The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA International Atomic Energy Agency

2023-21

Workshop on Topics in Quantum Turbulence

16 - 20 March 2009

Bottlenecks

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Quantum *vs*. Classical Turbulence in Superfluids

Introductory talk @ ICTP, Trieste, March 17, 2009

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Weizmann Institute of Science, Israel

Because superfluid eddies can only be formed from quantized vortex lines, one might expect quantum turbulence to be very different from its classical counterparts. But that's not necessarily so.

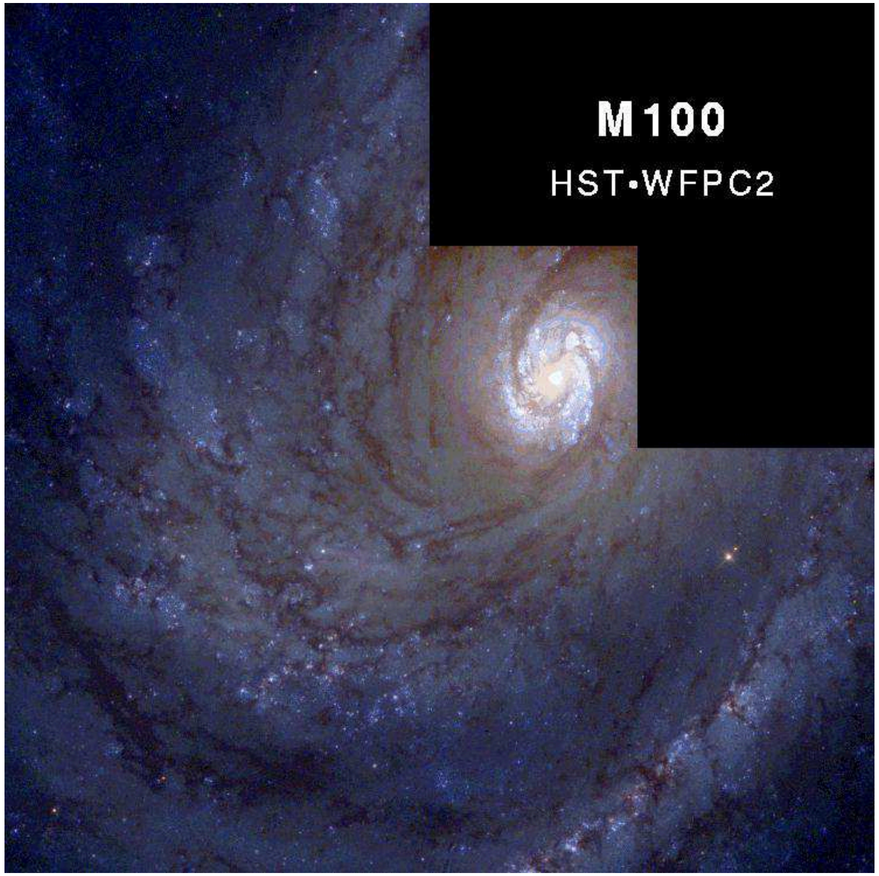
W.F. Vined and R.J. Donnelly, 2007

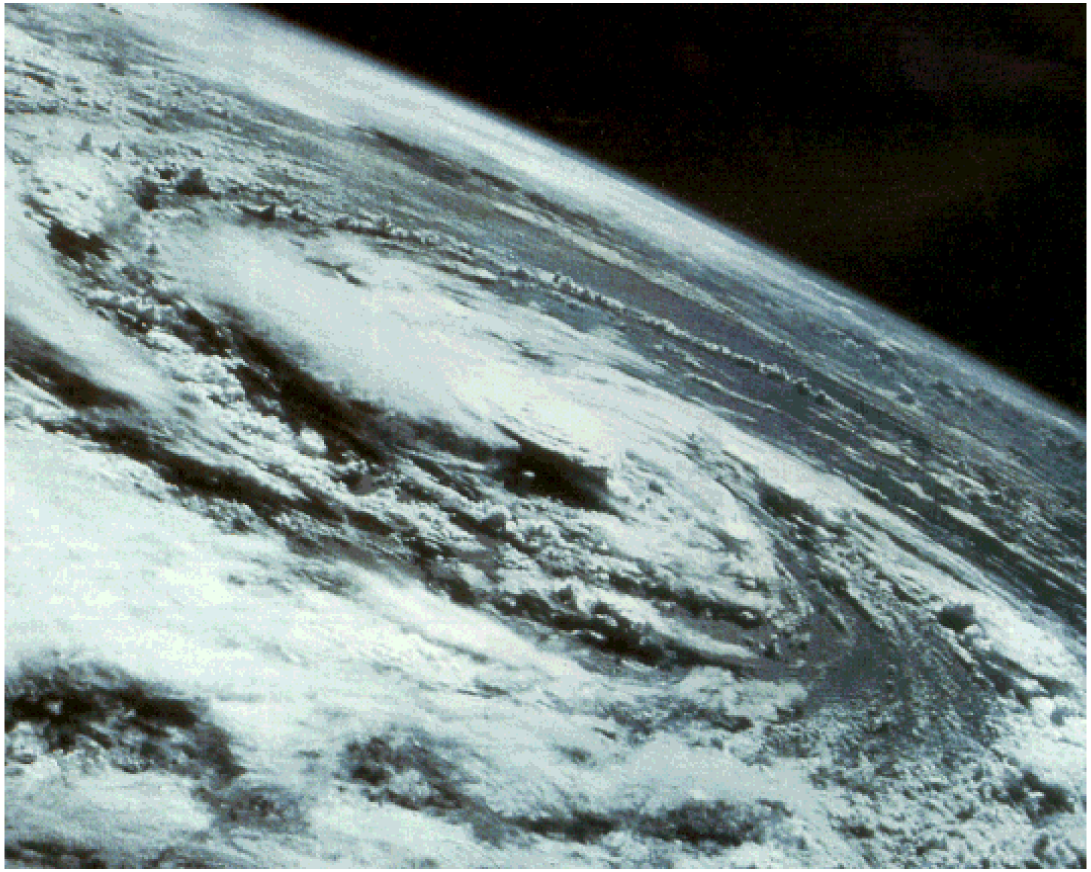
References: V.S.L'vov (VSL), S.V.Nazarenko (SVN), O.Rudenko (OR)

- VSL, SVN & Volovik, *JETP Letters*, **80**, 535 (2004): ⇒ Superfluid turbulent spectra
- VSL, SVN, & Skrbek, *J. Low Temp. Phys.* **145**, 125 (2006): ⇒ Superfluid turbulence
- VSL, SVN & OR, *Phys. Rev. B* **76**, 024520 (2007): ⇒ Bottleneck scenario
- VSL, SVN & OR, *J. Low Temp. Phys.,* **153**, 140 (2007): ⇒ Bottleneck theory
- Eltsov, Golov, Graaf Hanninen, Krusius, VSL & Solntsev, *Phys. Rev. Letters*, **99**, 265301 (2007); ⇒ Helsinki ^{3}He experiment
- Eltsov, Graaf, Hanninen, Krusius, Solntsev, VSL, Golov & Walmsley, *Prog. in Low Temp. Phys.* **XVI**, (2007): ⇒ Review of ^{3}He & ^{4}He exp's, Turbulent Front theory.
- W.F. Vinen and R. J. Donnelly, *Physics today*, April (2007) ⇒ Overview on superfluid turbulence
- Walmsley, Golov, Hall, Levchenko & W. F. Vinen, *Phys. Rev. Letters*, **99**, 265302 (2007) ⇒ Manchester ^{4}He experiment.
- Kozik & Svistunov, *Phys. Rev. Lett.* **92**, 035301 (2004): ⇒ Kelwin wave spectrum

Turbulence in the Universe

from Galactic scales $(10^{18} \div 10^{16})$ Km to Planetary scales $(10^{4} \div 10^{3})$ Km

