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Scattering theory
Part II

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SCATTERING by PARTICLES of SIMPLE SHAPES.   MIE THEORY

One can deal with this kind of problem by decomposing the e m field in elementary solutions of Maxwell equations. That is in particular solutions suitable for the shapes of the scatterers.

For instance. One considers **spherical waves** for spherical scatterers, **cylindrical waves** for cylinders.

The case of the sphere is dealt with by the " Mie Theory" : the incoming plane wave, the scattered field outside the sphere, and the internal field are represented by series of elementary solutions.
By applying the continuity of tangential electric and magnetic fields at the sphere surface the amplitudes of this elementary solutions are found .
In principle other shapes can be dealt with. However the cases of spheres and cylinders are the most usually found.

(An extension of the Mie Theory to particle different has been introduced by the so called Extended Boundary Condition method).

1. Principles and formalisms of the Mie theory.
For the sake of simplicity, without loosing generality, the constants for the external medium are εo, μo, while those of the internal medium are ε, μo.
The propagation constants are correspondingly : $k0 = \omega (\varepsilon o, \mu o)^{0.5}$, $k = \omega (\varepsilon, \mu o)^{0.5}$,

The elementary forms of spherical waves are found by applying the theory of Debye Potentials.

Spherical co-ordinates r, θ, ϕ, with centre $r = 0$ at the sphere centre.

A scalar function $f(r, \theta, \phi)$ obeying the equation :

1) $\Delta^2 f + k^2 f = 0$

with Δ^2 Laplacian in spherical co-ordinates, is the starting point.

One then considers two vectors $\mathbf{M}(r, \theta, \phi)$ and $\mathbf{N}(r, \theta, \phi)$:

$\mathbf{M}(r, \theta, \phi) = \mathrm{curl}\,(f(r, \theta, \phi)\, r\, \mathbf{s})$ with **s** unit vector in the radial direction

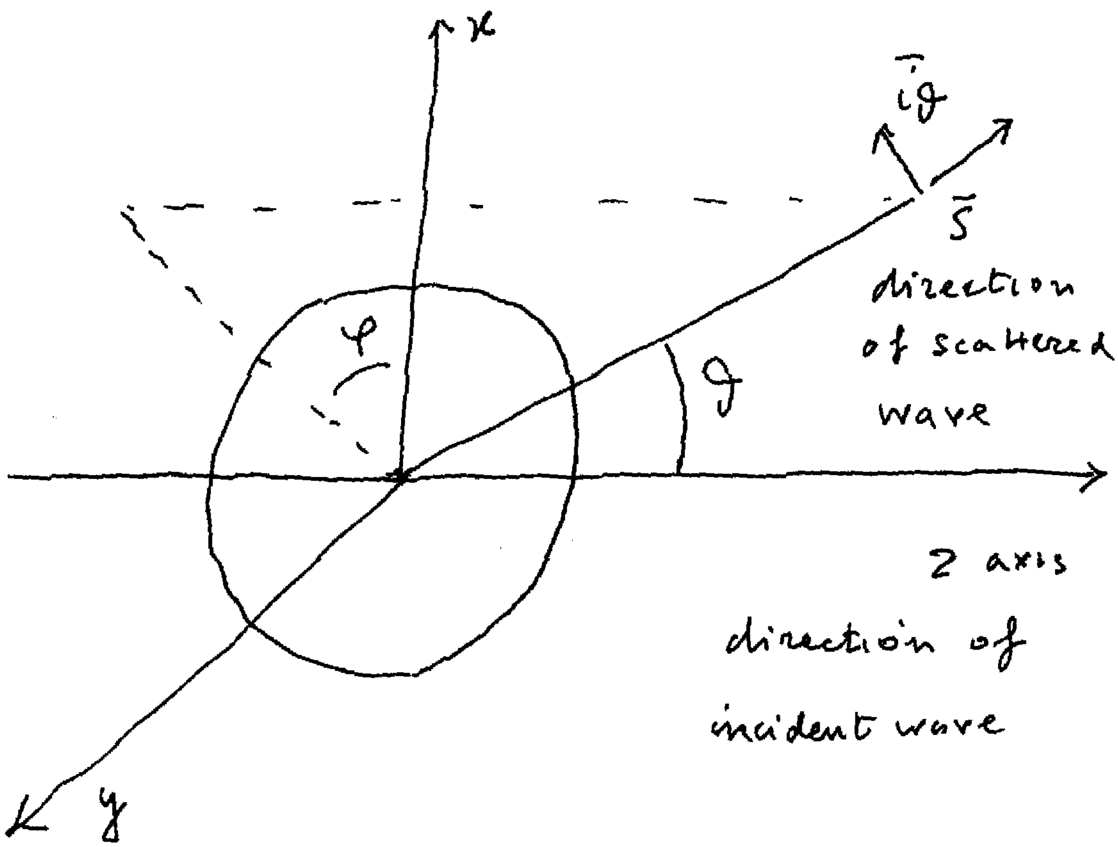
It can be shown that **M** obeys the equation

$\Delta^2 \mathbf{M}(r\, \theta, \phi) + k^2 \mathbf{M}(r, \theta, \phi) = 0$ (details available)

Then one considers the vector **N** :

$\mathbf{N}(r, \theta, \phi) = 1/k\ \mathrm{curl}\,\mathbf{M}$ **N** obeys the same equation as **M** :

$\Delta^2 \mathbf{N}(R, \theta, \phi) + k^2 \mathbf{N}(R, \theta, \phi) = 0$



φ angle with the axis x of the projection of $\bar{s}$ on the x y plane

$\vec{i}_\vartheta$ unit vector in the scattering plane directed with ϑ increasing

$\vec{i}_\varphi = \bar{s} \wedge \vec{i}_\vartheta$ $\qquad$ $\vec{i}_\varphi = \bar{s} \times \vec{i}_\vartheta$

SCATTERING PLANE:

plane of the z axis and $\bar{s}$

One has : curl $\mathbf{N}$ = k $\mathbf{M}$ curl $\mathbf{M}$ == k $\mathbf{N}$ $k = 2\pi/\lambda$

One sees that the **M** and **N** vectors are in an equivalent relations like the **E** and **H** fields.
Then, with opportune constant factors, they can represent electric and magnetic vectors in regions without sources.
Since $\mathbf{M} = \Delta f X\ \mathbf{r}$ (Δ gradient X vector product), one can see that **M** has no radial component (along the radial direction **s**), while **N** has a non zero radial component.

Then : if **M** is used for representing Electric fields one has **Transverse Electric** waves, with non-transverse Magnetic field which presents radial components.
While if **N** is used for representing Electric fields one has non transverse Electric waves presenting radial components, and **Transverse Magnetic** waves.

Thus ,with opportune factors, a general Electric field is a sum of both **M** and **N** independent vector solutions, with the corresponding magnetic field.

Combinations of the two (independent) positions represent general solutions of the system of differential equations of second order.

Every particular case under examination determines the particular coefficients for the expansion of the field in terms of the elementary vectors (eigenvectors and eigenvalues).

For the fields one can put

(1)

$\mathbf{H} = -i\,\omega\varepsilon\,\mathbf{M}$ $\mathbf{E} = k\,\mathbf{N}$ magnetic transverse case

And

$\mathbf{E} = i\,\omega\mu 0\,\mathbf{M}$ $\mathbf{H} = k\,\mathbf{N}$ electric transverse case

The two positions are linearly independent. A linear combination of the two gives the general solution for the field (in a region with no source).
The fields obey the basic equations

curl $\mathbf{H} = -i\,\omega\,\varepsilon\ \mathbf{E}$ and curl $\mathbf{E} = i\,\omega\,\mu_0\ \mathbf{H}$

From the relationship

$\mathbf{M}$ = curl (f (r, θ, ϕ) $\mathbf{r}$) ($\mathbf{r} = r\ \mathbf{s}$)

$\mathbf{M} = (1/\sin\theta)\ \delta f/\delta\phi\ \ \mathbf{i}\theta\ -\ \delta f/\delta\theta\ \mathbf{i}\phi$

i θ and **i** ϕ unit vectors **i** ϕ **r** X **i** θ X vector product

The expressions for the **M, N** vectors.

