



*The Abdus Salam
International Centre for Theoretical Physics*



2023-12

Workshop on Topics in Quantum Turbulence

16 - 20 March 2009

What Starts a Cascade on Vortex Lines? Curvature Growth versus Kelvin Waves

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**What Starts a Cascade on Vortex Lines?
Curvature Growth versus Kelvin Waves**

Prof. R.M. Kerr, Warwick University

3D anti-parallel Euler generates vortex waves
Simulation of anti-parallel vortex collapse.

ICTP Trieste 17 March 2009

Contrast to New work from Barenghi group in Newcastle for bundles of Gross-Pitaevskii vortices.

Basic question: **Can interacting vortex filaments generate a cascade?**

My answer: **It is possible.**

Outline of talk:

- Advances in simulation of the anti-parallel geometry.
- My new emphasis based on the 3D numerical DNS: **spirals**.
- **Inspiration:** Question of cascade in superfluid turbulence.
- A new vortex model: anti-parallel Biot-Savart + local induction approximation
- Question of torsion τ in Frenet-Serret formulation

<http://www.mae.cornell.edu/fdrl/research/wingvortex.html>



- Crow instability between anti-parallel vortices. **Vortices attract**
- **Spirals** appear in contrails.

If I had the time there are two topics I would like to discuss,

- Relationships between classical and quantum turbulence.
 - Decay rates, scales, types of cascades, etc..
 - Intervortex line spacing and Taylor microscale.
 - What type of cascade? It certainly isn't Richardson.
 - Growth of velocity where vorticity singular: $u_\omega \sim \frac{1}{(T_c - t)^{1/2}}$

Paoletti et al, PRL 2008 superfluid hydrogen particle experiments.

Kerr, Phys. Fluids 2005 classical Euler simulations.
- Vortex line dynamics: What must vortices do to start a cascade if the cascade isn't Richardson.

Past vortex line models seem incapable. Examples:

 - Schwarz tangle of with motion induced only by LIA and with artificial reconnections.
 - Pumir and Siggia smoothed Biot-Savart calculations.
 - de Waele and Aarts truncated Biot-Savart plus LIA calculations.

Relationships between classical and quantum turbulence.

- 1) **Decay rates.**
- 2) What type of **cascade?** The classical cascade certainly **isn't Richardson** and instead involves vortex dynamics.
- 3) **Growth of velocity** where vorticity singular: $u_\omega \sim \frac{1}{(T_c - t)^{1/2}}$ (Kerr, PF, 2005)
Like superfluid Paoletti et al (2008)
- 4) Consider the **inter-vortex line spacing** $b \sim L^{-1/2}$ so that $b^2 L = 1$.

b is the **Kolmogorov scale** η of classical turbulence if $\nu_{eff} \sim \kappa$.

Take dissipation $\epsilon \sim \kappa \overline{\omega^2} = \kappa^3 L^2$

Then $\eta_\kappa = (\kappa^3 / (\kappa^3 L^2))^{1/4} \sim L^{-1/2} = b$

- 5) Classical: Instead think about circulation κ_η about a classical Kolmogorov vortex of diameter $\eta = (\nu^3 / \epsilon)^{1/4}$. Then $\kappa_\eta \sim \overline{\omega} \eta^2 \sim (\epsilon / \nu)^{1/2} (\nu^3 / \epsilon)^{1/2} \sim \nu$.
- 6) For energy E think in terms of a structure function, with ℓ_0 fixed large length scale:

$$E \sim \epsilon^{2/3} \ell_0^{2/3} \sim (\kappa \overline{\omega^2})^{2/3} \ell_0^{2/3} = \kappa^2 L^{4/3} \ell_0^{2/3}$$

This gives $\overline{U} \sim \sqrt{E} \sim \kappa L^{2/3} \ell_0^{1/3}$.

- **Whatever we learn about quantum turbulence might be the key to understanding classical turbulence**
- If we could understand how quantum turbulence produces phonons, maybe we will learn why classical turbulence creates sound.
- **Reducing sound** has major engineering applications in the UK.

- The **Local Induction Approximation** (via Hasimoto) has Kelvin waves but **they do not cascade**. Two choices for a cascade:

1 A **Kelvin wave cascade** which depends on higher-order self-induction terms.

Is this the origin of superfluid turbulence?

2 **Biot-Savart terms could supercede this**

But can a Biot-Savart interaction generate its own cascade?

- Proposed model of two anti-parallel vortices uses:

A recent quaternionic formulation of the Euler equations.

Coupled with the local induction approximation.

As before, local induction by itself cannot generate a cascade.

In particular the problem with sharp reconnections (à la Schwarz) becomes evident.

- You just get Kelvin waves.

- But with the Biot-Savart quaternionic terms, the **direction of propagation** of nonlinear waves is **reversed**.
- This creates of very small fluctuations that could lead to **reconnection**.

End of talk.

Abstract

- In many superfluid situations: (Experimental and simulations of a 3D hard-core interaction model)

It is found that the vortex line length in **superfluid turbulence decays** in a manner **consistent with**:

- **Classical turbulence** $E(t) \sim t^{-2}$. (Originally due to S. Patterson, director of Scientific Computing at NCAR about 1979 and first mentioned in my 1981 PhD thesis.)

- **Decay mechanisms for superfluids that have been proposed:**

- **Kelvin wave emission along lines.**
- **Phonon radiation at small scales.**
- **Vortex ring emission.**

For rings to form: There must be rapid reconnection induced by Biot-Savart interactions.

- In this presentation such an interaction will be proposed.

It uses a recent quaternionic formulation of the Euler equations, coupled with the local induction approximation.

- As in earlier Biot-Savart work, local induction by itself cannot generate a cascade, even if there are sharp reconnections.
- But a reversal in the direction of propagation of nonlinear waves allows:

Fast reconnection and finite energy release as vortex rings.

- This is a model of a model, **new full Biot-Savart calculations are needed.**

However, this work demonstrates **how much resolution is really needed** to properly resolve Kelvin waves and other sharp features.

