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Scattering theory
Part I

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INTRODUCTION

The pages have been prepared for showing some basic points relevant to the subject.
The purpose is to use transparencies, leaving a full copy to the students.
A second copy will be also with me at Trieste.
Also some mathematical details, absent in the text, can be provided by added pages for the interested students.

INDEX

The text is divided in parts, each distinguished by a sign.

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SYMBOLS

Vector $\vec{V}$ or **V** bold or $\underline{V}$

Scalar V normal characters

$\vec{u}$ unit vectors

examples $\bar{x}$, $\bar{y}$, $\bar{z}$
$\vec{s}$, $\vec{i}_\theta$, $\vec{i}_\varphi$

$\bullet$ dot product.

$\times$ vector product. also $\wedge$

Differential Operators

$\vec{\nabla}$ gradient

$\vec{\nabla}\,\bullet$ divergence (div)

∇^2 Laplacian $\vec{\nabla}\cdot\vec{\nabla}$

$\nabla\times$ curl

$\nabla\times\nabla\times$ curl curl $= \vec{\nabla}\,\vec{\nabla}\,\bullet - \nabla^2$

NOTE: in part of the text the simbles Δ and Δ^2 are used in place of the more usual ∇ and ∇^2 (question of computer)

$\vec{\nabla}$ rectangular co-ordinates x, y, z

$$\vec{x}\frac{\partial}{\partial x} + \vec{y}\frac{\partial}{\partial y} + \vec{z}\frac{\partial}{\partial z}$$

$\vec{x}$ $\vec{y}$ $\vec{z}$ unit vectors

$\vec{\nabla}$ spherical co-ordinates r, ϑ, φ

$$\underline{i}_r \frac{\partial}{\partial r} + \underline{i}_\vartheta \frac{1}{r}\frac{\partial}{\partial \vartheta} + \vec{i}_\varphi \frac{1}{r\sin\vartheta}\frac{\partial}{\partial \varphi}$$

$\vec{i}_r, \vec{i}_\vartheta, \vec{i}_\varphi$ unit vectors

$\vec{\nabla} \cdot \vec{V}$ (divergence of vector $\vec{V}$)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 V_r\right) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(V_\vartheta \sin\vartheta\right) + \frac{1}{r\sin\vartheta}\frac{\partial V_\varphi}{\partial \varphi}$$

---

curl $\vec{V}$ $\quad \vec{\nabla} \times \vec{V}$

$$\frac{1}{r\sin\vartheta}\left(\frac{\partial}{\partial \vartheta}\left(V_\varphi \sin\vartheta\right) - \frac{\partial V_\vartheta}{\partial \varphi}\right)\vec{i}_r +$$

$$+ \frac{1}{r}\left(\frac{1}{\sin\vartheta}\frac{\partial V_r}{\partial \varphi} - \frac{\partial}{\partial r}\left(r V_\varphi\right)\right)\underline{i}_\vartheta +$$

$$+ \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r V_\vartheta\right) - \frac{\partial V_r}{\partial \vartheta}\right)\underline{i}_\varphi$$

SYM 3

$$\vec\nabla \times \vec\nabla \times \vec V = \vec\nabla\, \vec\nabla \cdot \vec V - \nabla^2 \vec V$$

$\nabla^2 =$ Laplacian Operator

rectangular coordinates x, y, z

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \vec\nabla \cdot \vec\nabla$$

spherical coordinates $\nabla^2 f =$

$$\cancel{\nabla^2 f =} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta \frac{\partial f}{\partial\vartheta}\right) +$$

$$+ \frac{1}{r^2 \sin^2\vartheta}\frac{\partial^2 f}{\partial\varphi^2}$$

with f function of r, ϑ, φ

SCA1 Copia TRIESTE

Scattering.

When an electromagnetic wave (e m wave) propagates in an otherwise homogeneous medium the presence of a local inhomogeneity, such as the presence of a finite obstacle or a local change of electromagnetic properties (real or complex refractive index), part of the incident power is deducted from the incident wave and is lost by absorption or/and is taken away by scattered field. This effect is generally called as "**scattering**".

One can distinguish between : A - a linear effect, when the scattered field maintains the frequency of the incident wave, and B - non-linear effect when there is a change of frequency.
The mathematical description is different.
We now limit ourselves to the first case A.

The case A finds itself in a variety of practical problems. We can list some examples.

1). Propagation of radiation in an atmosphere with the presence of inhomogeneities such as water or ice particles (tropospheric clouds which present spherical (or similar shape) water droplets, ice clouds with ice particles of different shapes : columns, plates, rosettes of regular or slightly irregular shapes).

2). Propagation in biological tissues (human or animal muscles, skin, bones).

3). Industrial materials (wood, food, tissue)

A particular, though very important case, is that of the scattering by a medium with many suspended particles, for which the scattering of the incident field can be separately dealt with for each single particles. For this it is necessary that the particles are positioned at a sufficient distances from each other (a commonly criterion could be that each particle is in the far field region of the field scattered by the other particles).
We now add that it is frequent to meet the case when it is important to take into account that the field scattered by each particle on turn acts on the other articles, and it is on turn scattered. This is the effect called as **Multiple Scattering** for which a possible detector receives radiation coming from"single scattering" , "double scattering" , third order scattering t", and so on. That is a realistic description could to take into account as many "orders of scattering" as it is opportune.

The case of **Single Scattering** by a single particle is the basic one. We can start by consider this case.

As a simple scheme we can think of a homogeneous not absorbing medium like that which is assumed in such problems as propagation of e m radiation in the atmosphere This is a very good approximation when one considers a very clean atmosphere.
In such a medium let us consider the simple case of an em (electromagnetic) wave (for instance with optical wavelength) impinging on a particle whose physical-chemical properties are different from those of the surrounding medium.

The physical effect can be schematized as:

The em field both inside and outside the particle becomes different from that of the "undisturbed" wave.

If the undisturbed incident wave is represented by the incident field **Ei** , then the resulting field outside the particle can be written as :

1) Ei + Es

where **Es** is generally indicated as **Scattered field.**

Es depends on several causes. The sum (1) is named as **diffraction** field. and is considered in such problems as diffraction by apertures in screens.

Coming back to the single particle, the internal field within the particle is again different from that which would have been in the homogeneous medium at the same position. It is generally named as Internal Field (let us use the symbol **Eint** for it).

The particle will be named as **Scatterer,**

It is useful to define some terms.
We now consider the important cases of linear problems, that is when the scattered and internal fields maintain the same frequency of the incident one.
We consider the case of fields oscillating with time at frequency $f = \omega / 2\pi$.
The time dependence is expressed by a factor $\exp(-i\omega t)$. The form of a generic electric field is therefore.

