The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA International Atomic Energy Agency

**2023-6**

**Workshop on Topics in Quantum Turbulence**

*16 - 20 March 2009*

# Modelling Kelvin Wave Turbulence

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# Modelling Kelvin Wave Turbulence

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March 17, 2009

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# Outline


Kelvin Waves In QT

Biot-Savart vs LIA

Truncated LIA

Differential Approximation Model

Future Research

Summary

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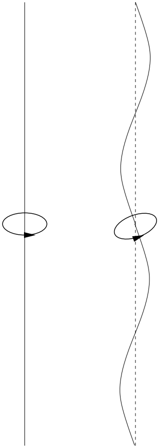
- Kelvin waves play an important role in the transfer of energy through scales in zero temperature quantum turbulence.
- Kelvin waves are generated by vortex reconnections.
- In the absence of viscosity, Kelvin waves extend the energy cascade of the macroscopic flow to smaller scales until phonon dissipation can remove the energy out of the system.



$$\omega = \frac{\kappa k^2}{4\pi}[\ln(1/ka) + C]$$

Role of Kelvin waves in the general picture of QT.

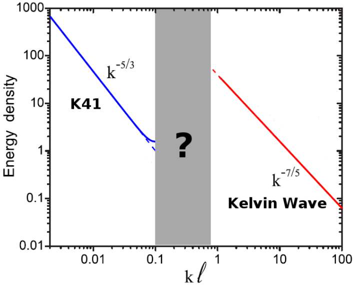
- Large scale flow behaves like classical K41.
- At intermediate scale - some 'crossover' occurs to Kelvin waves.
- Small scale Kelvin waves appear on vortex lines, that continue the cascade until phonon emission disperses energy.

L'vov *et al* (2007)

- Suppressed vortex reconnections in polarised tangles.
- Thermodynamic "warm up" at crossover - bottleneck effect.

Kozik and Svistunov (2007)

- Three distinct intermediate cascades involving specific reconnection mechanisms.



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At a macroscopic level, the superfluid vortex filament is a classical object whose dynamics is often described by the Biot-Savart equation (BSE)

$$\dot{\mathbf{s}} = \frac{\kappa}{4\pi} \int \frac{d\mathbf{s}_0 \times (\mathbf{s} - \mathbf{s}_0)}{|\mathbf{s} - \mathbf{s}_0|^3}. \tag{1}$$

However, the Biot-Savart equation contains a singularity as $\mathbf{s}_0 \to \mathbf{s}$, and one must invoke a cutoff at $a = |\mathbf{s} - \mathbf{s}_0|$.

- Cutoff poses an important problem!
    1. The type of cutoff determines the shape of the vortex core.
    2. The vortex core shape has an $\mathcal{O}(1)$ effect on the dispersion relation of the Kelvin wave.
    3. Which in turn, effects the nonlinear wave dynamics!

The BSE dynamics of the vortex filament admits a Hamiltonian formulation under a simple geometrical constraint: the position $\mathbf{s}$ of the vortex is represented in two-dimensional parametric form as $\mathbf{s} = (x(z), y(z), z)$, where $z$ is a given axis.

From a geometrical point of view, this corresponds to small perturbations with respect to the straight line configuration.

In terms of the complex canonical coordinate $w(z,t) = x(z,t) + iy(z,t)$,

$$H_{bs}[w] = \frac{\kappa}{4\pi} \int dz_1 dz_2 \frac{1 + Re(w'^*(z_1) w'(z_2))}{\sqrt{(z_1 - z_2)^2 + |w(z_1) - w(z_2)|^2}}, \qquad (2)$$

(see *E. Kozik and B. Svistunov, Phys. Rev. Lett.* **92**, *035301 (2004)*.)

An enormous simplification, both for theoretical and numerical purposes, is obtained by means of the so called local induction approximation (LIA).

$$\dot{\mathbf{s}} = \beta \mathbf{s}' \times \mathbf{s}'', \tag{3}$$

where $\beta = (\kappa/4\pi)\ln(\ell/a)$.

