The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA International Atomic Energy Agency

2017-6

Preparatory School to the Winter College on Optics in Environmental Science

26 - 30 January 2009

Basics of Image Theory, including images with coherent and incoherent radiation, Spectra, OTF, MTF

Consortini A.

Universita' di Firenze
Italy

The Abdus Salam ICTP, Trieste, Italy

PREPARATORY SCHOOL, 26-30 January 2009

Winter College on Optics in Environmental Science, 2-13 February 2009

ANNA CONSORTINI
Department of Physics
University of Florence, Italy

IMAGES

Mirrors

Lenses
- traditional
- others such as aspherical or graded index

Systems, more or less sophisticated

Methods:

1 - Geometrical optics:

simple
rays
allows accounting for aberrations
neglects diffraction

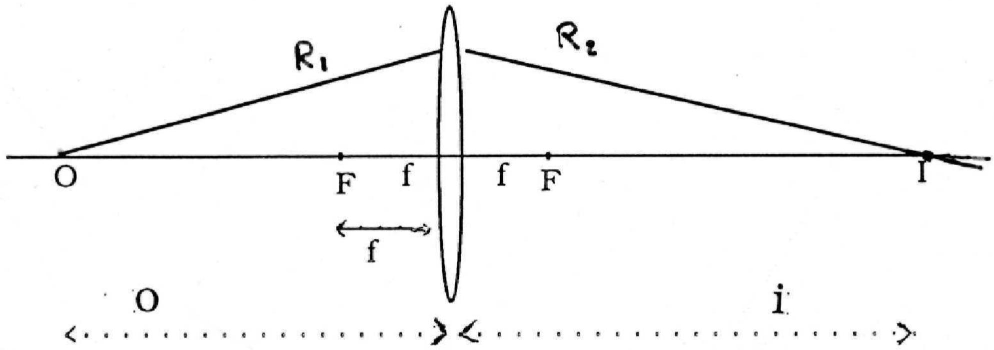
2 - Wave optics:

allows accounting for:
- diffraction
- aberrations

direct by using diffraction formulas
• scalar approximation
development based on linear systems theory
approximate results

1 - GEOMETRICAL OPTICS

Recall some fundamentals by thin lens



1) $$\frac{1}{o}+\frac{1}{i}=\frac{1}{f}$$

paraxial approximation, gaussian optics

2) $$\frac{1}{f}=(n-1)\left[\frac{1}{R_1}-\frac{1}{R_2}\right]$$ f focal length

Sing conventions

- $o>0$ if on left lens side
- $i>0$ if on right side
- Lens radius >0 if center on right side
  - in fig $R_1>0$, $R_2<0$
- $f>0$ converging lens $\quad$ $f<0$ diverging lens

When $o\rightarrow\infty$, i=Focus. Perfect lens makes parallel rays converge to (or diverge as from back) focus.
Ex: $f>0$: the lens makes rays converge. It transforms plane wave into spherical converging wave, see below.

Images: Real or Imaginary

In general, for lenses or systems of elements:

From source to image optical path along each ray the same (Fermat principle). Phase along each ray the same; at image point positive interference.

In paraxial approximation, one image point corresponds to source point; the rays from source only have one cross point. In general rays do not all cross at the same point; aberrations. (Here we neglect magnification and image reversal)

In addition diffraction effect. Images by systems without aberrations are called diffraction limited.

2 - WAVE OPTICS

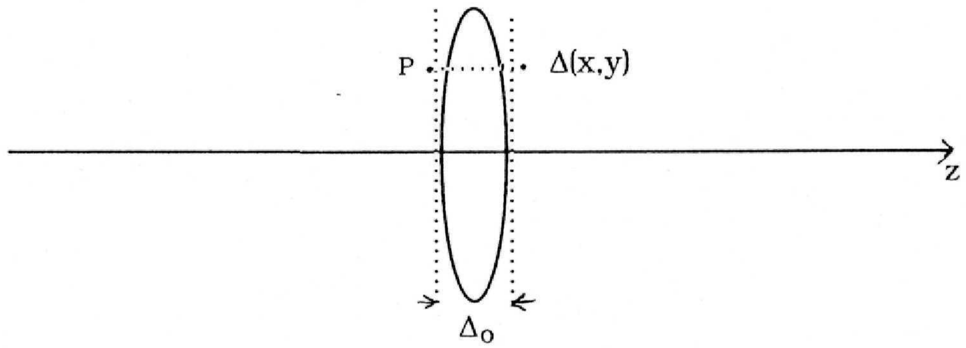
MONOCHROMATIC ILLUMINATION

THIN LENS

Lens introduces phase effect on impinging wave u(P) where P -coordinates x,y- point on entrance plane,