$\mathbf{E}(P,t) = \mathbf{E}(P)\exp(-i\omega t)$. where $\mathbf{E}(P,t)$ is a vector function of the position P .

We have to take into consideration that **in many papers or textbooks a factor exp (i ω t). is considered instead of exp(-i ω t).** All the formalism is then maintained with the simple change of the sign of the imaginary unit **i.**
A plane TEM electromagnetic wave which is propagating in a homogeneous medium along the z direction with ε, μ dielectric and magnetic constants of the medium (ε_0, μ_0 for a medium assuming the properties of the vacuum) can be of the form :

2a) $\mathbf{E}(P)\exp(-i\omega t).\exp(ikz)$ with $k = \omega(\varepsilon\mu)^{1/2}$ and the corresponding magnetic field is :

2b) $\mathbf{H} = 1/Z \quad \mathbf{z} \times \mathbf{E}$ with Z specific medium impedance $Z = (\mu/\varepsilon)^{1/2}$

The sign X indicates vector product. z : unit vector in the direction z

SMAVY
SCA3

By the presence of a scatterer a spherical wave with electric field **Es** is produced in the medium outside the scatterer. Let us consider it as a homogeneous medium.

One can derive from the general laws of electromagnetism that, at a sufficient distance r from the scatterer **Es** assumes the properties of a TEM spherical wave diverging from the particle.

TEM (Transverse Electro Magnetic) means that the scattered field has only components perpendicular to the radial line connecting a reference point within the particle to the point where **Es** is considered.

In this case a general expression for the field propagating in the radial direction **s** at sufficient distance r from the source is

3) $\mathbf{E}(P,t) = \mathbf{A}(P)\ \exp(-i(\omega t - kr)) / r$.
with the vector **A** (P) perpendicular to the radial direction **s.** $k = 2\pi/\lambda$
r is the distance from a reference point in the region of the scatterer (source). **s** unit vector in the radial direction.

A can have any form of polarization (linear, circular etc.).

In this case the magnetic field **H** is related to **E** by the common relationship for the TEM wave :

4) $\mathbf{H} = 1/Z\ \ \mathbf{s} \times \mathbf{E}$ with Z specific medium impedance $Z = (\mu / \varepsilon)^{0.5}$

The Poynting vector $\frac{1}{2}\,\mathbf{E} \times \mathbf{H}^*$ is in the radial direction **s** . It represents the (average) flux of power per unit area.

From now on we assume that the constants of the medium are $\mu 0$, $\varepsilon 0$ (those of the vacuum).
If the scattered field is due to a finite oscillating source whose size can be represented by a linear dimension D, the distance r at which the produced field assumes the forms **3,4** is related to the wavelength λ of the field, the linear dimension D of the scatterer by the simple qualitative relationship :

5) $$r >> D^2 / \lambda \ (*)$$

(>> generally means a factor of (say) several units.
In this case one says that the scattered field **Es** is in the "Far Field" (or simply **Es** is the Far Field).
When the relationship 5 is not respected **Es** has less simple properties with respect to Eq.5.
A component of E or H along the radial direction r can appear.

The characteristics of the divergent wave (intensity and polarization) depend on many factors : wavelength, physical and geometric characteristics of the scatterer, direction of scattering.

(*)M. N Mischenko, L.Travis, A.Lacis. Scattering, Absorption, and Emission of light by Small particles. Cambridge University Press. 2002, Revised Electronic Version. NASA Goddard Institute for Space Physics. Considerations in Sect. 3.3 indicate that this condition can be considered to be sufficient for considering independent scattering by randomly distributed scatterers., at least for particles not much smaller than the wavelength.

SCA4

One can see that in the far field one can represent the scattering effect by a matrix relationship between two component of the transverse incident wave , say Ex and Ey for an incident wave propagating in the z direction , and the two components , say Es1 and Es2 of the far field. For this we choose two perpendicular directions **u1** and **u2** (unit vectors) perpendicular to the propagation direction of **s** .The matrix is named as **Amplitude Matrix, or Jones Matrix.**
The relationship is then :

6)

$$\begin{vmatrix} Es1 \\ \\ Es2 \end{vmatrix} = \exp(i\,k\,r) / r \begin{vmatrix} f11 & f12 \\ \\ f21 & f22 \end{vmatrix} \begin{vmatrix} Ex \\ \\ Ey \end{vmatrix}$$

For this form of the matrix the elements f11, f12 ,f21, f22 have the dimension of length. They depend on the direction of scattering.

One has to note that in some textbooks the matrix relationship is presented in a different, but equivalent, form in which the elements are adimensional, due to the presence of a different form of the factor (e.g. exp(i k r) / k r))

A simple but important case (encountered for instance in the remote sensing of the atmosphere) is that of spherical scatterers (e.g. water droplets of rain or fog). In this case one can choose the directions **x** and **u1**in the "plane of scattering" (that defined by the incident and scattering directions) and **y** and **u2** perpendicular to this plane. Then, because of this choice, the symmetry situation simplifies the matrix. Now only the elements f11 and f22 of the matrix are different from zero)
For a more general case other matrix elements are non-zero.
When the Stokes Vector formalism is used, representing power propagation and polarization of the incident and the scattered wave, the relationship between the four elements of the Stokes vector of the waves are again related by a matrix formalism (4X4 Mueller Matrix or Phase matrix):

SCA5

Definition of the Stokes Vector and form of the Mueller (phase) matrix.

For the Stokes Vector (symbol **V**) one finds different definitions. They are equivalent since the Vector is very often used for relationships between a couple of quantities, and different normalisation of the Stokes vector elements are of no importance.

A common definition regards the Electric vector components of a TEM wave propagating in the z direction. The Electric field **complex** amplitudes are generally of the form

7) $Ex = |Ex| \exp(-i\psi)$ $\quad Ey = |Ey| \exp(-i\phi)$ $\quad$ (a factor $\exp(-i(\omega t - kz))$ is not shown)

The four "parameters" I, Q,U ,V of the Vector can be defined as :

8) $I = |Ex|^2 + |Ey|^2$ $\quad Q = |Ex|^2 - |Ey|^2$ $\quad U = 2\,\mathrm{Re}(Ex\,Ey^*)$ $\quad V = 2\,\mathrm{Im}(Ex\,Ey^*)$

(the apex *in dicate complex v conjugate)
It can be seen that I is proportional to the modulus of the Poynting Vector, that is power per unit area carried by the wave. (the factor 1 / 2 Z is not explicit)

Q represents the difference of power between the two electric components

Thus the Vector shows all the characteristics of the TEM wave : propagation of power and polarization .
Another equivalent form would be with the 3rd and 4th elements as

$U = 2\,|Ex|\,|Ey| \cos\delta$ $\quad V = 2\,|Ex|\,|Ey| \sin\delta$ $\quad$ as can be verified from the above forms.