They can be derived from the solutions of the equation for the scalar function $f(r, \theta, \phi)$:

$$\Delta^2 f(r, \theta, \phi) + k^2 f(r, \theta, \phi) = 0$$

This equation can be dealt with by the common method of setting $f(r, \theta, \phi)$ as the product of a function of r, a function of θ, and a function of φ. One obtains elementary solutions of the type :

(2)

$$f_{mn}(r, \theta, \phi) = (\pi / (2 k r))^{1/2} Z_{n+1/2}(kr) P_{mn}(\cos \theta) \cos(m \phi) \quad \text{or}$$

$$(\pi / (2 k r))^{1/2} Z_{n+1/2}(kr) P_{mn}(\cos \theta) \sin(m \phi)$$

where $(\pi / 2 kr)^{1/2} Z_{n+1/2}(kr)$ is indicated as " spherical Bessel function".

Figures on the m next page

In Eq.2 $n = 0, 1, 2 \ldots\ldots\ldots$ m integer lower or not greater than n

And $Z_{n+1/2}(kr)$ (Bessel functions of fractional order)

$P_{mn}(\cos \theta)$ **associate Legendre functions of first kind of degree n and order m**
See annexed page 11

From the scalar functions (2) one obtains the expressions for the **M** and **N** vectors, then the forms of the fields by derivations.

The **incident field and the internal field must not have singularity at the origin (the centre of the sphere).**
Thus for these fields the $\mathbf{Z}_{n+1/2}(kr)$ function must be the Bessel function of fractional order of first kind. That is the functions $\mathbf{J}_{n+1/2}(kr)$. (**J** capital letter)

The function $(\pi / (2 k r))^{1/2} J_{n+1/2}(kr)$ is generally indicated as

$J_n(kr)$ (j non capital letter)

The scattered field must obey the general property that at great distance (in the Far Field) it behaves as a transverse radial spherical wave, with amplitude factor 1 / r .

M4a

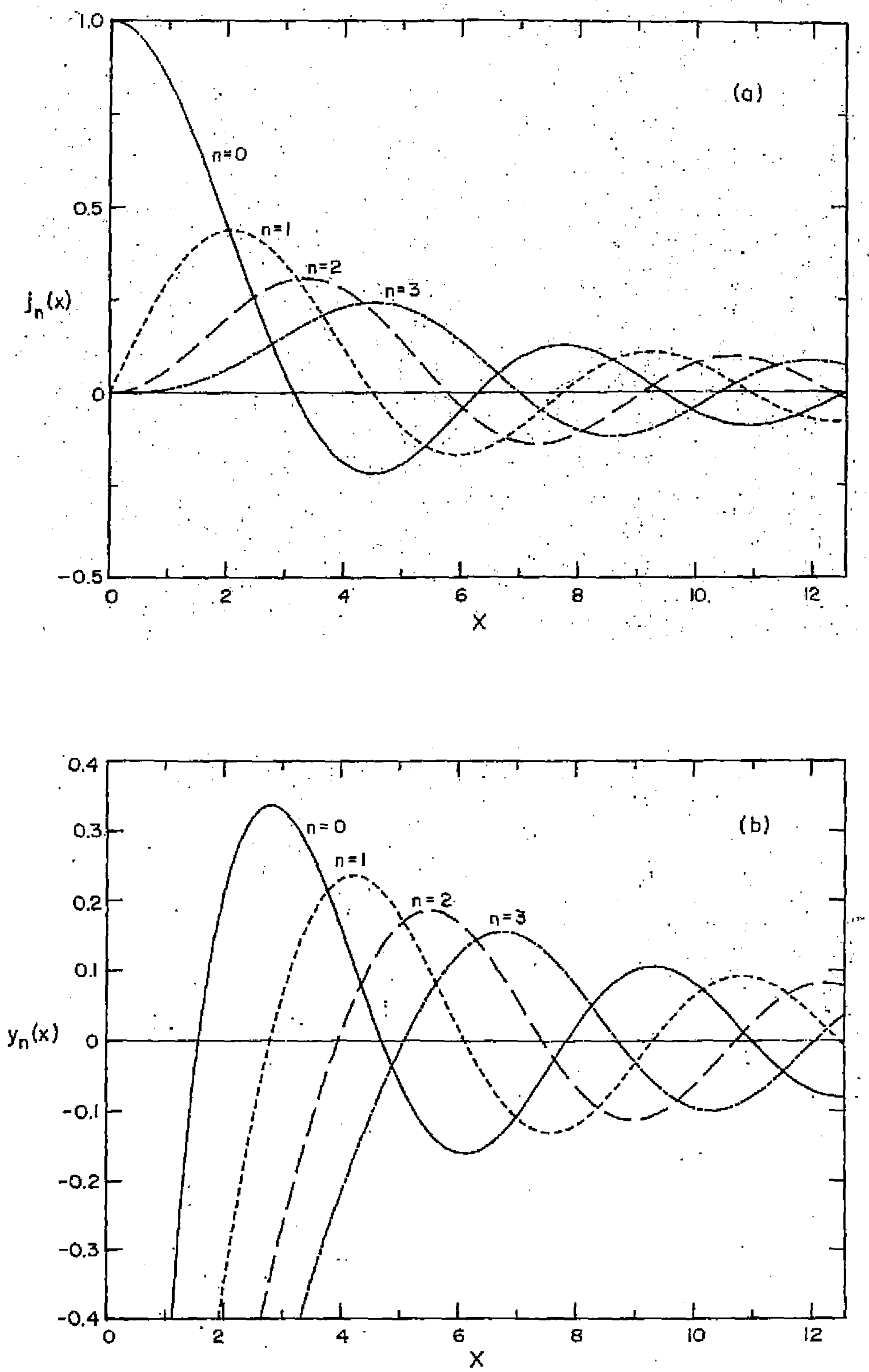


Figure 4.2 Spherical Bessel functions of the first (a) and second (b) kind.

That is the $\mathbf{Z}_{n+1/2}$ function must be of the Bessel function of third kind, which at great distance from the origin behaves as :

$$H_{n+1/2} = (2/\pi\kappa r)^{1/2} \exp(i(kr - \pi(n+1)/2))$$

(See the factor $(\pi/2\,kr)^{1/2}$ in Eq.2)

The spherical function $(\pi/2\,kr)^{1/2}\ Z_{n+1/2}(kr)$ if one chooses the Bessel function of third kind is generally indicated as

$$h_n(kr) \quad \text{(h non capital letter)}$$

For completeness : one has two independent functions for the spherical Bessel function of third kind.

Their symbols are (a)

$H^{(1)}_{n+1/2}$ whose behaviour at great distance from the origin tends to

$$(2/\pi\kappa r)^{1/2} \exp(i(kr - \pi(n+1)/2))$$

and (b)

$H^{(2)}_{n+1/2}$ whose behaviour at great distance from the origin tends to

$$(2/\pi\kappa r)^{1/2} \exp(-i(kr - \pi(n+1)/2))$$

The form (a) is used to represent a spherical wave diverging from the origin if the factor $\exp(-i\omega t)$ is used for the time factor.

The form (b) is used to represent a spherical wave diverging from the origin if the factor $\exp(i\omega t)$ is used for the time factor .

An apex (1) should be employed in the $h_n(kr)$ functions ov the previous pages.

da Bohren e Hufman pag. 89

Eq. 4

$$\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\left[\sin\vartheta\,\frac{dP_n^m(\vartheta)}{d\vartheta}\right]+\left[n(n+1)-\frac{m^2}{\sin^2\vartheta}\right]P_n^m(\vartheta)=0$$

(4)

$$\mathbf{M}_{emn}=\frac{-m}{\sin\theta}\sin m\phi\,P_n^m(\cos\theta)z_n(\rho)\hat{\mathbf{e}}_\theta-\cos m\phi\frac{dP_n^m(\cos\theta)}{d\theta}z_n(\rho)\hat{\mathbf{e}}_\phi, \tag{4.17}$$

$$\mathbf{M}_{omn}=\frac{m}{\sin\theta}\cos m\phi\,P_n^m(\cos\theta)z_n(\rho)\hat{\mathbf{e}}_\theta-\sin m\phi\frac{dP_n^m(\cos\theta)}{d\theta}z_n(\rho)\hat{\mathbf{e}}_\phi, \tag{4.18}$$

$$\begin{aligned}\mathbf{N}_{emn}=&\frac{z_n(\rho)}{\rho}\cos m\phi\,n(n+1)P_n^m(\cos\theta)\hat{\mathbf{e}}_r\\&+\cos m\phi\frac{dP_n^m(\cos\theta)}{d\theta}\frac{1}{\rho}\frac{d}{d\rho}[\rho z_n(\rho)]\hat{\mathbf{e}}_\theta\\&-m\sin m\phi\frac{P_n^m(\cos\theta)}{\sin\theta}\frac{1}{\rho}\frac{d}{d\rho}[\rho z_n(\rho)]\hat{\mathbf{e}}_\phi;\end{aligned} \tag{4.19}$$

$$\begin{aligned}\mathbf{N}_{omn}=&\frac{z_n(\rho)}{\rho}\sin m\phi\,n(n+1)P_n^m(\cos\theta)\hat{\mathbf{e}}_r\\&+\sin m\phi\frac{dP_n^m(\cos\theta)}{d\theta}\frac{1}{\rho}\frac{d}{d\rho}[\rho z_n(\rho)]\hat{\mathbf{e}}_\theta\\&+m\cos m\phi\frac{P_n^m(\cos\theta)}{\sin\theta}\frac{1}{\rho}\frac{d}{d\rho}[\rho z_n(\rho)]\hat{\mathbf{e}}_\phi,\end{aligned} \tag{4.20}$$

TOP

where the r-component of $\mathbf{N}_{mn}$ has been simplified by using the fact that P_n^m satisfies (4.4). Any solution to the field equations can now be expanded in an infinite series of the functions (4.17)–(4.20). Thus, armed with vector harmonics, we are ready to attack the problem of scattering by an arbitrary sphere.