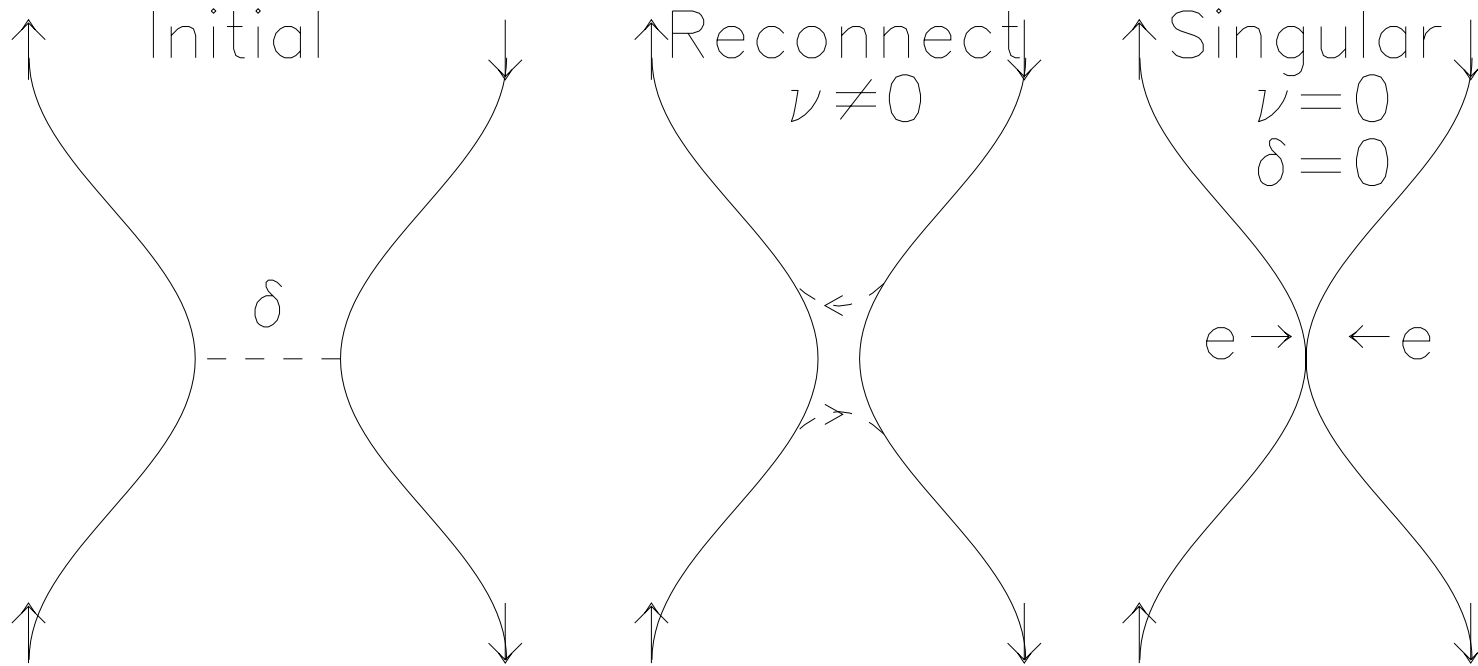
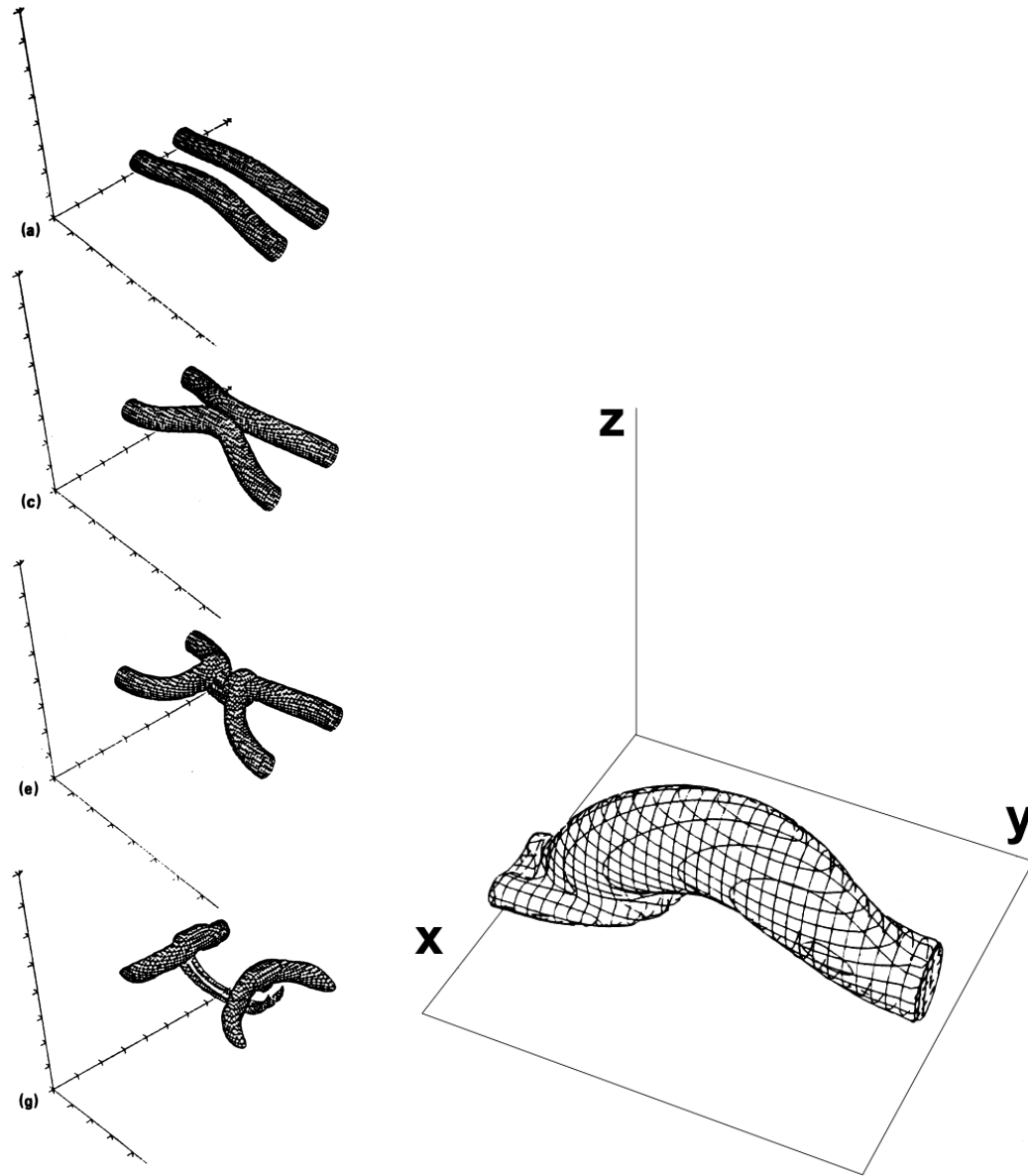


Diagram of the interaction of anti-parallel vortices. From an initial condition of anti-parallel vortices separated at their closest approach by δ , if $\nu \neq 0$ there is reconnection that forms new vortices indicated by the dashed curves. However, if $\nu = 0$, a singularity can form when $\delta = 0$ if the vortices are pushed together by the self-induced strain indicated by e .



Steps in **idealized anti-parallel** vortex reconnection taken from a low resolution, low Reynolds number calculation Melander and Hussain, 1989. From top to bottom, the first two frames show the anti-parallel vortex tubes being pushed together by self-interaction through the law of Biot-Savart. The third frame shows that reconnection has progressed to form two new tubes orthogonal to the original tubes. In the bottom frame the new tubes are separating.

Mathematical Issues

- Outstanding modern mathematical problem, \$1 million prize:
- For 3D incompressible **Navier-Stokes** with finite energy, etc.
 - Find an example of a singularity of 3D incompressible Navier-Stokes
 - Prove that Navier-Stokes is regular.
 - Folk-belief: Navier-Stokes is regular
- Related: 3D incompressible **Euler**, what is known:
 - There is only one unquestionable analytic bound: Beale, Kato, Majda (1984):

$$\int \|\omega\|_{\infty} dt \rightarrow \infty$$

(This has since also been proven using the more robust BMO norm)

- Folk-belief: Unknown.
- Constantin, Fefferman, Majda (1996) and Deng, Hou, Yu (2005) have further relations on time integrals of **curvature** and **velocity**:

$$\int \|\nabla \xi\|_{\infty}^2 dt \quad \int \sup |u|^2 dt \quad \int \|\nabla \xi\|_{\infty} |u| dt$$



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Physica D: Nonlinear Phenomena

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3D Euler about a 2D Symmetry Plane

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Initial results from new calculations of interacting anti-parallel Euler vortices are presented with the objective of understanding the origins of singular scaling presented by Kerr (1993) and the lack thereof by Hou and Li (2006). Core profiles designed to reproduce the two results are presented, new more robust analysis is proposed, and new criteria for when calculations should be terminated are introduced and compared with classical resolution studies and spectral convergence tests. Most of the analysis is on a $512 \times 128 \times 2048$ mesh, with new analysis on a just completed $1024 \times 256 \times 2048$ used to confirm trends. One might hypothesize that there is a finite-time singularity with enstrophy growth like $\Omega \sim (T_c - t)^{-\gamma_\Omega}$ and vorticity growth like $\|\omega\|_\infty \sim (T_c - t)^{-\gamma}$. The new analysis would then support $\gamma_\Omega \approx 1/2$ and $\gamma > 1$. These represent modifications of the conclusions of Kerr (1993). Issues that might arise at higher resolution are discussed.

PACS numbers: 47.10.A-, 47.11.Kb, 47.15.ki

I Introduction

One definition of solving Euler's three-dimension incompressible equations [1] is determining whether or not they dynamically generate a finite-time singularity if the initial conditions are smooth, in a bounded domain and have finite energy. The primary analytic constraint that must be satisfied [2] is:

$$\int_0^T \|\omega\|_\infty dt \rightarrow \infty \quad (1)$$

where $\|\omega\|_\infty$ is the maximum of vorticity over all space. To date, Kerr (1993) [3] remains the only fully three-dimensional simulation of Euler's equations with evidence for a singularity consistent with this and related constraints [4] Growth of the enstrophy production and stretching along the vorticity, plus collapse of positions, supported this claim [3]. Additional weaker evidence related to blow-up in velocity and collapsing scaling functions was presented later [5].

There is only weak numerical evidence supporting these claims [6,7] In a recent paper, as described in one of the invited talks of this symposium, Hou and Li (2006) [8] found evidence that the above scenario failed at late times.

This contribution will first comment on four issues raised at the symposium, then present preliminary new results. The four issues are:

- How should spurious high-wavenumber energy in spectral methods be suppressed?

- What criteria should be used to determine when numerical errors are substantial?
- What effect do the initial conditions have on singular trends? A cleaner initial condition is proposed.
- We introduce a new approach for determining whether there is singular behavior of the primary properties and the associated scaling. This is applied to both new and old data.

All calculations will be in the following domain: $L_x \times L_y \times L_z = 4\pi \times 4\pi \times 2\pi$ with free-slip symmetries in y and z and periodic in x with up to $n_x \times n_y \times n_z = 1024 \times 256 \times 2048$ mesh points. Using these symmetries only one-half of one of the anti-parallel vortices needs to be simulated.

The “symmetry” plane will be defined as xz free-slip symmetry through the maximum perturbation of the initial vortices and the “dividing” plane will be defined as the xy free-slip symmetry between the vortices.

II. How should spurious high-wavenumber energy in spectral methods be suppressed?

A generic difficulty in applying spectral methods to localized physical space phenomena is the accumulation of spurious high-wavenumber energy that leads to numerical errors.

Discriminating between early vortex filament methods

- **Schwarz (1978-1985)** primarily used a tangle of vortices moving under their own LIA (local induction approximation) motion.
- **Pumir and Siggia, PF 1987** smoothed out the Biot-Savart interactions.
- **de Waele and Aarts, PRL 1996** took Schwarz' full Biot-Savart model:

New vortex model

- Interacting anti-parallel vortex filaments.

It has two parts:

- Non-local interaction between two anti-parallel, mirrored vortices.

Using only Biot-Savart interaction near the symmetry plane.

New quaternionic formulation (due to Gibbon) to generate time-derivative of curvature (in Gibbon, Holm, Kerr, Roulstone).

But this is not enough.

- Need out-of-plane self-contribution. Therefore use:
- Local induction approximation of a filament.
 - This is converted into NLS (Hasimoto transformation) of a wavefunction ψ .
 - Whose real and complex parts, with an additional scalar gauge, give the direction and strength of curvature of the filament.