δ is the difference of phase between Ex and Ey. ($\delta = \phi - \psi$) :

(for instance $\delta = 0$ indicates linear polarization, $\delta = \pi/2$, with $|Ex| = |Ey|$ circular polarization, etc)

HOWEVER it is always opportune to check the forms of the definition of the Stokes Vector in every book or paper under examination.

In the so-called **Modified** Stokes Vector VM the two first elements are

9) $I1 = |Ex|^2$, $I2 = |Ey|^2$ ad the 3rd and 4th parameters as in the previous definition.

The forms (8) and (9) of the Stokes Vector are defined by considering a monochromatic wave with fixed frequency.
From the form (8) one derives the simple relationship :

10) $I^2 = Q^2 + U^2 + V^2$ which is typical for "monochromatic" totally polarized waves.
For a general case of non- monochromatic waves (with spectral extension), the relationship 10) is not strictly valid.

For instance for "Non Polarized" waves the elements U and V are null, since the phase δ has equal probability for any value, and all elements are defined by a probability law.

The linear relationship between the Stokes Vectors of the scattered (index s) and the incident (index i) waves is shown by a 4X4 Matrix (Phase matrix , or Mueller matrix) which we will indicate as **M**

One can easily see the relationship between the element of the Phase Matrix **M** and those of the Amplitude Matrix. For instance by considering the **Modified Stokes vector** of position 9. (I1,I2,U,V), and indicating the Stokes Vector of the incident wave as **Vi** and that of the scattered waver in a certain direction as **Vs**, one has :

:

Vs = $1/r^2$ **M Vi**

$$\begin{vmatrix} I_1^s \\ I_2^s \\ U^s \\ V^s \end{vmatrix} = \frac{1}{r^2} \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \begin{vmatrix} I_1^i \\ I_2^i \\ U^i \\ V^i \end{vmatrix}$$

The elements of the **M** matrix depend on the directions of scattering.. Then in case of the spherical symmetry of the scatterer the Phase Matrix can be simplified to having only 6 elements different from zero. For this one choses the unit vectors as indicated above :
directions **x** and **u1**in the "plane of scattering" (that defined by the incident and scattering directions) and **y** and **u2** perpendicular to this plane
In this case the non zero elements are a11, a22, a33, a44, a34, a43 . If the orientation of the axes for the incident wave is different, one can reach this situation by a rotating matrix applied to the Stokes Vector.

ELEMENTS OF THE PHASE MATRIX amn

a11 = $|f11|^2$ a12 = $|f12|^2$ a13 = Re { f11 f12*} a14 = -Im {f11 f12*}

a21 = f21 a22 = f22 a23 = Re { f21 f22*} a24 = - Im { f21 f22*}

a31 = 2 Re{ f11 f21*} a32 = 2 Re{ f12 f22*} a34 = Re{ f11 f22* - f12 f21*}

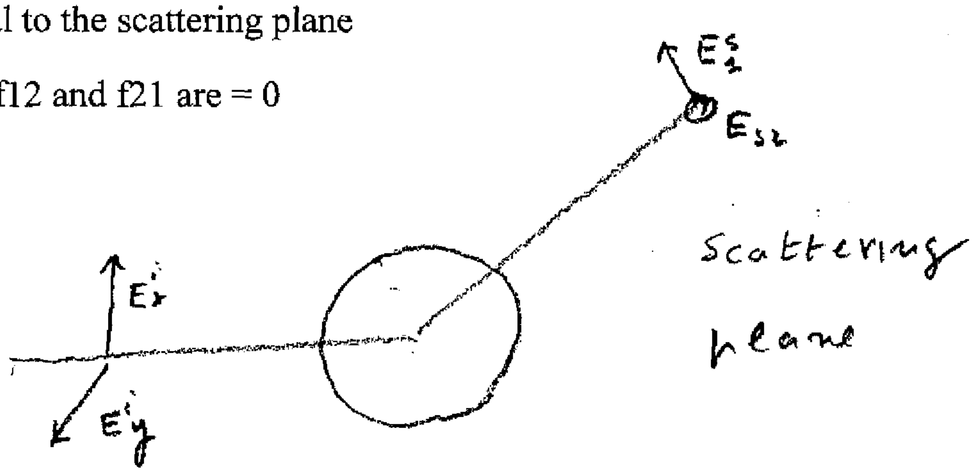
a34 = - Im{ f11 f22* + f12 f21*}

a41 = 2 Im { f21 f21*} a42 = 2 Im { f21 f22*} a43 = Im{ f11 f22* + f12 f21*}

a44 = Re{ f11 f22* + f12 f21*}

The matrix has only six elements different from zero if the scatterer is spherically symmetric and the components of the incident and scattered field are parallel or normal to the scattering plane

Then f12 and f21 are = 0



After page

SCA 7

BALANCE of POWER.
Now one can see that the scattered divergent spherical wave **Es**, while propagating far from the scatterer, carries power.

This power can be obtained by considering the Far Field and integrating the flux of power from a spherical surface containing the scatterer ,using the Poynting Vector $\mathbf{P} = (1/2\ Z)\ E^2\ \mathbf{s}$ (watt / m^2), which is directed as **s,** normal to this surface in the Far Field.
Let us define this total scattered power as Ws.

Moreover, if the medium inside the scatterer is absorbing (non-purely dielectric), the field inside the particle produce dissipation of power. Let us indicate this internal dissipated power as Wi. This happens if the relative dielectric constant ε_r is complex : $\varepsilon_r = \varepsilon r + i\ \varepsilon i$. One can see that the presence of εi. is equivalent to the presence of a density of electric current J

NEXT ANNEXED PAGE 7A

It is immediate to deduct that the power carried initially by the incident wave is reduced for this effect by an amount Wr, and

Wt = Ws + Wi

Given the Poynting vector **P** (W/m^2) of the incident undisturbed wave , one can write a proportionality relationship :

$$
\begin{aligned}
Ws &= \sigma s\ P \\
 & \qquad\qquad \text{(Watt)} \\
Wi &= \sigma i\ P
\end{aligned}
$$

Where the factor σs is indicated as Scattering Cross Section.
and σi is indicated as Absorption Cross Section.

The sum $\sigma s + \sigma i$ is the Total Cross Section, or Extinction Cross Section, , σe (all with dimension of area.

SCA 7a

APPENDIX ABSORPTION BY COMPLEX DIELECTRIC CONSTANT.