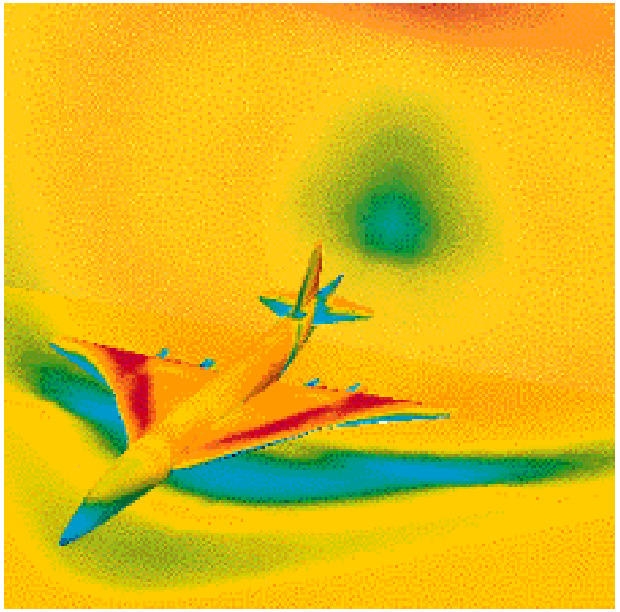




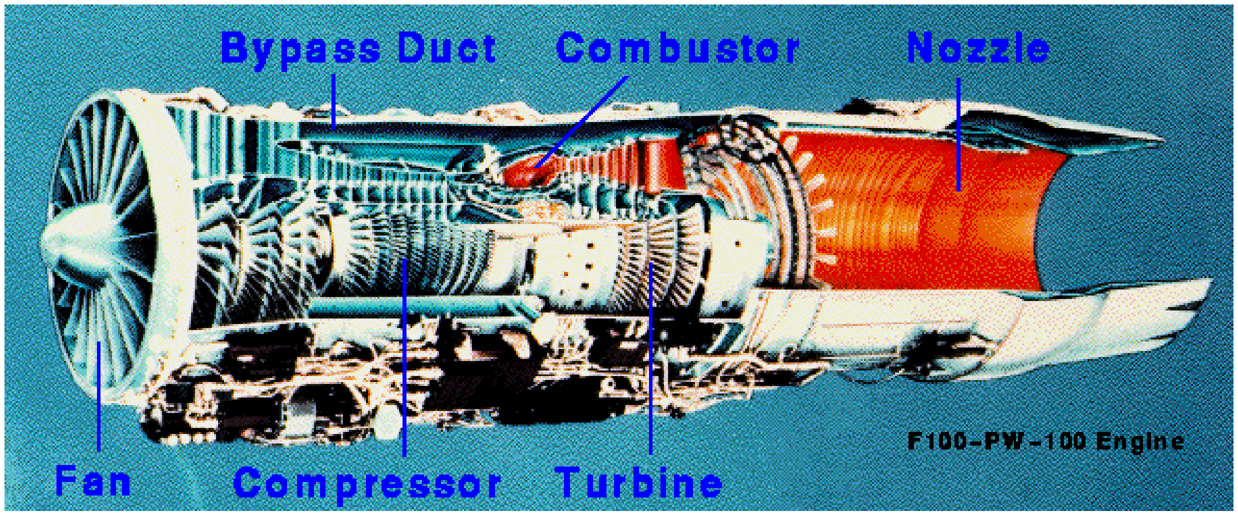
Spiral Galaxy M-100 in Coma Benerices. Distance $\sim 6 \cdot 10^{7}$ light years (Left) and Tropical Hurricane Gladis, Oct. 1968 (Right)

- **Turbulence on human scales (meters)**: Ottadalen, Norway, Aug.2003

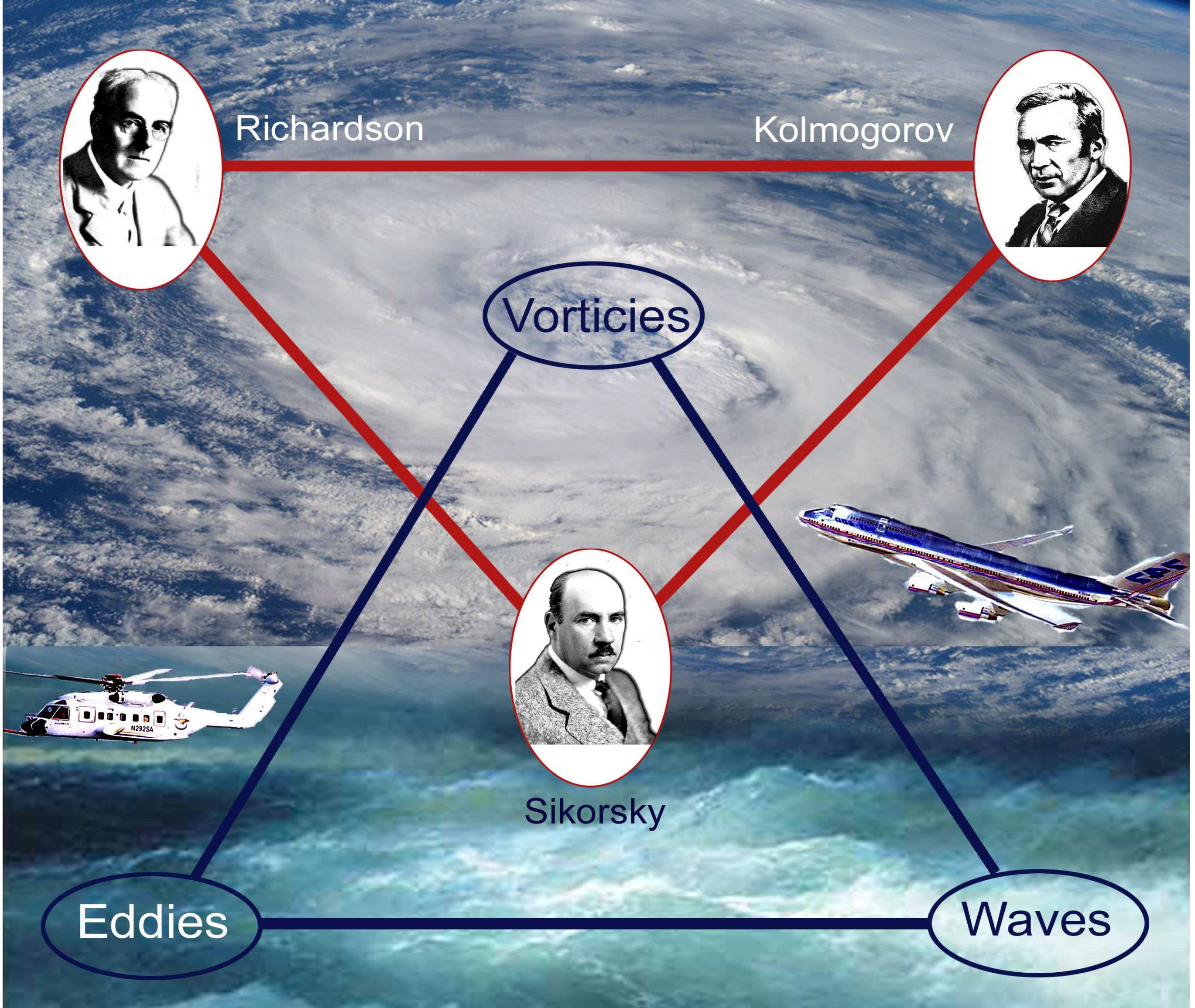
- **Some engineering aspects of turbulence**



Turbulent Boundary Layer:
Computer simulated turbulent air pressure: sonic boom behind supersonic aircraft Lockhid 3A



Turbulent fuel combustion in an aircraft engine


Richardson
Kolmogorov
Vorticies
Sikorsky
Eddies
Waves
N292SA

Basic models of hydrodynamics:
The Euler Equation

The Euler equation for $\boldsymbol{v}(\boldsymbol{r},t)$ is the 2nd Newton's law for the fluid particle:

$$\rho\left[\overbrace{\frac{\partial \boldsymbol{v}(\boldsymbol{r},t)}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}}^{\text{Fluid particle Acceleration}}\right] - \underbrace{(-\boldsymbol{\nabla}p)}_{}^{} = 0\,,$$

(Pressure Force: $(-\boldsymbol{\nabla}p)$)

Leonard Euler, 1741.

Basic models of hydrodynamics:
The Euler and Navier-Stokes equations

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Leonard Euler, 1741.

The Navier-Stokes equation accounts for the viscous friction:

$$\rho\,\Big[\frac{\partial \boldsymbol{v}(\boldsymbol{r},t)}{\partial t} + \underbrace{(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\Big] + \boldsymbol{\nabla} p}_{\text{Nonlinear interaction}} = \underbrace{(\rho\,\nu)\,\Delta\boldsymbol{v}}_{\text{viscous friction}}\,,$$

Claude L.M.H. Navier, 1827,
George Gabriel Stokes, 1845.

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Claude L.M.H. Navier, 1827,
George Gabriel Stokes, 1845.

Osborne Reynolds (1894) introduced "Reynolds number" $\mathcal{R}e$

$$\mathcal{R}e = \frac{\blacksquare}{\blacksquare} \simeq \frac{u\nabla v}{\nu\Delta v} \simeq \frac{LV}{\nu}$$

as a measure of the nonlinearity of the NSE.