$$u_{out}(P) = t(P)\, u_{in}(P) \qquad t(P) = e^{i\Phi(P)}$$

t(P) thickness function



Δ_o $\Delta(P)$

$$\Phi(P) = k\, n\, \Delta(P) + k\left[\Delta_o - \Delta(P)\right]$$

Simple computation (Goodman)

$$\Delta(P) = \Delta_o - R_1\left(1-\sqrt{1-\frac{x^2+y^2}{R_1^2}}\right) + R_2\left(1-\sqrt{1-\frac{x^2+y^2}{R_2^2}}\right)$$

Paraxial approx:

$$1-\sqrt{1-\frac{x^2+y^2}{R_1^2}} \cong 1-\frac{x^2+y^2}{2\,R_1^2}$$

$$\Phi(P) = i\,k\,n\,\Delta_o - i\,k(n\text{-}1)\,\frac{x^2+y^2}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

$$\Phi(P) = i\,k\,n\,\Delta_o - i\frac{k}{2f}\left(x^2+y^2\right)$$

If $u_{in}(P) = 1$, unit amplitude plane wave, (point source at infinity) one has

$$u_{out}(P) = e^{i\,k\,n\,\Delta_o}\; e^{\frac{-ik}{2f}\left(x^2+y^2\right)}$$

First term constant phase delay of no importance

Second term: quadratic approximation, at z=0, to a spherical wave converging towards the focus behind the lens, if f >0 (and then diverging) or diverging from the lens as if originating from focus before the lens if f<0.

Example f>0,

$$r = \sqrt{(f-z)^2 + x^2 + y^2} \cong (f-z) + \frac{1}{2(f-z)}\left(x^2 + y^2\right)$$

Result:

In paraxial approximation lens adds quadratic phase term, i.e. transforms plane into spherical wave.

In first approximation this can be extended to plane waves impinging at small angles. In this case the wave is focused at a point on the focal plane.

In general case: although the lens has spherical surfaces, the wavefront departs from spherical shape. Aberrations.

PUPIL FUNCTION: To take into account the finite dimensions of apertures (and also aberrations): pupil function. Will be useful in the sequel.

For systems without aberrations (diffraction limited)

$$P(x,y) = \begin{cases} 1 & \text{inside lens aperture} \\ 0 & \text{outside} \end{cases}$$

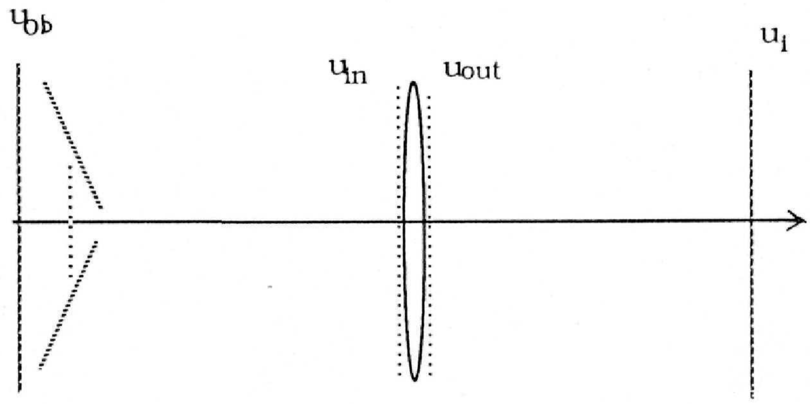
Note: here x,y point on the pupil.
Some authors include on the pupil function the field on the pupil due to a source point.