Dielectric constant of the medium $\varepsilon = \varepsilon 0 (\varepsilon r + i \varepsilon i)$

One has $\mathbf{D} = \varepsilon 0 \mathbf{E} + \mathbf{P}$

P density of polarization $\mathbf{P} = (\varepsilon - \varepsilon 0) \mathbf{E} = \varepsilon 0 (\varepsilon r + i \varepsilon i - 1)$

εr , εi real and imaginary part of relative dielectric constant

One has for the current density $\mathbf{J} \quad - i \omega \mathbf{P} = - i \omega \varepsilon 0 (\varepsilon r + i \varepsilon i - 1)$

wa density of absorbed power

One has

$$wa = 1/2 \; \mathrm{Re} \{ \mathbf{E} \cdot \mathbf{J}^* \}$$

$$= 1/2 \; \varepsilon 0 \, \varepsilon i \; \mathbf{E}\,\mathbf{E}^* \qquad \varepsilon i \text{ positive}$$

The absorption cross section : $\sigma a \int_V wa \, dv \, / Pi$

With the integral over the particle volume and Pi is the modulus of the Poynting vector of the incident wave

σa =

7 a

SCA 8

Passage of the incident wave in a medium with a random suspension of scatterers.

Consider a medium where scatterers are distributed at random in space.
First let us assume that they are equal and , if non spherical, with equal orientation of their axes.
Let N the number per unit volume of these particles. Each of them presents the same extinction cross section σe.
The Poynting vector P of the incident wave :

$\mathbf{Pi} = (1/2Z)\ Ei^2\ \mathbf{s_0}$ (Watt per square meter , $\mathbf{s_0}$ = unit vector in the direction of propagation)

is reduced by the passage through the medium.
If one can assume that each of these scatters or absorbs independently (They must be at sufficient distance from each other : one can put this condition as that they are in the Far Field of each scattering field from the others, although this condition is questionable and it can vary).
Then power per square meter of the incident wave lost per unit distance is $P\ N\ \sigma e$.

Then one can write the relationship:

$$dPi/dz = -\ Pi\ N\ \sigma e$$

With the consequence that there is an exponential reduction of the incident power for a path from $z = 0$ to a generic z :

$$Pi(z) = Pi(0) \exp(-\ N\ \sigma e\ z). = Pi(0) \exp(-c_e\ z).$$

The quantity $N\ \sigma = c_e$ is named as Linear Extinction Coefficient of the medium.
c_e (unity of length $^{-1}$) .

In case one has a mixture of different scatterers, each with its σ_{ei} and number Ni per unit volume the expression for the Extinction Coefficient of the medium comes from the sum

$c_e = \Sigma_i\ \sigma ei\ Ni$ with, immediately, the extension of the above exponential relationship.

SCA9

The "**Phase Function**" .

Generally the Poynting vector of the scattered wave depends on the direction **s**. The scattered power per unit solid angle Wu has a different value according to the direction. By considering as unitary the total scattered power Ws, one can normalize the power per solid angle to the total value, and define the phase function p(**s**) as .

p(**s**) = Wu(s) /Ws

p(s) has the dimension of solid angle to minus 1.

Obviously : the integral of p(**s**) over the total solid angle = **1**

NOTE : In some texts or papers the phase function is differently defined :

p(**s**) = 4 π Wu (s) / Ws and one has to check which definition is applied.

In case of a medium containing particles of the same particles (size, shape, orientation, internal medium) the phase function also represents power scattered per unit volume in each direction per unit solid angle divided by the total scattered power by the unit volume.

Given a particular kind of particle (size, physical and geometrical characteristics, orientation in space of its shape) one can define a particular phase function.
If the particles are of different, and of different number density the overall scattering, absorption , extinction coefficient of the medium come from averaging :

We indicate as Ni the number density (m-3) of particles of kind i.

As for the phase function the overall phase function p_t (s) can be written from averaging

p_t (s) = Σ_i Ni p_i(s) σ_{is} / Σ_i Ni σ_{is} where the sums are over the total numbers of particles of kind i per unit volume. That is p_t (s) represents power scattered in a certain direction per unit solid angle normalized to total power.

Σ_i Ni σ_{is} is the sum of powers scattered per unit volume due to the i components of the medium. Per each of them the extinction is expressed by the proper extinction coefficient , and the total power scattered per unit length of the medium, comes from integrating p_t (**s**) over the total solid angle 4 π.

Integral over 4 π. of p_t (**s**) d ω = 1

One can find that the definition of the phase function is simplified if one assumes an approximate azimuthal symmetry around the direction of incidence . Thus the phase function is substituted for by an average about the azimuth angle. This is acceptable for spherical particles and the scattering direction near the forward direction . However, it is valid for non polarized waves for which there is no distinction between different azimuth directions.

SCA10

Some simple methods for dealing with scattering.

Rayleigh - Debye Scattering. (R.D. approximation)

The Rayleigh-Debye scattering approach (also named as Rayleigh-Gans-Debye) is considered acceptable in the case of a small particle for which the phase of the internal **E** field is assumed to be simply that of the incident undisturbed wave. Moreover it is assumed that the complex amplitude of the incident field inside is not altered by the presence of the particle.

To accept this point the strong inequality is assumed :

1) $2k\, a\, (m-1) \ll 1 \qquad k = 2\pi/\lambda$

where a is the maximum linear extension of the particle. m is the internal refractive index.

We recall the relationship (with ε dielectric constant of a medium and ε_0 that of the vacuum)

div $\mathbf{D} = \varepsilon\, \mathbf{E} = \varepsilon_0 E + \mathbf{P}$ (in some formalism instead of **P** one finds $4\pi\, \mathbf{P}$)

with **P** density of electric polarization (electric moment per unit volume)

2) $\mathbf{P} = (\varepsilon - \varepsilon_0)\, \mathbf{E}$ $\qquad \varepsilon_0$ dielectric constant of vacuum

Given a single element of volume d V inside the particle, then the field internal to the element is assumed to be equal to that obtained if dV were a small spherical homogeneous dielectric particle, isolated in a medium, on which there was the incident external undisturbed wave. That is for dV the Rayleigh approximation is applied.

Thus, for each element the internal moment is calculated. The general properties of continuity of the electrostatic potential and its normal derivative tangential are sufficient to obtain the internal field and the corresponding density of electric moment.