I versori sono indicati con il simbolo $\hat{e}$

$\rho = KR$.

Le espressioni (3) sono adimensionali

Per i campi dovremo essere costanti moltiplicative opportune.

Suitable linear combinations of the elementary solutions for the outer field (sum of incident and scattered fields) , and the elementary ones for internal field give the formal solutions.
A constraint which determines the solution is added : the external field must present tangential components (summed of incident and scattered fields) equal to the one of the internal field : both for the magnetic and the electric field.

Now one can consider the incident field. It can be expanded in series of the elementary functions. Details are shown in the ANNEX EXPANSION . From this it can be seen that the only acceptable value for m is

$$m = 1.$$

After the expansion there results a series of linear equations, each **separately for each n**, whose solutions give the constant factors in the series expansions of the incident, internal and scattered field.
The separation is due to the orthogonality properties of the **Mn**m and **Nmn** vectors.

The index n is made to assume values from 1 to a "sufficient" number.

From now on we assume that the sphere is homogeneous, although the Mie formalism can be extended for spherically stratified spheres.

Due to the spherical symmetry of the problem one can start by considering an incident wave with linear electric polarization along the co-ordinate x.(Case X) The results can be directly extended to other kind of polarization of the incident wave. For instance if the incident **E** field is linearly polarized along the y axis, the scattered field **Es** can be obtained by the relationship.

Es (ϕ, polarization x) = **Es** ($\phi + \pi / 2$, polarization y)

which means that when the incident E field is polarized as y (Case Y) the scattered Es field can be obtained from the expressions for the case X by changing
ϕ to $\phi + \pi / 2$.

It is obvious that in the forward direction ($\theta = 0$)the scattered field is the same for the two case " a" and "b" .

At this point one can always reduce the problem to consider the **scattering plane Σ,** that is the plane defined by the incidence direction and the scattering direction.
Let us define as direction z that of the incident wave propagation.
Then one calls Eix the incident field component in this plane, and as Eiy the component perpendicular to this plane.

Let us consider the scattered field in the far field region.
It is easy to deduce by symmetry that the scattered field component in the scattering plane Σ the scattered field component (field Es1) is only due to Eix, while the scattered field component perpendicular to Σ (field Es2) is only due to Eiy,

If one starts with a different orientation of the x y axis one can always use a rotation matrix to come at the situation above.

The linearity of the equations implies that the two components of the incident fields are related to the two components of the scattered field by linear relationships.

For the simpler situation the Amplitude Matrix has only diagonal elements

$$\begin{vmatrix} E1s \\ \\ E2s \end{vmatrix} \quad \exp(ikr) / r \quad \begin{vmatrix} f11 & 0 \\ & \\ 0 & f22 \end{vmatrix} \quad \begin{vmatrix} Eix \\ \\ Eiy \end{vmatrix}$$

In order to maintain the diagonal characteristics of the matrix, one understands that the consideration of a different reference plane needs that one refers the incident field to different couples of axes. For this it is needed to apply a rotation matrix to the .incident field .

As already explained tthe scattered field in the Far Field region, as well as the internal field can be obtained by solving the groups of 4 equations connecting the tangential electric and magnetic components of the incident , scattered, and internal fields.
Without loss of generality we can limit us to consider then case (X) and the case (Y) separately.

We here skip details which are given in Bohren and Huffman book. The results are shown :

For the case (X) The scattered field in the far region is along the iθ unit vector. One has :

$Es = \sum_n$ over n between 1 and infinite

$\sum_n En/(ikr) ((i\, a_n \xi'_n \tau_n - b_n \xi_n \pi_n))\, i_\theta \qquad (\tau_n (\theta) \quad , \quad \pi_n (\theta))$

i_θ unit vector in the scattering plane in the direction of increasing the scattering angle.

$\xi_n (kr) = kr\, h_n^1(kr)$ apex ' = derivative with respect to the argument

$En = (2 n+1) (n (n +1)) Ei$

.

Case (Y) : one has for the scattered field

$Es = \sum_n$ over n between 1 and infinite

$\sum_n En/(ikr) ((- i\, a_n \xi'_n \pi_n + b_n \xi_n \tau_n))\, i_\phi \qquad (\tau_n (\pi/2) \quad , \quad \pi_n (\pi/2))$

$En = (2 n+1) (n (n +1)) Ei$

$\pi_n = P^1_n(\cos \theta) / \sin \theta \qquad \tau_n = d / d\theta \; P^1_n(\cos \theta)$

Now we assume that the magnetic permeability of the medium and sphere is the same, while the relative dielectric permittivity of the particle is m time the permittivity of the medium. (m can be complex if the particle shows conductivity)

By also putting

$\Psi n\,(kr) = k\,r\,j_n(kr)$ one has :

an =
$= (m\,\psi_n (m\,x)\ \psi'_n (x) - \psi_n (x)\,\psi'_n (x))) / (m\,\psi_n (mx)\ \xi' n(x) - \xi_n(x)\,\psi'_n (mx))$

bn =
$= (\psi_n (m\,x)\ \psi'_n (x) - m\,\psi_n (x)\,\psi'_n (x))) / (\psi_n (mx)\ \xi'_n(x) - m\,\xi_n(x)\,\psi'_n (mx))$

Where the parameter x is
$x = 2\,\pi\,a / \lambda$ (a radius of the sphere), and the apex ' in an and bn indicates derivative with respect of the argument.

The functions ψ_n and ξ_n are the Riccati Bessel functions :

$$\psi_n(\rho) = \rho\, j_n(\rho)$$

$$j_n^{(1)}(\rho) = (\pi/2\,\rho)^{-1/2}\ J_{n+1/2}^{(1)}(\rho)$$

$$\xi_n(\rho) = \rho\, h_{1\,n}^{(1)}(\rho)$$

$$h_{1\,n}^{(1)}(\rho) = (\pi/2\,\rho)^{-1/2}\ H_{1\ n+1/2}^{(1)}(\rho)$$

with apex (1) in the forms $h_{1\,n}(\rho)$ and $H_{1\ n+1/2}(\rho)$

Legendre Functions. $P^m{}_n(\cos\theta)$ Equation :

$$(1-x^2)\, d^2/dx^2\, P^m{}_n(x) - 2\, d/dx\, P^m{}_n(x) + (n(n+1) - m/(1-x^2))\, P^m{}_n(x) = 0$$

$x = \cos\theta$.

When $m = 0$ $P^m{}_n(\cos\theta)$ is simply written as $P_n(\cos\theta)$.

$$P_n(x) = (1/(n!\,2^n)\, d^n/dx^n\, ((x^2-1)^n)$$

$$P^m n(x) = (1-x^2)^{m/2}\, d^m Pn/dx^m$$

$P_0(\cos\theta) = 1$

$P_1(\cos\theta) = \cos\theta$

$P_2(\cos\theta) = 3/2\ \cos^2\theta - 1/2$

$P_3(\cos\theta) = 5/2\ \cos^3\theta - 3/2\cos\theta$

$P^1{}_1(\cos\theta) = \sin\theta$

$P^1{}_2(\cos\theta) = 3\sin\theta\cos\theta$

$P^2{}_2(\cos\theta) = 3\ \sin^2\theta$

Extension of the Mie scheme to non spherical particles.

A thorough description of the so called "T Matrix" method or "Extended Boundary Conditions" method can be found in :
M. N Mischenko, L.Travis, A.Lacis. Scattering, Absorption, and Emission of light by Small particles. Cambridge University Press. 2002, Revised Electronic Version. NASA Goddard Institute for Space Physics.

