated circular source.

$$W(\vec{r}_1{}', \vec{r}_2{}'; \omega) = \begin{cases} I_0\, \delta(x_2'-x_1')\,\delta(y_2'-y_1') & , |\vec{r}_{\frac{1}{2}}{}'| \le a \\ 0, & |\vec{r}_{\frac{1}{2}}{}'| > a \end{cases}$$

where $\vec{r}_{\frac{1}{2}}{}' = (x_{\frac{1}{2}}', y_{\frac{1}{2}}', 0)$

Using the VC-Z theorem:

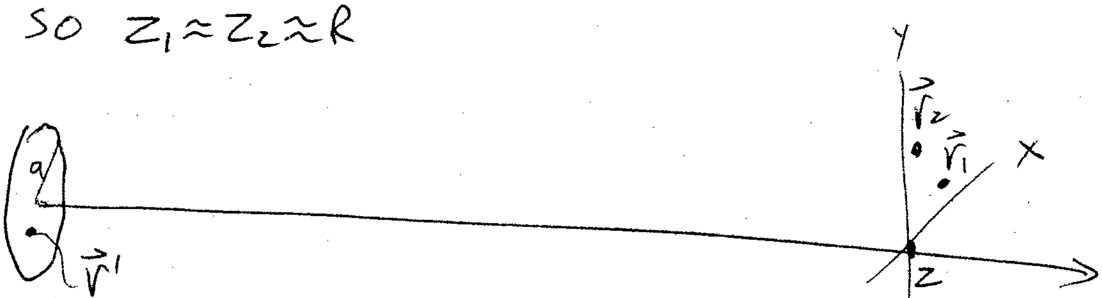
$$W(\vec{r}_1, \vec{r}_2; \omega) = \iint_{x'^2+y'^2 \le a^2} I_0\, K^*(\vec{r}\,', \vec{r}_1; \omega)\, K(\vec{r}\,', \vec{r}_2; \omega)\, dx'dy'$$

Use the far-field propagator

$$K(\vec{r}\,', \vec{r}; \omega) \approx \frac{z e^{ik|\vec{r}|}}{i\lambda |\vec{r}|^2} e^{-ik\frac{(x'x+y'y)}{|\vec{r}|}}$$

Assume that $\vec{r}_1$ & $\vec{r}_2$ are at the same distance from the origin: $|\vec{r}_1| = |\vec{r}_2| = R$.

Also assume that we are in the "paraxial" region, so $z_1 \approx z_2 \approx R$



Then

$$K^*(\vec{r}\,', \vec{r}_1; \omega)\, K(\vec{r}\,', \vec{r}_2; \omega) \approx \frac{z^2 \cancel{R}\, \cancel{e^{ikR}}\, \cancel{e^{-ikR}}}{\lambda^2\, \cancel{i(i)}\, R^{\cancel{4}\,2}} e^{-\frac{ik}{R}[x'(x_2-x_1)+y'(y_2-y_1)]}$$

$$= \frac{e^{-ik\frac{\underline{x}'\cdot\underline{\Delta x}}{R}}}{\lambda^2 R^2}, \quad \text{where } \underline{\Delta x} = \underline{x}_2 - \underline{x}_1$$

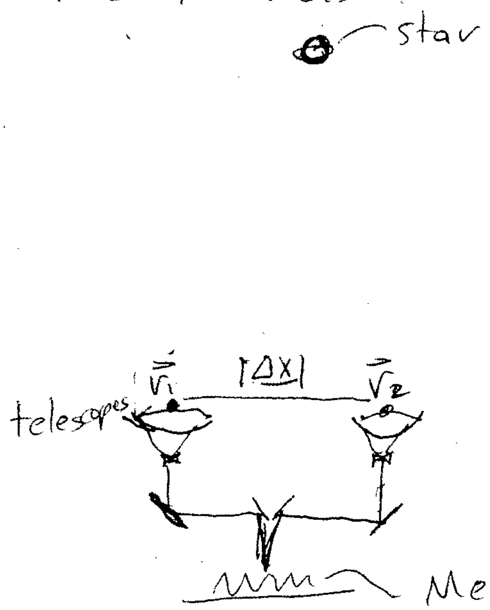
$$= (x_2-x_1, y_2-y_1)$$

Then

$$W(\vec{r}_1, \vec{r}_2; \omega) \approx \frac{I_0}{\lambda^2} \, 2\pi \frac{\alpha^2}{4} \underbrace{\left[\frac{J_1\left(\frac{k\alpha}{2} |\Delta x|\right)}{\frac{k\alpha}{2} |\Delta x|} \right]}_{\text{Airy pattern!}}$$

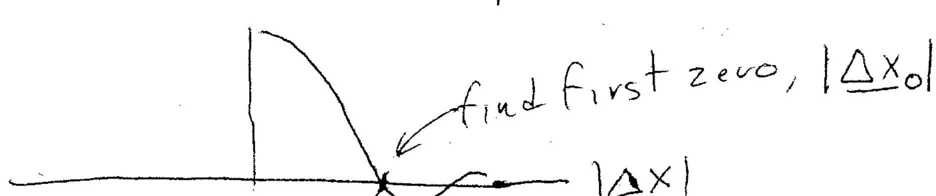
That is, the coherence as a function of point separation is an Airy pattern.

Distant stars are effectively like uncorrelated disks with very small subtended angle α. Interferometry can be used to measure α. This is the basis of the Michelson stellar interferometer.



This way, he measured the angular size of the star Bietedgeuse

Measure W for different separation of the telescope:



$$3.832 \cdot \frac{2}{k\alpha} = \frac{1.22\lambda}{\alpha} \Rightarrow \alpha = \frac{1.22\lambda}{|\Delta x_0|}$$