- **End Result:** NLS + image Biot-Savart

$$i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial s^2} - \frac{1}{2}|\psi|^2\psi + \text{image Biot-Savart}$$

Or view it as: **Symmetry plane Biot-Savart** plus LIA

Velocity on the symmetry plane from **one nearly 2D vortex** with cylindrical Rosenhead regularization of uniform radius a **plus curvature term**:

$$\mathbf{u} = \frac{\kappa}{2\pi} \left[\mathbf{1} + \frac{|\mathbf{s}''| \mathbf{x}_n}{2} \right] \frac{(-\mathbf{y}, \mathbf{x})}{\mathbf{r}^2 + \mathbf{a}^2} + \frac{\kappa |\mathbf{s}''|}{8\pi} \left[\log \frac{\epsilon^2}{\mathbf{r}^2 + \mathbf{a}^2} - 2 \right] (-\mathbf{n}_y, \mathbf{n}_x) \quad (2)$$

- **NOTE:** κ =circulation. \mathbf{s}'' =curvature
- We need **two** such vortices: one with $\mathbf{n}_1 = (\mathbf{n}_x, \mathbf{n}_y)$ at $y_1 = d$ and another at $y_2 = -d$ with $\mathbf{n}_2 = (\mathbf{n}_x, -\mathbf{n}_y)$.
- To this we need to add out-of-plane **stretching** at y_1 due to vortex at y_2 : $\boldsymbol{\alpha}(\mathbf{y}_1) = -\nabla \cdot \mathbf{u}_2(\mathbf{y}_1)$
- And an equation for the curvature, which is 1D NLS+stretching+local Biot-Savart because:

1D Hasimoto NLS complex wave function ψ is related to curvature by

$$\psi = (\mathbf{s}''_x + i\mathbf{s}''_y) e^{i\theta_1} \quad \text{where} \quad \mathbf{s}'' = (\mathbf{s}''_x, \mathbf{s}''_y) \quad (3)$$

and θ_1 is a gauge that is constant along the filament. That is:

$$i \frac{\partial \psi}{\partial t} = -G(t) \left(\frac{\partial^2 \psi}{\partial s^2} + \frac{1}{2} |\psi|^2 \psi \right) + \text{image Biot-Savart}$$

where $G(t) \sim C_G \frac{\kappa}{4\pi} \log \frac{\epsilon}{a}$ and $\epsilon \sim c_\epsilon |\mathbf{s}''|^{-1} \sim c_\epsilon R$ (radius of curvature).

- $C_G \sim \mathcal{O}(1)$ but $c_\epsilon \sim 20$.

Local Biot-Savart terms

- From Gibbon (2002) (quaternions for Euler), rate-of-change of the direction of vorticity is:

$$\frac{D}{Dt} \mathbf{s}' = \boldsymbol{\chi} \times \mathbf{s}' \quad \text{where} \quad \boldsymbol{\chi} = \mathbf{s}' \times \mathbf{S} \mathbf{s}' \quad (4)$$

- Then take the arclength derivative to get (Gibbon *et al.*, 2006) for curvature $\mathbf{s}'' = |\mathbf{s}''| \mathbf{n}$.

$$\frac{D \mathbf{s}''}{Dt} = (\boldsymbol{\chi} \times \mathbf{s}'),_s - \alpha \mathbf{s}'' \quad \text{or} \quad \frac{D \mathbf{s}''}{Dt} = \mathbf{u}_{,ss} - \alpha_{,s} \mathbf{s}' - 2\alpha \mathbf{s}'' \quad (5)$$

- For two curved vortex filaments:

$$\frac{D \mathbf{s}''_1}{Dt} = \underbrace{\mathbf{u}_{1,ss}}_{NLS} + \mathbf{u}_{2,ss} - \alpha_{2,s} \mathbf{s}' - 2\alpha_2 \mathbf{s}''$$

Understanding direction of wave/soliton propagation

Consider the following generalization of the Hasimoto wave function:

$$\psi(s) = |\mathbf{s}''|(s) \exp(i\theta) \quad \text{where} \quad \theta = \int_0^s \tau(\sigma) d\sigma + \theta_0 + \theta_1 \quad (6)$$

θ_1 is the gauge variable that relates the this wave function to the curvature.

- On the symmetry plane the following is obeyed:

$$\frac{d}{dt}\theta_1 = \frac{1}{2}|\mathbf{s}''|^2 \quad \frac{d}{dt}|\mathbf{s}''| = -|\mathbf{s}''|\tau' \quad \frac{d}{dt}\mathbf{n} = \frac{|\mathbf{s}''|''}{|\mathbf{s}''|}\mathbf{b} \quad \frac{d}{dt}|\mathbf{s}''|\mathbf{n} = -\tau'|\mathbf{s}''|\mathbf{n} + |\mathbf{s}''|''\mathbf{b} \quad (7)$$

- $\frac{d}{dt}|\mathbf{s}''|$ requires knowing $\tau' = d\tau/ds$
- **What are the meaning of τ and τ' ?**
- τ is the phase velocity:

$$v_{ph} = \frac{d}{ds}\theta = \tau$$

- **Sign of torsion** determines the direction of phase velocity.
- On the symmetry plane one finds that:

$$\frac{d\tau'}{dt} = \frac{|\mathbf{s}''|''''}{|\mathbf{s}''|} - \frac{(|\mathbf{s}''|'')^2}{|\mathbf{s}''|^2} + |\mathbf{s}''||\mathbf{s}''|'' - 2(\tau')^2$$

– $\max(|\mathbf{s}''|)$ is on the symmetry plane, therefore $|\mathbf{s}''|'' < 0$. Negative.

– $\max(-|\mathbf{s}''|'') < 0$. Positive definite. Therefore:

$$\underbrace{\frac{|\mathbf{s}''|''''}{|\mathbf{s}''|}}_{>0} \quad \underbrace{-\frac{(|\mathbf{s}''|'')^2}{|\mathbf{s}''|^2} + |\mathbf{s}''||\mathbf{s}''|'' - 2(\tau')^2}_{<0}$$

Which sign dominates?

Discriminating between early vortex filament methods

- **Schwarz (1978-1985)** primarily used a tangle of vortices moving under their own LIA (local induction approximation) motion. Artificial reconnections when lines crossed:
 $\tau' > 0$ and $|\mathbf{s}''|$ very large, so Kelvin waves propagate out on the vortex lines.
- **Pumir and Siggia, PF 1987** smoothed out the Biot-Savart interactions.
- **de Waele and Aarts, PRL 1996** took Schwarz' full Biot-Savart model:

Discriminating between early vortex filament methods

- **Schwarz (1978-1985)** primarily used a tangle of vortices moving under their own LIA (local induction approximation) motion. Artificial reconnections when lines crossed:

$\tau' > 0$ and $|\mathbf{s}''|$ very large, so Kelvin waves propagate out on the vortex lines.

Energy could only be removed by wave interactions on the vortex lines, and this was weak.

Instead he invoked mutual friction terms to remove energy.

- **Pumir and Siggia, PF 1987** smoothed out the Biot-Savart interactions.

Attempt to represent the finite-core effects without explicitly invoking LIA. $\mathbf{G}(\mathbf{t}) = \text{constant}$.

But line attraction stalled within many core diameters. No reconnections and no Kelvin waves.

Reason: **Smoothing suppresses curvature/NLS terms.**

- **de Waele and Aarts, PRL 1996** took **Schwarz (1988)** full Biot-Savart model:

Minus mutual friction. Regularization allows $\mathbf{G}(\mathbf{t})$ **time dependent.**

But **de Waele/Aarts** never carried these calculations beyond the rate of approach.

Why the differences? Pumir/Siggia versus **de Waele/Aarts**

- The differences all seem to come down to how the Biot-Savart interactions are regularized.

My model is a single framework in which to see effects the different regularizations have.

I conclude that the **de Waele/Aarts** approach has the **fewest artificial elements.**

But need $\mathbf{G}(\mathbf{t}) \sim \log(R_c/a_0)$ to get the experimentally observed **velocity growth:**

$$\mathbf{u} \sim (\mathbf{T}_c - \mathbf{t})^{-1/2}$$

Understanding direction of wave/soliton propagation

Consider the following generalization of the Hasimoto wave function:

$$\psi(s) = |\mathbf{s}''|(s) \exp(i\theta) \quad \text{where} \quad \theta = \int_0^s \tau(\sigma) d\sigma + \theta_0 + \theta_1 \quad (8)$$

θ_1 is the gauge variable that relates the this wave function to the curvature.

- On the symmetry plane the following is obeyed:

$$\frac{d}{dt}\theta_1 = \frac{1}{2}|\mathbf{s}''|^2 \quad \frac{d}{dt}|\mathbf{s}''| = -|\mathbf{s}''|\tau' \quad \frac{d}{dt}\mathbf{n} = \frac{|\mathbf{s}''|''}{|\mathbf{s}''|}\mathbf{b} \quad \frac{d}{dt}|\mathbf{s}''|\mathbf{n} = -\tau'|\mathbf{s}''|\mathbf{n} + |\mathbf{s}''|''\mathbf{b} \quad (9)$$

- $\frac{d}{dt}|\mathbf{s}''|$ requires knowing $\tau' = d\tau/ds$
- **What are the meaning of τ and τ' ?**
- τ is the phase velocity:

$$v_{ph} = \frac{d}{ds}\theta = \tau$$

- **Sign of torsion** determines the direction of phase velocity.
- On the symmetry plane one finds that:

$$\frac{d\tau'}{dt} = \frac{|\mathbf{s}''|''''}{|\mathbf{s}''|} - \frac{(|\mathbf{s}''|'')^2}{|\mathbf{s}''|^2} + |\mathbf{s}''||\mathbf{s}''|'' - 2(\tau')^2$$

– $\max(|\mathbf{s}''|)$ is on the symmetry plane, therefore $|\mathbf{s}''|'' < 0$. Negative.