- **Lewis Fry Richardson (1920) cascade model of turbulence:**

"Big whirls have little whirls

That feed on their velocity

And little whirls have lesser whirls

And so on to viscosity"

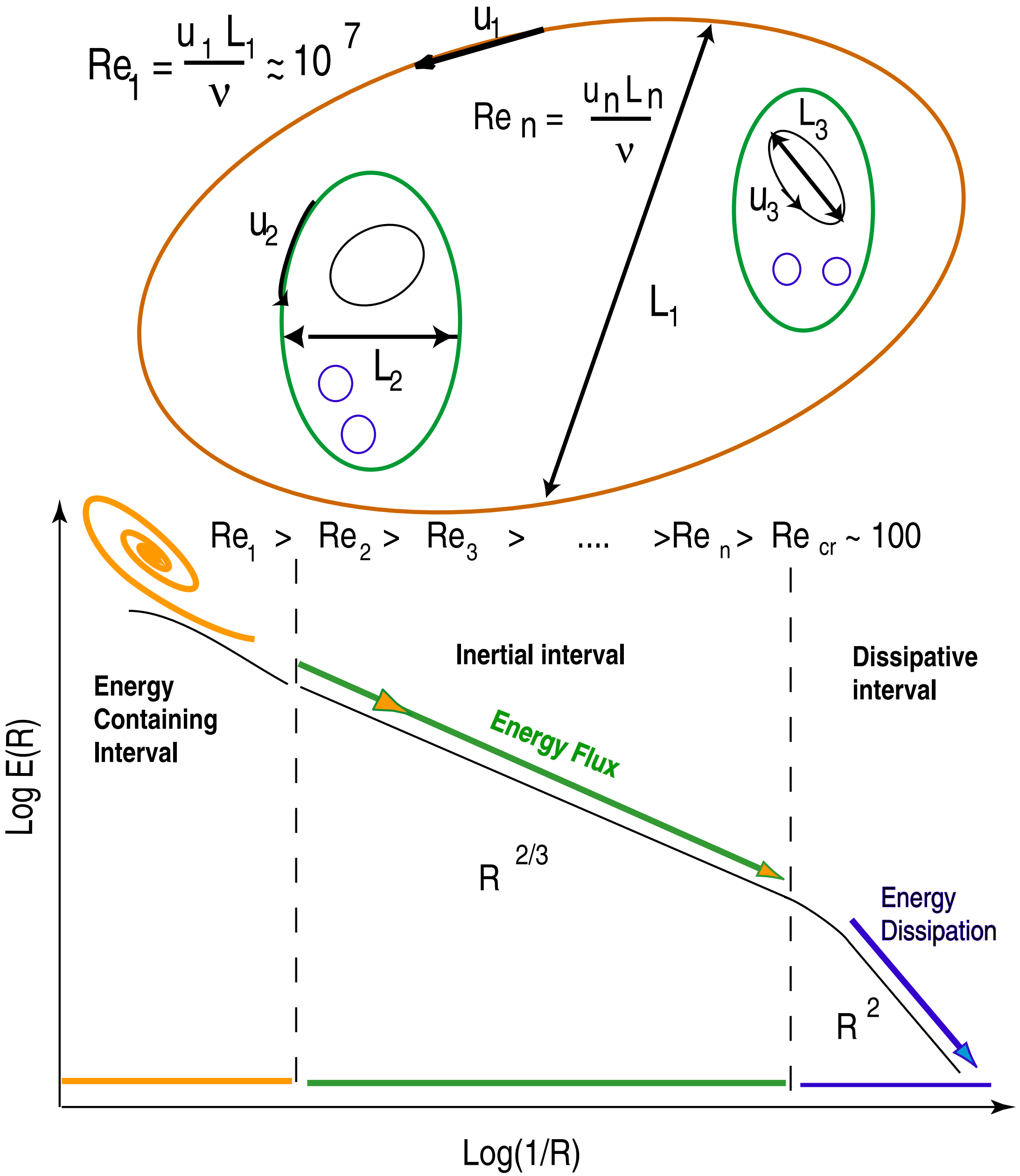
L.F. Richardson, paraphrase of J. Swift

⇐ Hurricane Bonnie, $V_{\mathrm{T}} \simeq 300\frac{\mathrm{m}}{\mathrm{s}}$,
Reynolds number at $H \simeq 500$m

$$\mathcal{R}e = \frac{V_{\mathrm{T}} H}{\nu} \simeq 10^{10} \gg \mathcal{R}e_{\mathrm{cr}} \sim 10^2$$

Unstable H, V_{T}-eddies create smaller H_1, V_1-eddies with $\mathcal{R}e > \mathcal{R}e_1 \gg \mathcal{R}e_{\mathrm{cr}}$. Their instability creates H_2, V_2-eddies of the second generation, end so on, until $\mathcal{R}e_n$ of the n-th generation eddies reaches R_{cr} and will be dissipated by viscosity: $\mathcal{R}e > \mathcal{R}e_1 > \mathcal{R}e_2 > \ \ldots \ > \mathcal{R}e_{n-1} > \mathcal{R}e_n > \mathcal{R}e_{\mathrm{cr}}$.

Andrei N. Kolmogorov-1941 cascade model of homogeneous turbulence:

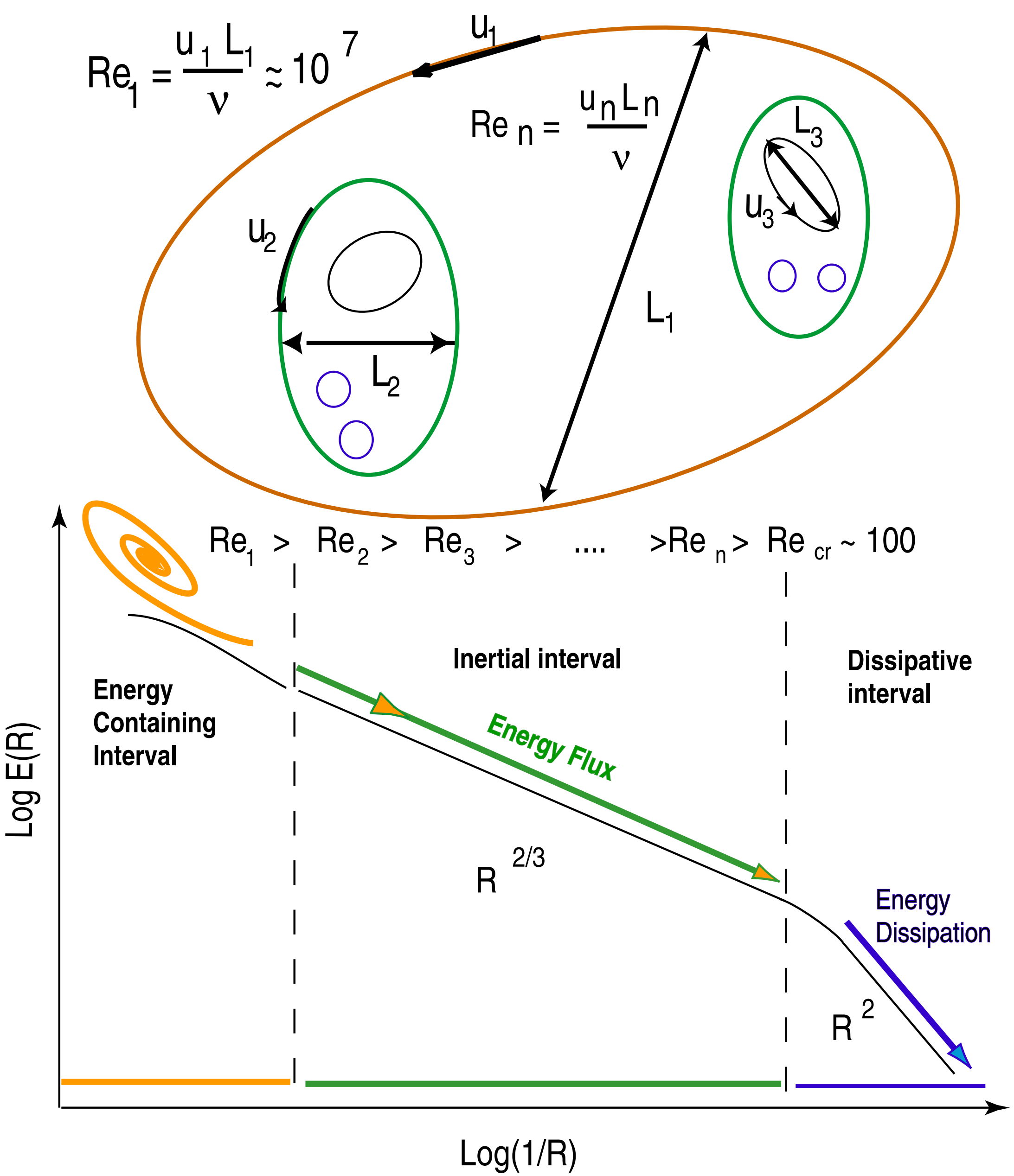


I. Universality of small scale statistics, isotropy, homogeneity;

II. Scale-by-scale "locality" of the energy transfer;

III. In the inertial interval of scales the only relevant parameter is the mean energy flux ε .

Andrei N. Kolmogorov-1941 cascade model of homogeneous turbulence:



I. Universality of small scale statistics, isotropy, homogeneity;

II. Scale-by-scale "locality" of the energy transfer;

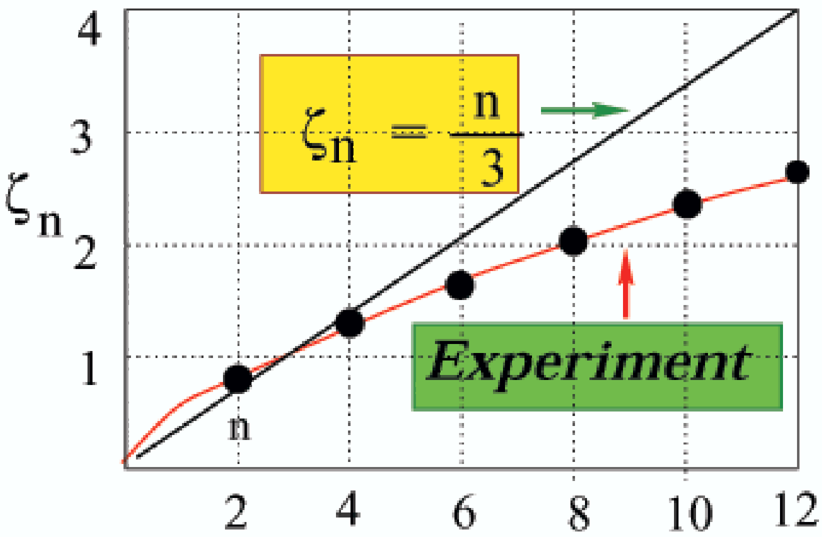
III. In the inertial interval of scales the only relevant parameter is the mean energy flux ε .

