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# Workshop on Topics in Quantum Turbulence

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## Computations based on the Gross-Pitaevskii (GP) model

M. Tsubota
*Osaka City University, Dept. of Physics Osaka*
*Japan*

Strada Costiera 11, 34014 Trieste, Italy - Tel. +39 040 2240 111; Fax +39 040 224 163 - sci_info@ictp.it, www.ictp.it

# Computations based on the Gross-Pitaevskii (GP) model

M. Tsubota (Osaka City University, Japan)

Thanks to M. Kobayashi, M. Machida, R. Numasato

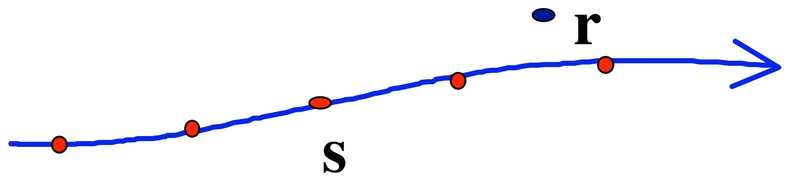
1. Introduction -Brief history-
2. Quantum turbulence in trapped Bose gas
3. Big simulation towards the classical-quantum crossover
4. Two-dimensional quantum turbulence

# 1. Introduction -Brief history-

## How to describe the vortex dynamics

Vortex filament model (Schwarz) **Biot-Savart law**

$$\mathbf{v}_s(\mathbf{r}) = \frac{\kappa}{4\pi}\int \frac{(\mathbf{s}-\mathbf{r})\times d\mathbf{s}}{|\mathbf{s}-\mathbf{r}|^3}$$




A vortex makes the superflow of the Biot-Savart law, and moves with this local flow. At a finite temperature, the mutual friction should be considered.

The Gross-Pitaevskii (GP) model for the macroscopic wave function

$$\Psi(r) = \sqrt{n_0(r)}e^{i\theta(r)}$$

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = \left[-\frac{\hbar^2\nabla^2}{2m} + V_{\mathrm{ext}}(\mathbf{r}) + g|\Psi(\mathbf{r},t)|^2\right]\Psi(\mathbf{r},t)$$

# J. Koplik and H. Levin, PRL71, 1375(1993)

They showed directly vortex reconnections for the first time.

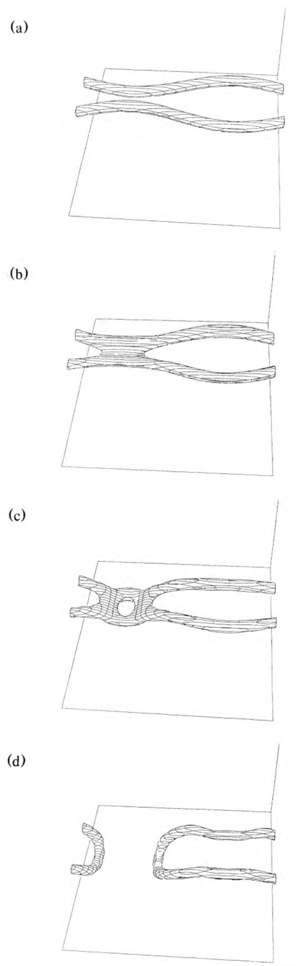



FIG. 1. Reconnection of antiparallel vortices: (a) initial configuration, (b) time 3.0, (c) time 10.0, and (d) time 20.

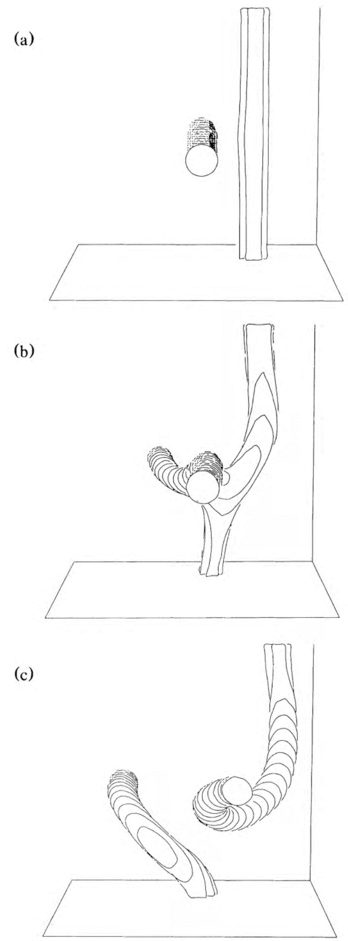



FIG. 2. Reconnection of 90° vortices: (a) initially, (b) time 7.0, and (c) time 20.

M. Leadbeater, T.Winiecki, D. C. Samules,C. F. Berenghi, C. S. Adams, PRL86, 1410(2001)

They showed sound emission at vortex reoconnections.

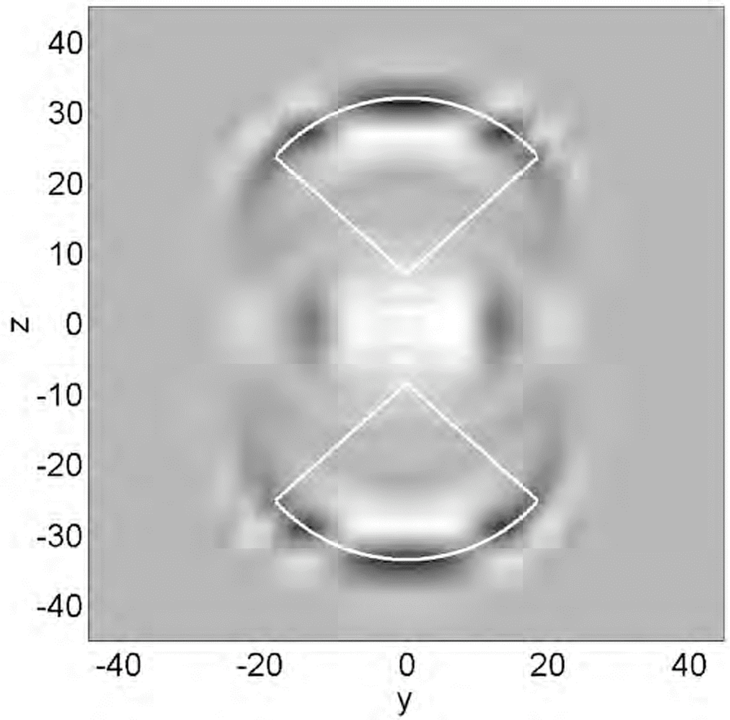
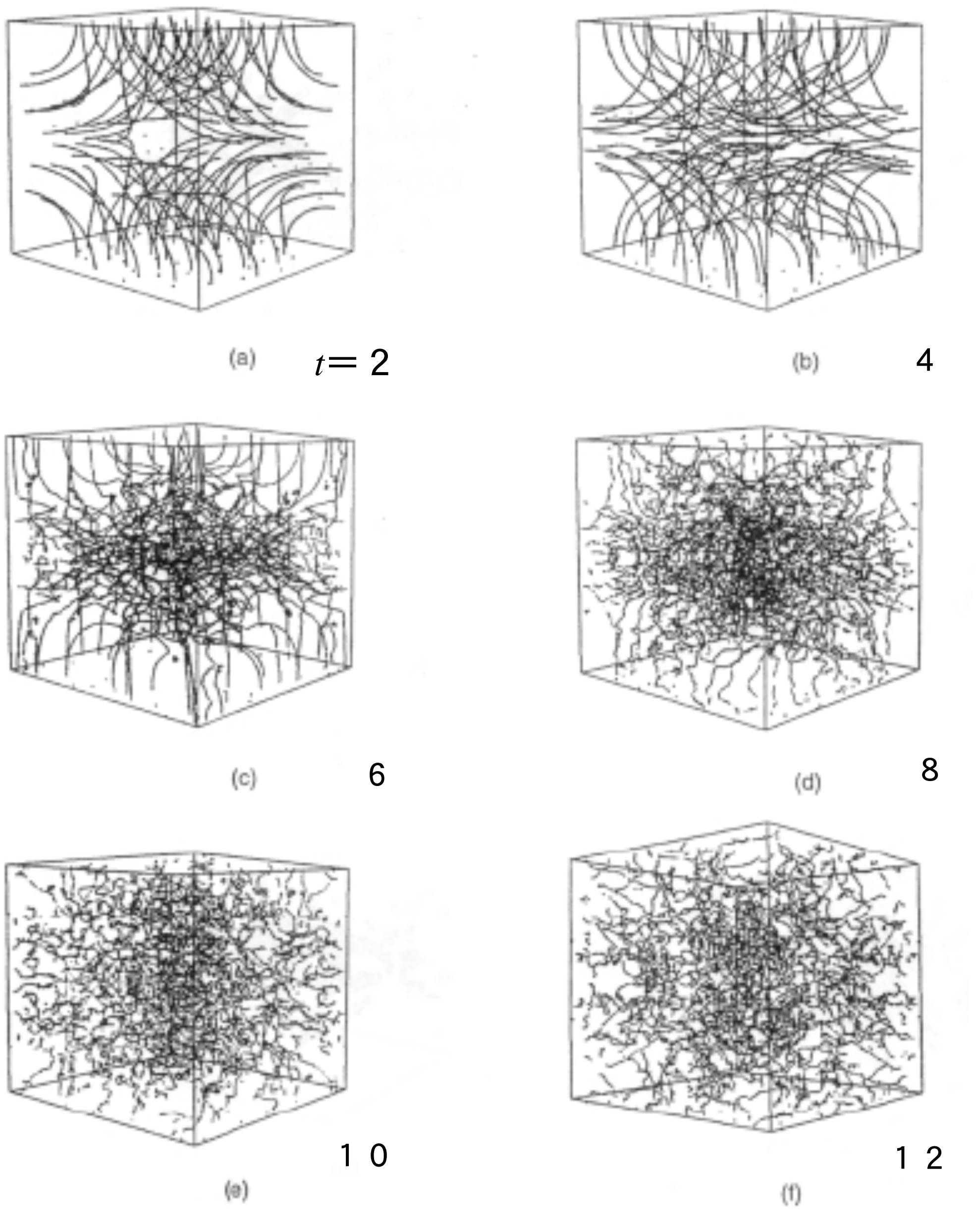



FIG. 4. Density cross-sections in the $x = 0$ plane at $t = 54$ for ring radii $R = 6$ with offset $D = 8$ (same parameters as in Fig. 3). The sound pulse appears as two arcs with radius of curvature, 25, suggesting that the reconnections occurred at $z = \pm 7$ and $t = 29$, consistent with Fig. 3. The angular spread is approximately equal to the reconnection angle ($\theta = 96°$ for this example). The white lines indicate the positions of the reconnections, $z = \pm 7$, the reconnection angle $\theta$ and the expected position of the sound pulse. Grey scale: black 0.95; white 1.025.

C. Nore, M. Abid and M.E.Brachet, Phys.Fluids 9, 2644(1997)




By using the GP model, they obtained a vortex tangle with starting from the Taylor-Green vortices.