– $\max(-|\mathbf{s}''|'') < 0$. Positive definite. Therefore:

$$\underbrace{\frac{|\mathbf{s}''|''''}{|\mathbf{s}''|}}_{>0} \quad \underbrace{-\frac{(|\mathbf{s}''|'')^2}{|\mathbf{s}''|^2} + |\mathbf{s}''||\mathbf{s}''|'' - 2(\tau')^2}_{<0}$$

Which sign dominates?

Notation problem: differential geometry versus superfluids

- The symbol for direction of the vorticity we are using is $\boldsymbol{\xi}$.

So what you call $\mathbf{s}' = \boldsymbol{\xi}$.

- The symbol you use for circulation κ is the standard symbol for the curvature of a line.

So that what you call $\mathbf{s}'' = \kappa_c \mathbf{n}$ where \mathbf{n} is the direction of the curvature and to avoid ambiguity I am calling the curvature κ_c .

- Hasimoto (1972) was a bit ambiguous about relationship between phase of ψ and direction of curvature \mathbf{n} .

If $\boldsymbol{\kappa}_c = \kappa_c \exp(i\theta_1)$ then $\psi = \kappa_c \exp(i\theta)$

where $\theta = \int_0^s \tau(\sigma) d\sigma + \theta_0 + \theta_1$

Understanding direction of wave/soliton propagation

Consider the following generalization of the Hasimoto wave function:

$$\psi(s) = \kappa_c(s) \exp(i\theta) \quad \text{where} \quad \theta = \int_0^s \tau(s') ds' + \theta_0 + \theta_1 \quad (8)$$

θ_1 is the gauge variable that relates the this wave function to the curvature in physical space in (12).

- On the symmetry plane the following is obeyed:

$$\dot{\theta}_1 = \frac{1}{2}\kappa_c^2 \quad \dot{\kappa} = -\kappa\tau' \quad \dot{\mathbf{n}} = \frac{\kappa_c''}{\kappa_c} \mathbf{b} \quad \frac{d}{dt}\kappa_c \mathbf{n} = -\tau' \kappa_c \mathbf{n} + \kappa_c'' \mathbf{b} \quad (\text{eq : symmp})$$

- $\dot{\kappa}_c$ requires knowing $\tau' = d\tau/ds$
- **What are the meaning of τ and τ' ?**
- τ is the phase velocity:

$$v_{ph} = \frac{d}{ds}\theta = \tau$$

- **Sign of torsion** determines the direction of phase velocity.
- On the symmetry plane one finds that:

$$\frac{d\tau'}{dt} = \frac{\kappa_c''''}{\kappa_c} - \frac{(\kappa_c'')^2}{\kappa_c^2} + \kappa_c \kappa_c'' - 2(\tau')^2 \quad \text{Which sign dominates?}$$

Which sign dominates?

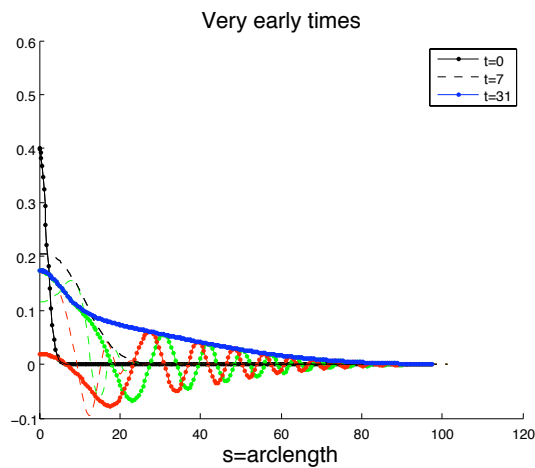
$$\frac{d\tau'}{dt} = \underbrace{\frac{\kappa_c''''}{\kappa_c}}_{>0} \quad \underbrace{-\frac{(\kappa_c'')^2}{\kappa_c^2} + \kappa_c \kappa_c'' - 2(\tau')^2}_{<0}$$

- $\max(\kappa_c)$ is on the symmetry plane $\Rightarrow \kappa_c'' < 0$ Negative definite at $s = 0$.
- $\max(\kappa_c'') < 0. \Rightarrow \kappa_c'''' > 0$ Positive definite at $s = 0$.
- If you have a sharp **artificial** reconnection, then $\kappa_c'''' > 0$ dominates, $\tau' > 0$ and waves propagate outwards.
- To get a singularity one needs $\tau' < 0$. **Why?**
 - a) It is necessary to add this to the non-local Biot-Savart terms in order to give the curvature a **tilt**.
 - b) If there is no **tilt** then stretching will saturate
 - c) And there won't be any singularity.
- But if there is stretching along the vortex filament:

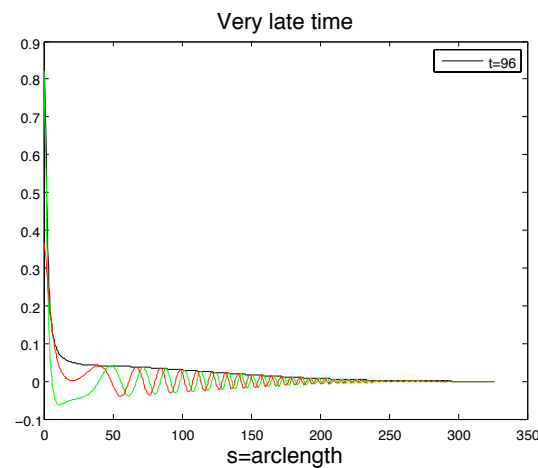
This will be due to the non-local anti-parallel Biot-Savart interaction.

 - d) Then the < 0 term could dominate at late times.
 - e) Then waves propagate inwards, colliding on the symmetry plane.

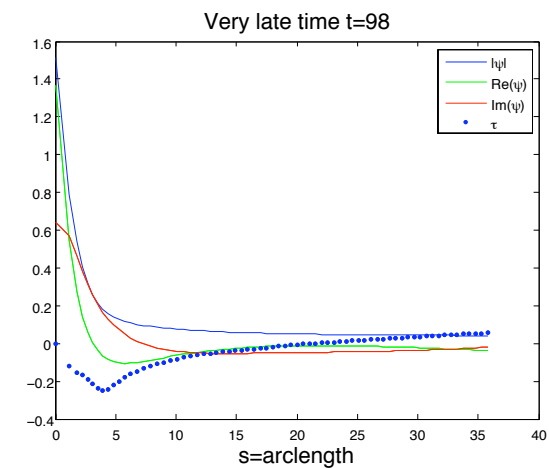
- Profiles of wave functions at many times. Figures show the initial emission of Kelvin waves.
- Absolute value curves are indicated in legends. Also shown for some times are: real (green) and imaginary (red).



Early times: $t = 0, 7, 31$.



Late time: $t = 96$.
Whole length to show:



Torsion: $t = 98, s < 36$.
No sign of waves.

- **Early times:** $t = 0, 7, 31$. ICTP Trieste 17 March 2009

Components at $t = 31$ indicated by real (green) and imaginary (red).

Note that in this phase with $\tau' > 0$, the peak (black to blue for $s = 0$) relaxes substantially.

- **Late time:** $t = 96$. Whole length to show:

Right propagating Kelvin wave train for $s > 50$.

Transition to something else for $s < 50$ where $\tau < 0$.

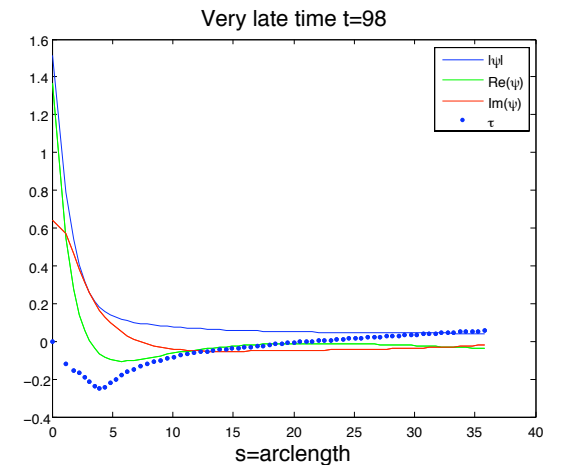
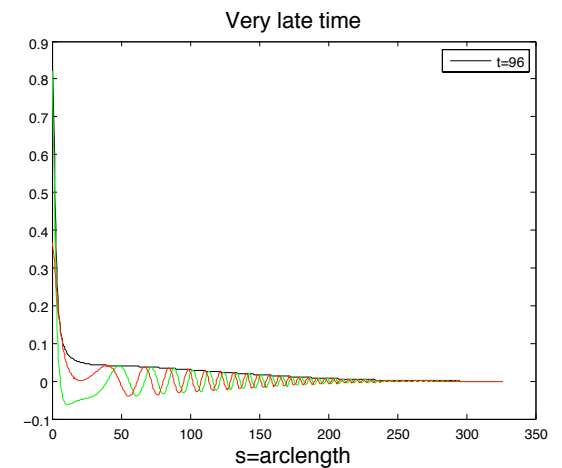
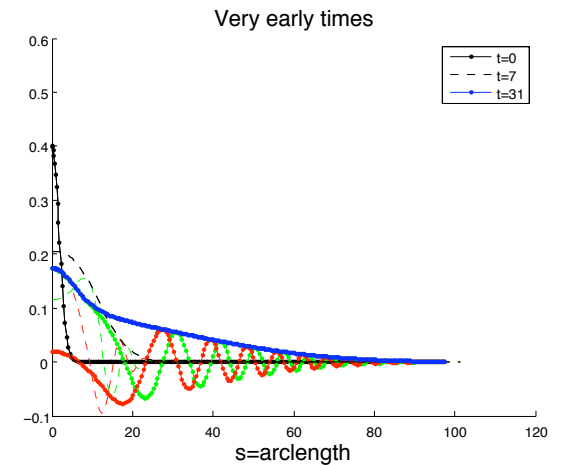
Blue $|\psi(s)|$. **Green** $Re[\psi(s)e^{-i\theta_0}]$. **Red**
Red $Im[\psi(s)e^{-i\theta_0}]$.