$\Rightarrow$ dimensional reasoning $\Rightarrow$

1. Turbulent energy of scale ℓ in inertial interval $E_\ell \simeq \rho\, \varepsilon^{2/3}\, \ell^{2/3}$,

2. Turnover and life time of ℓ-eddies: $\tau_\ell \simeq \varepsilon^{-1/3}\, \ell^{2/3}$

3. Viscous crossover scale
$$\eta \simeq \varepsilon^{-1/4}\, \nu^{3/4}, \quad N \sim \mathcal{R}e^{3/4} \ldots$$



n	n/3	ζ_n
2	0.67	0.70
4	1.33	1.20
6	2.00	1.62
8	2.67	2.00
10	3.33	2.36
12	4.00	2.68

Velocity difference across separation $\boldsymbol{r}$ gives velocity of "$\boldsymbol{r}$-eddies:"

$\vec{\boldsymbol{W}}_{\vec{r}} = \vec{v}(\vec{r}, t) - \vec{v}(0, t)$, – Longitudinal velocity: $W^{\ell}_{\vec{r}} = \vec{\boldsymbol{W}}_{\vec{r}} \cdot \vec{r}/r$

Longitudinal velocity structure functions $S^{\ell}_n(\vec{r}) = \langle \big(W_{\vec{r}}{}^{\ell}\big)^n \rangle \propto r^{\zeta_n}$.

In particular: $S_2(\vec{r})$ – Energy of $\vec{r}$ -eddies,

$S_3(r) = -\frac{4}{5}\, r$ (Kolmogorov-41) – Energy flux on scale r,

$S_4(r) - 3\, S_2^2(r)$ – Deviation from the Gaussian statistics,

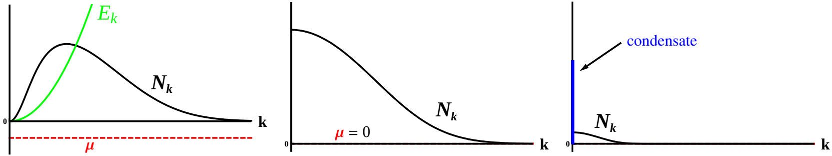
...

$S_{2n}(\vec{r})/S_2^n(r)$ – Statistics of very rare events

$$S^{\ell}_n(r) = C_n\, (\vec{r})^{n/3} \left(\frac{r}{L}\right)^{\zeta_n - n/3} , \quad L - \text{renormalization length} .$$

Bose condensation

- Superfluid behavior of ^{4}He originates from a coherent particle field – the condensate wave-function of ^{4}He atoms with $\boldsymbol{p} = 0$ is associated with Bose condensation of ^{4}He *zero-spin* atoms. $T_{\rm c} \approx 2.2$K.



$T_1 > T_2 > T_3$

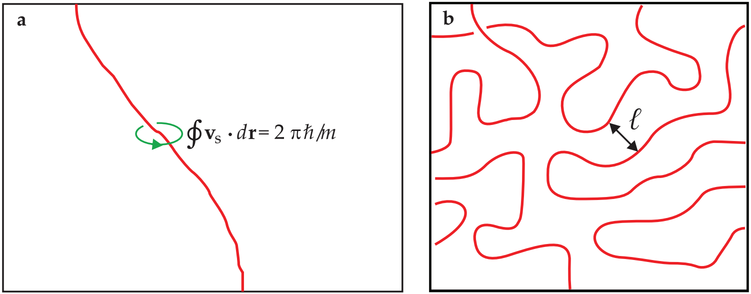
$$N_k \equiv \frac{4\pi k^2}{\exp(E_k - \mu)/T - 1}\,, \quad E_k = \frac{\hbar^2 k^2}{2m}\,, \quad N = \int_0^\infty n_k dk\ .$$

- Superfluidity of ^{3}He for $T < t_{\rm c} \approx 2 \times 10^{-3}$ K $\Rightarrow$ Bardeen-Cooper-Schrieffer condensate of Cooper-pairs of ^{3}He-atoms, $S = \frac{1}{2}$.

Quantization of vortex lines, core radius a_0 and intervortex distance ℓ

$$\oint \boldsymbol{v}_s \cdot d\boldsymbol{r} = n\,\kappa, \quad \text{where } \kappa = \frac{2\pi\hbar}{M} \quad \text{is the circulation quant.}$$

$M = 4$ for ^{4}He and $M = 6$ for a pair of ^{3}He atoms.



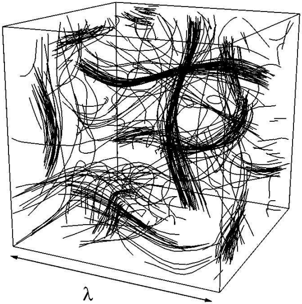
- ℓ is the mean intervortex distance,
- Vortex core radius $a_0 \simeq 1$ Å for ^{4}He & $a_0 \simeq 800$ Å at low p.

Two-fluid model (for scales $L \gg \ell$) & Mutual friction

"Coarse-grained" equation for the superfluid velocity $\boldsymbol{U}(\boldsymbol{r},t)$ [1]:

$$\frac{\partial \boldsymbol{U}}{\partial \tilde{t}} + (1-\alpha')(\boldsymbol{U}\cdot\boldsymbol{\nabla})\boldsymbol{U} + \boldsymbol{\nabla}\mu = -\Gamma\boldsymbol{U}\,,\ \Gamma \equiv \alpha\,\omega_{\rm ef}\ .$$

- Chemical potential μ serves as the pressure,
- $\alpha'(T)$ $\alpha(T)$ describe the mutual friction,
- Dissipative term Γ is taken in the simplified form, where $\omega_{\rm ef}$ is an effective vorticity.
- "Reynolds number" is $q^{-1} \equiv (1-\alpha')/\alpha$.



[1] E.B. Sonin, Rev. Mod. Phys. **59**, 87 (1987) and G. E. Volovik, JETP Lett. **78**, 533 (2002).

Kelvin waves of vortex line bending: $w(z) = x + i\,y$

- Energy (Hamiltonian) $\mathcal{H} = \dfrac{\kappa^2}{4\pi} \displaystyle\int \frac{\{1 + \mathrm{Re}\,[w'^*(z_1) w'(z_2)]\}\, dz_1 dz_2}{\sqrt{(z_1 - z_2)^2 + |w(z_1) - w(z_2)|^2}}$,

  - Hamiltonian form of Bio-Savart equations $i\,\kappa \dfrac{\partial w}{\partial t} = \dfrac{\delta \mathcal{H}\{w, w^*\}}{\delta w^*}$,
  - For small wave amplitudes, $w \ll \lambda$, $\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_4 + \mathcal{H}_6 + \dots$

- $\mathcal{H}_2 = \displaystyle\sum_k \omega_k |a_k|^2$, ($a_k = \sqrt{\kappa} w_k$) describes propagation of free KW

with the frequency $\omega_k = \dfrac{\kappa \Lambda}{4\pi} k^2$, $\Lambda = \ln\left(\dfrac{\ell}{a_0}\right) = \begin{cases} \simeq 15\,, & \text{for } ^4\text{He}\,, \\ \sim 10\,, & \text{for } ^3\text{He}\,. \end{cases}$

- $\mathcal{H}_4$ and $\mathcal{H}_6$ describe 4- and 6-wave interactions

$$\omega_1 + \omega_2 = \omega_3 + \omega_4\,, \quad \text{and} \quad \omega_1 + \omega_2 + \omega_3 = \omega_4 + \omega_5 + \omega_6$$

Effective 6-wave interaction coefficients $W_{\rm eff}$

$$\mathcal{H}_4 = \frac{1}{4}\sum_{1+2=3+4} T_{12,34} a_1^* a_2^* a_3 a_4 \,, \quad \mathcal{H}_6 = \frac{1}{36}\sum_{1+2+3=4+5+6} W_{12,34} a_1^* a_2^* a_3^* a_4 a_5 a_6$$

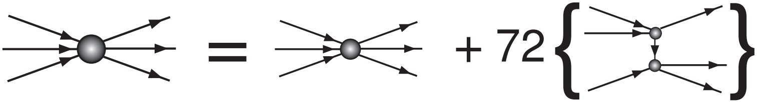
$$T_{12,34} = T^{\rm LIA}_{12,34} + \widetilde{T}_{12,34}\,, \qquad W_{123,456} = W^{\rm LIA}_{123,456} + \widetilde{W}_{123,456}\,.$$

LIA – Local Induction Approximation.

$$T^{\rm LIA}_{12,34} \simeq \Lambda\, k_1k_2k_3k_4 \sim \Lambda k^4\,, \qquad W^{\rm LIA}_{123,456} \simeq \Lambda\, k_1k_2k_3k_4k_5k_6 \sim \Lambda k^6\,,$$

$$\widetilde{T}_{12,34} \simeq k_1k_2k_3k_4 \sim k^4\,, \qquad \widetilde{W}_{123,456} \simeq k_1k_2k_3k_4k_5k_6 \sim k^6\,.$$