This approximation only takes into account the local contributions from neighbouring vortex elements.

When applied to Hamiltonian (2), the LIA procedure gives

$$H_{lia}[w] = 2\frac{\kappa}{4\pi}\ln\left(\frac{\ell}{a}\right)\int dz\sqrt{1+|w'(z)|^2}. \tag{4}$$

- LIA is an integrable system! No turbulent interactions.
- Theoretical insight in BSE is difficult and the evaluation of the integral is extremely computationally expensive.
- BSE is not exact because of the invoked cutoff; it is just another approximation.
- Ambiguity of the vortex core structure leads to leading order effects in Kelvin wave turbulence.

$$\omega_k = \frac{\kappa k^2}{4\pi}\left[\ln\left(\frac{1}{ka}\right) + C\right]$$

| Core Structure | $C$ |
|---|---|
| Uniform Vorticity | $-\gamma + \ln 2 + 1/4$ |
| Hollow Core | $-\gamma + \ln 2$ |
| BSE Model | $-\gamma - 3/2$ |

- Not clear if vortex core shape in BSE is any better than any other?
- Makes sense to find a '*minimal*' model with same scalings as BSE.

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We present a simple model for the modeling of Kelvin waves on a quantised vortex.
We truncate the LIA Hamiltonian to the first three terms:

$$\begin{aligned} H_{trunc}[w] &= H_0 + H_1 + H_2 = \\ &= \int dz \left(1 + \frac{1}{2}|w'|^2 - \frac{1}{8}|w'|^4\right). \end{aligned} \quad (5)$$

This model possesses the same scaling properties as the full BSE;

- It conserves energy and wave action, $\int dz|w|^2$.
- Gives rise to a dual-cascade six-wave system.
- Interaction coefficient has the same order of homogeneity as the one of the BSE.

Applying a Fourier transform to variable $w$, the truncated LIA Hamiltonian can be written in a wave action variable form from wave turbulence (WT) theory,

$$\begin{aligned} H_{trunc} &= \int \omega_k |w_k|^2 dk + \\ &+ \int dk_{1234} W_{1234} \delta^{12}_{34} w_1^* w_2^* w_3 w_4, \end{aligned} \tag{6}$$

with $\omega = k^2/2$, $W_{1234} = -\frac{1}{8} k_1 k_2 k_3 k_4$ and we used the standard notation $\delta^{12}_{34} = \delta(k_1 + k_2 - k_3 - k_4)$ and $dk_{1234} = dk_1 dk_2 dk_3 dk_4$.

WT theory deals with exact resonances of waves to form a kinetic equation of the evolution of wave action density $n_k = \langle w_k w_k^* \rangle$.

- Four-wave dynamics need to satisfy frequency and wavenumber resonance conditions.
- Impossible, need to go to higher order interactions.
- Use a quasi-identical transformation $w_k \to c_k$ to remove lower order nonlinearities - leads to six-wave dynamics.

$$H_c = \int \omega_k |c_k|^2 dk + \int dk_{123456} C_{123456} \delta^{123}_{456} c_1^* c_2^* c_3^* c_4 c_5 c_6, \qquad (7)$$

where $C_{123456} = -\frac{1}{16} k_1 k_2 k_3 k_4 k_5 k_6$. This six-order interaction coefficient $C_{123456}$ is obtained from coupling of two fourth-order vertices of $H_2$.