The non-sphericity of the scatterer does not make it possible to connect each n element in the expansions of the incident wave in terms of Vector Spherical Waves with the corresponding n element of the scattered field.

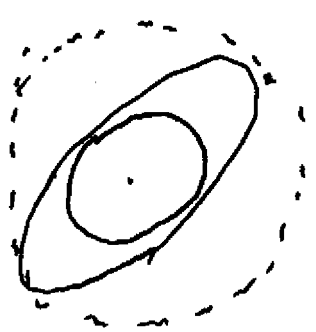
However, by considering a minimum radius of a sphere around the scatterer a matrix can be obtained connecting by linear relationships each n element of the scattered expansion to all the elements of the incident wave (a finite sum is reasonably stopped)

This is justified since inside the scattered one can determine a maximum sphere, at the surface of which one considers the expansion of the internal field, Then the fields internal to the sphere are connected to the fields in the space between the scatterer' surface and the external sphere by linear relationships of continuity. The same happens at the surface of the external sphere between field arriving from inside and total field at the external surface, which is the sum of the incident and scattered wave.

Details of the method are exposed in the quoted book,Part II, Sect.5.

Theory :
P.C.Waterman. a :Matrix formulation for electromagnetic scattering. Proc. IEEE 53. 805-839.1971.
b: Symmetry unitarity and geometry in electromagnetic scattering. Phys. Rev.D3 825-812. 1971.

For an application : WJ:Wiscombe, A.Mugnai. Single scatteing from non-spherical Chebyshev Particles. B NASA Ref. Publ1157 : GSFC Greenbelt. Md. 1986



ANNEX EXPANSION.

Expansion of the incident and scattered field in Vector Spherical Harmonics. Determination of the coefficients.

In practical problems of scattering by spherical particles the wave impinging on the spheres generally has the local characteristics of a plane wave. The radius of curvature of the a wave front is much greater than the scatterer's diameter.
(The same situation is taken into account in many practical problems of scattering by non-spherical particles).

Thus one has a plane wave as a start point.
It can be shown that a plane e m wave can be represented as a sum of spherical wave.
This is shown in text books (e.g. C: Bohren, D:Huffman. Absorption and scattering by small particles. Wiley 1983 Sect. 4.2).

Briefly: Due to the linearity to the problem, one can assume a linear polarization along the x direction

Ei = Ei exp(ikz) **x** , with the corresponding magnetic field

Hi = 1/ Z **z** X **Ei** . $Z = (\mu 0 / \varepsilon 0)^{1/2}$ External medium : vacuum
(X - vector product, propagation in the z direction)

In the quoted book on the basis of orthogonality of the spherical wave components, it is shown how to expand the incident field in series of elementary spherical waves.

Quoted book, Sect.4.2 : Expansion of a plane wave in **vector spherical harmonics**,

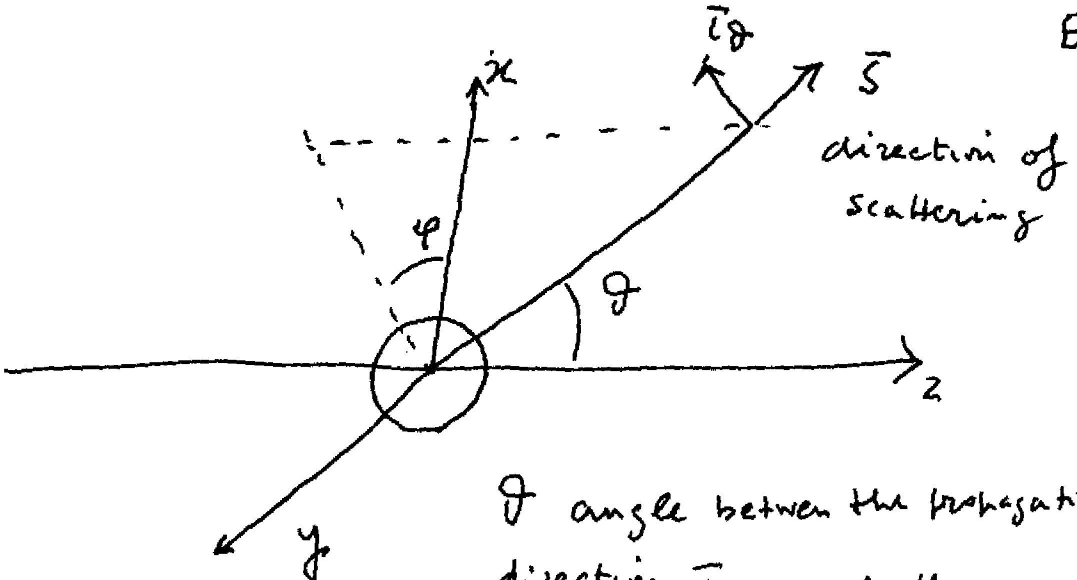
In the next page we copy the expressions for the vector components obtained from the formalism explained on the previous pages :
In this page :
$\rho = k r$.
$z_n(r)$ indicates the spherical Bessel functions, which is of the first kind for incident and internal field, and of third kind for scattered field.
As explained with this choice the incident and internal field have no singularity at the sphere centre, while the scattered field behaves as a spherical wave with 1 / r dependence on r in the Far Field.

The incident field is then written as (the sums to infinite)

Ei = $\Sigma_{m=0}$ $\Sigma_{n=m}$ (Bemn **M**emn + Bomn **M**omn + Aemn **N**emn + Aomn **N**emn).

A comparison with the table of the spherical functions between the expression of the incident field shows that in the expression of Ei only the terms with m = 1 are present. Also the parity shown by Ei indicates that Mo1n and Ne1n are present. Thus one has SEE NEXT PAGES

Ei = $\Sigma_{n=1}$ (Bomn **M**omn + Aemn **N**emn)



ϑ angle between the propagation direction $\bar{z}$ and the scattering direction $\bar{S}$

$\underline{i}_\vartheta$ unit vector in the plane of scattering Σ directed as ϑ increasing

$\underline{i}_\varphi$ unit vector in the plane $x\,y$ directed as φ increases

linear polarization of the incident electric field as $\bar{x}$

one has

$$\bar{x} = \sin\vartheta\cos\varphi\,\bar{S} + \cos\vartheta\cos\varphi\,\underline{i}_\vartheta - \sin\varphi\,\bar{i}_\varphi$$

EXP 3

~~[illegible]~~

Let us indicate as a, b, c the terms in the expression

From the table. ~~d~~ Representation of the incident vector $E_i \bar{x}$

$m = 1$ only

$\bar{M}_{emn}$ a : factor $\sin\varphi \, \underline{i}_\theta$

b : factor $\cos\varphi \, \underline{i}_\varphi$

$\bar{M}_{en}$ NO

M_{omn} a factor $\cos\varphi \, \underline{i}_\theta$

b factor $\sin\varphi \, \underline{i}_\varphi$

N_{emn} a factor $\cos\varphi \, \bar{S}$

b factor $\cos\varphi \, \underline{i}_\theta$

c factor $\sin\varphi \, \underline{i}_\varphi$

N_{omn} a $\sin\varphi \, \bar{S}$

b factor $\sin\varphi \, \underline{i}_\theta$

c factor $\cos\varphi \, \underline{i}_\varphi$

NO N_{omn}

for x polarization only factor

~~M_{emn}~~ M_{o1n} N_{e1n}

The constants Bomn and Aemn have to be defined by comparison with the incident field expression. An example is shown in APPENDIX to EXPANSION.

Then one obtains

Bo1n $= i^n$ Ei (2n+1) /(n (n+1))

Ae1n = -i Ei i^n (2n+1) /(n (n+1))

Then : **Ei** = Ei Σ_n i^n (2n+1) /(n (n+1)) (**Moin** -i **Ne1n**) the sum Σ_n from n = 1 to infinite

With k of the esternal medium and the spherical Besselfunctions of the first kind.
By the relationship between electric and magnetic field., one has for the incident Magnetic Field

Hi = (- k / ωμ0) Ei Σ_n (i^n (2(n + 1) / (n (n + 1)) (**Memn** + I **Nemn**).

The sum extended from n = 1 to infinite.

One has taken to take account that vector harmonics with different **m** are mutually orthogonal., because of the dependence on sin (m ϕ) or cos (m ϕ)

EXP 5

An analogous expansion is made for the internal field and the scattering field.
Spherical Bessel functions of the first kind are present in the expansion of the incident and internal medium and Spherical Bessel functions of the third kind are present for the scattered field.