**In order to study the Kolmogorov spectrum, it is necessary to decompose the total energy into some components. (Nore *et al.*, 1997)**

Total energy

$$E = \frac{1}{\int d\mathbf{x}\,\rho} \int d\mathbf{x}\,\Phi^* \left[ -\nabla^2 + \frac{g}{2} |\Phi|^2 \right] \Phi \qquad \Phi = \sqrt{\rho}\exp(i\theta)$$

$$E = E_{\text{int}} + E_q + E_{kin}$$

The kinetic energy $E_{kin} = \frac{1}{\int d\mathbf{x}\,\rho} \int d\mathbf{x} \left( |\Phi| \nabla\theta \right)^2$ is divided into

the compressible part $E_{kin}{}^{c} = \frac{1}{\int d\mathbf{x}\,\rho} \int d\mathbf{x} \left[ \left( |\Phi| \nabla\theta \right)^{c} \right]^2$ with $\text{rot}\left( |\Phi| \nabla\theta \right)^{c} = 0$

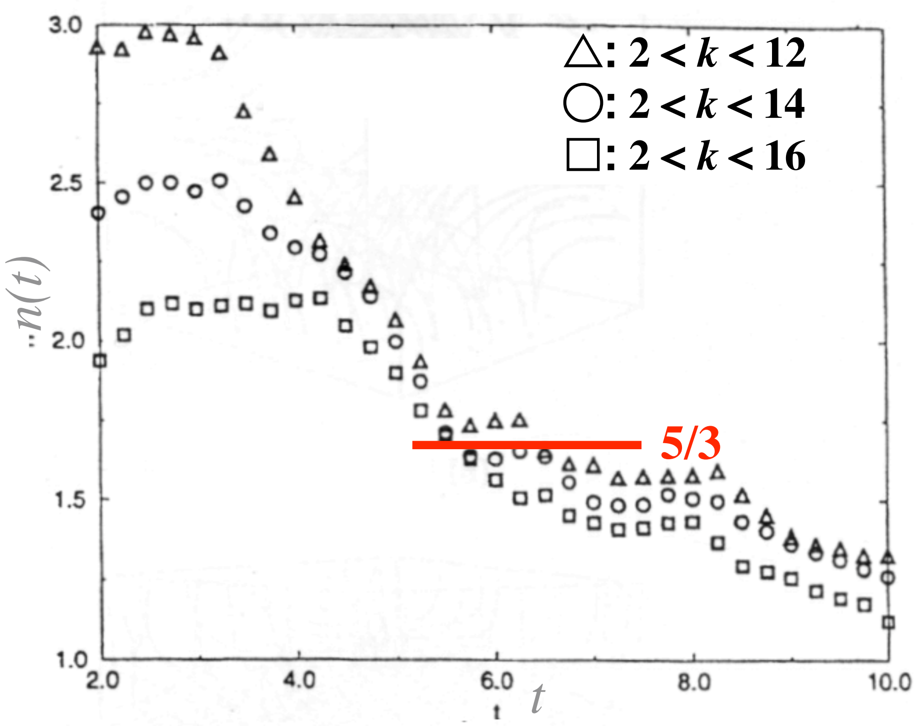
and

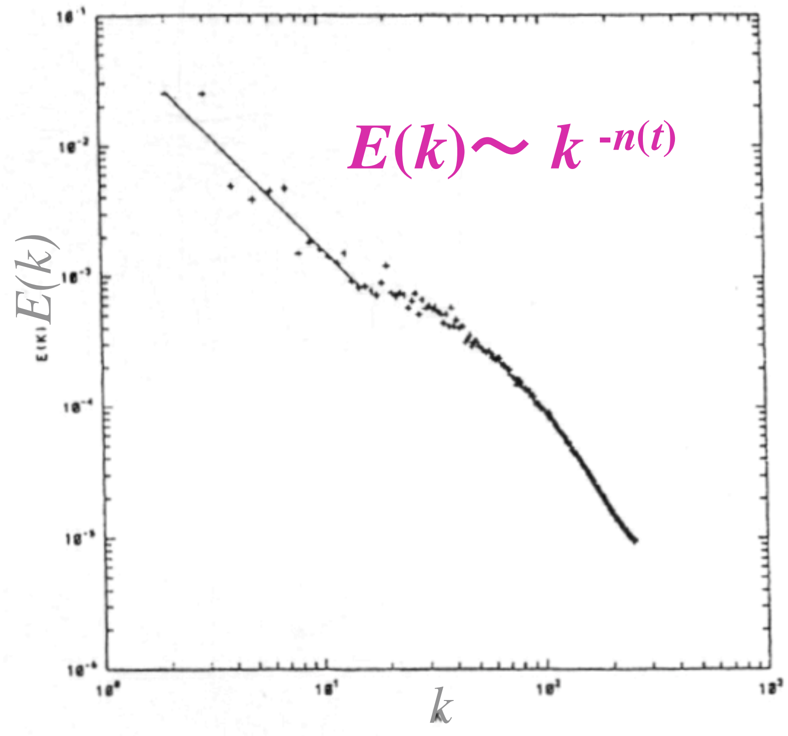
the incompressible part $E_{kin}{}^{i} = \frac{1}{\int d\mathbf{x}\,\rho} \int d\mathbf{x} \left[ \left( |\Phi| \nabla\theta \right)^{i} \right]^2$ with $\text{div}\left( |\Phi| \nabla\theta \right)^{i} = 0$

.

This incompressible kinetic energy $E_{\text{kin}}{}^{\text{i}}$ should obey the Kolmogorov spectrum.

C. Nore, M. Abid and M.E.Brachet, Phys.Fluids 9, 2644(1997)

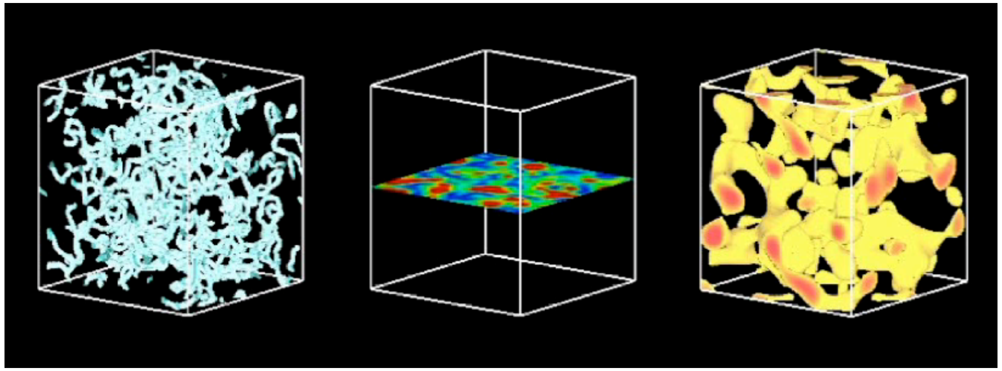


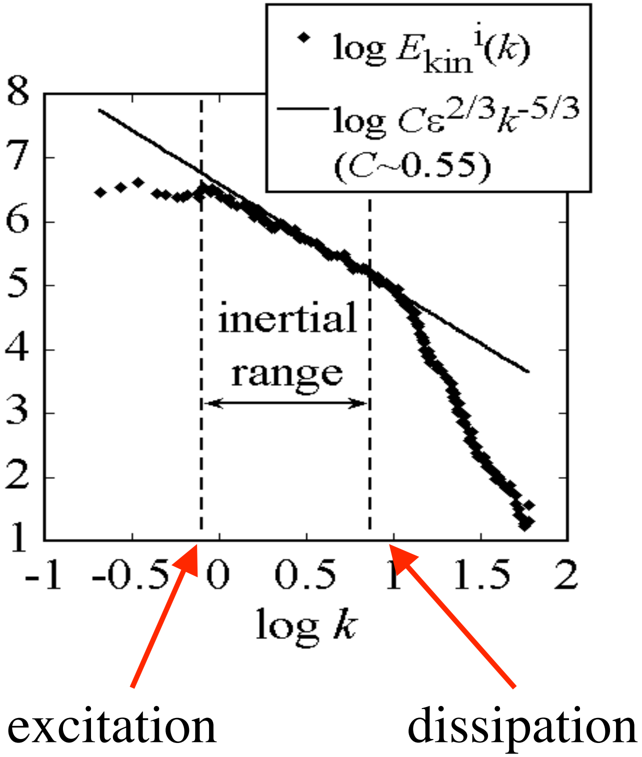




The right figure shows the energy spectrum at a moment. The left figure shows the development of the exponent *n(t)*. The exponent *n(t)* goes through 5/3 on the way of the dynamics.

In the late stage, however, the exponent deviates from 5/3, because the sound waves resulting from vortex reconnections disturb the cascade process of the inertial range.

**M. Kobayashi and M. Tsubota, Phys. Rev. Lett. 94, 065302 (2005), J. Phys. Soc. Jpn. 74, 3248 (2005)**



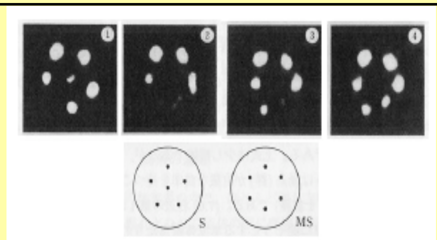
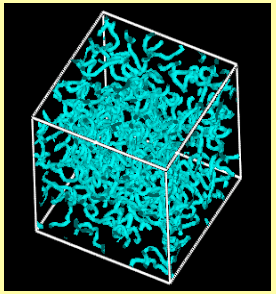
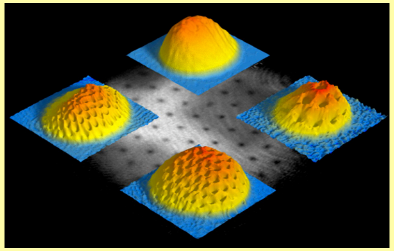



They obtained a statistical steady state by introducing large-scale excitation and small-scale dissipation. The state showed the Kolmogorov spectra in the inertial range.