Applying a WT closure yields a six-wave kinetic equation

$$\dot{n}_k = 18\pi \int dk_{23456} \, |C_{k23456}|^2 \, \delta^{k23}_{456} \, \delta(\omega^{k23}_{456}) \, f_{k23456}, \qquad (8)$$

where we have introduced

$$f_{k23456} = n_k n_2 n_3 n_4 n_5 n_6 \left[ \frac{1}{n_k} + \frac{1}{n_2} + \frac{1}{n_3} - \frac{1}{n_4} - \frac{1}{n_5} - \frac{1}{n_6} \right],$$

$$\delta(\omega^{k23}_{456}) = \delta(\omega_k + \omega_2 + \omega_3 - \omega_4 - \omega_5 - \omega_6),$$

and

$$\delta^{k23}_{456} = \delta(k + k_2 + k_3 - k_4 - k_5 - k_6).$$

A simple dimensional analysis of (8) gives

$$\dot{n}_k \sim k^{14} n_k^5, \tag{9}$$

which is the same form obtained from the full BSE (*Kozik and Svistunov* $(2004)$).

Important non-equilibrium *constant flux* solutions of the kinetic equation (Kolmogorov-Zakharov solutions) exist.

- By requiring the existence of a range of scales in which the energy flux is $k$-independent leads to the spectrum

$$n_k \sim k^{-17/5}. \tag{10}$$

- A similar argument can be applied to the wave action flux

$$n_k \sim k^{-3}. \tag{11}$$

We performed numerical simulations to verify the existence and scalings of the dual cascade behaviour of the truncated LIA model (*Boffetta et al* (2009)).

$$\dot{w} = \frac{i}{2}\left[w'\left(1 - \frac{1}{2}|w'|^2\right)\right]' - (-1)^p \nu \nabla^{2p} w - \alpha w + \phi. \qquad (12)$$

- The small scale dissipative term (with $p > 1$) physically represents the radiation of phonons.
- The large scale damping term can be interpreted as the friction induced by normal fluid at a rate $\alpha$.
- Energy and wave action are injected in the vortex filament by a white-in-time external forcing $\phi(z, t)$ acting on a narrow band of wavenumbers around a given $k_f$.

# Direct Energy Cascade

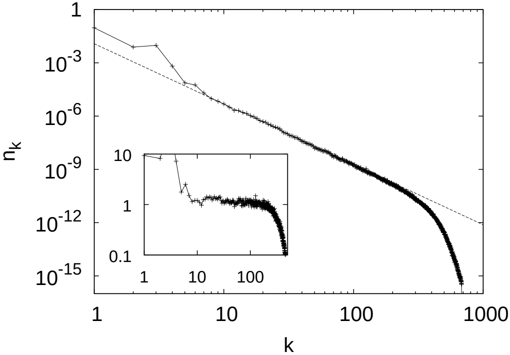


Figure: Wavenumber spectrum $n_k$ for a simulation of the direct cascade in stationary conditions at resolution $M = 2048$. Forcing is restricted to a range of wavenumbers $1 \leq k_f \leq 3$ and dissipation by phonon emission is modeled with hyperviscosity of order $p = 4$. The straight line represents the kinetic equation prediction $n_k \simeq k^{-17/5}$. The inset shows the spectrum compensated with the theoretical prediction.

# Inverse Wave Action Cascade

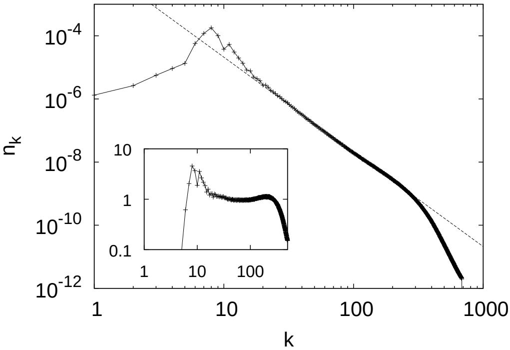


Figure: Wavenumber spectrum $n_k$ for a simulation of the inverse cascade in stationary conditions at resolution $M = 2048$. Forcing is restricted to a range of wavenumbers around $k_f = 300$ and dissipation by phonon emission is modeled with hyperviscosity of order $p = 4$. The straight line represents the kinetic equation prediction $n_k \simeq k^{-3}$. The inset shows the spectrum compensated with the theoretical prediction.