By taking into account the phase the $\delta(P)$ that the undisturbed incident field would have in a point P inside the particle, the internal field **E**int is given as (see : APPENDIX SPHERE-RAYLEIGH)

3) $- \mathbf{E}i \exp(i\, \delta(P))\ 3\, \varepsilon_0 / (\varepsilon_1 + 2\, \varepsilon_0)$ $\qquad \varepsilon_0$ dielectric constant of vacuum (as already stated we assume that the external medium has the characteristics of the vacuum)

The differential internal electric moment d **m** is given as :

4) $d\, \mathbf{m} = - \mathbf{E}i \exp(i\, \delta)\ (\varepsilon_1 - \varepsilon_0)\ 3\, \varepsilon_0 / (\varepsilon_1 + 2\, \varepsilon_0)\, dV$

SCA 11

If one considers an elementary electric moment d **m** one can write for the field **Es** in the far region, in the **s** direction of scattering due to this moment :

5) $d\,\mathbf{Es} = -k^2 / (4 \pi \varepsilon_0)\ (\mathbf{s} \wedge (\mathbf{s} \wedge d\,\mathbf{m}\ ) \exp(ikr) / r$ ^ **vector product**

By considering the scattered field, a more complete expression for the electric field **Es** produced by an oscillating electric moment d **m** in a homogeneous medium (ε_0, μ_0) is :

5') Es=

$= - k^2 / (4 \pi \varepsilon_0)\ (\mathbf{s} \wedge (\mathbf{s} \wedge d\,\mathbf{m}) \ +(1 - ikr) (\ (3\, d\,\mathbf{m} \cdot \mathbf{s})\ \mathbf{s}\ - d\,\mathbf{m}\ ) \ / (\varepsilon_0\ r^2)\ ) \exp(ikr) / r$

One can see that Eq.5 is the form of Eq.5' at an r distance from the elementary moment for which

5'') $r >> \lambda$.

where the far expression for the magnetic field H is simply if μ_0 / ε_0

$\mathbf{H} = 1/Z\ \mathbf{s} \wedge \mathbf{E}$ $Z = (\mu_0 / \varepsilon_0)^{0.5}$ if μ_0, ε_0 are the constants of the external medium.

In the far field (5,5'') one has a TEM electromagnetic spherical wave.

To obtain the scattered field one has to integrate over the particle volume. There is a phase factor which differs according to the position of the element of volume This factor has different values for different positions inside the particle.
SEE THE FIGURE.
Generally one can refer to a "central point" O inside the particle , for which

exp (ikr) = exp(ikr0). Also the phase of the incident field are referred to the position of O.

Then in the integration for each position an overall phase factor has to be taken into account. Let us consider the scattered field in the far region.
With reference to the phase of the element at the position O, the phase pertaining to an element at the position P is given as

$\exp(ik\,(P\text{-}O) \cdot \mathbf{z}) \exp(-\,ik\,(P\text{-}O) \cdot \mathbf{s})$ (incidence in the z direction)

The first phase factor is due to the different z of the element. The second factor is due to the different path length from the element to the scattered field point in the direction **s.**

An integration over the total volume of the particle gives the scattered field. If the partickle is homogeneous only the phase factor enters in the integration.

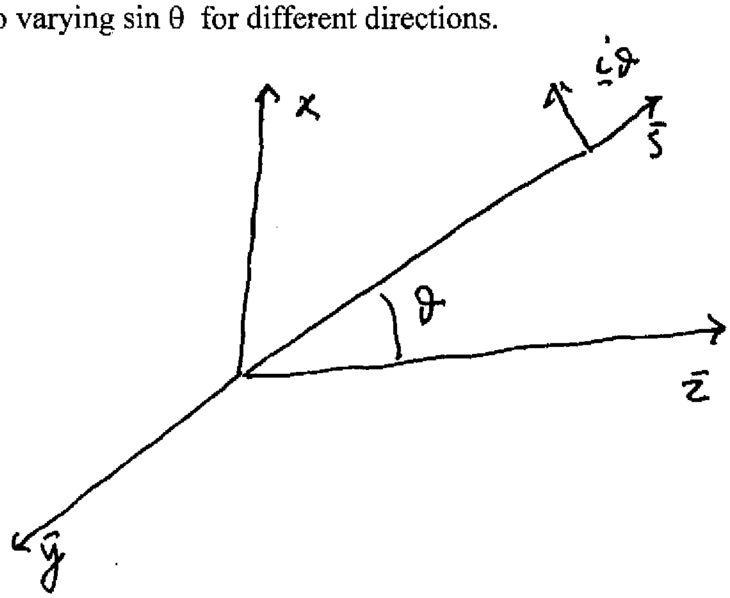
For a simplest example. Propagation of the incident field in the z direction. Polarization of **Ei** in the x direction.
Consider the case of a small sphere, when the scatterer is so small that the phases factors can be considered equal for the whole volume .
In the far field region, the scattered field from a n elementary moment m of linear polarization , small with respect to the wavelength has the expression .
SEE FIGURE

SCA12

$\underline{m} = m\,\bar{x}$

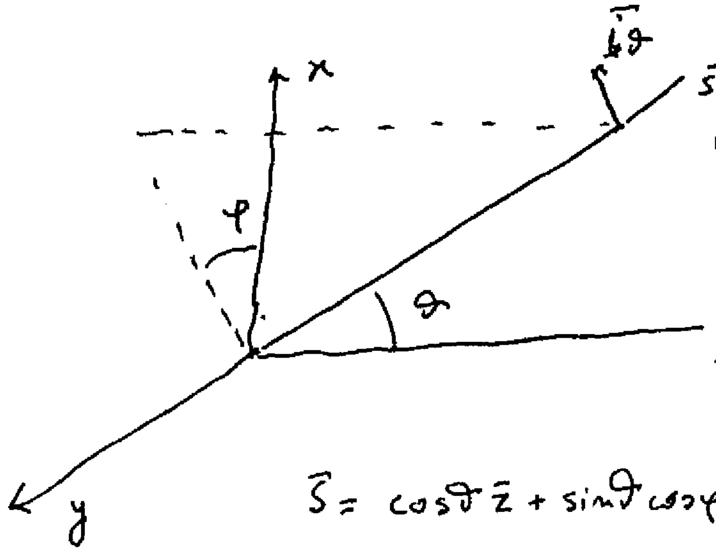
Es = $- \exp(ikr) / r\, K^2 m / (4\, \pi\, \varepsilon o)\, \sin\theta\, i_\theta$

Where the unit vector i_θ is directed in the plane of scattering and θ is the scattering angle .
m is the total internal electric moment given by the product of the volume times the density of internal moment, which is assumed to be uniform for the particle small with respect to the wavelength.
One sees that, given a scattering plane, an elementary moment directed as **x** (see figure corresponds to varying $\sin\theta$ for different directions.