# 2. Quantum turbulence in trapped Bose gas

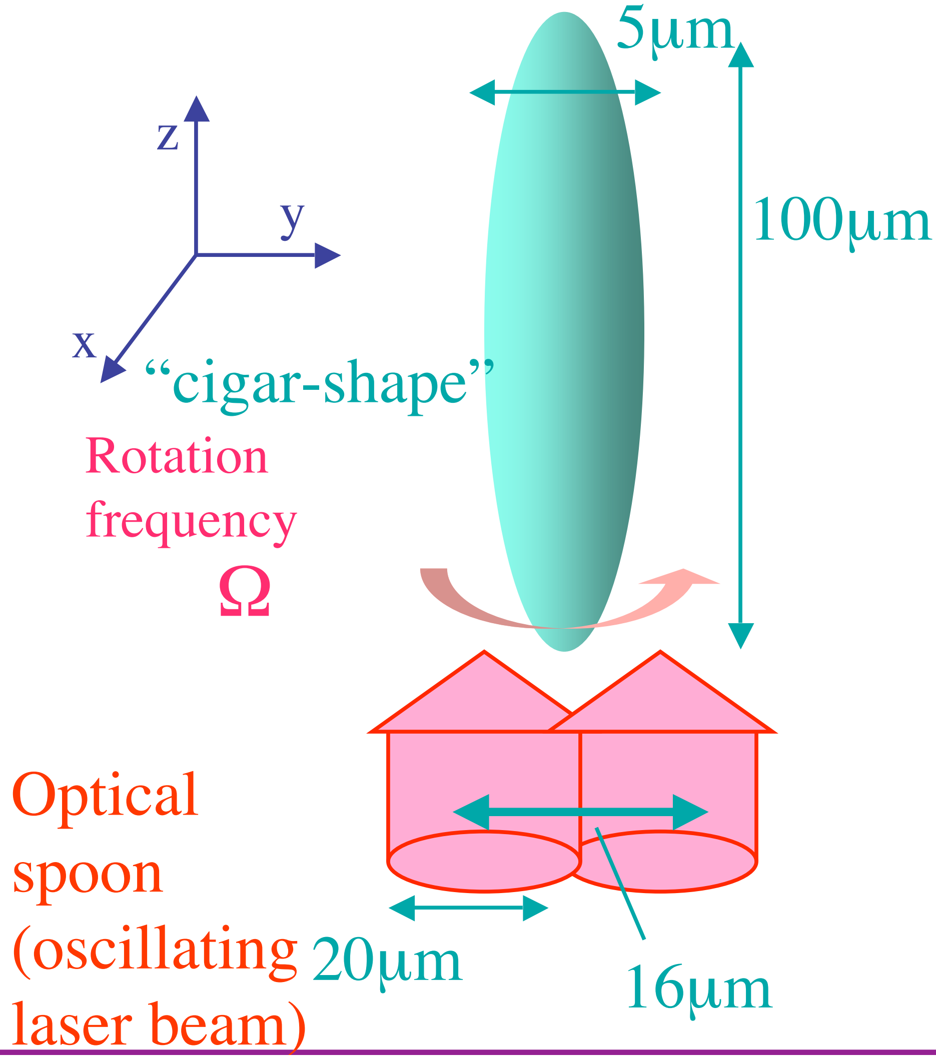
M. Kobayashi and M. Tsubota, Phys. Rev. A76, 045603 (2007)

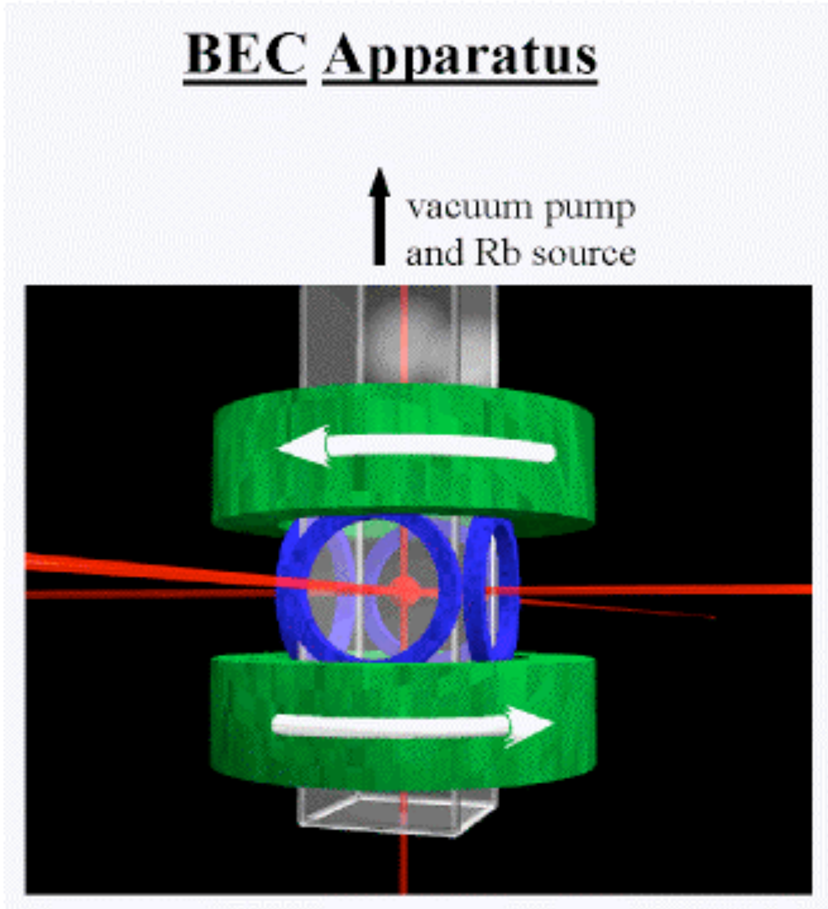
Two principal cooperative phenomena of quantized vortices are vortex arrays and vortex tangles.

| | Vortex array | Vortex tangle |
|---|---|---|
| Superfluid He |  Packard (Berkeley) |  |
| Atomic BEC |  Ketterle (MIT) | None |

# Usual setup of atomic BECs

K.W.Madison et.al Phys.Rev Lett **84**, 806 (2000)

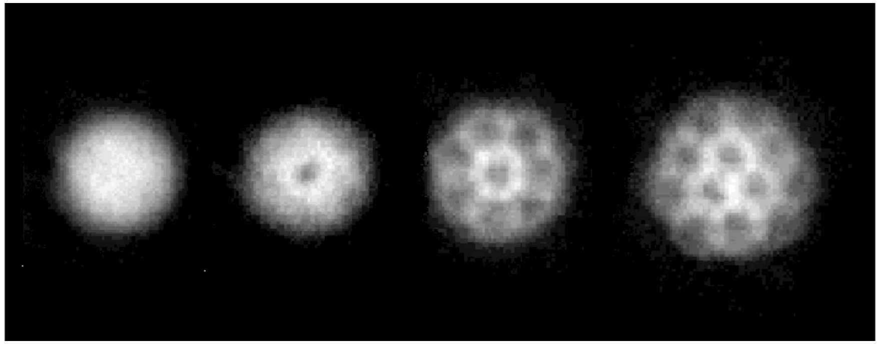


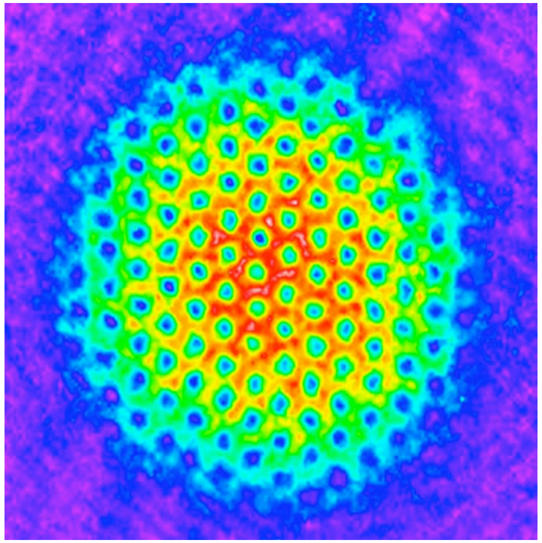




Laser cooling

# Observation of quantized vortices in atomic BECs

**ENS** K.W. Madison, et. al PRL **84**, 806

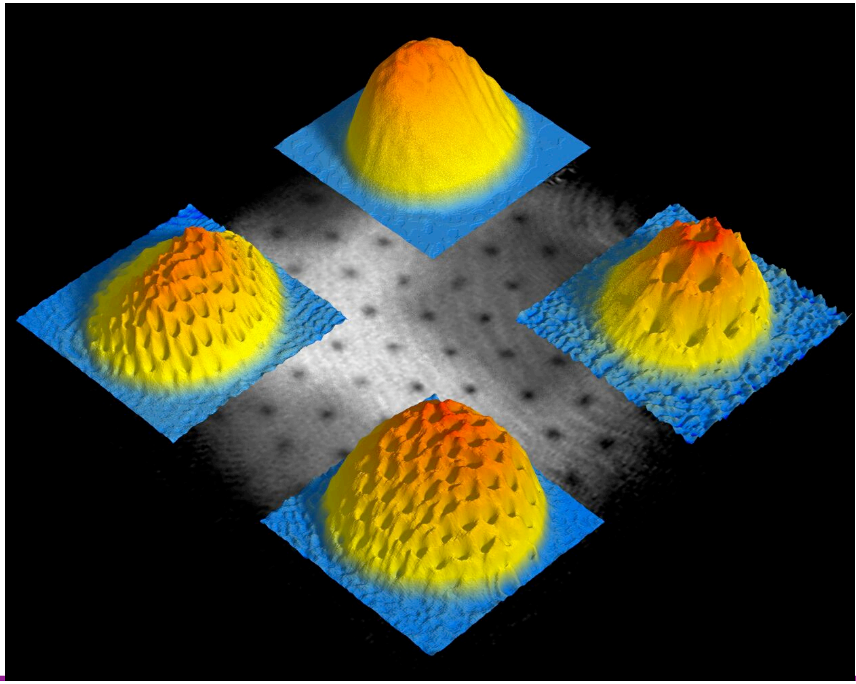




**JILA**

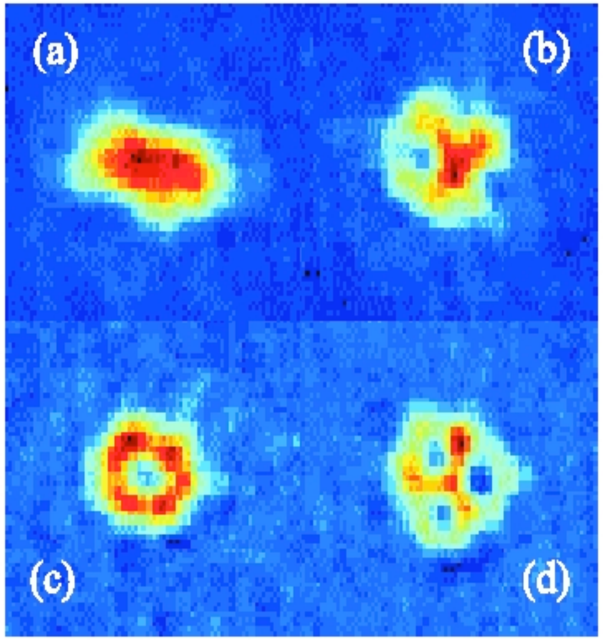
P. Engels, et. al PRL **87**, 210403 (2001)