$2^{\rm nd}$ order perturbation theory:



$$W_{\rm eff} = W_{123,456} + 72\left\{T^{\,2}_{12,34}/\omega_k\right\}, \quad \omega_k \simeq \Lambda k^2\,.$$

Complete integrability $\Rightarrow$ ∞ # integrals of motion $\Rightarrow$

$$W^{\rm LIA}_{\rm eff} \equiv 0 \quad \Rightarrow \quad W_{\rm eff} \sim k^6 \ll \Lambda k^6\,.$$

Six-wave Kinetic Equation (Kozik-Svistunov [9])

$$\frac{\partial n_k}{\partial t} = \frac{\pi}{12}\int |W^{\rm eff}_{k,1,2;3,4,5}|^2 \left[\mathcal{N}_{3,4,5;k,1,2} - \mathcal{N}_{k,1,2;3,4,5}\right]$$
$$\times\delta(\omega_k+\omega_1+\omega_2-\omega_3-\omega_4-\omega_5)\delta(k+k_1+k_2-k_3-k_4-k_5)\, dk_1 dk_2 \dots dk_5\,,$$

$\mathcal{N}_{1,2,3;4,5,6} \equiv n_1 n_2 n_3 (n_4 n_5 + n_4 n_6 + n_5 n_6)$ has stationary solutions:

$n_k \propto \omega_k^{-1}$ $\Rightarrow$ energy-equipartition ;

$n_k^{\rm KS} \simeq (\kappa^3\rho)^{-1/5}\,\epsilon^{1/5}\,|k|^{-17/5}$ $\Rightarrow$ constant-energy-flux ϵ along one vortex line.

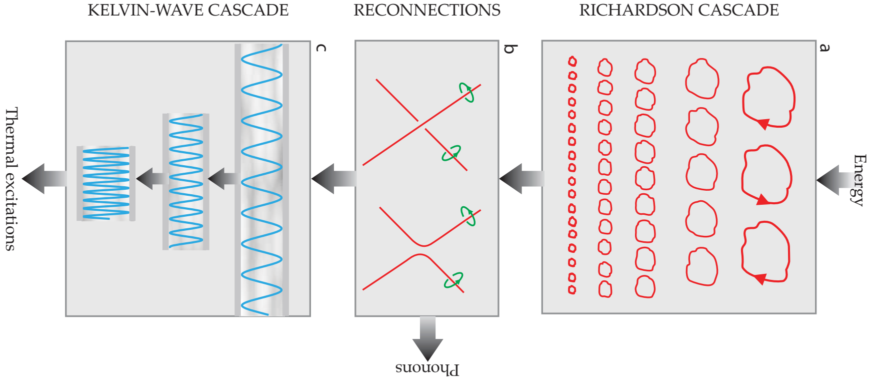
In 3D space with the vortex-line density ℓ^{-2} the KW energy density is

$$E^{\rm KW} = \int \mathcal{E}_k^{\rm KW} dk\,, \qquad \mathcal{E}_k^{\rm KW} = \ell^{-2}\omega_k n_k \simeq \Lambda \ell^{-2}(\kappa^3\rho)^{-1/5}\,\epsilon^{1/5}\,|k|^{-7/5}$$

With the flux of energy density (per unite mass) $\varepsilon = \epsilon/\rho\ell^2$ we got [3]:

$$\mathcal{E}_k^{\rm KW} \simeq \Lambda(\kappa^7\varepsilon/\ell^8)^{1/5}|k|^{-7/5}\,.$$

From Classical to Quantum energy cascade [7]



Very small scales: $L \ll \ell$.
Vortex lines can be considered as independent

Intermediate, Intervortex scales: $L \sim \ell$

Very large scales: $L \gg \ell$

Turbulent Energy Spectra in Superfluids

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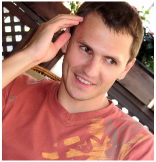


March, 2009 @ ICTP, Trieste

- *Large scale, Eddy-dominated energy spectra and short scale energy spectra of Kelvin-waves are described uniformly, within differential approximation.*
- *Suggested natural physical hypotheses allow us to describe energy spectra at intermediate region of scales, and at finite temperatures without fitting parameters.*
- *Our model is in reasonable qualitative agreement with available experimental results.*

Outline

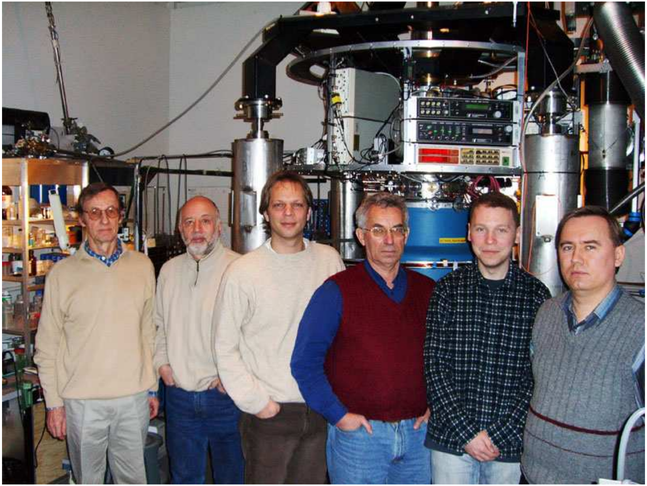
1. Story begins: Large– and small–scale turbulence in superfluids
   - Quasi-classical Two-fluid model & Mutual friction [7]
   - Kelvin waves of quantized vortex lines [9]

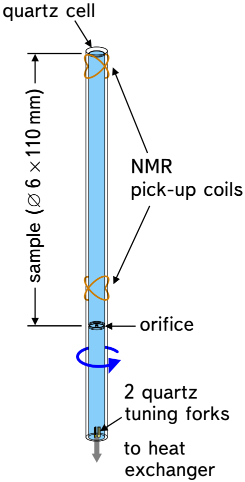
2. Helsinki ^{3}He-B experiment and bottleneck scenario
   - Turbulent front propagation and rate of energy dissipation [5]
   - Quasi-classical model of propagating turbulent front [6]
   - Eddy-wave Bottleneck scenario [3]

2. Manchester ^{4}He experiment and theory of bottleneck spectra
   - Manchester spin-down ^{4}He experiment [8]
   - Differential models for classical-quantum energy fluxes [4]
   - Bottleneck energy accumulation and effective viscosity [4]
   - Energy spectra at finite temperatures

Helsinki rotating (below 10^{-3}K) cryostat [5]

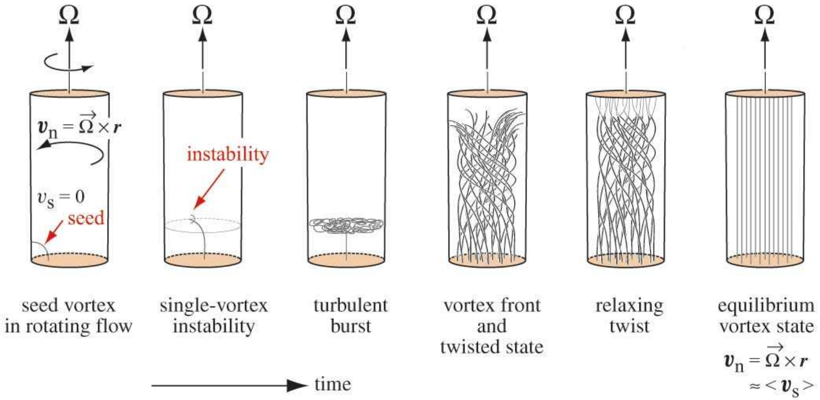
Low Temperature Laboratory, Helsinki University of Technology, Finland





M. Krusius, G. Volovik, R. de Graaf, VSL, R.E. Solntsev, V.B. Eltsov

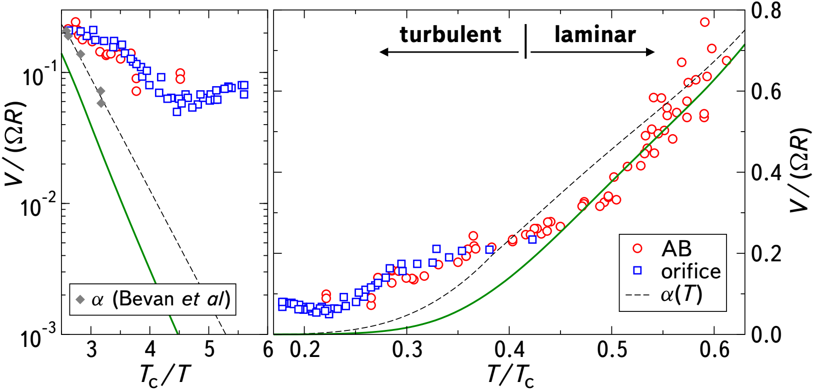
Vortex instability and turbulence in a rotating ^{3}He-B [5]



Resting (vortex-free) state of ^{3}He-B in rotating cell is meta-stable.
A seed vortex loop is injected in the vortex-free flow and the subsequent evolution is depicted. Different transient states are traversed, until the stable rotating equilibrium vortex state is reached.