$\varphi = 0$

In general:



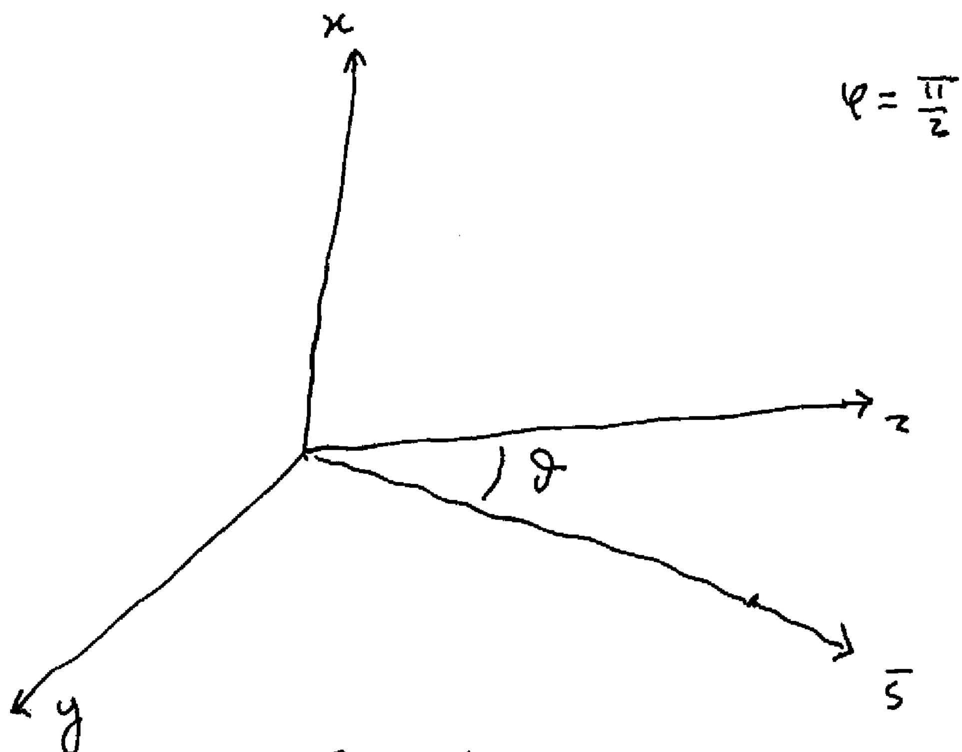
$i_\varphi = \bar{s} \wedge \bar{i}_\vartheta$

$\bar{S} = \cos\vartheta\, \bar{z} + \sin\vartheta \cos\varphi\, \bar{x} + \sin\vartheta \sin\varphi\, \bar{y}$

If the polarization of the incident field is with Ex and Ey different from zero, let us consider the plane y z as plane of scattering .

For a scattering direction **s** in this plane one sees that the factor with the trigonometric function sin for the field due to the component Ex is sin $\pi / 2$ for every direction in this plane. The factor differs for the field due to the Ey component for different directions in this plane.

That is the polarization of the scattered field generally differs with the direction of scattering. It is maintained if the scattering is in the z direction $\theta = 0$ for which $\theta = \pi / 2$ for both Ex and Ey.



$\hat{s}$ in the y z plane

SCH 14

AN APPLICATION:

An application of the Rayleigh method can be found in :K.Sarabandi, T.Senior : "Low-Frequency Scattering from a Cylindrical Structure at Oblique Incidence ".IEEE Trans on Geoscience and Remote Sensing . Vol.28,pp.879-xxx, 1990

In the internal of the small section cylinder the polarization of the medium is obtained by r the R.D.G. approximation. Cylinders also of non circular section are dealt with.

THE DISCRETE-DYPOLE APPROXIMATION.

In a paper by C.E.Dungey, C.Bohren. "Light scattering by non-spherical particles . a refinement to the coupled-dyèole method". J.Opt.Soc.A. Vol.8, pp 81-87. 1991
the so-called **Discrete-dipole approximation** for dealing with scattering by extended particles was recalled and discussed.

(See also B.Draine and P.J.Flatau (J.Opt.Soc.Am. Vol.11. pp-1491-1496. 1994))

The principle of the discrete dipole method is based on the decomposition of the scatterer into small spheres.
To each of them the internal moment is calculated as if they were isolated under the action of the incident undisturbed field.
The summed scattered field from them in an external far field is obtained by interference taking into account the phases due to the different positions.
This is the first order approximation.
However each sphere is in the summed first order scattered fields from the other particles. Thus one calculates the second order scattered field that each particle produces due to this effect.
One sees that the first order field by which one sphere acts on another one is **NOT** the far field. One must use a complete expression for it (the terms of the scattered field neglected for the far field must be included).

The relations for this second factor is then expressed by a matrix connecting the field on every particle to the fields of the other ones.

Thus the scattered secondary field from each particle can be calculated.
This is added to the first order scattered field.
The process can be repeated until one obtains some almost stable result.

Reference to Dungey's and Bohren's paper for a history of applications.

OT1

THE OPTICAL THEOREM FOR NON-SPHERICAL PARTICLES.

(from Il Nuovo Cimento C)

Starting with the case of a medium containing one type of particles, the relationship between the incident (i) and scattered (s) fields (transverse components in the far field) via the amplitude scattering matrix can be written as

$$\mathbf{E}^{s} = \mathbf{J}\,\mathbf{E}^{i} \qquad \text{with:} \qquad (1)$$

$$\mathbf{J} = \frac{\exp(ikr)}{r}\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

where the elements f_{mn} depend on the directions of arrival and scattering.

Given the incident field: Ei = E_0 **u** exp (ikz), with the unit vector **u** = ax **x** + ay **y** defining polarization. the optical theorem for the particle extinction cross section Ce can be written as:

$$Ce = (4\pi / k)\ \mathrm{Im}\,(\,f_{11}\,|ax\,|^2 + f_{12}\,ax\,ay + f_{21}\,ax\,ay^{*} + f_{22}\,|ay\,|^2\,) \qquad (2)$$

This is the explicit form of the relationship given by text books such as in [5], where the relative phase of incident and scattered field is not always evidenced.

In Eq.2 **ax** has been taken as reference for phase, and the matrix elements f_{mn} are taken in the forward direction. For non-spherical particles **Ce** depends on the direction of incidence.

OT2

For a medium with a suspension of identical particles with equal orientation the linear extinction coefficient of the medium is given as:

$\sigma_e = N\,Ce$, (3) with N number of particles per unit volume.

Eqs. (2) and 3 can be obtained as follows.

Let us define as x, y (**x, y**) axes of the reference system those along two directions for which the 2x2 scattering matrix J has the elements f_{11}, f_{12}, f_{21}, f_{22}, which are functions of the direction **s** with respect to the incident wave direction **z.** Let the incident wave field Ei (unitary amplitude) have the complex components ax, ay along the x and y axes respectively.

$(|ax|^2 + |ay|^2) = 1$. Thus polarization is defined).