- **Torsion:** $t = 98, s < 36$. No sign of waves.

Blue $\tau(s, 98)$.

Note that the first mesh point in s is at $s \approx 1$.

- For $s < 11$ and especially near $s = 0$: $\tau < 0$ where there is blow-up.
- For $s > 11$: $\tau > 0$ where there is Kelvin wave propagation away.



- **Top:** Deepening of the $\tau(s)$ profile with time. $t = 0.46$, $\tau > 0$ still. Then τ has an increasingly deep negative peak for: $t = 0.94, 96, 98$.

- **Middle:** Time development of symmetry plane and global NLS terms. $t = 0$ to 92, an overall view.

- Nothing much grows until $\tau' < 0$ **Green●**: about $t = 60$.

Red●: Enstrophy $\int |s''|^2 ds = \int |\psi|^2 ds$ growth.

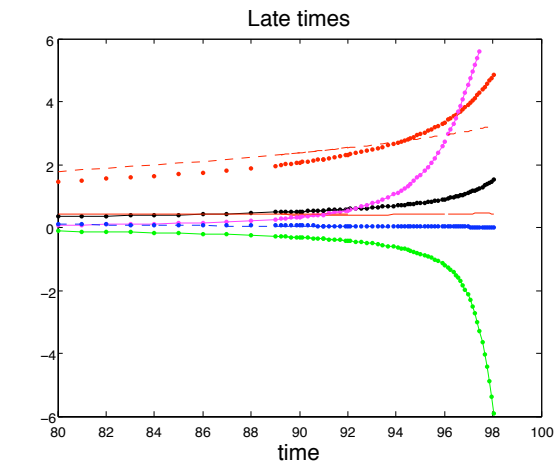
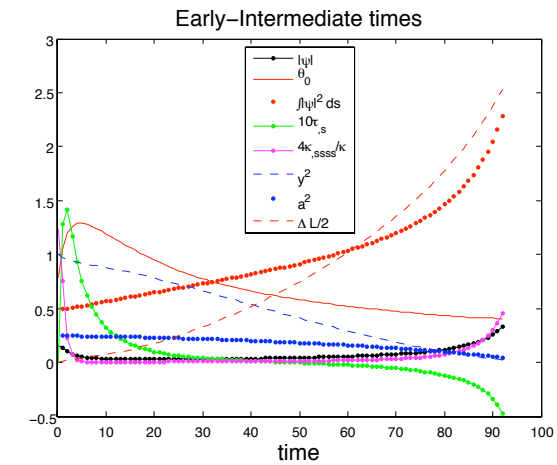
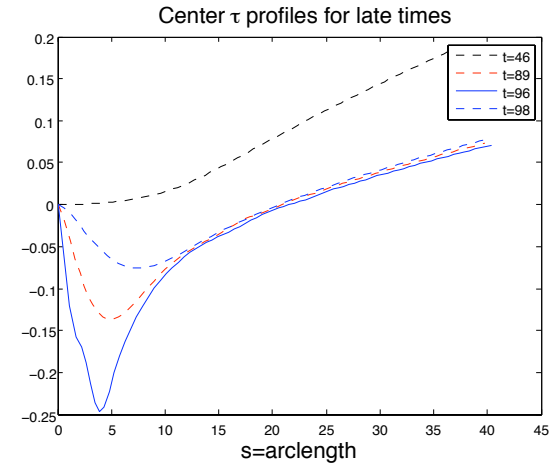
Red--: ΔL vortex line self-energy growth.

ΔL grows at expense of vortex pair interaction energy.

Black● curvature $|s''| = |\psi|$ on symmetry plane has not grown much.

- **Bottom:** Continuation in time with emphasis on growth. In symmetry plane $\tau'(0)$, $|s''|''''(0)/|s''|(0)$ grow. Global “enstrophy” grows but the self-energy change ΔL does not.

Black● curvature $|s''| = |\psi|$ on symmetry plane has finally taken off.



Thoughts on: Quantum fluids versus classical Implications for growth of L and enstrophy.

In a classical fluid: • Energy is strictly conserved.

- In **my Euler simulations** the enstrophy blows up as

$$\|\omega\|_2^2 \sim \frac{\kappa^{3/2}}{(T_c - t)^{1/2}}$$

- In **my Navier-Stokes**: after reconnection, after spectrum fills up low wavenumbers:

$$E \sim (t - T_0)^{-2}. \text{ Enstrophy } \|\omega\|_2^2 \sim (t - T_0)^{-3}.$$

In **my vortex filament model**:

- If vortex energy $E_\kappa \sim \kappa^2 L$, then L has a maximum related to available (initial) inter-vortex energy: $E_{b0} \sim \kappa^2 L_0 \log(b/a_0)$

- “Enstrophy” : $\int |\psi|^2 ds \rightarrow (T_c - t)^{-1/2}$

- Like my Euler calculations. But which is “enstrophy” in superfluid?

- $L \sim \int ds$ or • $\int |\psi|^2 ds$

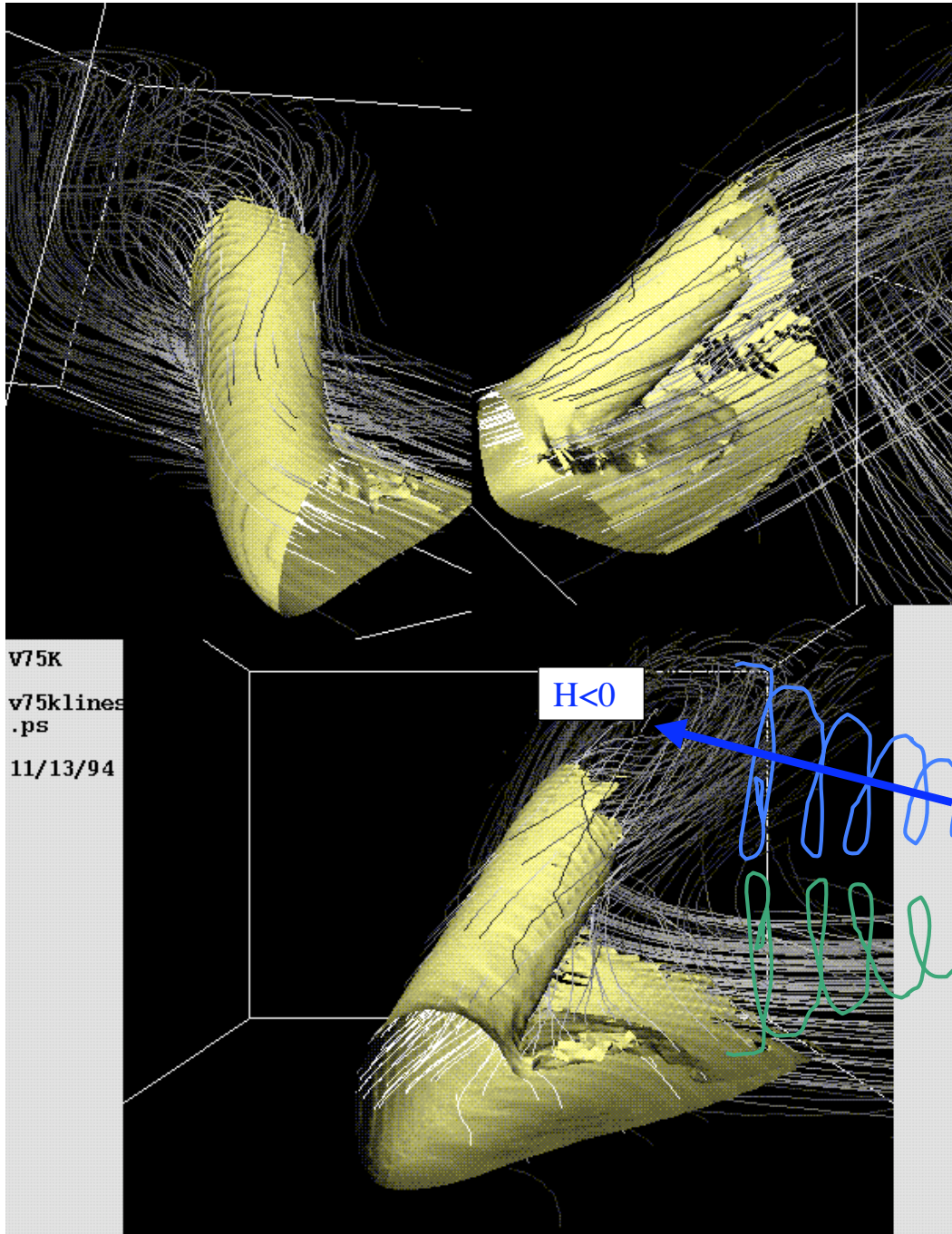
- The latter is more like the Kozik/Svistunov Hamiltonian??