3 - WAVE OPTICS

COHERENT IMAGING: OBJECT ILLUMINATED WITH MONOCHROMATIC COHERENT FIELD

The problem of images is: given the field distribution at the object find the field distribution at the image.

3-1 LENS AND PLANE OBJECT



u_{ob} field (complex amplitude) from object

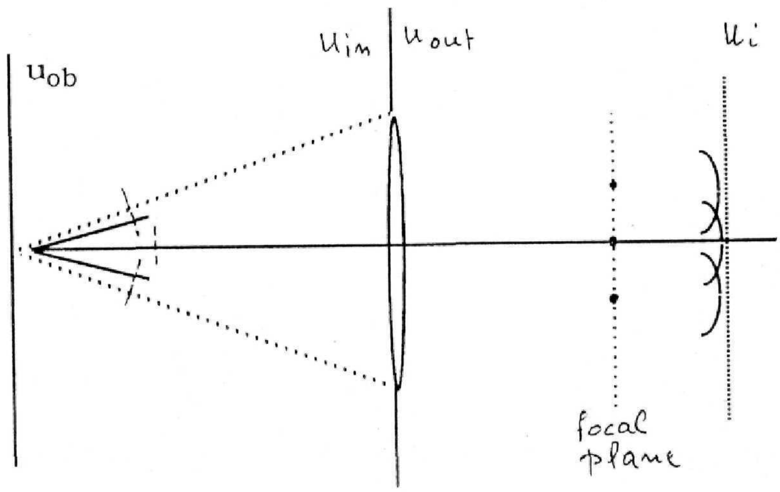
u_{in} field on input plane of the lens(entrance pupil),

u_{out} field after the lens (exit pupil),

u_i field on image plane (defined by geom. optics).

Problem: given u_{ob} find u_i

Typical diffraction problem: the field from object can be considered a diffraction field: from the object plane waves in all directions (Fourier components). u_{in} is result of interference on input plane. Field u_{out} at the output of lens can be obtained by multiplying u_{in} for the lens function t(P), and pupil function. Field u_i by using one of different formulas of diffraction theory, such as Kirchhoff's, which takes into account the finite dimension of the aperture.



Let us consider decomposition of diffracted field in plane waves (Fourier). Each plane wave is focused by the lens at focal plane. On each plane behind the lens all waves interfere; in the image plane interference is expected to "reproduce" the object field.

However the image is never equal to the object, because not all plane waves from the object enter the lens, but only those with angle respect to axis less than a/o, a=aperture radius, o=object distance from the lens. In addition evanescent waves are lost.

Due to the limited aperture, from a plane incident wave, source at infinity, one has an Airy diffraction pattern (see diffraction); not a simple point as expected from the lens.

3-2 IMAGES BY A SYSTEM - COHERENT CASE

Let us now think of a general imaging system, of which the lens is a particular case.
First, let us consider, in the source plane, an object constituted by a simple point (source point); in general, due to diffraction and aberrations, the image is not a point but a "pattern". A point source can be represented by a Dirac delta function.

Let $h(x,y;x_o,y_o)$ denote the field at point x,y in the image produced by a source point located at point x_o,y_o in the source plane. In a first approximation and no aberrations, $h(x,y;x_o,y_o)$ is the Airy diffraction pattern. In general it is a diffraction pattern.

Function $h(x,y;x_o,y_o)$ which represents the <u>impulse response</u> of the system is called the <u>Spread</u> <u>Function</u>

Let us assume to have an extended source. Let $u_{ob}(x_o,y_o)$ be the complex amplitude distribution density (surface density). Each element $dx_o dy_o$ gives a contribution to the field at x,y, given by

$$u_{ob}(x_o,y_o)\ h(x,y;x_o,y_o)\ dx_o dy_o$$

The field $u_i(x,y)$ at point x,y on image plane, due to the object is obtained by integrating over all the object

$$3) \quad u_i(x,y) = \int_{-\infty}^{\infty}\int u_{ob}(x_o,y_o)\, h(x,y;x_o,y_o)\, dx_o dy_o$$

Typically the object will be of finite dimension and the integrand different from zero on a finite area.