The coefficients are then determined by imposing continuity conditions at the surface of the sphere. For the orthogonality properties of the spherical vectors four equations are separately valid for each n. These conditions regard the components along the unit vectors i_ϕ and i_θ, for the Electric and the Magnetic vectors. Thus for each index n one has 4 equations connecting the couples of coefficients of the scattered and internal fields.
By indicating the scattered field as Es

One obtains

$\mathbf{Es} = \Sigma_{n=1} \quad En \; (-bn\, \mathbf{Mo1n} + i\, an\, \mathbf{Ne1n}).$

am , bm
from the equations

And by the relationship between **Hs** and **Es**

$\mathbf{Hs} = (k / \omega\mu0) \; \Sigma_{n=1} (an\, \mathbf{Me1n} + i\, bn\, \mathbf{No1n}).$

With $En = \; (i^n(2(n+1)/(n\,(n+1))Ei$

And with the spherical Bessel function of third kind

From the table of the Vector Spherical Harmonics one observes that the term in $\mathbf{e_r}$ (radial unit vector) of **Nemn** is going to zero at great distance from the origin. That is the scattered E field becomes Transversal.

This can be seen since the term $\mathbf{e_r}$ has a factor that tends to $1 / (kr)^2 \exp(i\, kr + \alpha)$,while in the other components of **Nemn** a factor $1 / r$ prevails with distance.
The scattered field assumes the behaviour of a spherical TEM wave diverging from the origin (the centre of the sphere).

Here: r is written as R

Esempio

onda incidente $E_0 \underline{i} e^{+ikz} = E_0 \underline{i} e^{+ikR\cos\vartheta}$

$$= E_0 \left(\operatorname{sen}\vartheta \cos\varphi \, \underline{s} + \cos\vartheta \cos\varphi \, \underline{i}_\vartheta - \operatorname{sen}\varphi \, \underline{i}_\varphi \right) e^{ikR\cos\vartheta}$$

$$= \sum_{n=1}^{\infty} \left(B_{o1n} \, \underline{M}^{(1)}_{o1n} + A_{e1n} \, \underline{N}^{(1)}_{e1n} \right)$$

(le funzioni sferiche sono le $j_n(kR)$)

Calcolo di B_{o1n}

$$B_{o1n} = \frac{\int_0^{2\pi}\int_0^{\pi} \underline{E}_{inc} \cdot \underline{M}^{(1)}_{o1n} \operatorname{sen}\vartheta \, d\vartheta \, d\varphi}{\int_0^{2\pi}\int_0^{\pi} \left| M^{(1)}_{o1n} \right|^2 \operatorname{sen}\vartheta \, d\vartheta \, d\varphi}$$

Denominatore :

$$\int_0^{2\pi}\int_0^{\pi}\left|M^{(1)}_{o1n}\right|^2 \operatorname{sen}\vartheta \, d\vartheta \, d\varphi =$$

$$\int_0^{2\pi}\int_0^{\pi}\left[\frac{m^2}{\operatorname{sen}^2\vartheta}\cos^2\varphi\left(P_n^1(\cos\vartheta)\, j_n(kR)\right)^2 + \right.$$

$$\left. + \operatorname{sen}^2\varphi\left(\frac{d P_n^1(\cos\vartheta)}{d\vartheta}\, j_n(kR)\right)^2\right]\operatorname{sen}\vartheta \, d\vartheta \, d\varphi$$

$$= \pi\int_0^{\pi}\left(j_n(kR)\right)^2\left[\frac{\left(P_n^1(\cos\vartheta)\right)^2}{\operatorname{sen}^2\vartheta} + \left(\frac{d P_n^1(\cos\vartheta)}{d\vartheta}\right)^2\right]\operatorname{sen}\vartheta \, d\vartheta$$

$$\frac{1}{\operatorname{sen}\vartheta}\frac{d}{d\vartheta}\left(\operatorname{sen}\vartheta\,\frac{dP_n^1}{d\vartheta}\right)P_n^1+\left(n(n+1)-\frac{1}{\operatorname{sen}\vartheta}\right)P_n^1P_n^1=0$$

$$\frac{1}{\operatorname{sen}\vartheta}\frac{d}{d\vartheta}\left(\operatorname{sen}\vartheta\,\frac{dP_n^1}{d\vartheta}\,P_n^1\right)-\frac{dP_n^1}{d\vartheta}\,\frac{dP_n^1}{d\vartheta}+$$

$$+\left(n(n+1)-\frac{1}{\operatorname{sen}^2\vartheta}\right)P_n^1P_n^1=0$$

$$\frac{dP_n^1}{d\vartheta}\,\frac{dP_n^1}{d\vartheta}+\frac{1}{\operatorname{sen}^2\vartheta}P_n^1P_n^1=$$

$$=\frac{1}{\operatorname{sen}\vartheta}\frac{d}{d\vartheta}\left(\operatorname{sen}\vartheta\,\frac{dP_n^1}{d\vartheta}P_n^1\right)+n(n+1)P_n^1P_n^1$$

Integrale da eseguire:

$$\pi j_n^2\int_0^\pi\left[\frac{(P_n^1)^2}{\operatorname{sen}^2\vartheta}+\left(\frac{dP_n^1}{d\vartheta}\right)^2\right]\operatorname{sen}\vartheta\,d\vartheta=$$

$$=\pi(j_n)^2n(n+1)\int_0^\pi P_n^1P_n^1\operatorname{sen}\vartheta\,d\vartheta=$$ × Normalizzazione

$$=\pi(j_n)^2n(n+1)\frac{2}{2n+1}\frac{(n+1)!}{(n-1)!}$$

Numeratore.

$$\int_0^{2\pi}\int_0^{\pi} \underline{E}_{inc} \cdot \underline{M}^{(1)}_{o1n} \,\mathrm{sen}\vartheta\, d\vartheta\, d\varphi =$$

$$= E_0 \int_0^{2\pi}\int_0^{\pi} e^{+ikR\cos\vartheta} j_n(kR) \left\{ \frac{1}{\mathrm{sen}\vartheta} \cos^2\varphi \, P_n^1(\cos\vartheta) \cancel{\cos\vartheta} + \right.$$

$$\left. + \frac{dP_n^1(\cos\vartheta)}{d\vartheta} \mathrm{sen}^2\varphi \right\} \mathrm{sen}\vartheta \, d\vartheta\, d\varphi =$$

$$= E_0 \int_0^{\pi} \pi\, j_n(kR) \left\{ P_n^1(\cos\vartheta)\cos\vartheta + \frac{dP_n^1}{d\vartheta} \mathrm{sen}\vartheta \right\} e^{+ikR\cos\vartheta} d\vartheta$$

$$= E_0 \int_0^{\pi} \pi\, j_n(kR) \frac{d}{d\vartheta} \left(P_n^1(\cos\vartheta) \cdot \mathrm{sen}\vartheta \right) e^{+ikR\cos\vartheta} d\vartheta$$

$$P_n^1(\cos\vartheta) = - \frac{dP_n}{d\vartheta}$$

$$- \frac{d}{d\vartheta}\left(\mathrm{sen}\vartheta \frac{dP_n}{d\vartheta} \right) = n(n+1)\,\mathrm{sen}\vartheta\, P_n$$

Numeratore

$$n(n+1)\,\pi\, j_n(kR) \int_0^{\pi} P_n(\cos\vartheta)\, e^{+ikR\cos\vartheta} \operatorname{sen}\vartheta\, d\vartheta$$

Formula di Gegenbauer:

$$j_n(\rho) = \frac{(-i)^n}{2} \int_0^{\pi} P_n(\cos\vartheta)\, e^{+ikR\cos\vartheta} \operatorname{sen}\vartheta\, d\vartheta$$

Numeratore:

$$E_0\, n(n+1)\,\pi\,\left(j_n(kR)\right)^2 \frac{2}{(-i)^n}$$

$$B_{o1n} = \frac{n(n+1)\,\pi\,(j_n(kR))^2\; 2\,(2n+1)(n-1)!}{(-i)^n\;\pi\,(j_n)^2\; n(n+1)\; 2\,(n+1)!}$$

$$B_{o1n} = E_0 (-i)^{-n} \frac{2n+1}{n(n+1)} \qquad \Bigg\| \qquad A_{e1n} = -i\,E_0\, i^{+n} \frac{2n+1}{n(n+1)}$$

SCS 1

CROSS SECTIONS. PHASE FUNCTION. PHASE (MUELLER) MATRIX

In the case of the sphere the formalism to obtain the extinction cross section and the scattering cross sections is particularly simple. Let us have an e m wave propagating in the z direction, with electric field components Ex and Ey.