Summary

Can Biot-Savart filament interactions could generate a cascade?

- They can if torsion (wave speed) complements Biot-Savart interactions.
- This requires that on the symmetry plane $\tau' < 0$.
 - They can't if torsion $\tau >$, that is phase velocity points out.
Kelvin waves will kill any trend to a singularity.
 - $\dot{\tau}' < 0$ unless $|\mathbf{s}''|''''$ (κ_c'') is very large.
 - * In particular this is the problem with sharp reconnections (á la Schwarz).
- In the new model the vortex stretching reduces to the sign of κ_c'''' making $\dot{\tau}' < 0$ possible.
- Vortex lines stretch significant by absorbing energy from the inter-vortex potential. This could help create long vortices throughout the superfluid, as in Golov's spin-down cube.
- Energy dissipation??
 - Reconnection into vortex rings could drain energy.
 - This could turn into rarefaction waves then phonons, then heat.
 - Or just go to the domain walls.

End of talk.

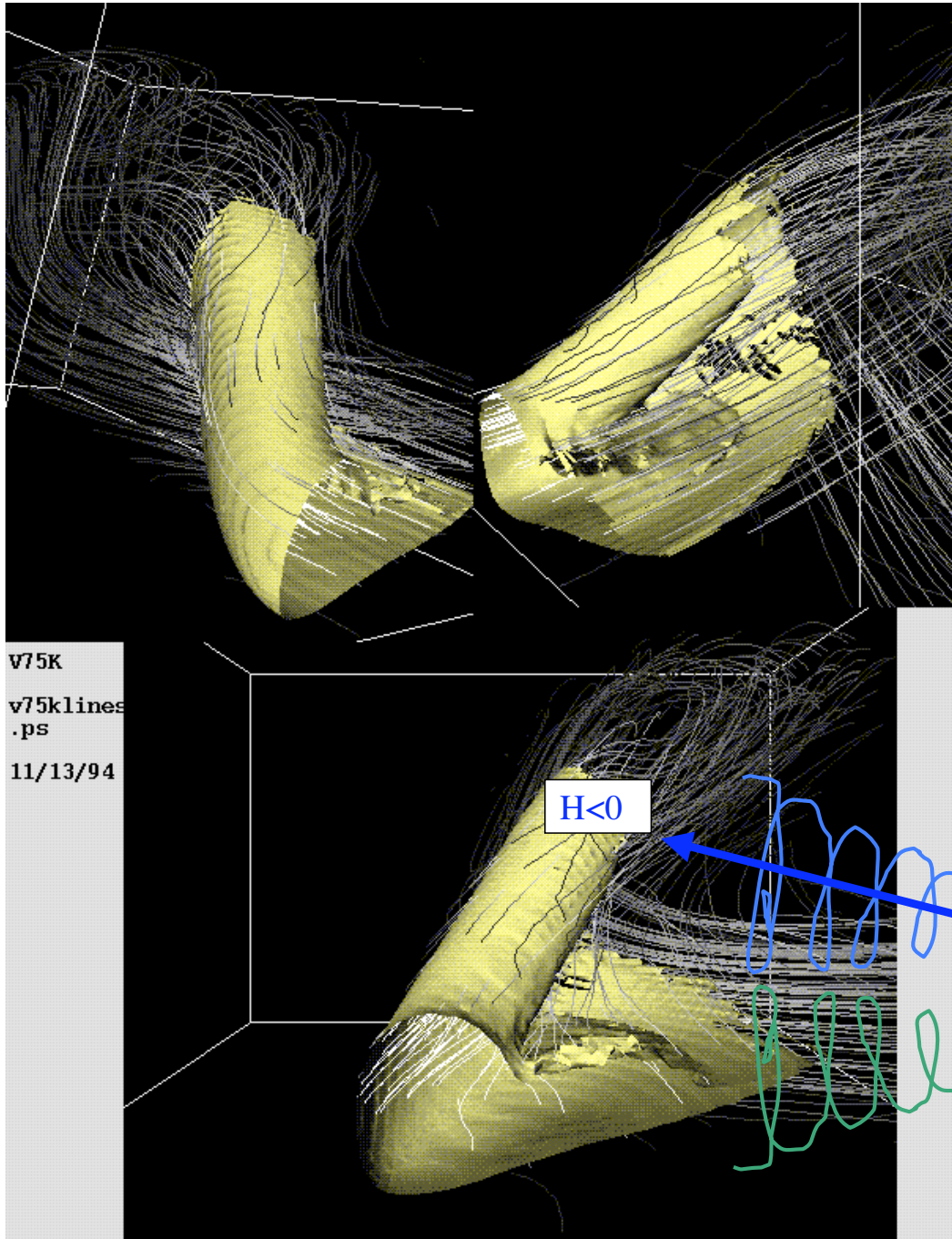


Helicity density $h = \mathbf{u} \cdot \boldsymbol{\omega}$.
Scalar product of velocity and vorticity.

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H<0

H>0



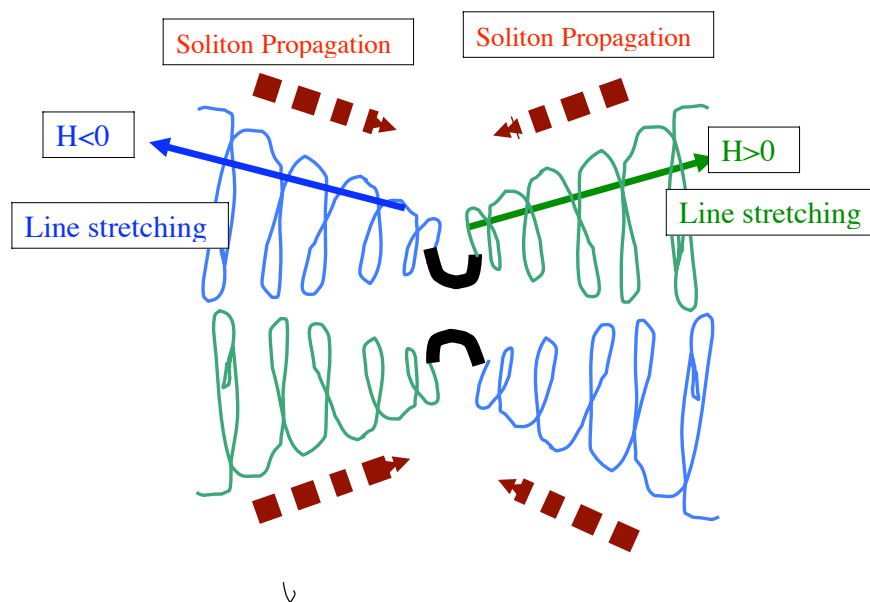
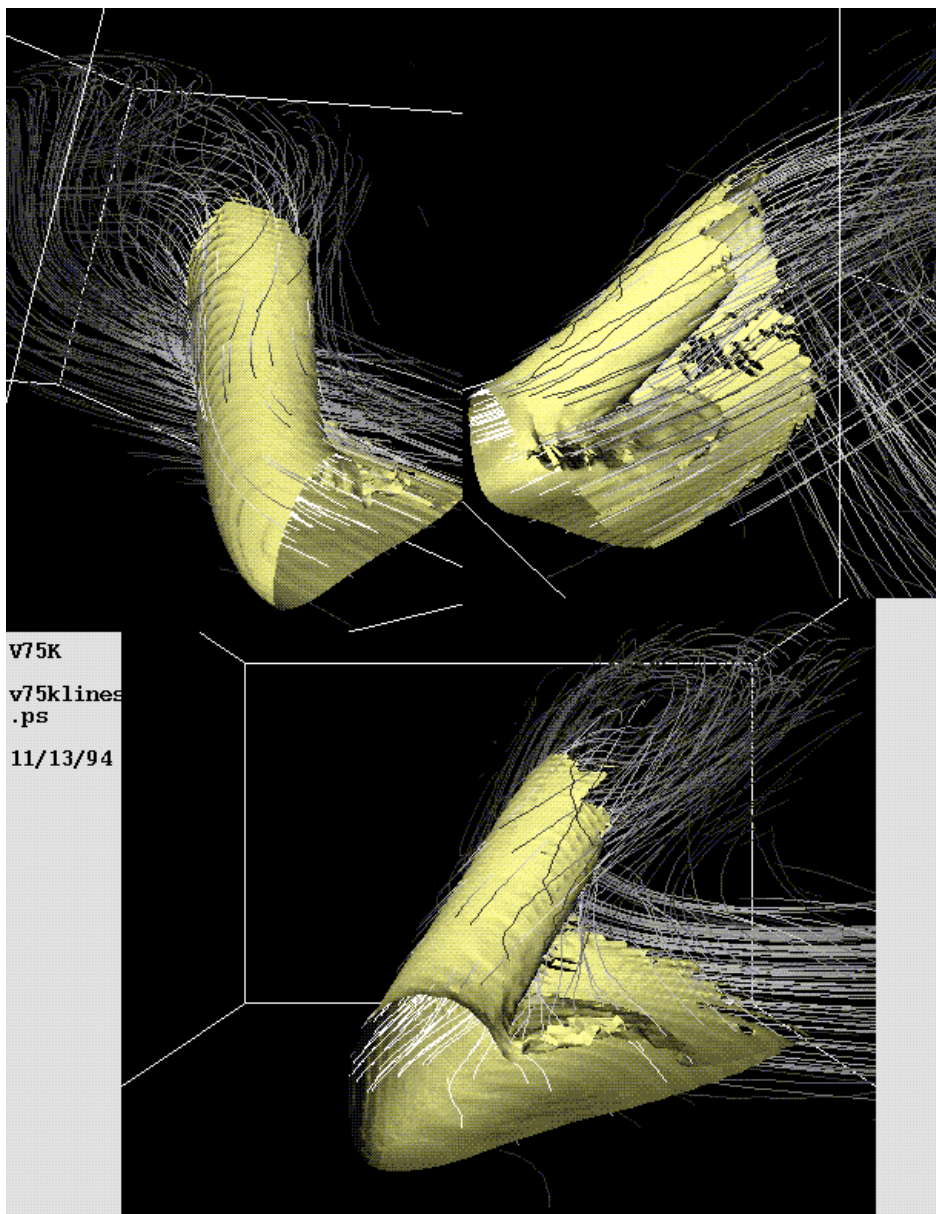
Helicity density $h = \mathbf{u} \cdot \boldsymbol{\omega}$.