The fact that one can easily write the total field at x,y as the integral (sum) of the contributions produced by the different points of the object is direct consequence of the linearity of Maxwell's equations. According to this linearity the total field at x,y is the superposition (interference) of the contributions from the different elements of the object.

Linearity implies use of the basic elements of linear systems. They are used here, when necessary.

If Spread Function only depends on coordinates difference

$$4) \quad h(x,y;x_o,y_o) = h(x-x_o, y-y_o)$$

the system is called <u>isoplanatic</u> (or space-invariant). In practice isoplanatism means that the system "response" is independent of the object location on the source plane.
For a isoplanatic system one has

$$5) \quad u_i(x,y) = \int_{-\infty}^{\infty}\int u_{ob}(x_o,y_o)\, h(x-x_o, y-y_o)\, dx_o dy_o$$

As already stated, in general $h(x-x_o,y-y_o)$ is a diffraction pattern, not a simple Dirac function as in geometrical optics approximation. Therefore the field at point x,y is affected by all source points and not only by the corresponding point of the object. This means that, due to diffraction (and aberrations), the image is a smeared version of the object. In the integral we recognize a (bidimensional) convolution operation which is the mathematical formulation of this fact.

In convolution formalism the integral can be written

6) $$u_i(x,y) = u_{ob}(x,y) \otimes h(x,y)$$

where $\otimes$ denotes the convolution operation[1].

Well known theorem, called the convolution theorem, states that the spectrum of a convolution of two functions is the product of the spectra of the two functions. In formula

7) $$U_i(u,v) = H(u,v)\, U_{ob}(u,v)$$

Capital letters denote the spectra; note that they are bidimensional Fourier transforms,

Fourier transform	of	
$U_{ob}(u,v)$	$u_{ob}(x,y)$	object
$U_i(u,v)$	$u_i(x,y)$	image
$H(u,v)$	$h(x,y)$	spread function

---

[1] By definition $u \otimes v = \int\int_{-\infty}^{\infty} u(x_o, y_o)\, v(x-x_o, y-y_o)\, dx_o dy_o$

H(u,v), Fourier transform of the Spread Function, is called the Optical Transfer Function, OTF or, as in linear systems theory, the System Transfer Function, or simply the Transfer Function. Sometimes the adjective Coherent is added to avoid confusion with the case of incoherent radiation, see below.
Eq. 6 is very important, both for theory and applications, because in the realm of spectra the convolution becomes a simple product and allows optical images to be exploited by the techniques commonly used in systems applications, such as filtering in electric systems.

3-3 SPREAD FUNCTION OF A SOURCE POINT of unit amplitude and a thin lens without aberrations (diffraction limited)

From a source point on the axis spherical wave. In paraxial approximation, the field incident at point x,y on the lens, at distance r_o from the source, is (see diffraction lessons)

$$u_i = \bar{a}\,\frac{e^{ikr_o}}{r_o} \cong \frac{\bar{a}}{o}\, e^{iko+ik(x^2+y^2)/(2o)}$$

Here $\bar{a}$ includes the constant phase term and o denotes source-lens distance. The field u_{out} at the output plane of the lens is

$$u_{out} = \frac{\bar{a}}{o} e^{iko+i\,k\,n\,\Delta_o}\; e^{\frac{-ik}{2f}(x^2+y^2)+\frac{ik}{2o}(x^2+y^2)}$$

The quadratic phase term is an approximation to a spherical wave converging to a point at distance i from the lens

$$-\frac{1}{i}=\frac{1}{o}-\frac{1}{f}$$

as from geometrical optics. The field u_i at a point x_i,y_i is obtained by using any diffraction formula, e.g. the paraxial form of Huygens-Fresnel principle (see lessons on diffraction). One obtains

$$u_i=c\iint_{aperture} e^{\frac{ik}{2}\left(\frac{1}{o}-\frac{1}{f}\right)(x^2+y^2)+\frac{ik}{2i}\left[(x-x_i)^2+(y-y_i)^2\right]}dxdy$$

where complex constant c takes into account constant amplitude and phase terms. By developing the squares the phase term can be rewritten as

$$i\Phi=\frac{ik}{2}\left(\frac{1}{o}-\frac{1}{f}+\frac{1}{i}\right)(x^2+y^2)+\frac{ik}{2i}\left[x_i^2+y_i^2-2xx_i-2yy_i\right]$$