Velocity of the Front Propagation in rotating ^{3}He-B [5]

High Temperature ($T > 0.4T_c$) **LAMINAR REGIME**



Vortex state behind the front is twisted: $\Rightarrow$ Free energy difference and front velocity are reduced by some factor: see green line.

Quasi-classical model of turbulent front propagation [6]

Global kinetic energy balance:

$$\frac{\pi}{4} V_{\rm f}\,\Omega^2 R^4 = 2\pi \int_0^R r\,dr \int_0^{\Delta(r)} dz \left[\frac{\widetilde{b}\,K^{3/2}(z,r)}{\Delta(r)} + \Gamma K(z,r)\right] .$$

$K \equiv \frac{1}{2}\langle |u|^2\rangle$ turbulent kinetic energy density per unite mass, $\Delta(r)$ -effective front width (outer scale of turbulence) $\tilde{b} \equiv (1-\alpha')\,b_{\rm cl}$, in classical turbulent boundary layer $b_{\rm cl} \approx 0.27$,
$\Gamma = \alpha\omega_{\rm eff}$ – mutual friction damping.

Ignoring z-dependence of K and Γ one gets:

$$V_{\rm f}\,\Omega^2 R^4 = 8J\,, \quad J \equiv \int_0^R r\,dr\,[\widetilde{b}\,K^{3/2}(r) + \alpha\,\omega_{\rm eff}(r)\Delta(r)K(r)\,.$$

Quasi-classical model of turbulent front propagation [6]

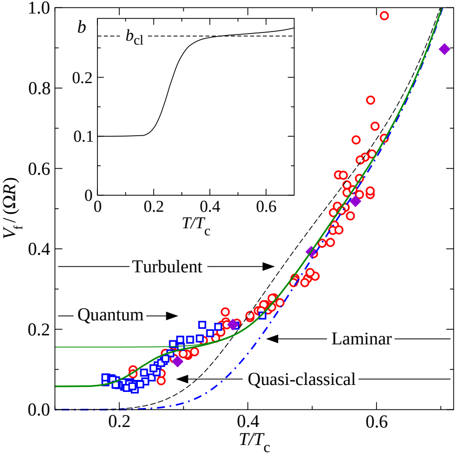
Model comparison with Helsinki LTL experiment [5]

• Accounting for spatial turbulent diffusion of kinetic energy toward the centerline in the radial energy balance gives $\Delta(r) \propto r$, $K(r) \propto r^2$.

• Also one naturally can suggest $\omega_{\rm eff}(r)$ is r-independent.

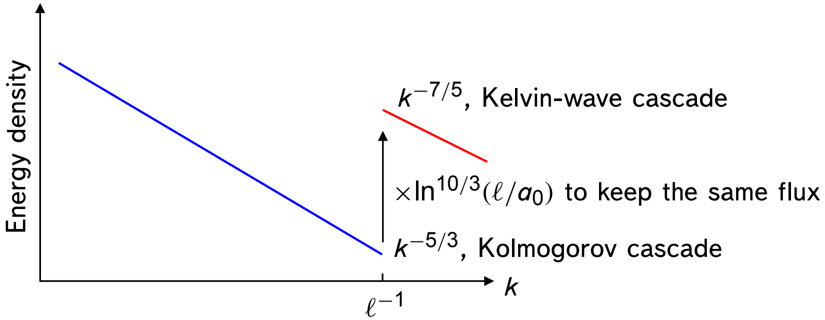
Taking $\Delta(r)\omega_{\rm eff}(r) = a\,\Omega r$, ($a \simeq 0.5$), $K(r) = c(\Omega r)^2/2$ ($c \simeq 1$) one gets Eqs. $\Downarrow$, show by ——— $\Rightarrow$

$$\frac{V_{\rm f}}{\Omega R} \simeq \frac{4\,c}{5}\left[b(1-\alpha')\sqrt{\frac{c}{2}} + \alpha\, a\right]$$



Below 0.25 $T_{\rm c}$ data deviates down!

Eddy-wave Bottleneck scenario [2]



quantum crossover scale

$$\mathcal{E}_k^{\mathrm{K41}} \simeq \varepsilon^{2/3}|k|^{-5/3} \xrightarrow{\varepsilon = const} \mathcal{E}_k^{\mathrm{KW}} \simeq \Lambda \left(\kappa^7 \varepsilon / \ell^8\right)^{1/5} |k|^{-7/5} \, .$$

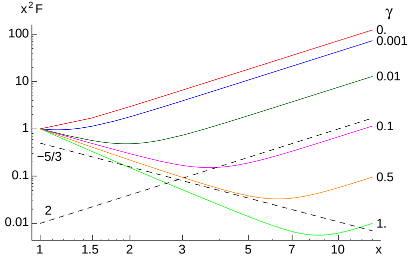
$$\Lambda \equiv \ln(\ell/a_0) \gg 1 \, .$$

Eddy-wave Bottleneck scenario [2]

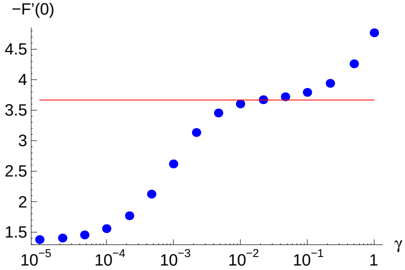
Balance of the energy density [1D energy spectrum $E = \int E_k dk$] in k-space:

$$\frac{d\varepsilon(k)}{dk} = -E_k\,, \qquad \varepsilon(k) \equiv (1-\alpha')\sqrt{k^{11}E_k}\,\frac{d(E_k/k^2)}{8\,dk}\,,$$

Here $\varepsilon(k)$ is taken in the Leiht-Nazarenko differential model



Resulting energy spectra E_k for different values of mutual friction parameter Γ



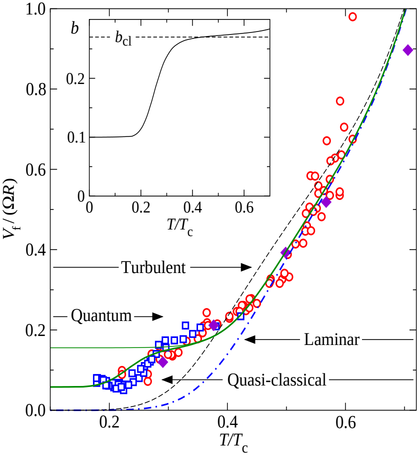
Resulting energy influx $\varepsilon(k_0)$ for fixed value E_{k_0} vs. Γ gives Γ-dependence of b in estimate $\varepsilon(k_0) = b(1-\alpha')K^{3/2}/\Delta$

Eddy-wave Bottleneck scenario vs. LTL experiment

- Using the same estimate

$$\frac{V_f}{\Omega R} \simeq \frac{4\,c}{5}\left[b(T)(1-\alpha')\sqrt{\frac{c}{2}}+\alpha\,a\right],$$

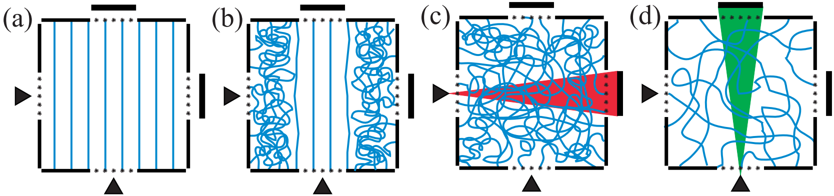
but with temperature dependent b(T), (see insert), which accounts for the bottleneck effect, one achieves a reasonable description of the temperature dependence of the front velocity in the quantum turbulent region, see green line ——⇒



Achieved agreement is an evidence that our model adequately reflects main physical features of the front propagation in ^{3}He-B in laminar, quasi-classical turbulent and quantum-turbulent regimes.

Manchester cube $(4.5\mathrm{cm})^3$ spin-down ^{4}He experiment

Measuring the time-decay of the vortex line density by negative-ion scattering



$\Uparrow$ Cartoon of the vortex configurations $\Uparrow$.

(a) Regular array of vortices at $\Omega =$const.;
(b) Immediately after stopping rotation;
(c) Homogeneous turb.: $\Omega t \sim 30 \div 300$;
(d) Almost decayed turbulence: $\Omega t > 10^3$.

Shaded areas indicate the paths of probe ions when sampling the vortex density.