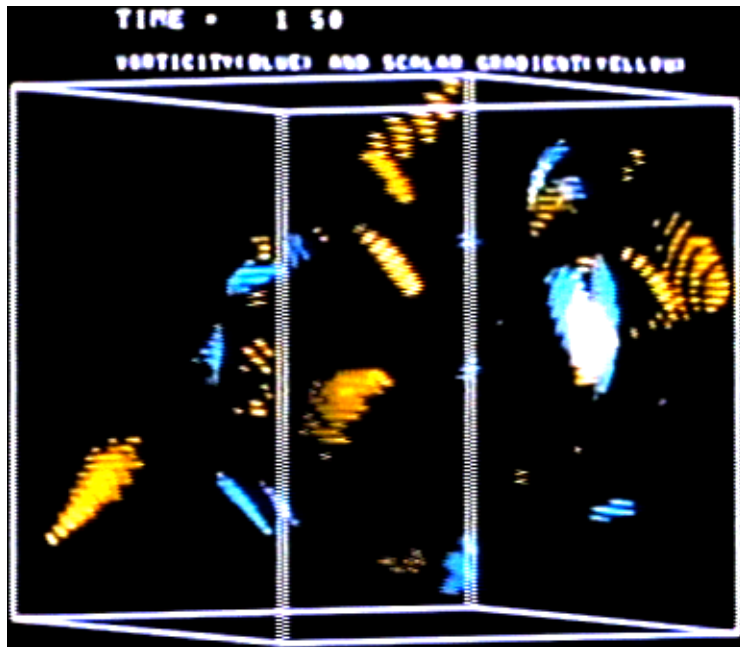
- Does fluid coming out of symmetry plane imply that waves propagate outward?
- No: waves and fluid move in opposite directions on vortex filaments.

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How is collapse preserved?

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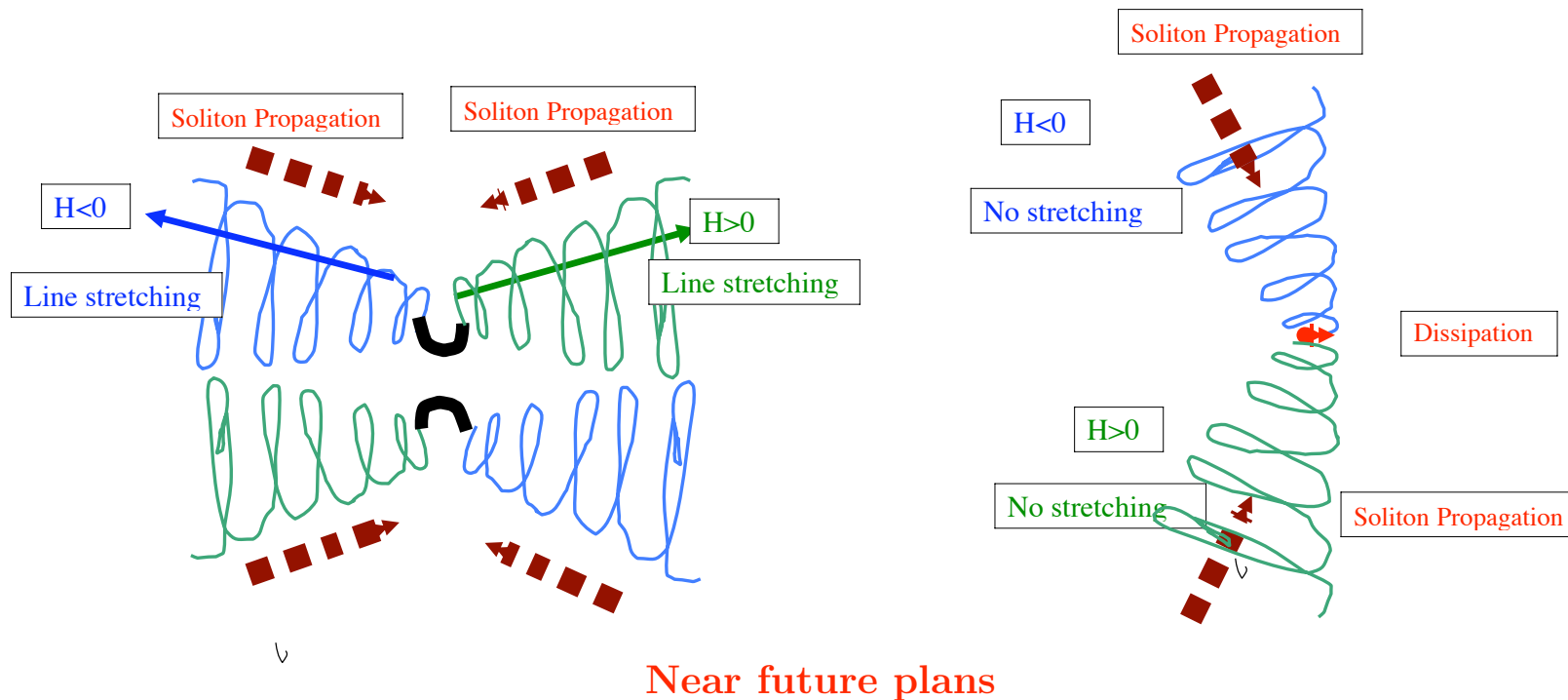
Three images of 3D forced 64^3 turbulence in a box. Lines show vector directions for quantities above a threshold. In all three frames the primary field, vorticity, is shown using blue lines. Left frames are at $t = 1.50$, but different perspectives and different secondary fields. Yellow is for scalar gradients and red is for compressive strain, White is the overlap between the scalar or compressive fields and two interacting regions of vorticity.

Before I emphasized the difference in the center between the bottom two frames due to reconnection of the two central vortices.

Today I want to emphasize the spiraling or interwoven vortices just above and below this event.



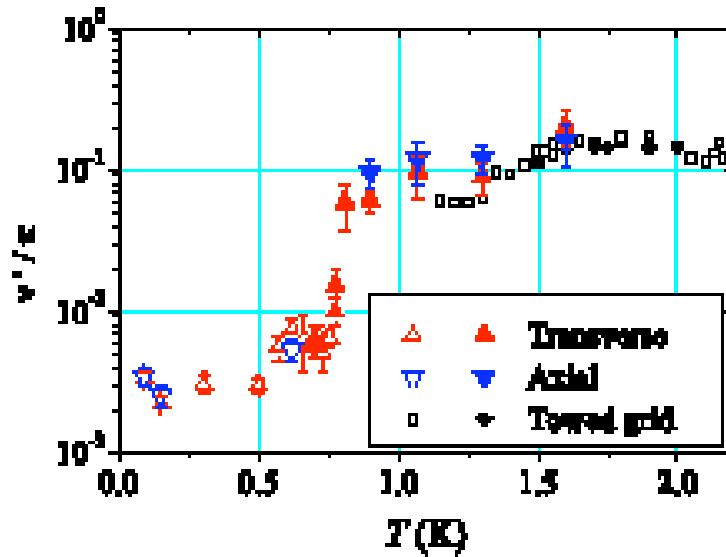
What happens after reconnection?