On image plane first term in parenthesis is zero. Let us assume that also Fraunhofer condition is satisfied and neglect first and second term in square brackets. The integral reduces to

$$u_i=c\iint_{aperture} e^{\frac{-ik}{2i}\left[2xx_i+2yy_i\right]}dxdy$$

This integral was evaluated in the diffraction section, when the field diffracted from an aperture uniformly illuminated was calculated in the Fraunhofer approximation (pg 2.9-2.13). In terms of energy the result is the well known Airy pattern. Therefore, in the considered limits, and apart from a complex constant factor, the spread function of a source point is the diffraction pattern, in the Fraunhofer region, of a uniformly illuminated aperture. In other words, the spread function of a source point at finite distance is proportional to that of the source at infinity. Note also that when the source distance is infinite the pattern location is on the focal plane. This corresponds to the well known fact that in general, a lens transfers on its focal plane the Fraunhofer patter of the field on its aperture.

SPREAD FUNCTION = FOURIER TRANSFORM OF THE PUPIL FUNCTION.

VALID IN GENERAL. Easy to understand by plane wave development.

CUT OFF FREQUENCY

3 - 4 LINE SPREAD FUNCTION

Let the object be a line, infinitely thin and concident with the y axis:

$$u_{ob}(x_o,y_o)= \delta(x_o)$$

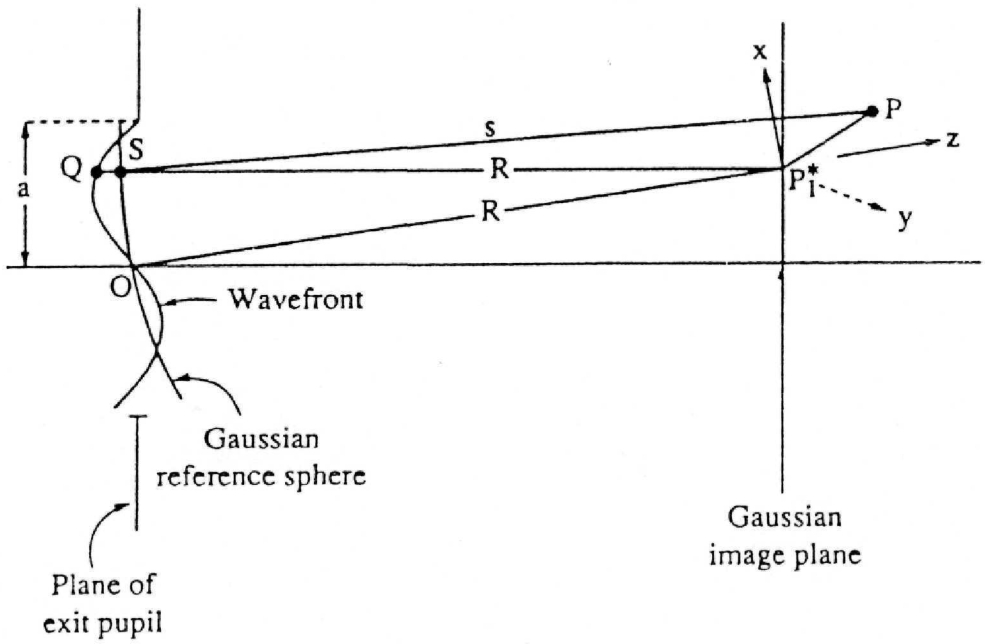
From 5) one obtains

$$u_i(x)= \int_{-\infty}^{\infty} h(x,y_o)\, dy_o$$

3 - 5 ABERRATIONS

According to Wolf function W(x,y) is the departure from spherical shape in the exit pupil. Phase distortion due to aberrations kW(x,y).

Effect of aberration on diffraction patter: lowering the maxima, filling the zeros and rising the minima.