Let W be the power per unit area.

W is the sum of the powers relevant to the two polarization components :

W = Wx + Wy.

The cross section for each polarization is the same. Scattered power is the sum of powers scattered by each of them.

Therefore let us assume linear polarization Incident electric vector along the x direction. (unit vector **i**)

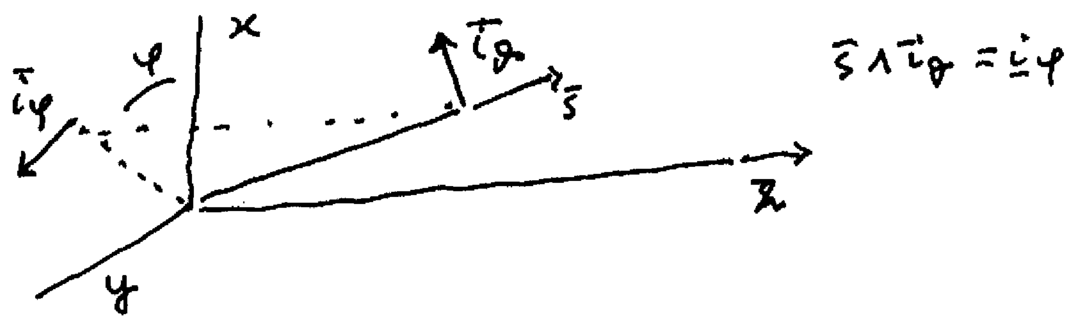
From the general Optical Theorem one has : _

$$*) \quad \sigma e = (4\pi / k)\, \mathrm{Im}\{ \mathbf{f} \cdot \mathbf{i} \} \qquad (\,.\text{ dot product})$$

where **f** is defined from the relation : **Es** = Ei exp (i k r) / r **f**. **f** scattering amplitude.

In Eq. **(*) f** is taken in the forward direction in the

In the direction z the angle $\phi = 0$, the angle θ is 0 . The unit vector **i** θ is - **x** .



In the forward direction one has :

$$\pi n\,(0) = n\,(n+1)\; / 2 \qquad \tau n\,(0) = n\,(n+1)\; / 2$$

Then .

Es = **x** Σ Ei i exp(ikr) / (kr) (2n + 1) / 2 (an + bn) Then the extinction x cross section is :

$$\sigma e \quad = (2\,\pi / k^2)\, \Sigma_n\, (2n+1)\;\; \mathrm{Re}\,\{an + bn\}$$ where the sum over n is from 1 to infinite.

If the sphere is not absorbing the scattering cross section σs is equal to the extinction cross section σe.

However, a different expression can be obtained for σs if there is absorption.

σs can be calculated by considering the scattered power and the integration of the power flux though a spherical surface of large radius around he sphere.

Scs 2

One obtains =

$$\sigma s = (\pi / k^2)\, \Sigma_n\ 2\,(2n+1)\ (|an|^2 + |bn|^2)$$

A common representation of σe is given by the ratio Q

$$Q = \sigma e \ / \pi a^2$$

with a radius of the sphere. That is the ratio between σe and the geometrical cross section of the sphere.

A typical representation is shown by the figure. It refers to a sphere of water, at different wavelength, as a function of the ratio $R = a / (2\pi\ \lambda)$. The behaviour of Q is typical : at small values of R (the so called "Rayleigh " zone) there is a rapid increment of Q. Then there is an intermediate zone (the so called Mie zone) where there is an oscillating behaviour. Then Q tends toward a value Q = 2. This last zone is in agreement with the so called "Optical paradox" , by which the cross section tends to doubling the geometrical g cross section.
However the paradox is only apparent. A simple scheme assuming a geometrical "stop" of "rays" arriving at the at the "obstacle" does not take into account the diffraction of the wave at the line of the particle border.

The Phase function.

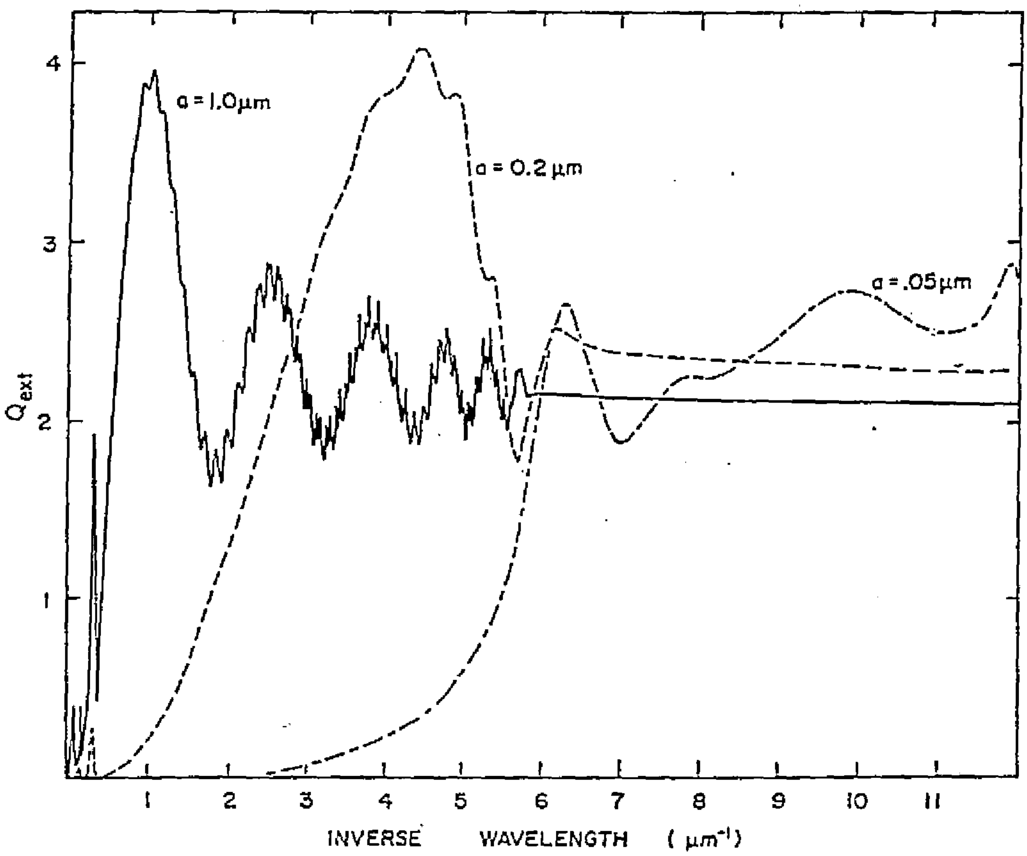
The scattered power per unit solid angle in a certain direction is different if incident polarization is in the scattering plane or normal to this plane.
However, one can average power scattered in all the azimuthal direction, and there is no dependence on polarization. With this average one can consider a "phase function" defined as the power scattered per unit solid angle in a certain direction making at angle Θ with the incident direction, divided by the total scattered power.
That is the phase function provides the scattered power per unit solid angle, averaged over the azimuthal direction. (There would be a certain difference if one considers scattered power per unit solid angle at different incident polarization with respect to the scattering plane).

If one needs to take into account this difference one should refer to the a scattering matrix (the Mueller matrix).
The Figures show several cases of phase ~~**matrix**~~ function, as a function of the scattering angle, for different values of the ratio a/λ One sees that when this ratio increases the phase function becomes more and more peaked in the forward direction. This effect is of the same type as that of the diffraction by an aperture in a screen).

(**) It is needed to take into account that some books and papers define the phase function by a different normalization. The meaning is not changed. There is the only change by a constant factor)

As already explained the Phase matrix (Mueller matrix) has a simpler form in the case of a sphere and when the reference axes for the incident wave are : one in the scattering plane and one normal to the plane.

SCS 3



Bohren
Huffman

Figure 4.6 Extinction efficiencies for water droplets in air; plotting increment = 0.01 μm^{-1}.