**MIT** J.R. Abo-Shaeer, et. al Science **292**, 476



**Oxford**

E. Hodby, et. al PRL **88**, 010405 (2002)



# Dynamics of vortex lattice formation by the GP model

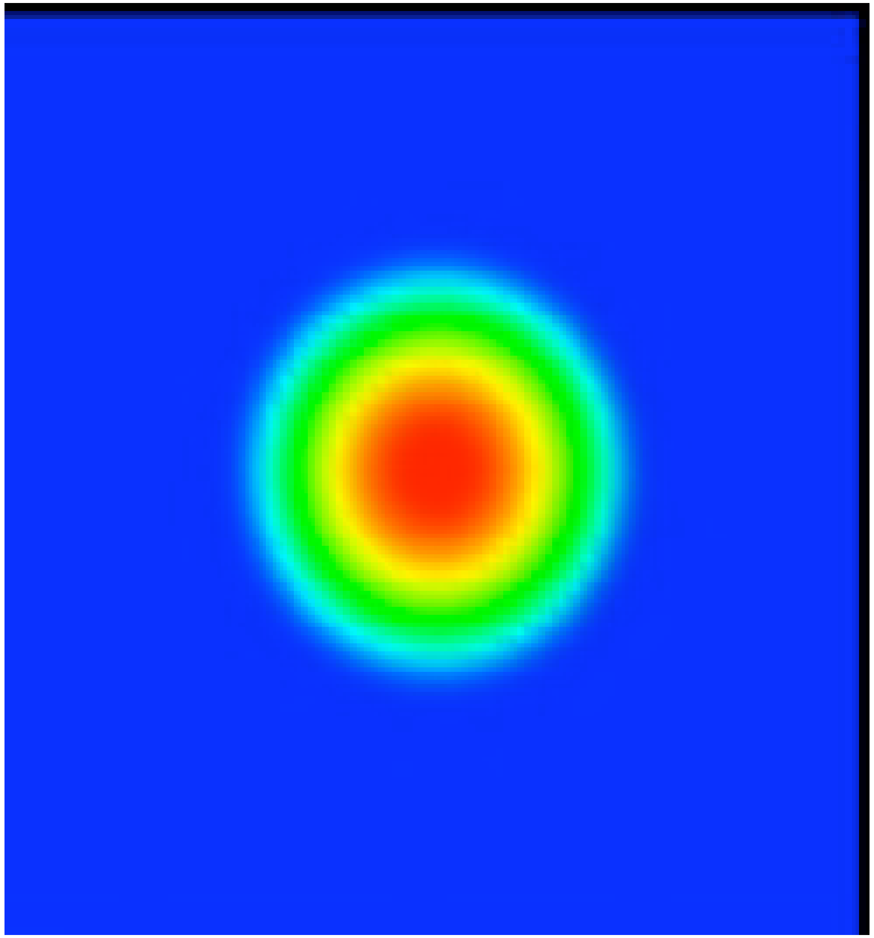
Time development of condensate density $n_0$

$$\Omega = 0.7\omega_\perp$$

M. Tsubota, K. Kasamatsu and M. Ueda, Phys. Rev. A **65**, 023603 (2002)

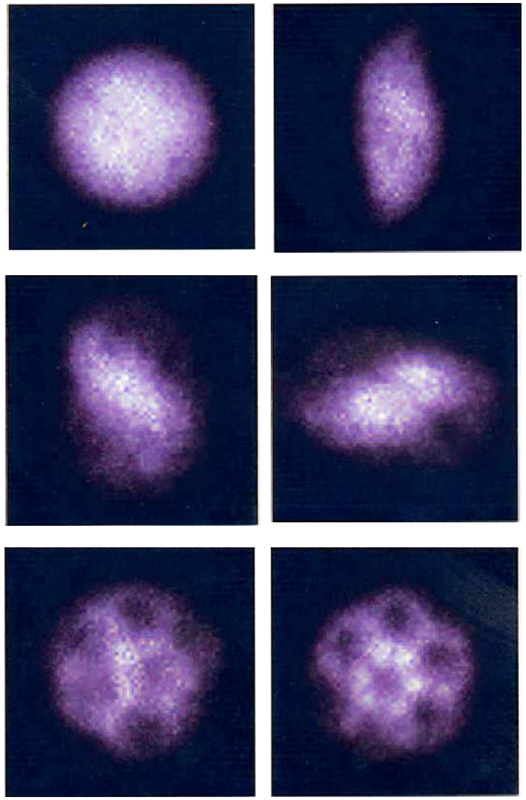
$$V_{\mathrm{trap}}(r) = \frac{1}{2}m\omega_\perp^2 r^2$$

$$\Psi(r) = \sqrt{n_0(r)}e^{i\theta(r)}$$



Experiment

K.W. Madison et.al., PRL **86**, 4443 (2001)

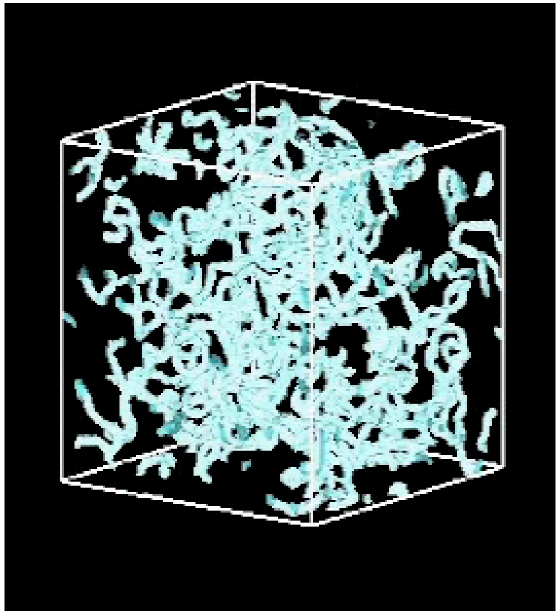


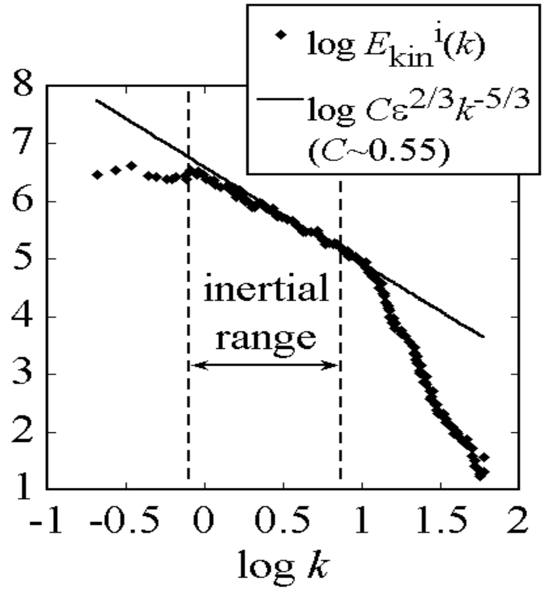
# Is it possible to produce turbulence in a trapped BEC?

M. Kobayashi and M. Tsubota, Phys. Rev. A76, 045603 (2007)

(1) We cannot apply any flow to this system.

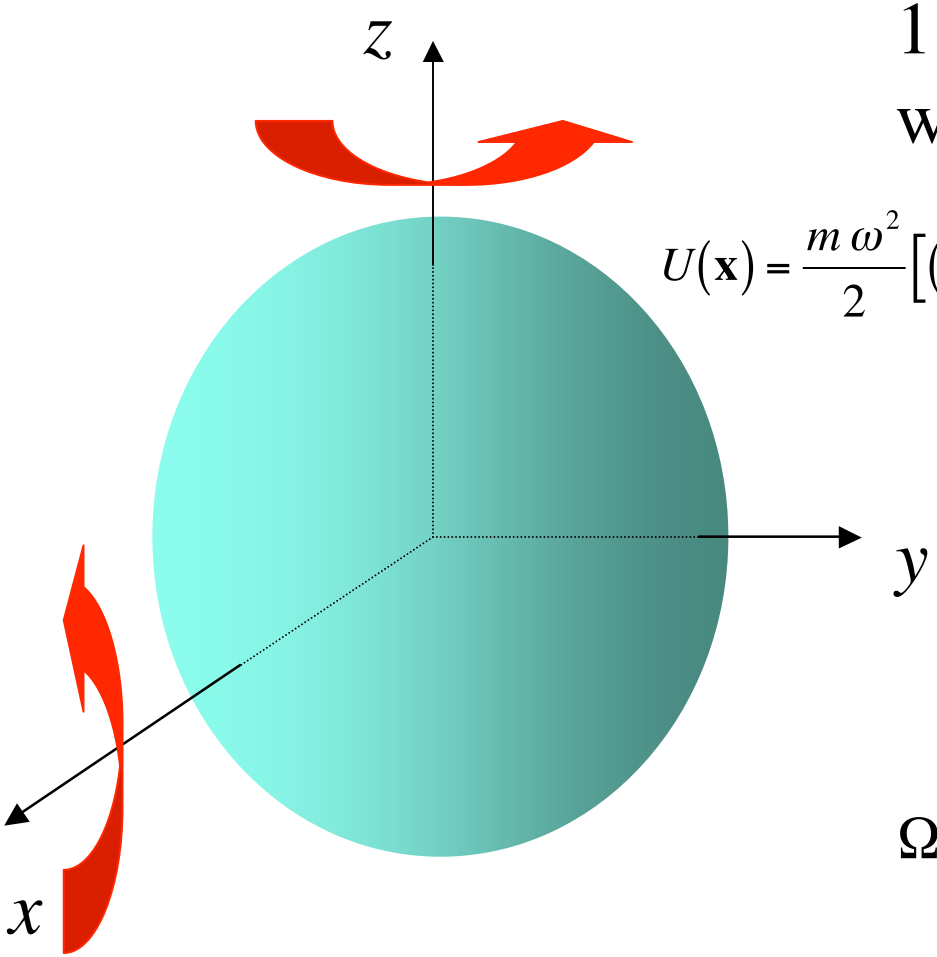
(2) This is a finite-size system. Can we make turbulence with a sufficiently wide inertial range?






The coherence length is not much smaller than the system size. However, we could confirm Kolmogorov's law.

# How to produce turbulence in a trapped BEC




1. Trap the BEC in a weak elliptical potential.

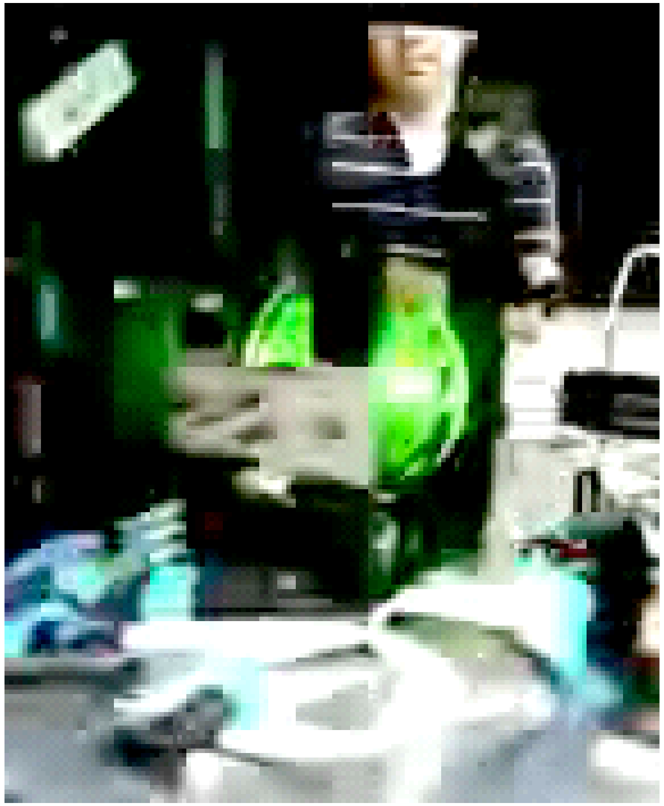
$$U(\mathbf{x}) = \frac{m\,\omega^2}{2}\left[(1-\varepsilon_1)(1-\varepsilon_2)x^2 + (1+\varepsilon_1)(1-\varepsilon_2)y^2 + (1+\varepsilon_2)z^2\right]$$