W(x,y) from Born and Wolf

Quality of a system is described by Strehl ratio for source point:

$$S = \frac{\text{Intensity in the (nominal) central maximum}}{\text{Theoretical intensity with no aberrations}}$$

General definition suitable also for partial coherence and for aberrations introduced by propagation in random media, such as turbulent atmosphere.

An image is well corrected if S not less than 0.8.

It can be shown, for small enough aberrations, that intensity at point P is

$$i(P) \cong 1 - k^2 (\Delta\Phi_P)^2$$

where $(\Delta\Phi_P)^2$ mean square deformation of the wavefront and k wavenumber. It follows that condition $S \geq 0.8$, requires $|\Delta\Phi| \leq \lambda/14$.
(Criterion by Rayleigh $\lambda/4$ for spherical aberration)

Best focus.

Point on axis. Aberrations function W_a, with respect to a sphere centered on best focus is expanded in terms of r, position on pupil, or ($\Omega \approx r/d_i$ field angle)

$$W_a = a_4 r^4 + a_6 r^6 + \ldots = W_4 + W_6 + \ldots$$

W_4 primary aberrations, W_6 secondary aberrations and so on.
For points off axis in general there are also angles, total power of primary aberrations is always 4, including angles. Primary Seidel aberrations.
(Extended theory: Born and Wolf, Goodman,see also R.W. Ditchburn for instrument applications)

EFFECT OF ABERRATIONS ON OTF

Generalized Pupil Function including W

$$P(x,y) = \exp(ikW) \quad \text{inside pupil}$$
$$= 0 \quad \text{outside}$$

W, phase factor

W does not affect total intensity, but adds phase factor to the different (spatial) frequencies

Blurring of the image.

Example in terms of rays (recall source point): the normal to the wavefront, (ray) changes direction and the rays no longer have a common point.

In terms of plane wave development of the field, each wave has a change in phase, and they are no longer focussed at the same point.

In general lowering of maxima, disappearing of zeros, increase of minima

3 - 6 IMAGING - INCOHERENT CASE

The most common light found in nature, emitted by bodies much larger than the wavelength, is incoherent radiation. The emitting atoms of a body, emit randomly, in time and space, wave trains which are completely uncorrelated, unless the atoms are very near each other, with respect to wavelength. Only the laser emits coherent radiation. At a point outside an emitting source (not a laser) the field is constituted by many wave trains with random phase, which interfere with each other but continuously and rapidly change. One cannot think of a "wave", as in the case coherence, but rather of energy. For incoherent radiation one has to deal with the modulus square of the field.

Although the general case is partial coherence, both in time and space, we will consider here only the limiting case of incoherent quasi-monochromatic light, as the case corresponding to the coherent monochromatic one already considered.

Quasi-monochromatic light has a bandwidth $\Delta\nu$ which is very small with respect to the central frequency ν, that is $\Delta\nu/\nu \ll 1$.

Quantity of interest here is the average value of the intensity in a long time with respect to the period of oscillation (infinite time). In practice the response time of the eye or of typical instruments. In this case approximating time and space incoherent radiation with monochromatic (time coherent) radiation ν, is a good approximation. Frequency =central frequency of incohrerent radiation, Therefore the radiation is only spatially incoherent.

The instantaneous intensity $I_{inst}(P)$ at a point P is the field square (see diffraction). The space coherence of a field is described by the field correlation function $B_u(P,P')$ defined as

8) $$B_u(P,P') = \langle u(P)\, u^*(P')\rangle$$

Asterisk as usual denotes complex conjugate and brackets infinite time average. The average intensity I(P) is given by (assume homogeneity)

$$I(P) = \lim_{P' \to P} \langle u(P)\, u^*(P')\rangle$$

For spatially incoherent radiation:

9) $$B_u(P,P') = I(P)\, \delta(P-P')$$

The intensity in the image of incoherent radiation is

$$I(x,y)=\langle u_i(x,y)u_i^*(x,y)\rangle=\int_{-\infty}^{\infty}\int\int_{-\infty}^{\infty}\int \langle u_{ob}(x_o,y_o)u_{ob}^*(x'_o,y'_o)\rangle$$

$$h(x-x_o,y-y_o)h^*(x-x'_o,y'-y_o)\, dx_o dy_o dx'_o d'y_o$$

where average and integral operations have been interchanged and the fact that the impulse response does not depend on time has been taken into account. This relationship holds for partially coherent light and could be further developed.