SCS4

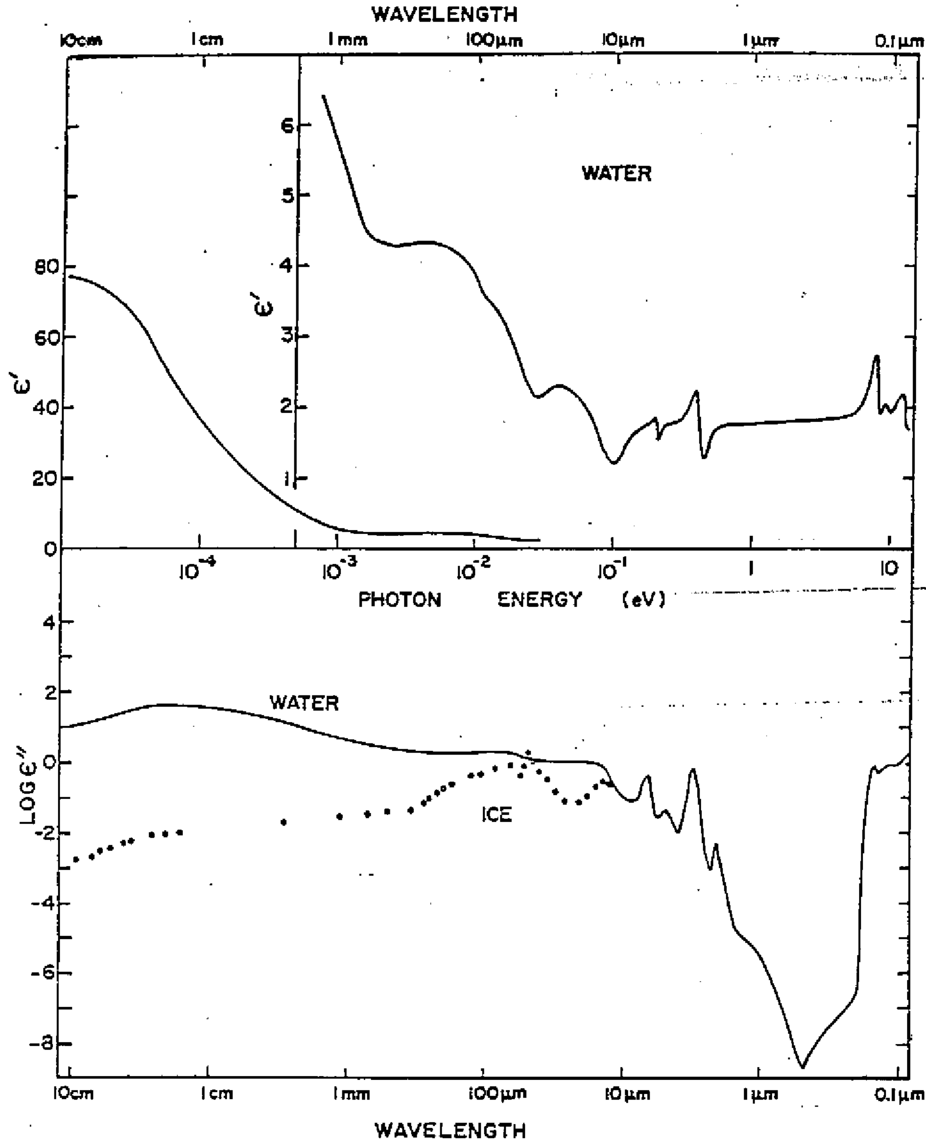


Figure 10.3 Dielectric functions of water (Hale and Querry, 1973). ε'' for ice is taken partly from Irvine and Pollack (1968) and partly from an unpublished compilation of the optical constants of ice, from far ultraviolet to radio wavelengths, by Stephen Warren (to be submitted to *Applied Optics*).

Also: see Visible and near ultraviolet absorption spectrum of liquid water

R. Litjens et al

Applied optics Vol 38 1216 - 1223 1999

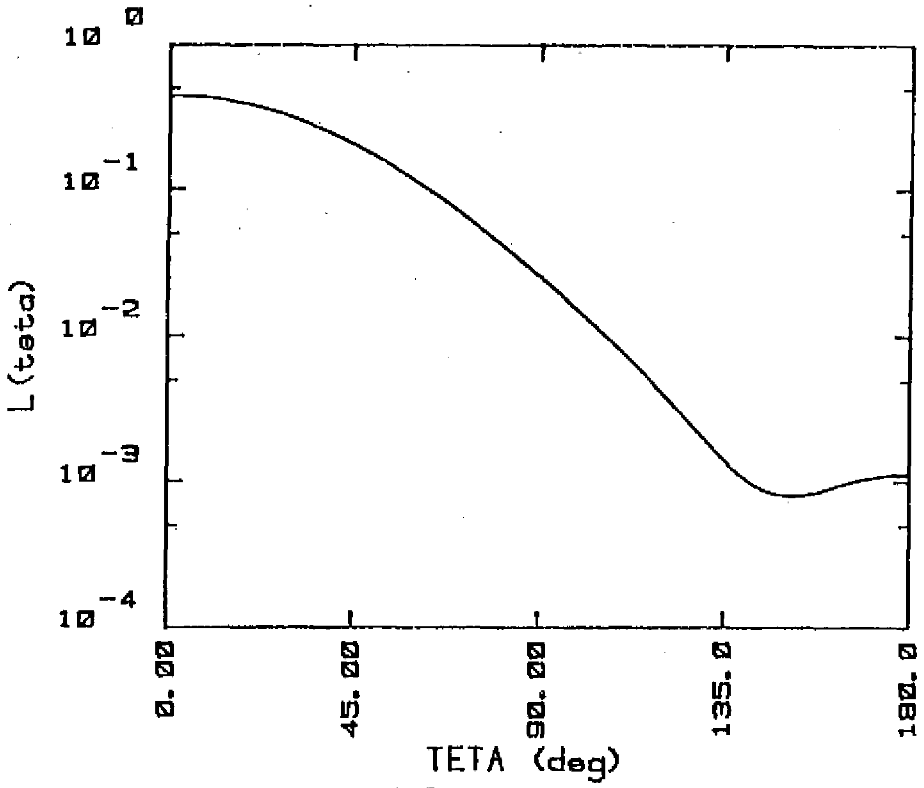
C. F. Bohren D. R. Huffman
Absorption and Scattering of Light by small Particles J. Wiley 1983

$\lambda = 0.4\ \mu m$

per $\varepsilon'' \ll \varepsilon'$ $\quad n = n' + i n''$

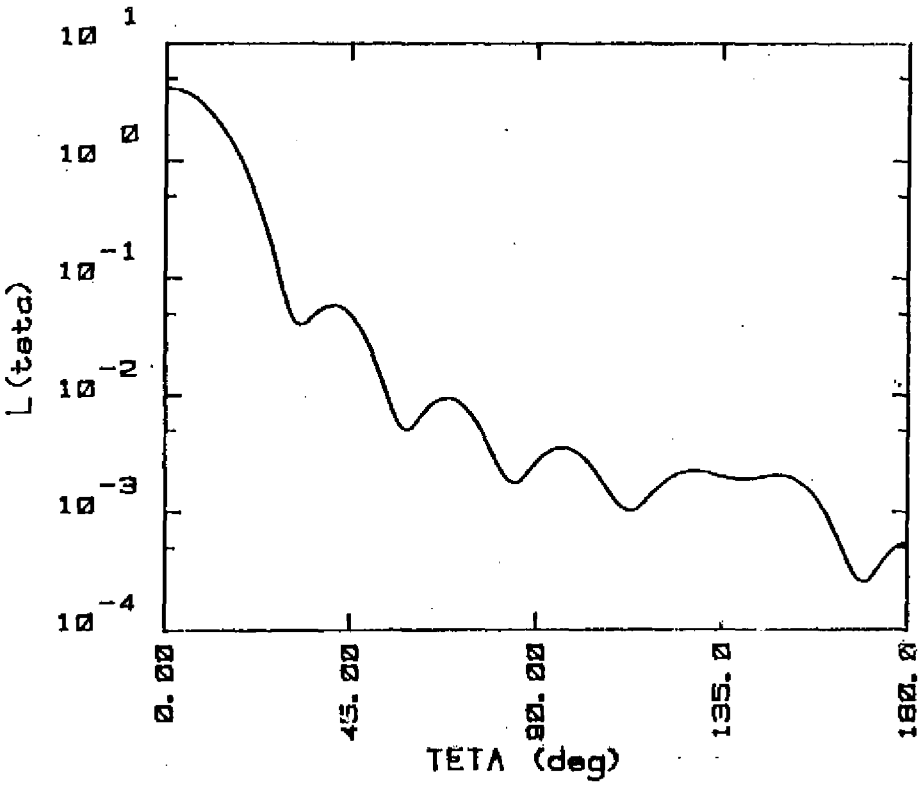
$$n'' = \varepsilon''/2\sqrt{\varepsilon'} \approx \varepsilon''/2n'$$