2. Rotate the system first around the *x*-axis, then around the *z*-axis.

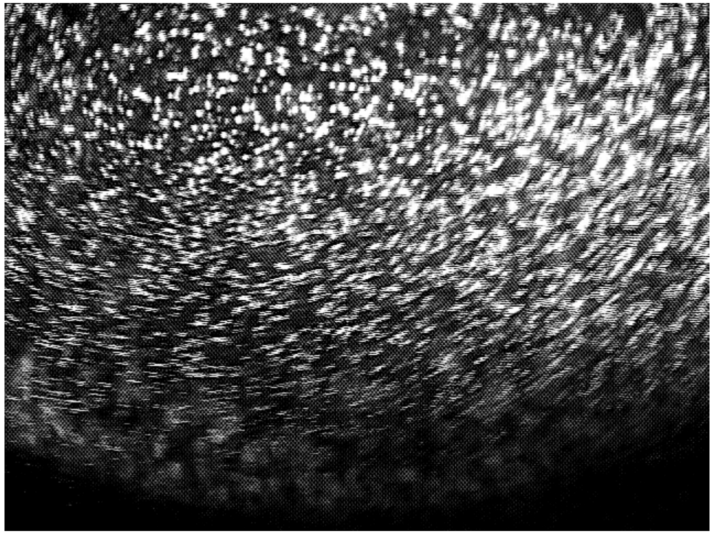
$$\Omega(t) = \left(\Omega_x,\ \Omega_z \sin\Omega_x t,\ \Omega_z \cos\Omega_x t\right)$$

# Actually, this idea has been already used in CT.

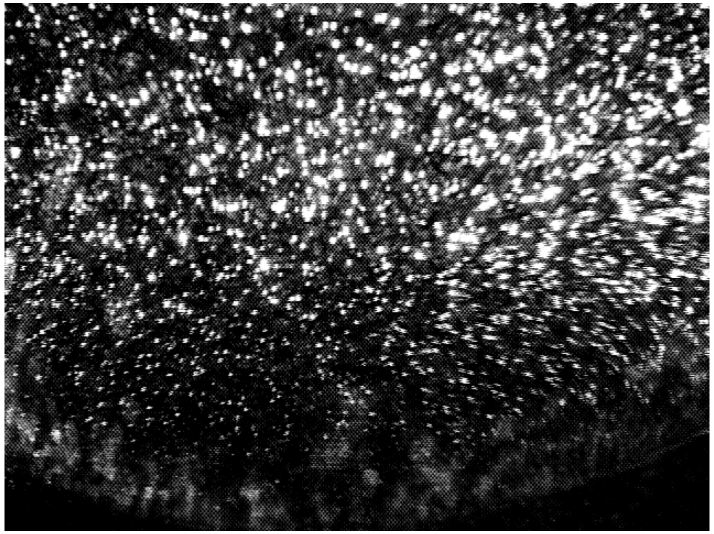
S. Goto, N. Ishii, S. Kida, and M. Nishioka, Phys. Fluids 19, 061705 (2007)



Rotation around one axis



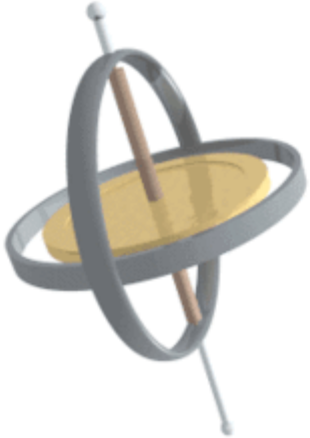
Rotation around two axes



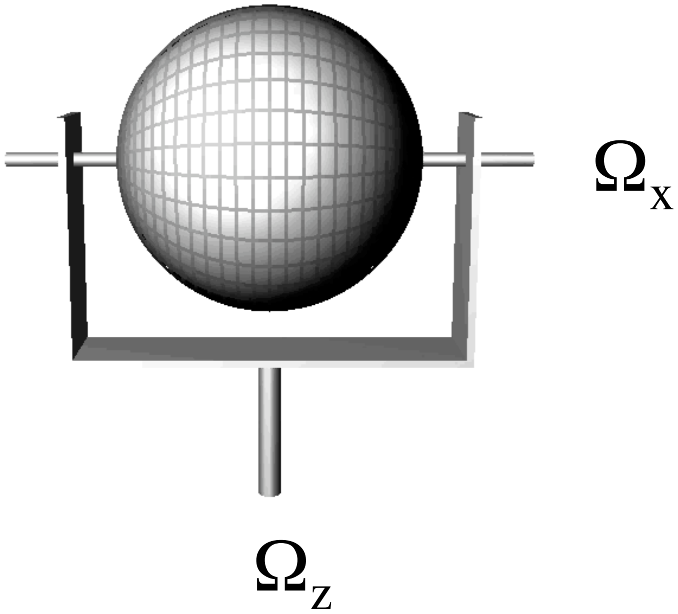
## *Are these two rotations represented by their sum? No!*

# Precession

**Spin axis itself rotates around another axis.**

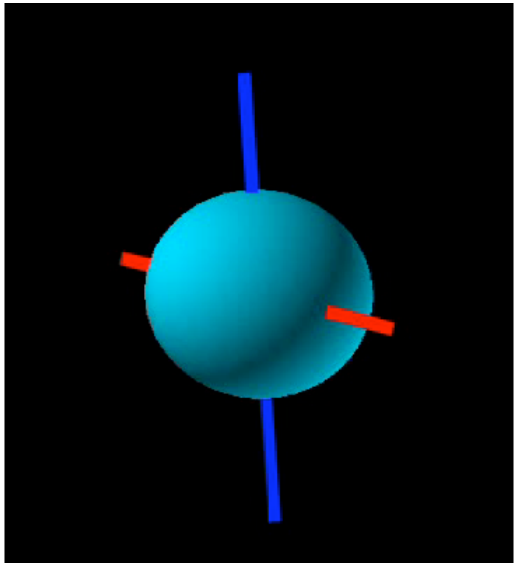


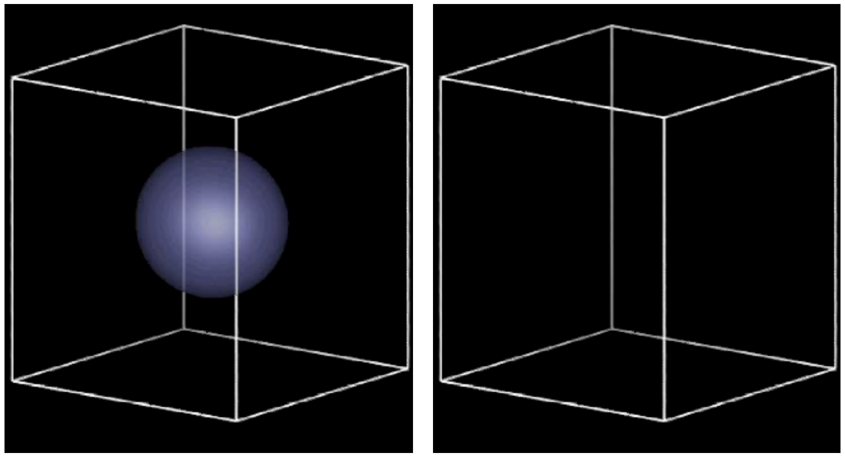
Precessing motion of a gyroscope




We consider the case where the spinning and precessing rotational axes are perpendicular to each other. Hence, the two rotations do not commute, and thus cannot be represented by their sum.

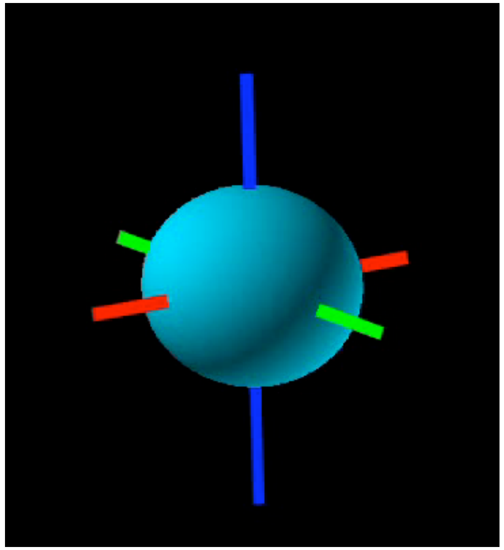
## Two precessions ($\omega_x \times \omega_z$)