In the case of complete incoherence, introduction of Eq. 9 gives[2] the important final result:

$$10)\qquad I(x,y) = \int\int_{-\infty}^{\infty} I(x_o,y_o)\ |h(x-x_o,y-y_o)|^2 dx_o dy_o$$

Conclusions for INCOHERENT CASE:

-Intensity

-Convolution relationship between source intensity and (incoherent) point spread function

-The incoherent point spread function is the modulus square of the coherent spread function.

Example. The incoherent spread function of a source on the axis of a thin lens free from aberrations (already considered for the coherent case): is the Airy function, (Airy function is defined as the modulus square of the Fraunhofer diffraction pattern, see lessons on diffraction) centered on the geometrical image point.

---

[2] Recall that $\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)$

Let $\mathbb{I}_o$ and $\mathbb{I}_i$ and $\mathbb{H}$ Fourier Transforms of the intensities of object, image and spread function respectively. By convolution theorem:

11) $$\mathbb{I}_i = \mathbb{H}\,\mathbb{I}_o$$

$\mathbb{H}$ is called Incoherent Optical Transfer Function; its modulus:

MODULATION TRANSFER FUNCTION, MTF.

Generally normalization to 1 at zero frequencies, where there is the maximum (see e.g. Goodman).

General relationship between incoherent, $\mathbb{H}$ (normalized), and coherent, H, transfer functions:

12) $$\mathbb{H}(f_x, f_y) = \frac{\int_{-\infty}^{\infty}\int H(u,v)H^*\left(u+f_x, v+f_y\right)du\,dv}{\int_{-\infty}^{\infty}\int \left|H(u,v)\right|^2 du\,dv}$$

valid for systems both with and without aberrations.

For coherent systems one has (see diffraction)

$$H(u,v) = P(\lambda i u, \lambda i v)$$

λ wavelength, i image distance from the lens.

For incoherent system, introduction of H(u,v) into Eq. 12 shows that $\mathcal{H}$ (normalized) is the spatial autocorrelation function of the pupil function:

$$\mathcal{H}(f_X,f_Y) = \frac{\int_{-\infty}^{\infty}\int P(\lambda iu,\lambda iv)P\left(\lambda iu+f_X,\lambda iv+f_Y\right)dudv}{\int\int_{-\infty}^{\infty} P(u,v)dudv}$$

Recall P(x,y) real function of modulus one. Denominator=pupil area. Numerator the common area of pupil and displaced pupil.

FOR INCOHERENT SYSTEMS THE SPECTRUM IS DIFFERENT THAN FOR COHERENT

In particular it has a larger widht (due to convolution of the pupil function)

Consequence the same system gives different images with coherent or incoherent radiation.

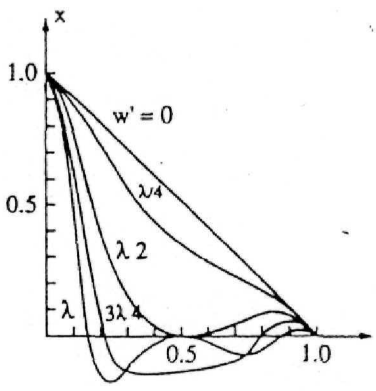
Advantages and disadvantages depend also on the object.

<u>Effect of aberrations</u> on incoherent systems

ALWAYS DECREASE MTF

In general lower the contrast of each spatial frequency component, leaving the cut off unchanged. However the higher frequencies can be severely reduced, so that, in practice cut off can be much lower than in the diffraction limited case.

Aberrations can also give rise to negative values of OTF in some ranges of frequencies. Consequence: constrast reversal in image, that is intensity maxima can become zeros and viceversa.
Typical example of this case is defocusing error



Defocus OTF for a square pupil.

4 - RESOLVING POWER

1- Rayleigh criterion (v diffraction)

2- OTF or MTF half width

3 Degrees of freedom of images

Superresolution