SCS5



D

$\tau = 0.15\mu$



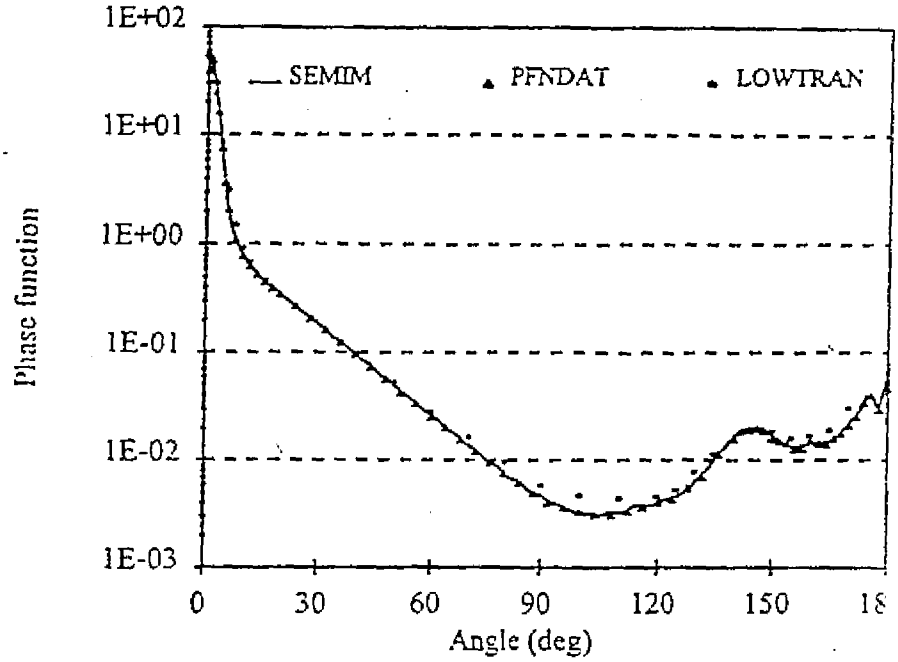
C

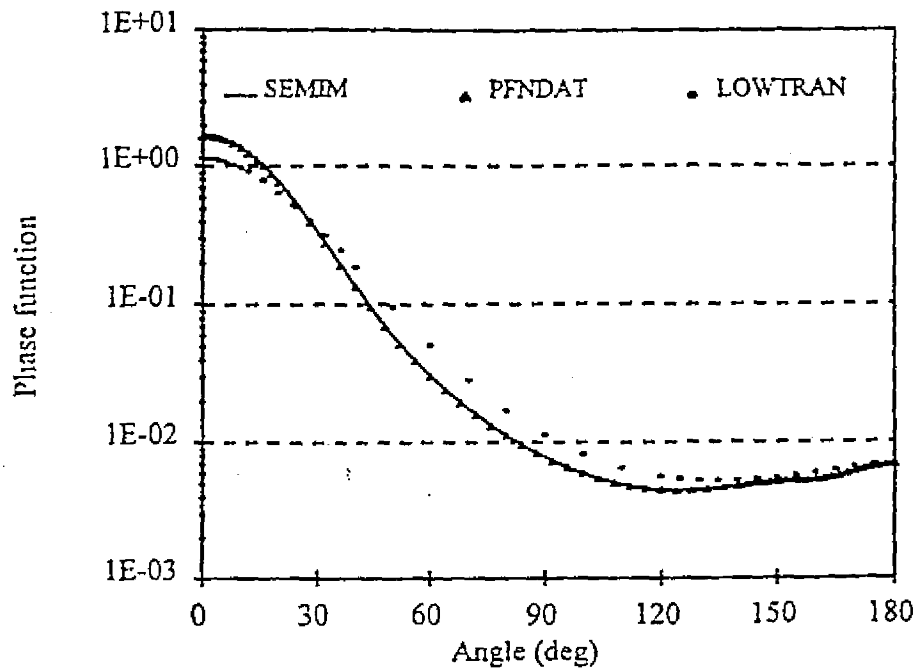
$\tau = 0.55\mu$

λ_{HeNe}/m_w

$\frac{m}{m_w} = 1.2$

Scs 6





Comparison of calculated scattering phase functions of radiation fog in the visible and in the infra red (a - 0.55 μm, b - 4.5 μm) with calculated phase function by two other groups.

Scs 7

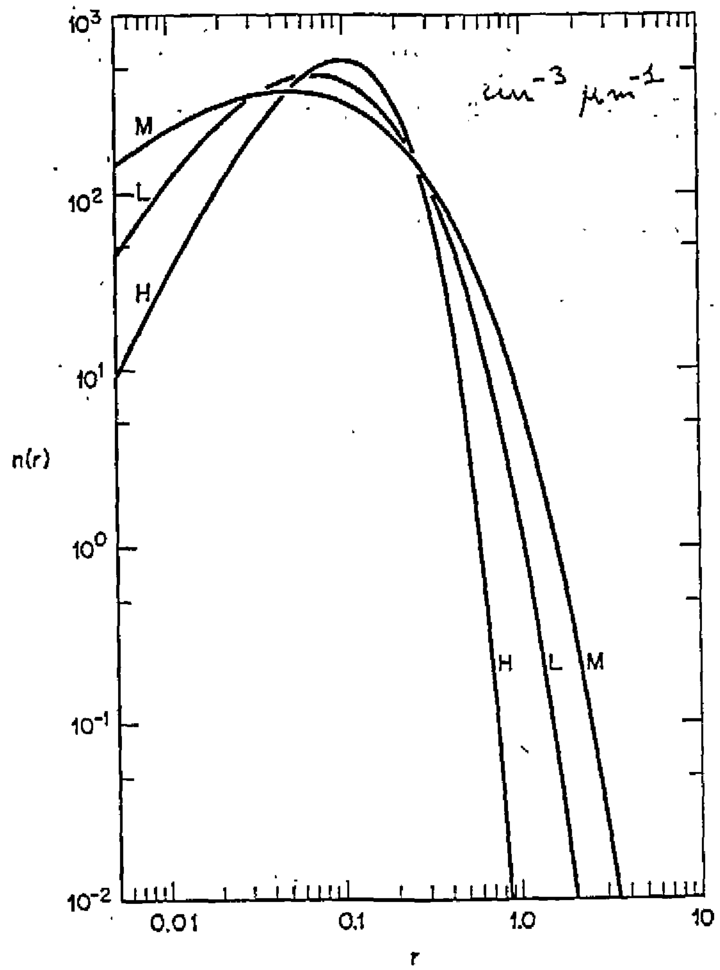


Fig. 20—Haze-type distribution functions used. The units for the radius r and for the unit volume in $n(r)$ depend on the particular model (see Table 5).

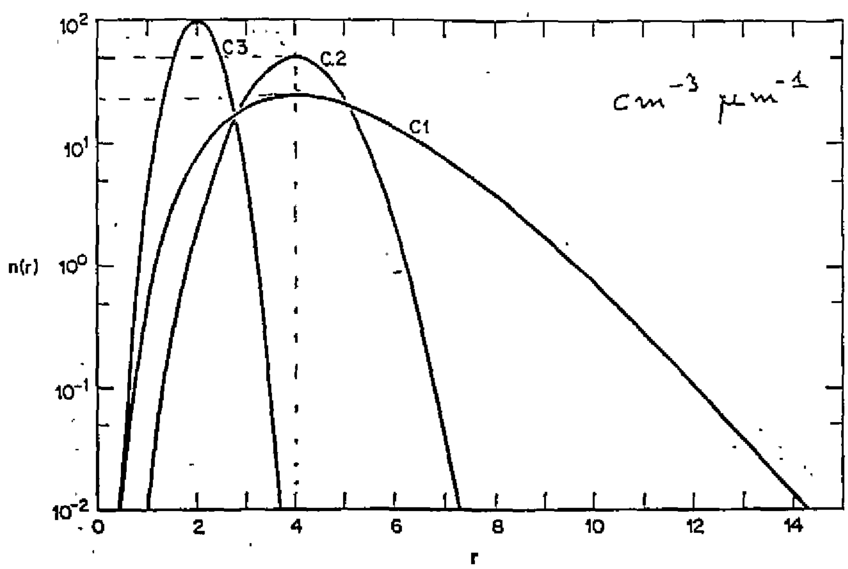


Fig. 21—Cloud-type distribution functions. The scale for the radius r is linear here, rather than logarithmic as in Fig. 20.