Walmsley, Golov, Hall, Levchenko and Vinen,
PRL, **99**, 265302 (2007)

Manchester spin-down ^{4}He experiment [8]

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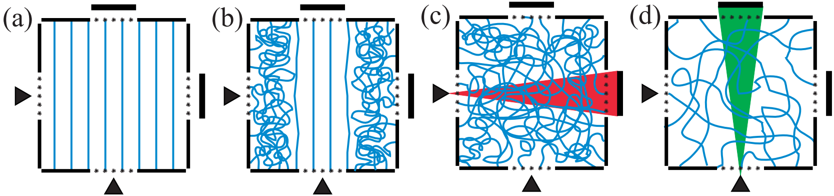
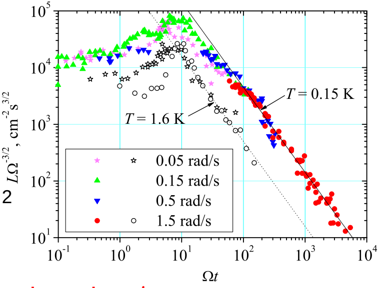


⇑ Cartoon of the vortex configurations ⇑



Vortex line density ($L\,\Omega^{-3/2}$) *vs.* ($\Omega\, t$) ⇒.

$$\frac{dE(t)}{dt} = \varepsilon(t) = \nu' \left\langle |\omega|^2 \right\rangle , \ \left\langle |\omega|^2 \right\rangle = (\kappa L)^2 ,$$

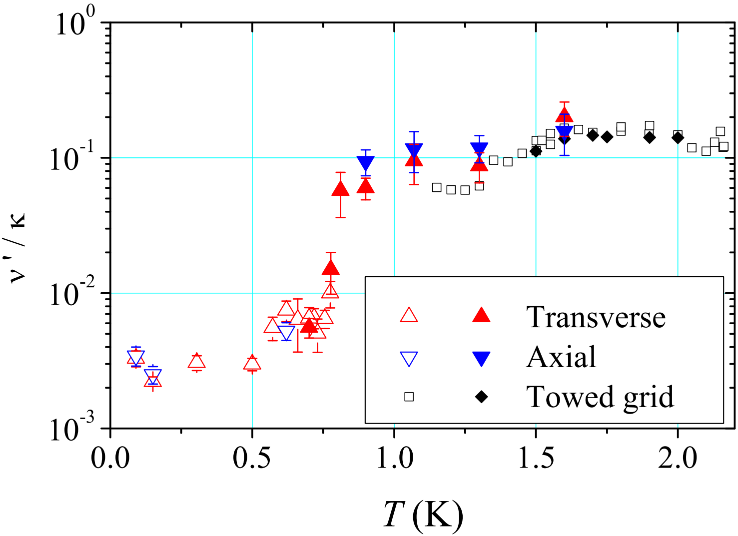
Turb. Energy $E \propto \varepsilon^{2/3} \Rightarrow E(t) \propto (t - t_*)^2$

$$\Rightarrow L(t) \propto 1/[\kappa\sqrt{\nu'(t-t_*)^3}] \Rightarrow$$

Data on $L(t)$ allows to measure effective viscosity ν'

Temperature dependence of the effective viscosity ν'

Manchester spin-down experiment from $\Omega = 1.5$ rad/s in superfluid ^{4}He [8]



Differential model for classical⇒quantum energy flux

Leiht-Nazarenko differential model for Classical hydrodynamic (HD) energy flux [3]

$$\varepsilon_k = -\frac{1}{8}\sqrt{k^{13}F_k}\,\frac{dF_k}{dk}\,, \qquad F_k = \frac{\mathcal{E}_k^{\rm HD}}{k^2}\,,$$

F_k – 3-dimensional spectrum of turbulence.

• Generic spectrum with a constant energy flux is the solution to Eq. $\varepsilon_k = \varepsilon = \text{const}$:

$$F_k^{\rm HD} = \left[\frac{24\,\varepsilon}{11\,k^{11/2}} + \left(\frac{T}{\pi\rho}\right)^{3/2}\right]^{2/3} \Rightarrow \begin{cases} (24/11)^{2/3}\ \varepsilon^{2/3}\,k^{-11/3}\,, \\ T/\pi\rho\,. \end{cases}$$

Low k region: K41 spectrum $\mathcal{E}^{\rm HD} \propto \varepsilon^{2/3}k^{-5/3}$,
Large k region: energy equipartition with an effective temperature T.

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Differential model for classical⇒quantum energy flux [4]

L'vov-Nazarenko-Rudenko [4] differential model for quantum Kelvin-wave energy flux

$$\varepsilon_{\rm KW}(k) = -\frac{5}{7}\frac{(k\ell)^8\mathcal{E}^4_{\rm KW}(k)}{\Lambda^5\kappa^7}\frac{d\mathcal{E}_{\rm KW}(k)}{dk}\,.$$

- Equation $\varepsilon_{\rm KW}(k)=\varepsilon=\text{const}$ has the solution

$$\mathcal{E}_{\rm KW}(k)=\Big[\frac{\Lambda^5\kappa^7}{\ell^8}\frac{\varepsilon}{k^7}+\Big(\frac{T}{\pi\rho}\Big)^5\Big]^{1/5}\Rightarrow\begin{cases}\Lambda\left(\kappa^7\varepsilon/\ell^8\right)^{1/5}k^{-7/5}\,,\\ T/\pi\rho\,.\end{cases}$$

Low k region: Kozik-Swistunov spectrum [9] of Kelvin waves (KW),
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Differential model for classical⇒quantum energy flux

Unified model for the eddy-wave total energy flux: Basic ideas

- Fluid motion is approximated as a mixture of “pure" HD and KW motions, with the portions of the energy density $g(k,\ell)$ & $1-g(k,\ell)$:

$$\mathcal{E}_k^{\rm HD}=\mathcal{E}_k\, g\,,\quad \mathcal{E}_k^{\rm KW}=\mathcal{E}_k\,(1-g)\,,\quad g(k,\ell)=\begin{cases}1\,, & k\ell \text{ small}\,,\\ 0\,, & k\ell \text{ large}\,.\end{cases}$$

- To find $g(dk,\ell)$ we define here HD and KW energies via velocities of k-bent, ℓ-separated parallel vortex lines $\boldsymbol{v}_{j,k}(\boldsymbol{r})$

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After cumbersome calculations and controlled approximations this finally gives analytical formula for the blending function, that depends ONLY on $x=k\,\ell$

$$g(x)=g_0[0.32\,\ln(\Lambda+7.5)x]\,,\ g_0(x)=\left[1+\frac{x^2\exp(x)}{4\pi(1+x)}\right]^{-1}.$$

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All above reasoning finally give in the stationary case:

$$\varepsilon = \varepsilon(k) = -\Big\{\frac{1}{8}\sqrt{k^{11} g(k\ell)\mathcal{E}(k)} + \frac{5}{7}\frac{(k\ell)^8 k_*^2 [1-g(k\ell)]^4 \mathcal{E}(k)^4}{\Lambda^5 \kappa^7}\Big\} \times \frac{d}{dk}\Big\{\mathcal{E}(k)\Big[\frac{g(k\ell)}{k^2} + \frac{1-g(k\ell)}{k_*^2}\Big]\Big\}, \quad k_* = \frac{2}{\ell} . \qquad \text{(LNR)}$$

This is an ordinary differential equation, that allows to find energy spectrum $\mathcal{E}(k)$ (at given ε and ℓ) in the entire region of scales: classical HD, quantum KW and crossover scales $k\ell \sim 1$.

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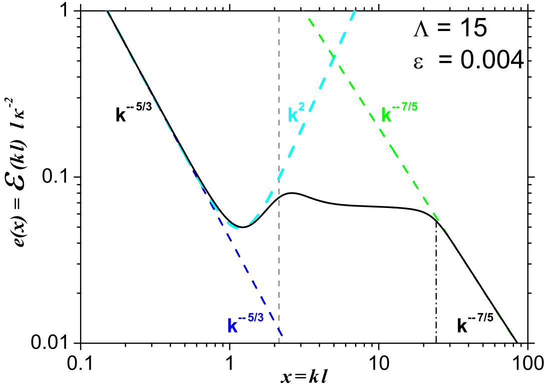
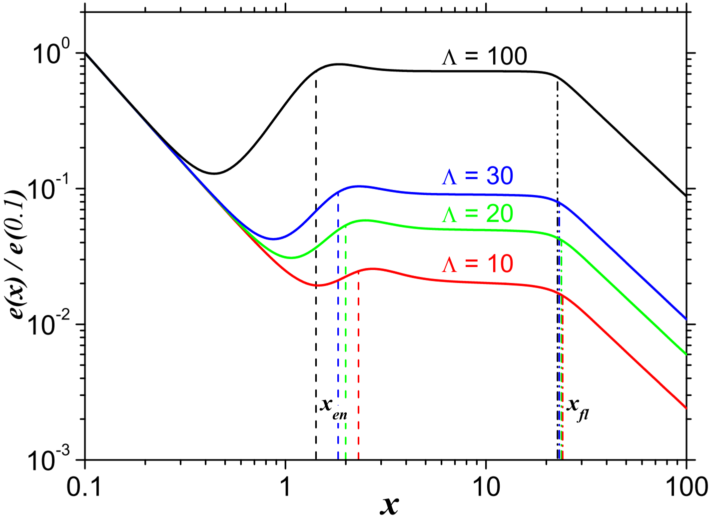


Black line – Spectrum $\mathcal{E}(k\ell)$ from solution of Eq. (LNR) $\varepsilon(k\ell) = \varepsilon$;
Dashed blue line – K41 HD energy spectrum $\mathcal{E}_{\rm HD}(k\ell) \propto k^{-5/3}$;
Dashed cyan line – general HD spectrum, including equilibrium $\propto k^2$;
Dashed green line – Energy spectrum of Kelvin waves $\mathcal{E}_{\rm KW} \propto k^{-7/5}$.
Vertical dashed lines – Left: energy crossover, $\mathcal{E}_{\rm HD}(k\ell) = \mathcal{E}_{\rm KW}(k\ell)$,
Right: flux crossover, $\varepsilon_{\rm HD}(k\ell) = \varepsilon_{\rm KW}(k\ell)$

Total energy spectra for different Λ

as the solutions of Eq. (LNR) $\varepsilon(k\ell) = \varepsilon$ for self-consistent values of ε: see next slide



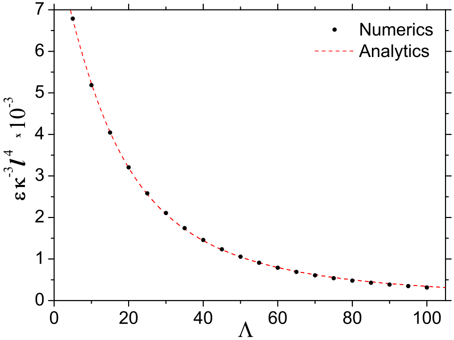
Self-consistent estimate of ε and effective viscosity ν'

Eq. (LNR) gives $\mathcal{E}(k)$ at fixed ε and ℓ, related by: $(\kappa^2/\ell^4) = \langle|\omega|^2\rangle$

$$= 2\int_0^\infty k^2 g(k\ell)\mathcal{E}(k)\,dk\ .$$

This allows to numerically determine ε and ℓ as a function of Λ, see black dots on Fig. and the analytical fit:

$$\epsilon = \frac{\nu'}{\kappa} = \frac{8.65}{10^3 + 45.8\Lambda + 1.98\Lambda^2}\,,$$



shown as red dashed line.