Consider an elementary layer (**width dz**) where the scatterers are contained, and the scattered field on an x y plane at a distance D (plane D) from the layer. In a generic point on this plane the scattered field has also a component parallel to the z axis. A part from this component, which is here neglected, the component dE'^s in the D plane of the scattered field, due to a volume element dx dy dz of the layer, is related to the components of the incident field by linear relationship

OT3

dE'^{s} = (4)

$$\exp(ik(\rho^2+D^2)^{1/2})\ (\rho^2+D^2)^{1/2}\ ((ax\ f_{11}' + ay\ f_{12}')\ \mathbf{x} + (ax\ f_{21}' + ay\ f_{22}')\mathbf{y})\ N\ dx\ dy\ dz$$

ρ = (x2 + y2)1/2 ,

N number of particles per unit volume. The parameters f_{mn}' In Eq.4 are linear coefficients, depending on positions x,y and D. They connect the field components relative to the same x,y axes both for the incident and scattered fields. In the forward direction they coincide with the f_{mn}.

dE'^{s} **of Eq.4 has to be added to the incident wave.**

OT4

If the layer is extended laterally (actually to infinite) one can apply **the principle of stationary phase** to obtain by integration the field scattered by the whole layer:

$\mathbf{dE'^s}$ = $(2\ i\ \pi/\ k)\ \exp\ (ikD)\ ((axf_{11} + ayf_{12})\ \mathbf{x} + (axf_{21} + ayf_{22})\ \mathbf{y})\ N\ dz$ (5)

where the f_{mn} parameters are the elements of the scattering matrix **J** taken in the forward direction. **The total field is thus obtained as.**

$\mathbf{E} = \mathbf{E}^i + d\,\mathbf{E'}^s$

The applied principle takes into account the value of the integrand at the point where the phase is stationary, that is at $\rho = 0$. Due to the applied principle $\mathbf{dE}^s$ is parallel to $\mathbf{E}^i$.

Since the added $d\mathbf{E}^s$ is uniform over the plane at D, **it can be considered as the field of a plane wave travelling in the z direction and added to the undisturbed wave.**

OT5

As for power W per unit area at the distance D, taking into account the differential, one has (apart from a constant factor 1/(2 Z), with Z medium specific impedance)

$$W = |\mathbf{E}^i + d\,\mathbf{E}^s|^2 = |\mathbf{E}^i|^2 + 2\,\mathrm{Re}\,(\mathbf{E}^i \cdot d\,\mathbf{E}^{s*}) \qquad (6)$$

By considering Eq. (5) one has the increment per unit width of the medium

(dz = 1)

$$\mathbf{d\,W = -(4\,\pi / k)\,N\,Im\,(f_{11}^*\,|\,ax\,|^2 + f_{12}^*\,ax\,ay^* + f_{21}^*ax^*\,ay + f_{22}^*\,|\,ay\,|\,2)}$$

$$\mathbf{= (4\,\pi / k)\,N\,Im\,(f_{11}\,|\,ax\,|^2 + f_{12}^*\,ax^*\,ay + f_{21}ax\,ay^* + f_{22}\,|\,ay\,|^2)}$$

(with f_{mn} in the forward direction).

If the considered incident power is unitary, one obtains the linear extinction coefficient, **From Eq.7 one can verify that the extinction cross section Ce for the particles in the medium is in agreement with Eqs. (2) and (3).**

$$\mathbf{Ce = (4\pi / k)\ Im\,(f_{11}\,|ax|^2 + f_{12}\,ax\,ay + f_{21}\,ax\,ay^* + f_{22}\,|ay|^2)} \qquad (2)$$

And the linear extinction coefficient of the medium is:

$\boldsymbol{\sigma_e = N\,Ce}$, (3) with N number of particles per unit volume.

OT6

2. Polarization of the propagating field.

It is always possible to define **two polarization states** for which polarization is maintained. In [5,6] one finds that the effect can be related to the properties of the Mueller **(4x4)** matrix **M** pertaining to the medium: [5,6]).

These polarization states are obtained in [6 Sect. 3.8] by considering the **M matrix**, and imposing that the four elements of the Stokes vector of the propagating beam are attenuated in the same proportion. **This note** shows that they can also be obtained by considering the properties of the 2X2 **J** matrix of the scatterers.

One can impose that the the polarization of the scattered field in the forward direction is equal to that of the incident field. That is the ratio of complex amplitude components is the same as that of the incident field.

$$(\mathbf{a}_x \mathbf{f}_{11} + \mathbf{a}_y \mathbf{f}_{12}) / (\mathbf{a}_x \mathbf{f}_{21} + \mathbf{a}_y \mathbf{f}_{22}) = \mathbf{a}_x / \mathbf{a}_y$$

We thus obtain a second order equation for the ratio a_x / a_y, and the E_x and E_y component of the incident field. By simple algebraic passages one obtains the two modes, which, apart from constant factors to be introduced for normalization

$$\mathbf{E}_1 = \mathbf{x} - 2\,\mathbf{y}\; f_{21} / (f_{22} - f_{11} + R)$$

$$\mathbf{E}_2 = \mathbf{y} + 2\,\mathbf{x}\; f_{12} / (f_{22} - f_{11} + R)$$

with: $R = ((f_{22}-f_{11})^2 + 4\, f_{12}\, f_{21})^{1/2}$, and the elements f_{mn} of the **J** matrix are taken in the forward direction.

One can see that Eqs. (8a) and (8b) for the polarization modes (directly obtained from the **J** matrix elements) correspond to those of [6, sect. 3.8,].

OT7

3. Polydispersion of particles.

Since the scattered field components are taken in the strictly forward direction, the expressions for the modes and the linear extinction coefficient of the medium are obtained from those for the monodispersions, with the following substitutions (i: index for the kind of particle):

$$g_{11},\ g_{12},\ g_{21},\ g_{22} \quad \text{in the place of} \quad f_{11},\ f_{12},\ f_{21},\ f_{22} \tag{9}$$

with:: $g_{mn} = \Sigma_i\, Ni\, f_{mni}$ and N_i number of particles of kind i per unit volume. The extinction coefficient becomes:

$\sigma e = \Sigma\, Ni\, Cei$ (indexes i for particles with different particles and orientation). The polarization modes are obtained from eqs. (8a) and (8b) by substituting the quantities g_{mn} in place of f_{mn}. One can see that Eqs. (8a) and (8b) for the polarization modes (directly obtained from the **J** matrix elements) correspond to those of [6, sect. 3.8] where they were obtained by considering the extinction matrix for Stokes vectors.

For Eq.9 it has been taken into account that in the forward direction the relative phases contributions to scattered field from the particles are only due to the f_{mn} matrix elements, and not to differences in paths.

Acknowledgement. The study has been carried out in the framework of the INTAS program. Contract 05-100000-8024

OT8

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A generalization of the form for the optical theorem to the case of a scatterer in the near field of a source : evanescent components of the incident field are taken into account :

Physical Review E 71, (2005). pp. 056610 : 1-10 D.R.Lytle II, P.Scott Carney, J:C:Schotland, E.Wolf.