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The Differential Approximation Model (DAM) provides an interesting tool for analysing the dual cascade behaviour of Kelvin wave turbulence (*Nazarenko* (2005)).

- Preserves the main scalings of the kinetic equation closure.
- Several *reduced* versions to simplify the analysis.
- Contains physically relevant forcing and dissipation terms.
    1. Forcing by reconnections: $+\lambda\omega^{-2}$.
    2. Friction: $-\alpha\omega n_\omega$.
    3. Dissipation by phonons: $-\nu\omega^{9/2}n_\omega^2$.

For the Kelvin wave spectrum, the general model is

$$\dot{n}_\omega = F(n_\omega, \omega) + \lambda\omega^{-2} - \alpha\omega n_\omega - \nu\omega^{9/2}n_\omega^2 \tag{13}$$

In this talk, I will only consider the *complete* DAM.

The *complete* version of DAM contains

- Direct energy cascade $n_\omega \sim \omega^{-17/10}$.
- Inverse wave action cascade $n_\omega \sim \omega^{-3/2}$.
- Thermodynamical solutions $n_\omega \sim \frac{T}{\omega+\mu}$.

The nonliner term $F(n_\omega, \omega)$ for the *complete* model is

$$F(n_\omega, \omega) = \omega^{1/2} \partial_\omega^2 \left( n_\omega^6 \omega^{21/2} \partial_\omega^2 \left(1/n_\omega\right)\right). \tag{14}$$

Analytical solutions can only be found in the *reduced* DAMs, but are appropriate predictions for the *complete* DAM,

$$n_\omega = 10^{1/5} \omega^{-3/2} \left(\epsilon_0 \omega^{-1} - \frac{2}{3} \lambda \omega^{-3/2} - \eta_0\right)^{1/5}, \tag{15}$$

with flux

$$\epsilon_\omega = \epsilon_0 - \lambda \omega^{-1/2}. \tag{16}$$

$\eta_0$ and $\epsilon_0$ are the asymptotic values of the wave action and energy fluxes respectively.

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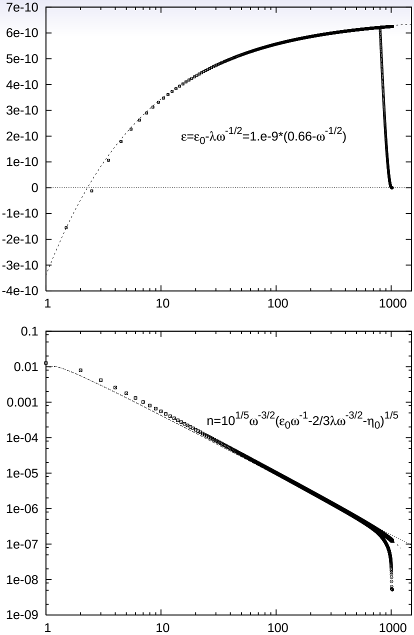


Figure: Complete DAM: the energy flux $\varepsilon$ (top) and the spectrum, respectively compared with the analytical predictions (16) and (15) . In each picture are shown the results of the simulations with and without $(\nu = 0)$ phonon dissipation. The number of points in the simulation is $M = 1024$, the forcing amplitude is $\lambda = 1 \times 10^{-9}$, viscosity $\nu = 1 \times 10^{-12}$ and the asymptotic values of the energy and wave action fluxes are $\epsilon_0 \simeq 0.66 \times 10^{-9}$ and $\eta_0 \simeq 0.75 \times 10^{-12}$.