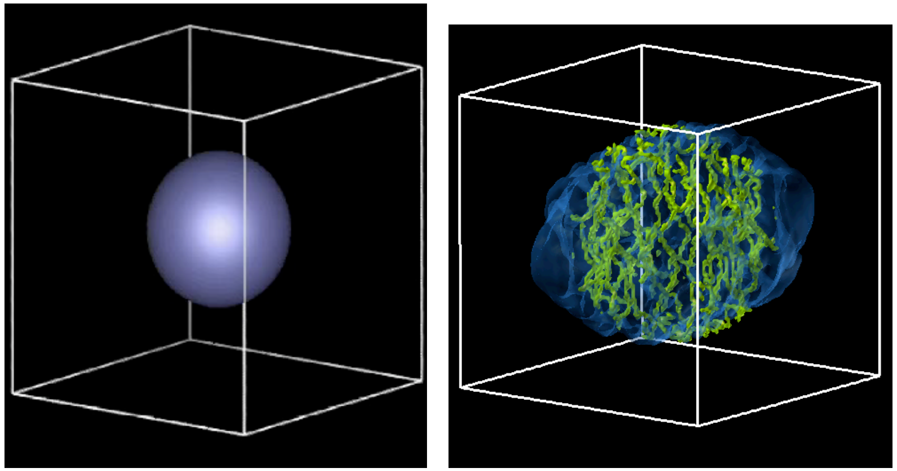




Condensate density | Quantized vortices

## Three precessions ($\omega_y \times \omega_x \times \omega_z$)

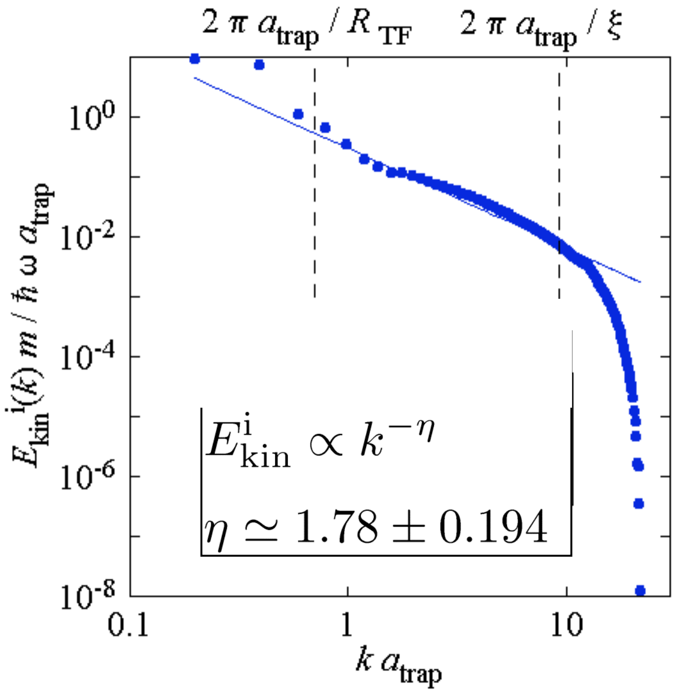




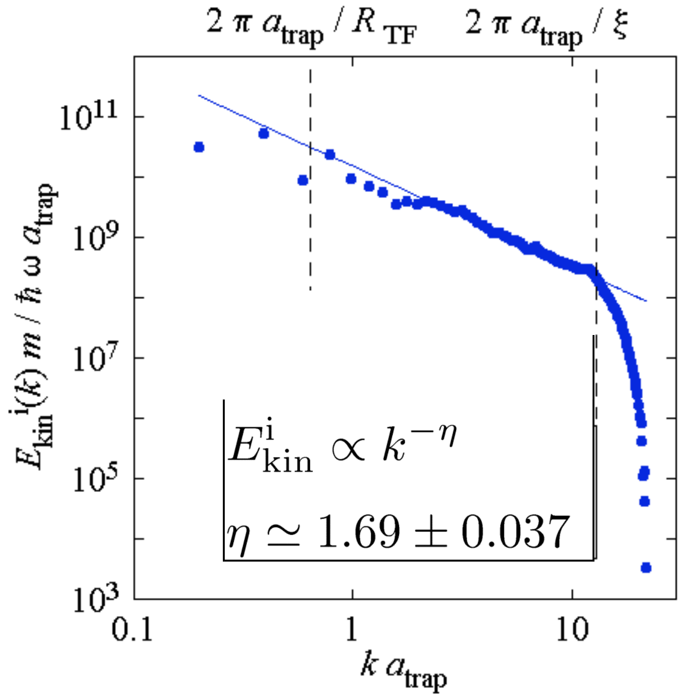
Condensate density | Quantized vortices

# Energy spectra for two cases

Two precessions




Three precessions

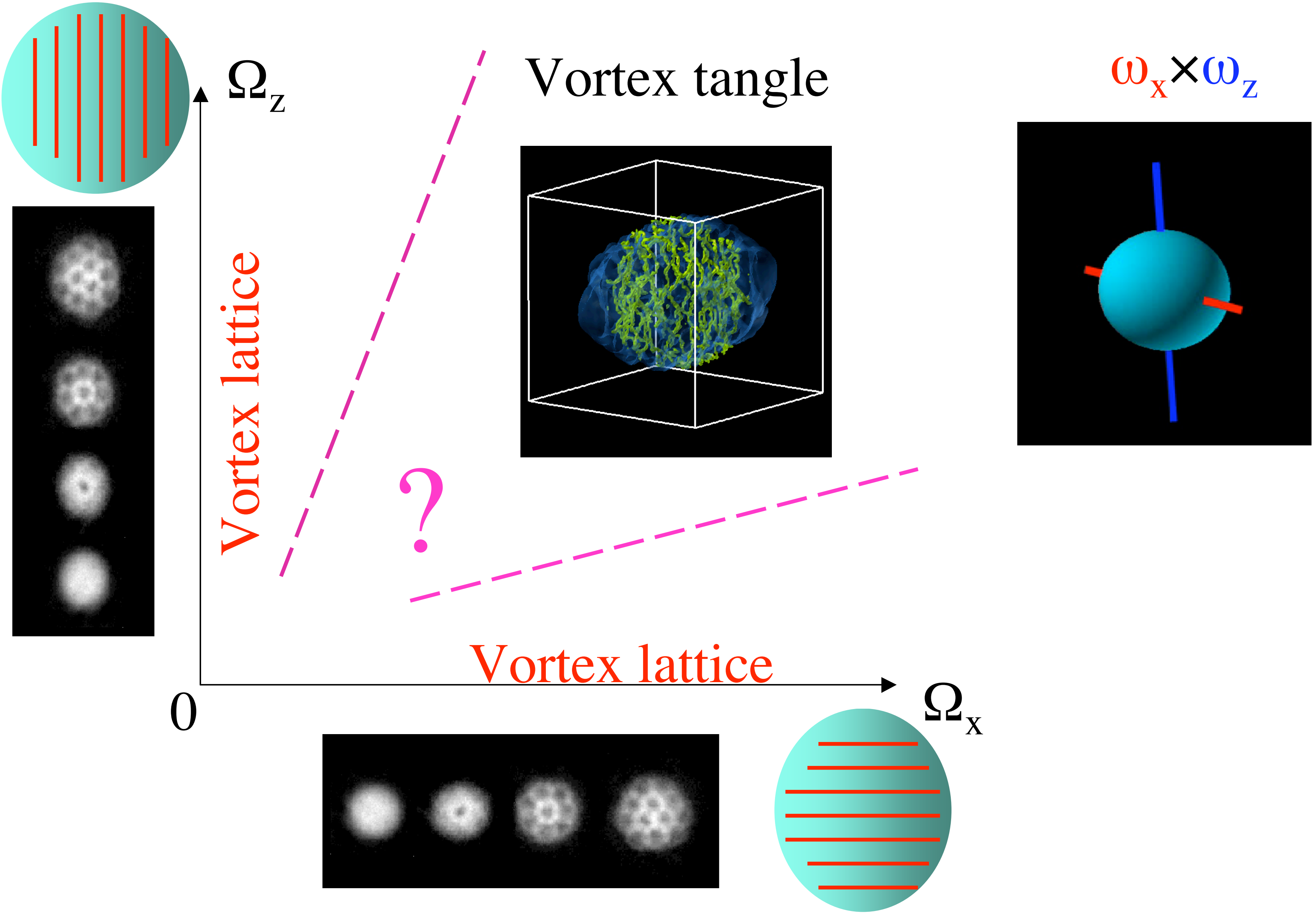



Three precessions produce more isotropic QT, whose $\eta$ is closer to 5/3.

# What can we learn from QT of atomic BEC?

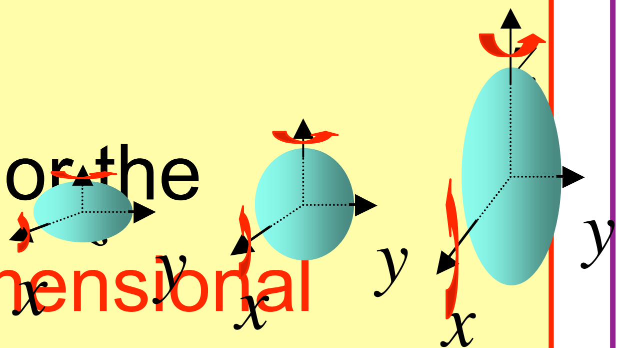
- Controlling the transition to turbulence by changing the rotational frequency or interaction parameters, etc.
- We can visualize quantized vortices. We can consider the relation of real space cascade of vortices and the wavenumber space cascade (Kolmogorov's law).
- Changing the trapping potential or the rotational frequency leads to dimensional crossover (2D↔3D) in turbulence.

# Controlling the transition to turbulence

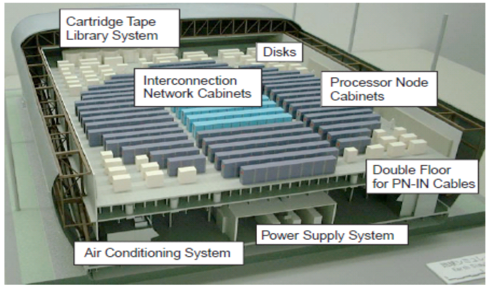


# What can we learn from quantum turbulence of atomic BEC?

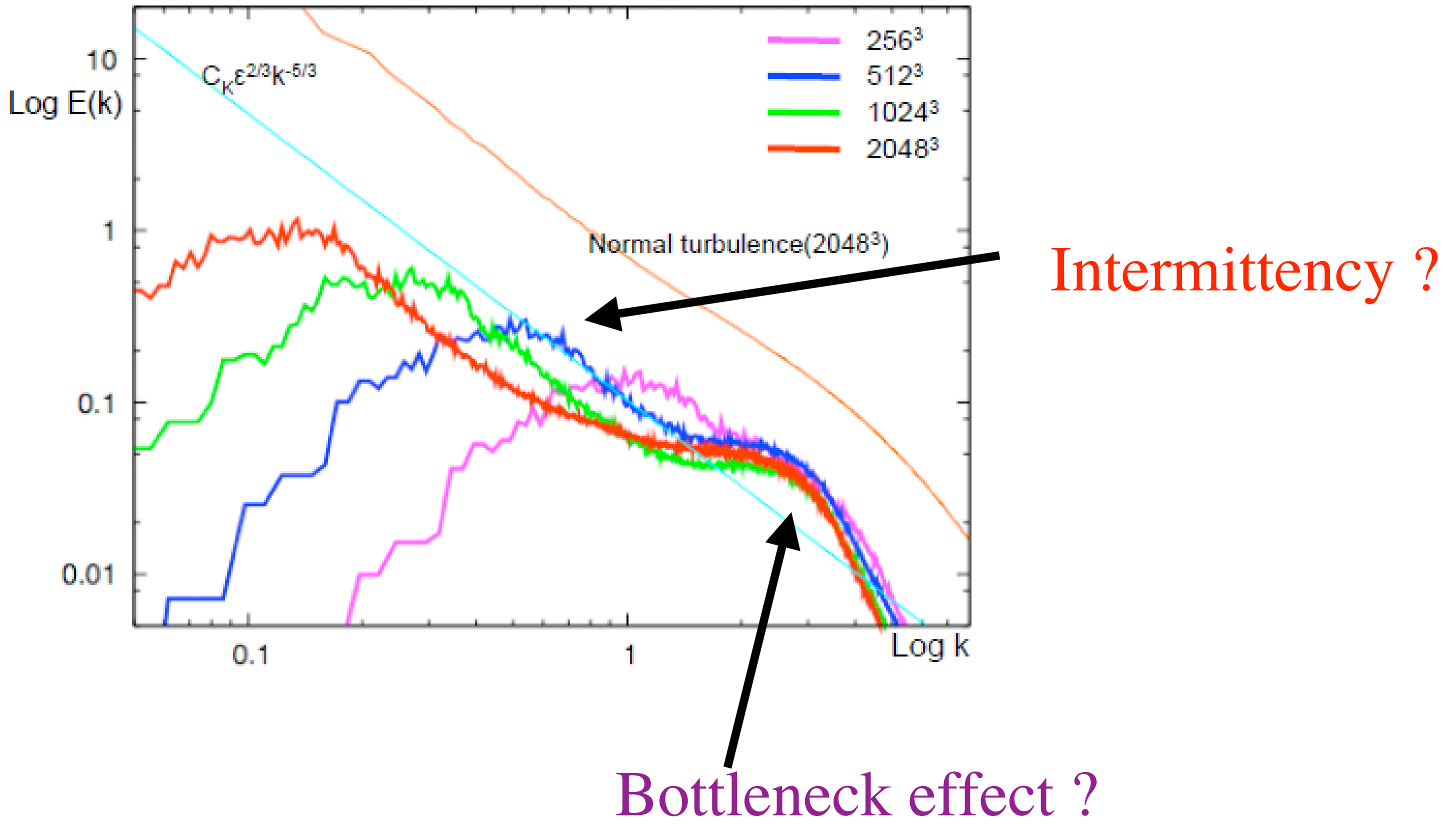
- Controlling the transition to turbulence by changing the rotational frequency or interaction parameters, etc.
- We can visualize quantized vortices. We can consider the relation of real space cascade of vortices and the wavenumber space cascade (Kolmogorov's law).
- Changing the trapping potential or the rotational frequency leads to dimensional crossover (2D↔3D) in turbulence.