For ^{4}He value of $\Lambda \simeq 15$ we got $\nu'_{\rm theor} \approx 0.004\,\kappa$, which is quite close to Manchester spin-down experimental value $\nu'_{exp} \approx 0.003\,\kappa$. Having in mind that our model does not contain fitting parameters, we consider this agreement as more than satisfactory.

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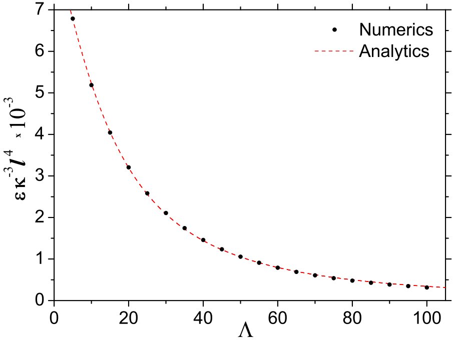

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Energy spectra at finite temperatures: UNDER CONSTRICTION

LNR model for the energy balance equation at finite temperature:

At finite temperatures ($\alpha \neq 0$) the energy balance equation includes:

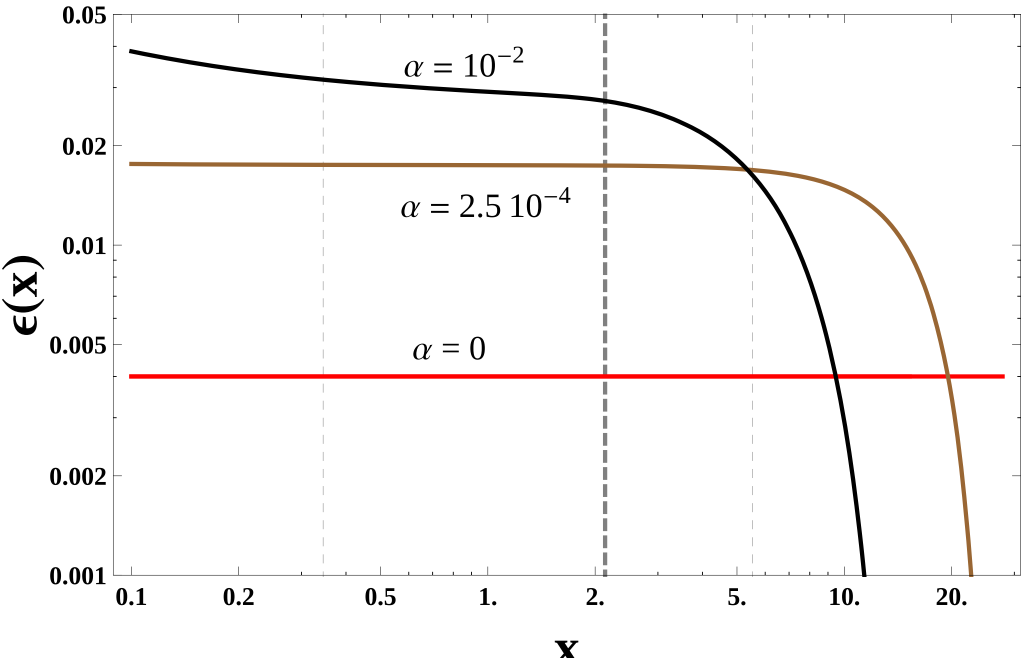
- Large-scale (eddy-dominated) dissipation and
- dissipation due to Kelvin waves

$$\frac{\partial \mathcal{E}(k,t)}{\partial t} + \frac{\partial \varepsilon(k)}{\partial k} = -\alpha\, \mathcal{E}(k,t)\Big\{ g(k\,\ell)\sqrt{\langle|\omega|^2\rangle} + \kappa k^2 \Big[1 - g(k\,\ell)\Big]\Big\}\,.$$

Here LNR (ours) model for the energy flux reads:

$$\varepsilon(k) = -\Big\{\frac{1}{8}\sqrt{k^{11} g(k\ell)\mathcal{E}(k)} + \frac{5}{7}\frac{(k\ell)^8 k_*^2 [1-g(k\ell)]^4 \mathcal{E}(k)^4}{\Lambda^5 \kappa^7}\Big\} \times \frac{d}{dk}\Big\{\mathcal{E}(k)\Big[\frac{g(k\ell)}{k^2} + \frac{1-g(k\ell)}{k_*^2}\Big]\Big\}, \quad k_* = \frac{2}{\ell}\,.$$

Stationary (numerical) solutions of this equation (at $\Lambda = 15$) and different α are as follows:



0.05
0.02
0.01
0.005
0.002
0.001
ϵ(x)
0.1
0.2
0.5
1.
2.
5.
10.
20.
x
$\alpha = 10^{-2}$
$\alpha = 2.5\ 10^{-4}$
$\alpha = 0$

Energy spectra at finite temperatures: UNDER CONSTRICTION

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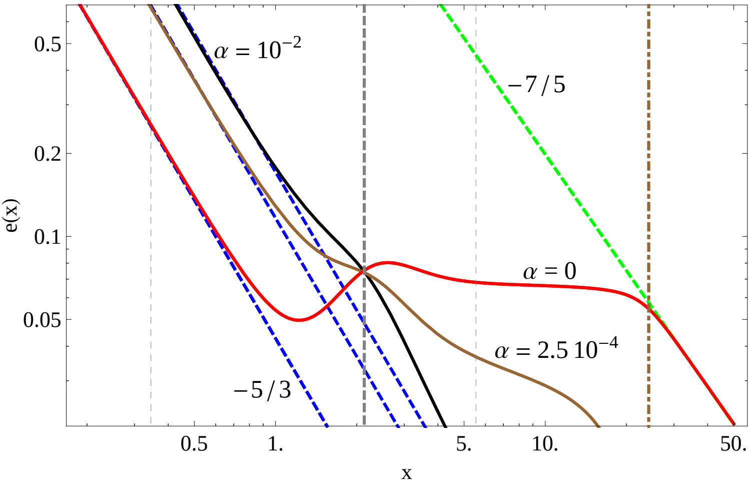
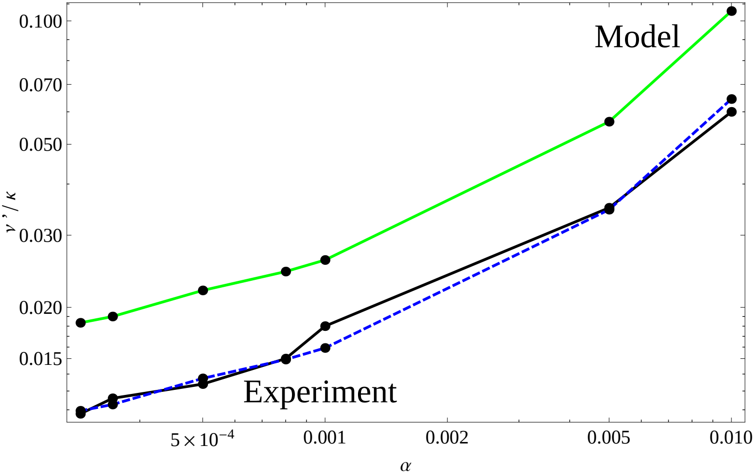


Energy spectra at finite temperatures: UNDER CONSTRICTION

Temperature dependence of the effective viscosity:



Summary and road ahead

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Our theory adequately reflects main physical features of superfluid ^{3}He and ^{4}He turbulence in laminar, quasi-classical turbulent, quantum-turbulent and crossover regimes.

• Our feeling is that
The model assumptions and approximations we made do not essentially affect the resulting physical picture of superfluid turbulence.

• The theory predicts not only the temperature dependence of ν' but the Entire energy spectrum at zero and finite temperatures, consisting (at $T \to 0$) of: K41 HD energy spectrum with constant energy flux $\propto k^{-5/3}$, HD equilibrium $\propto k^2$, a KW equilibrium $\simeq$ const. and a KW-spectrum with constant energy flux, $\propto k^{-7/5}$.

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THE END