OT9

PARTICULAR SIMPLE CASE

IN THE CASE OF A SPHERICAL (HOMOGEOUS) PARTICLE THE OPTICAL TEOREM HAS A SIMPLE FORM.
IF THE POLARIZATION OF THE INCIDENT FIELD IS EXPRESSED BY THE VECTOR .

$$a_x \mathbf{x} \quad + a_y \mathbf{y}$$

PROPAGATION IN THE Z DIRECTION

x **y** unit vectors $\qquad |ax|^2 + |ax|^2 = 1$

ONE HAS :

$$CE = 4 P / K \ \mathrm{Im}\{ f11\, |ax|^2 + f22\, |ay|^2 \}$$

THE EXTINCTION POWER IS THE SUM OF THOSE RELEVANT TO THE TWO COMPONENTS OF THE INCIDENT FIELD.

OT10

Small dielectric particles and their cross section dependence on polarization deserves particular attention. For a simplest example let us consider the case of small dielectric ellipsoids. For small dielectric rotational ellipsoids the scattering properties can be obtained by the electrostatic approximation.
If the electrostatic approximation is adopted, the cross section of a dielectric particle cannot be simply obtained from the optical theorem, since scattered radiation in the forward direction is in phase with the exciting radiation.
However, we can remember the case of a small dielectric sphere. For this particle the scattered power and the cross section are obtained by taking into account the internal electric moment and the emitted power. On the other hand, the Optical Theorem gives the same cross section if one considers the first imaginary term in the expansion of scattered field in spherical wave. Then, analogously, for small dielectric ellipsoids the cross section can be obtained by considering the electric moment **M** internal to the particle, and the relevant radiated power. W

Let us now consider the z-axis as the narrow beam direction.
The scattered field for such particles can be obtained by considering the internal field and the relevant components of the electrical moments as are obtained by electrostatic method (Ref.11)
We now take into account the case where two groups of equal ellipsoids with different orientation are present in the medium. For each group fig. 1 shows the considered geometry. A group has the major axis along the x axis. The direction of the ellipsoid major axis for the second group is in the x y plane at an angle of 45degrees with the x axis.
NOTE : more details on the derivation of the **J** matrix elements for ellipsoids and the cross sections are given in Ref. 10, following Refs.11,12)

In order to calculate the linear extinction coefficient of the medium a factor equal to the number of identical particles per unit volume (or group of particles for the case of interest) is sub-intended. For a generic orientation of the ellipsoid major axis, the Fig. 1 indicates the geometry.
The Stokes vector is here taken in the form I, Q, U, V.

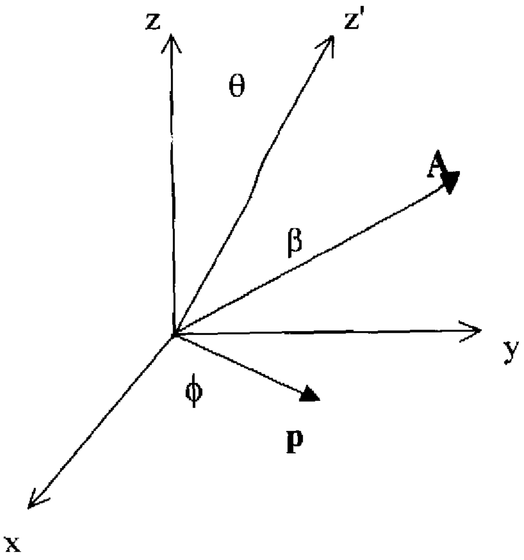

Fig.1 The propagation direction of the incident radiation is along z. The scattering direction is along z', and is on the plane y z (scattering plane) The vector **A** is along the mayor axis of the dielectric ellipsoid. It makes the angle β with the direction z. The unit vector **p** is along the projection of **A** on the plane perpendicular to z. **p** makes the angle φ with the axis x.

For our cases $\theta = 0$ (forward scattering)

Here we present one case in detail. Other cases can be found in Ref. 10
The polarization modes are obtained for a couple of dielectric ellipsoids.
One ellipsoid has its major axis along the x axis and the second, of equal size and shape, has its major axis in the x y plane, with an angle of 45°with the x axis. In our case . $\theta = 0$ (forward direction), $\phi = 0$ and $\pi/4$, $\beta = \pi/2$
For a general orientation of the ellipsoid, the elements the f_{mn} elements of the **J** matrix in the forward direction can be obtained (Ref.12) as:

JT11

The following table shows an example of ratio between scattered powers for small dielectric ellipsoids

Ratio between the scattered powers relevant to two orthogonal linear polarizations.
Dielectric Rotational Ellipsoid . Ratio between major axis A and minor axis B : A/B = 4
Relative dielectric constant $\varepsilon_r = 1.5$
Propagation of the beam along the z direction
Position of the axis A:
In the x z plane, making an angle θ with the x axis
W1 = power scattered with linear polarization of the incident beam along the x direction
W2 = power scattered with linear polarization of the incident beam along the y direction

D1 =1.039 (*) **D2 = 1.235 (*)**

θ	W1/W2
0	1.413
$\pi/9$,	1.364
$2\pi/9$	1.242
$\pi/3$	1.103
$4\pi/9$	1.012
$\pi/2$	1

same results for θ and $\pi - \theta$

(*) D1 and D2 are the parameters according to the formalism for the scattering by dielectric ellipsoids in J.Stratton. Electromagnetic Theory (Ref.7)

Slyngy, OT 12

PRINCIPLE OF THE STATIONARY PHASE.

It is a method for dealing a single or double integral when the integrand shows an oscillatory behaviour which becomes very rapid and tends to present a greater and greater number of oscillations when a parameter increases.

A single integral of the type

$$I(k) = \int_a^b g(x) \exp(ik f(x))\, dx$$

where f (x) and g (x) are regular, with a, b, k real numbers.

The situation occurs when k tends to an infinite value.
One understands that the integrand function shows oscillations more and more frequent, so that a in a greater and greater part of the x interval the contribution to the integration is cancelled. But If there is a zone where the phase is stationary this zone acy tually contributes to the intef gration results. The effect can be produced when the parameter k tends to infinite.

Say that this occurs at a point x0 of the integration interval.
It can be shown that the integration tends to the form :

I (k) >

$$g(x_0) \sqrt{\frac{2i\pi}{k f''(x_0)}} \exp(i k f(x_0))$$

Where the apex " indicates double derivation.

[illegible] OT13

The principle can be extended to a double integration of the type :

$$I(k) = \iint_{\Sigma} g(P) \exp(i k f(P)) \, d\Sigma$$

In this case the possibility of applying an analogous principle occurs if in the integration dominion there is an interval of the variables about the values x0, y0 where the phase is stationary.

It can be shown that the integral tends to :

$$I(k) \Rightarrow g(x_0, y_0) \frac{2 i \pi \exp(i k f(x_0, y_0))}{k \sqrt{f''_{xx}(x_0, y_0) \, f''_{yy}(x_0, y_0)}}$$