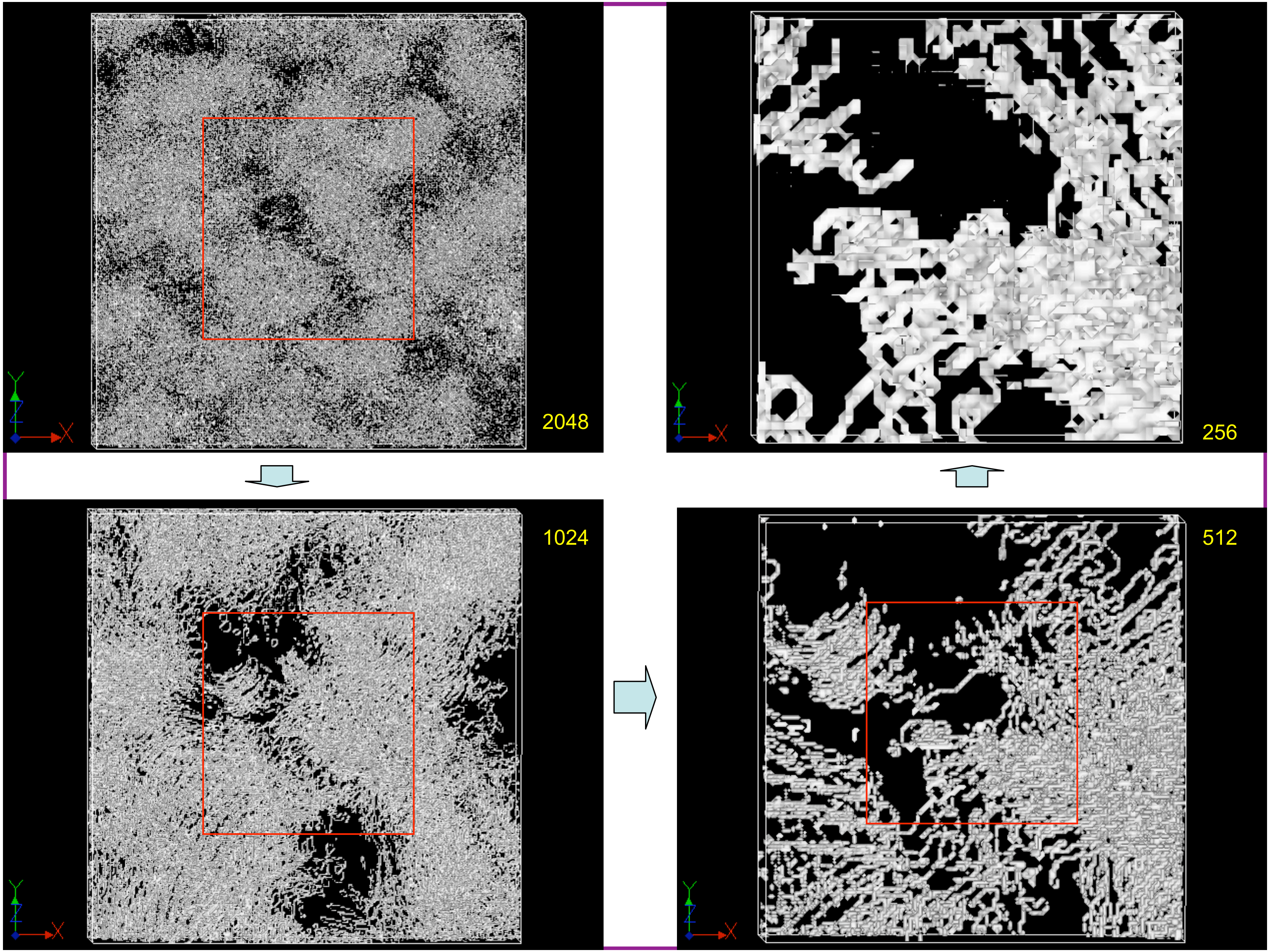


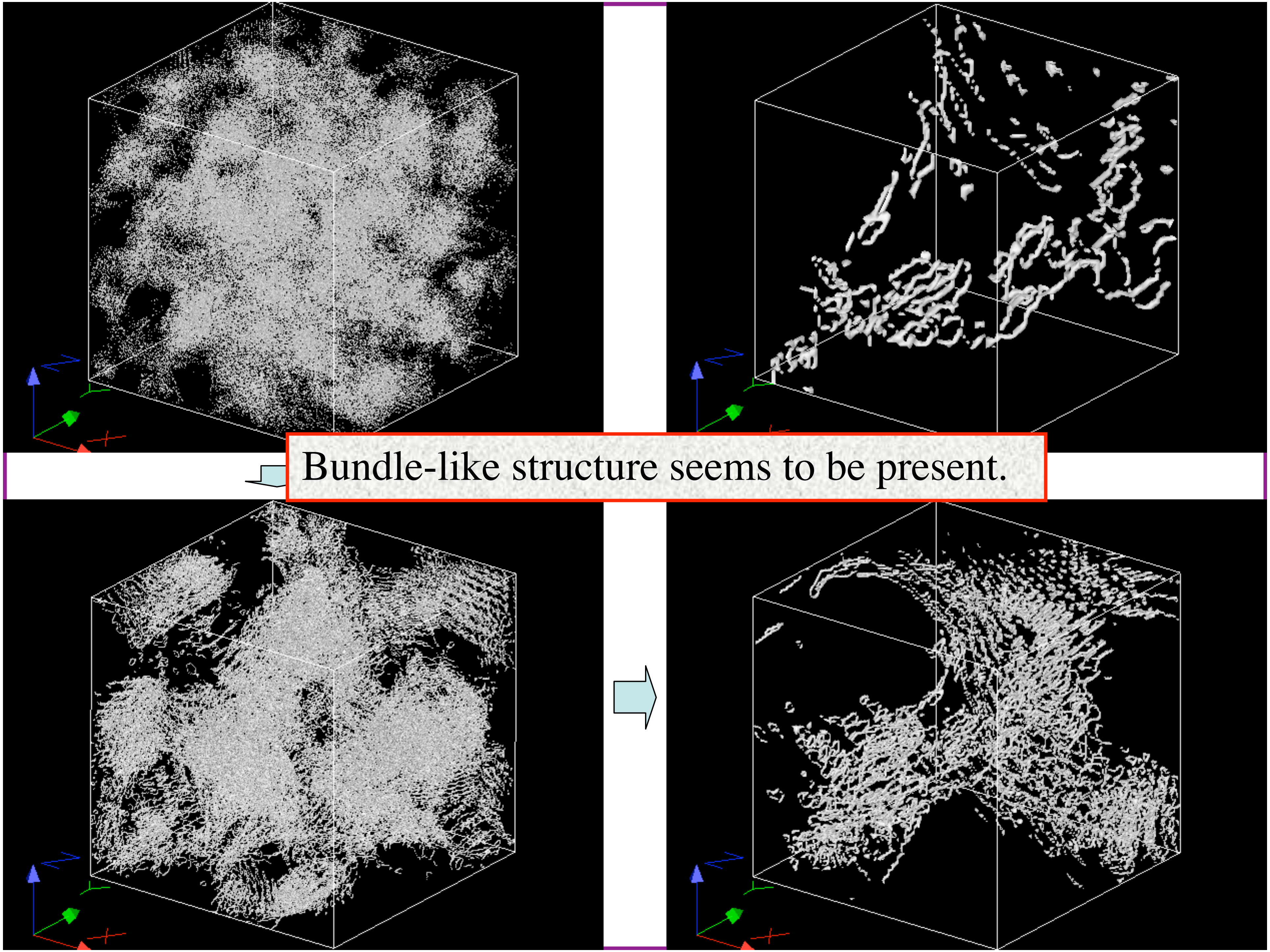
3. A bigger simulation of the GP model was recently made by M. Machida using the earth simulator (private communication).




$256^3$ (Kobayashi and Tsubota) → $2048^3$ (Machida)




2048
256
1024
512
Y
Z
X


Bundle-like structure seems to be present.

# 4. Two-dimensional quantum turbulence

Two-dimensional classical turbulence has different properties from three-dimensional one.

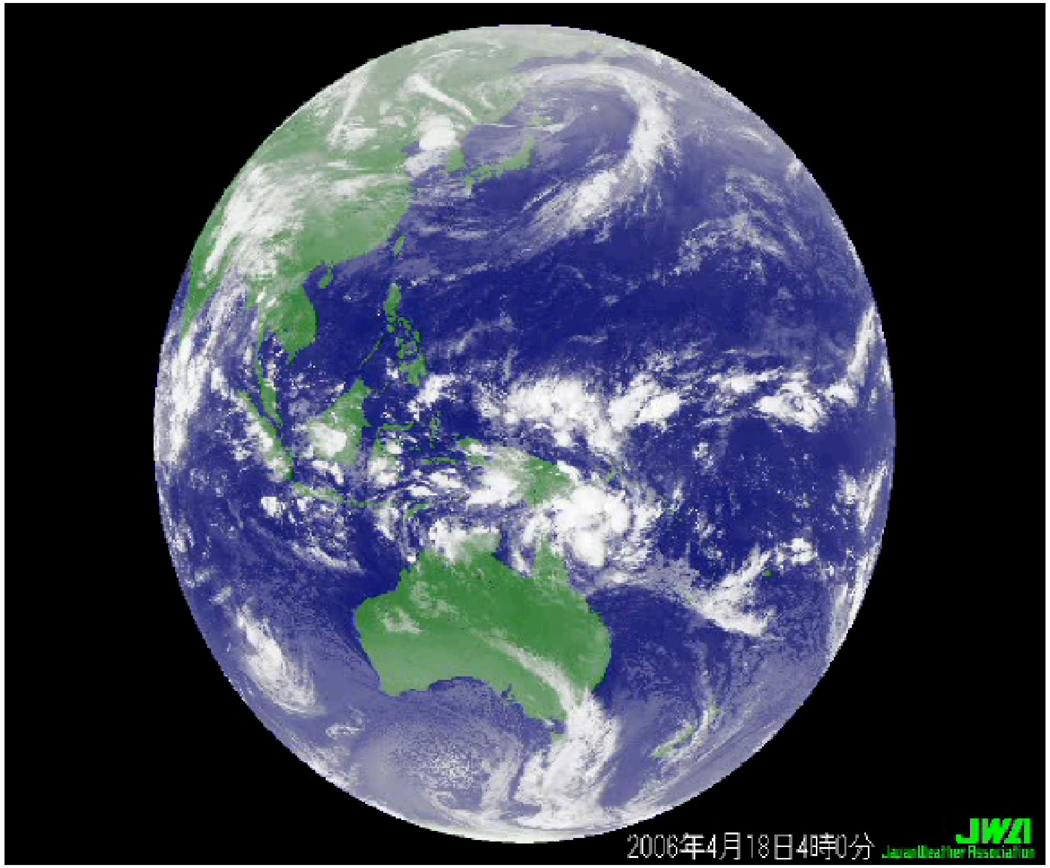
The most important difference is the presence of the inverse cascade.

Our interests are

- Can we obtain the inverse cascade even in quantum case?
- If so, the behavior should be reduced to the motion of quantized vortices. What occurs?

Two-dimensional quantum turbulence can be realized in superfluid helium and cold atoms.