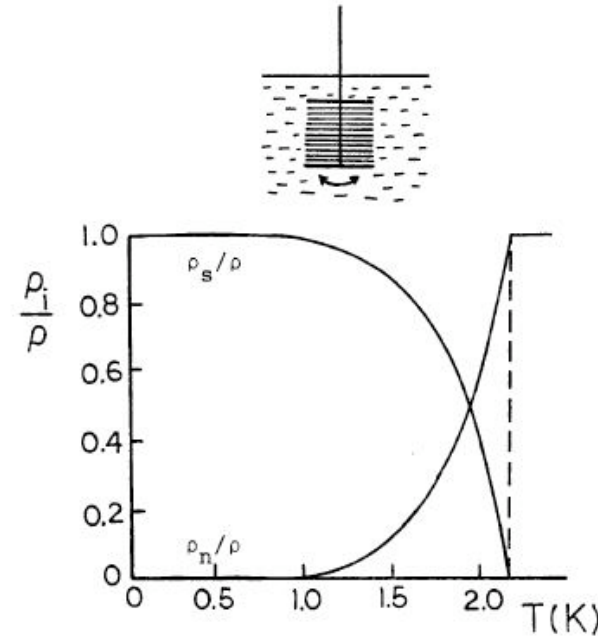
- Of course bigger and better simulations.
- **Analysis**
 - Scaling analysis
 - Does circulation κ collapse with $\|\omega\|_\infty$?
 - Velocity and circulation.
 - Pressure Hessian
 - Acoustic noise generation: Could small scales generated by the nearly singular events be the strongest source of acoustic noise?

Superfluid turbulence challenge

- There is a rapid decrease in the effective viscosity, coincident with where $\rho_n/\rho < .01$.



Left: FIG. 5 from Walmsley et al (2007). The effective kinematic ν' after a spin down from $\Omega = 1.5$ rad/s measured in the transverse (\triangle) and axial (∇) directions. Closed (open) triangles correspond to measurements with free ions (charged vortex rings). Error bars specify the uncertainty of fitting. Squares and diamonds: second sound measurements of grid turbulence [12,22].



Right: The original experiment by Andronikashvili for the densities of normal and superfluid based on Landau's two-fluid model.

- Mauer/Tabeling and Roche et al experiments support some type of -5/3 regime, therefore we believe there should be an energy cascade.
- An energy cascade implies the energy extraction is still at very small scales.

But as $T \rightarrow 0$, in the **ultra-cold, quantum** regime, with no viscous normal fluid, there are only two plausible mechanisms for the extraction kinetic energy.

1 Kelvin waves: Vinen (2000)

2 Break-down of Robert/Jones solitons or phonons of Leadbeater et al (2001) (Barenghi group).

- It has been implied that there is no simple mechanism connecting the large-scale dynamics $r > b$ to these small scales $r \ll b$, where b is the average inter-vortex spacing.

Symmetry plane Biot-Savart

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Velocity on the symmetry plane due to a cylindrical vortex of uniform radius a (in Rosenhead regularization) is:

$$\mathbf{u} = \frac{\kappa}{2\pi} \left[1 + \frac{|\mathbf{s}''|x_n}{2} \right] \frac{(-y, x)}{r^2 + a^2} + \frac{\kappa|\mathbf{s}''|}{8\pi} \left[\log \frac{\epsilon^2}{r^2 + a^2} - 2 \right] (-n_y, n_x) \quad (2)$$

which minus the regularization is equation (2.3.9) of Saffman's book. [Saffman (1995)] **This neglects effect of twisting (torsion) of vortex lines out of the symmetry plane.**

- **NOTE: κ =circulation. s'' =curvature,** following Frenet-Serret coordinates.
- To close : Diameter $\dot{a} = -\frac{\alpha}{2}a$. Separation $\dot{y} = u_y$. Stretching $\alpha = -\nabla \cdot \mathbf{u}$. Curvature $\dot{\kappa}$ = from quaternions.
- From Gibbon (2002) (quaternions for Euler), rate-of-change of the direction of vorticity is:

$$\frac{D}{Dt} \mathbf{s}' = \boldsymbol{\chi} \times \mathbf{s}' \quad \text{where} \quad \boldsymbol{\chi} = \mathbf{s}' \times \mathbf{S} \mathbf{s}' \quad (10)$$

- Then take the arclength derivative to get (Gibbon *et al.*, 2006) for curvature $\mathbf{s}'' = |\mathbf{s}''|\mathbf{n}$.

$$\frac{D\mathbf{s}''}{Dt} = (\boldsymbol{\chi} \times \mathbf{s}')_{,s} - \alpha \mathbf{s}'' \quad \text{or} \quad \frac{D\mathbf{s}''}{Dt} = \mathbf{u}_{,ss} - \alpha_{,s} \mathbf{s}' - 2\alpha \mathbf{s}'' \quad (11)$$

Local induction

$$\psi = (\mathbf{s}''_x + i\mathbf{s}''_y)e^{i\theta_1} \quad \text{where} \quad \mathbf{s}'' = (\mathbf{s}''_x, \mathbf{s}''_y) \quad (12)$$

- If the complex variable ψ satisfies

Then if $\alpha = 0$ (as in the local induction approximation) then (11) for $\dot{\mathbf{s}}''$ is equivalent to the nonlinear Schrödinger equation for ψ due to the **Hasimoto** transformation.

Non-local vortex pair

$$\dot{\psi} = i\psi'' + \frac{1}{2}\psi|\psi|^2 \quad (13)$$

- If $\tau = 0$, then for two interacting filaments a and b , using \mathbf{u} from (2), find $\mathbf{u}_{,ss}$ and α to get:

$$\frac{D\mathbf{s}''^{ab}}{Dt} = \mathbf{u}_{,ss}^{ba} - 2\alpha^{ba} \mathbf{s}''^{ab} \quad (14)$$

\mathbf{u} and α of b affect a and vice-versa.

- I am convinced all previous Biot-Savart **single** filament calculations made a mess of this.

Do superfluid vortex lines stretch?

- Stretching must satisfy conservation of energy and mass.

True if:

- There is conversion of vortex interaction energy into vortex self-energy.
- Vortex radius, defined by where $\rho_s \sim \bar{\rho}_s$, collapses as it stretches, conserving density deficit.
- This could from intensification of $\nabla\rho_s$ due to strong convergence of \mathbf{u} in density equation resulting from Madelung transformation.

$$i\hbar\psi_t(\mathbf{x}, t) = -\frac{\hbar}{2m}\Delta\psi(\mathbf{x}, t) + V_0\psi(\mathbf{x}, t)|\psi(\mathbf{x}, t)|^2 - E_v\psi(\mathbf{x}, t) \quad (15)$$

Setting $\hbar = 2m = V_0 = E_v = 1$ then

$$i\frac{\partial\psi}{\partial t} + \Delta\psi + (1 - |\psi|^2)\psi = 0 \quad (16)$$

Under the Madelung transformation the equations become

$$\left\{ \begin{array}{l} \frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\mathbf{v}) = 0, \\ \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \rho^{-1}\nabla p_b = -\nabla p_q = 0 \\ \text{where } p_b = \frac{\rho^3}{6} \quad \text{and} \quad p_q = \frac{|\nabla\rho|^2}{8\rho^2} + \frac{\Delta\rho}{4\rho} \end{array} \right. \quad (17)$$

Neglecting the “quantum pressure” p_q , this system reduces to the Euler equations of a barotropic fluid.

- **My model makes an arbitrary assumption in the regularization of the curvature equation to compensate for the unphysical overlap of the unphysical cylindrical vortex cores.**

Hasimoto transformation for curvature in an NLS equation

The Hasimoto transformation is derived from the local induction approximation for motion $\dot{\mathbf{X}}$ of a thin vortex filament described by:

$$\frac{\partial \mathbf{X}}{\partial t} = G\kappa \mathbf{b} \quad (18)$$

where

$$G = (\kappa/4\pi)[\log(\epsilon/r) + O(1)] \quad (19)$$

To which are added the Frenet-Serret equations of differential geometry

$$\begin{aligned} \mathbf{X}' &= \mathbf{s}' & , & \quad \mathbf{s}'' = |\mathbf{s}''| \mathbf{n} \\ \mathbf{n}' &= \tau \mathbf{b} - |\mathbf{s}''| \mathbf{s}' & , & \quad \mathbf{b}' = -\tau \mathbf{n} \end{aligned} \quad (20)$$

Here \mathbf{s}' =the direction of vorticity, $|\mathbf{s}''|$ =curvature, \mathbf{n} =direction of curvature, $\mathbf{b} = \mathbf{s}' \times \mathbf{n}$ is the bi-normal and τ =torsion where $|\mathbf{s}''| \mathbf{n} = (\mathbf{s}' \cdot \partial_s) \mathbf{s}'$.

Then the wave function is

$$\psi(s) = |\mathbf{s}''|(s) \exp(i \int_0^s \tau(s') ds' + i\theta_0 + i\theta_1) \quad (21)$$

- On the symmetry plane: $|\mathbf{s}''| \neq 0 \quad |\mathbf{s}''|' = 0 \quad \tau = 0 \quad \tau' \neq 0$
- Therefore
$$|\dot{\mathbf{s}}''| = -|\mathbf{s}''| \tau' \quad \dot{\mathbf{n}} = \frac{|\mathbf{s}''|''}{|\mathbf{s}''|} \mathbf{b} \quad \frac{d}{dt} |\mathbf{s}''| \mathbf{n} = -\tau' |\mathbf{s}''| \mathbf{n} + |\mathbf{s}''|'' \mathbf{b} \quad (22)$$
- I can show that given ψ and an extra gauge variable $\theta_1(t)$ at one point on the curve that a separate equation for \mathbf{n} does not need to be solved. $\dot{\theta}_1 = \frac{1}{2} |\mathbf{s}''|^2$.

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