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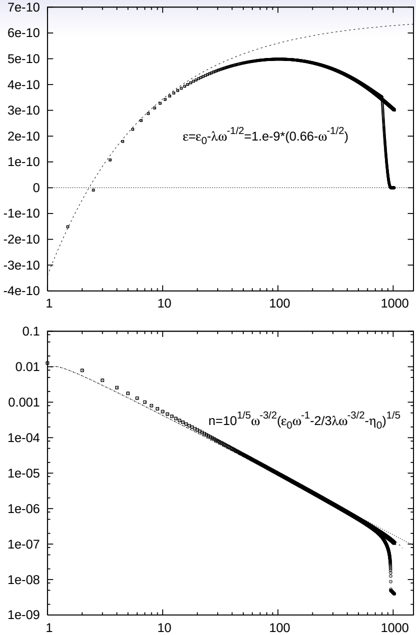


Figure: Complete DAM with $\alpha \neq 0$ and $\nu = 0$ or $\nu \neq 0$: a synthesis of the considerations of previous figure.

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We plan to develop an algorithm similar to a '*fast multipole method*' to approximate the far field calculation of the Schwarz equation (*Schwarz* (1988)), or more specifically the Biot-Savart integral.

- Work in collaboration with Carlo Barenghi (Newcastle), Sultan Alamri (Newcastle) and Sergey Nazarenko (Warwick).

The Schwarz equation is

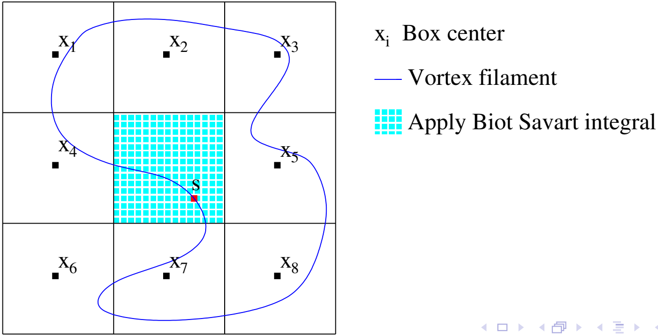
$$\dot{\mathbf{s}} = \mathbf{v}_{si} + \mathbf{v}_s + \alpha\mathbf{s}' \times (\mathbf{v}_n - \mathbf{v}_{si}) - \alpha'\mathbf{s}' \times \left[\mathbf{s}' \times (\mathbf{v}_n - \mathbf{v}_{si})\right] \quad (17)$$

where $\mathbf{s}' = d\mathbf{s}/d\xi$ and $\mathbf{v}_{si}$ is the self induced velocity of the vortex filament:

$$\mathbf{v}_{si} = \beta'\mathbf{s}' \times \mathbf{s}'' + \frac{\kappa}{4\pi}\int_{\ell'} \frac{d\mathbf{s}_0 \times (\mathbf{s} - \mathbf{s}_0)}{|\mathbf{s} - \mathbf{s}_0|^3}. \quad (18)$$

Explanation of algorithm:

- Split domain into a mesh of boxes.
- The calculation of the velocity of a section of a vortex filament is an integral over the total vortex array.
- However, only take the exact BS integral in a region around the vortex segment.
- Average the local vorticity of all other boxes, then add these to the BS calculation.
- Hopefully takes an $\mathcal{O}(N^2)$ algorithm down to $\mathcal{O}(N\ln(N))$.



What we hope to achieve:

- Decrease the computational time of calculating the Biot Savart equation with minimal loss of accuracy.
- Allows us to vastly increase the resolution of vortex filaments.
- See the presence of a Kelvin wave cascade.
- Apply a K$41$ background flow, and get some insight into the K$41$-Kelvin wave cross-over.

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- The unknown shape of the vortex core, questions the validity of any vortex line model.
- This means, truncated LIA is justified to the same extend as BSE.
- We present a simple model that incorporates the dual cascade behaviour of Kelvin wave turbulence.
- Need for further numerical insight into what happens at the K$41$-Kelvin wave crossover.
- Create a numerical scheme to speed up the calculation of the Schwarz equation.

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