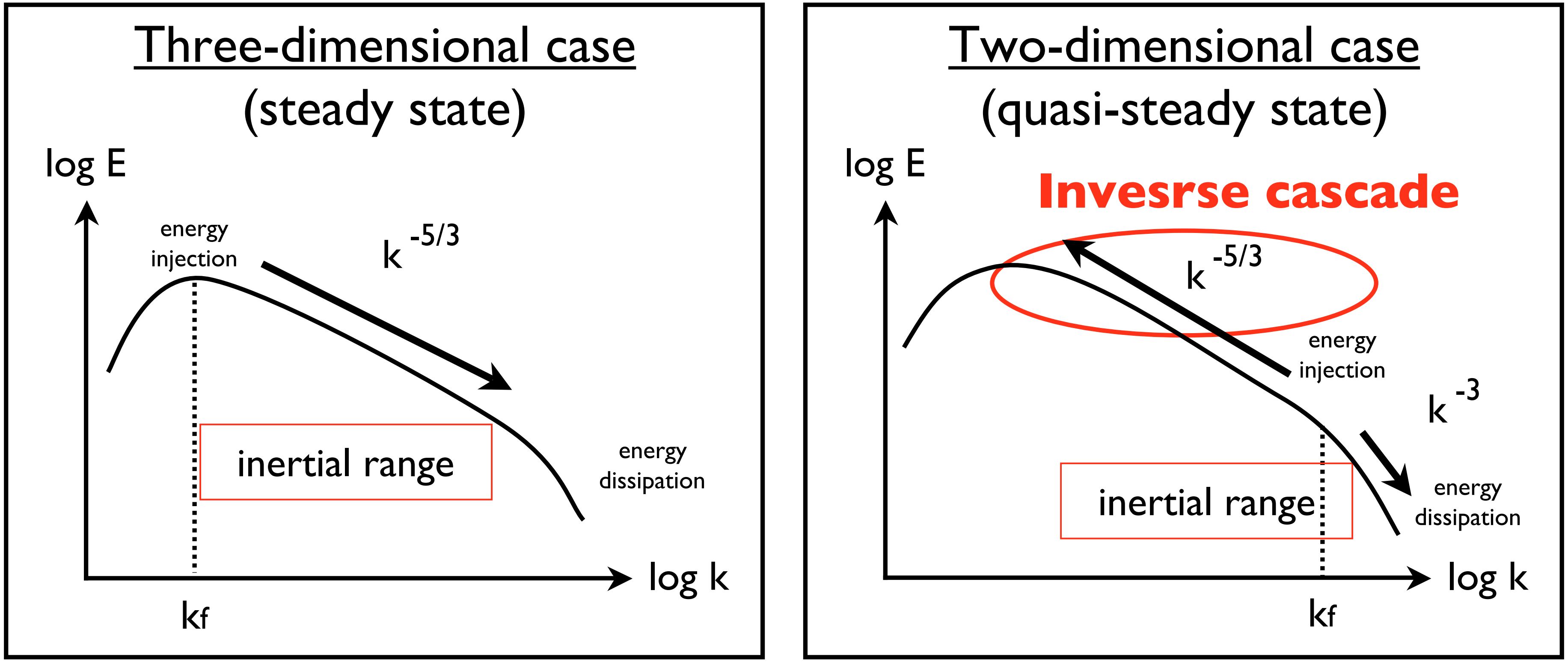
# Two-dimensional turbulence is very common in our daily life.



# The Inverse Cascade of Energy

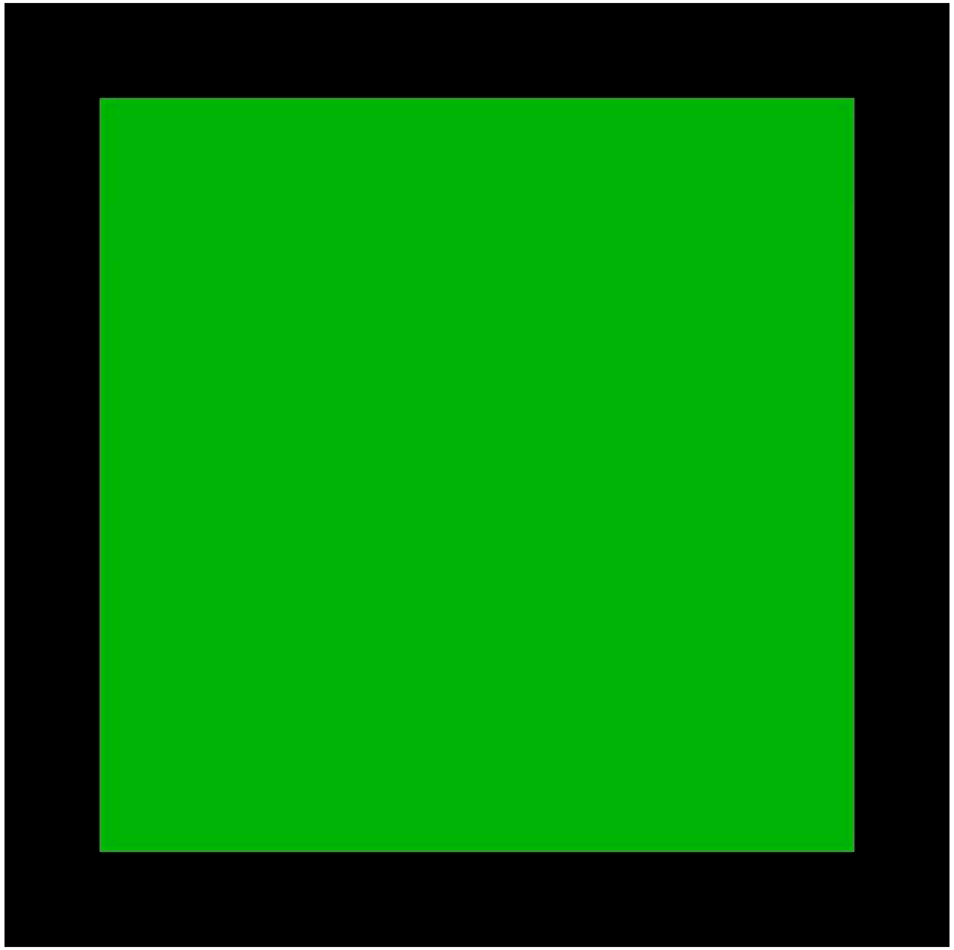
Inverse cascade is one of the most remarkable phenomenon of two-dimansional classical turbulence!

## Energy spectrum of fully developed turbulence



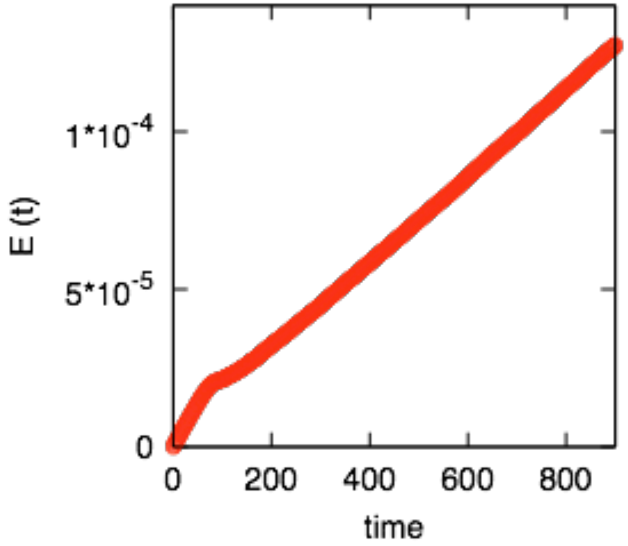

R. H. Kraichnan, Phys. Fluids **10**, 1417 (1967)

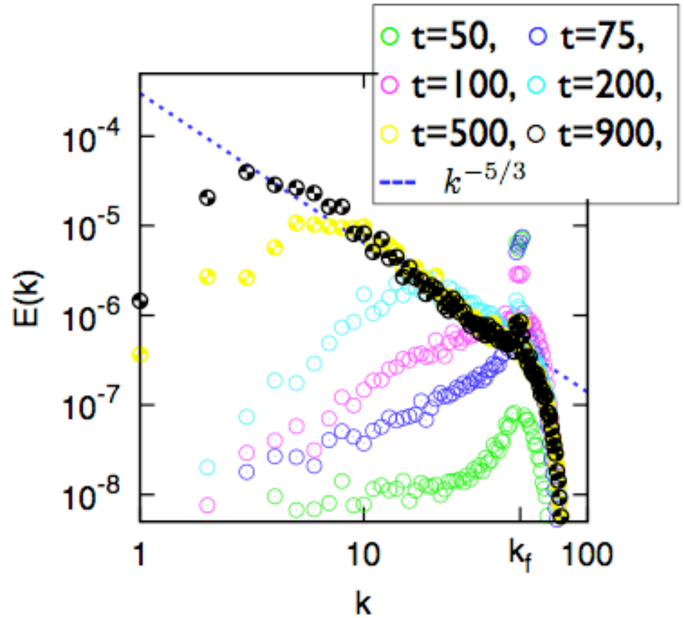
## Two-dimensional classical turbulence (the NS equ.)



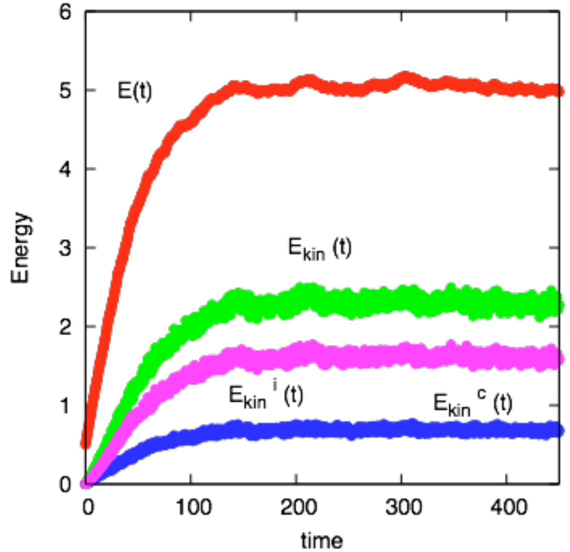
Vorticity distribution

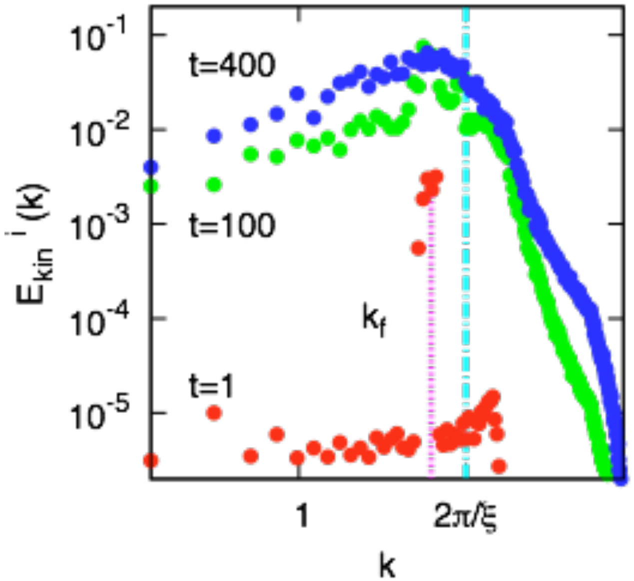
**Large-scale motion occurs with inverse cascade!**

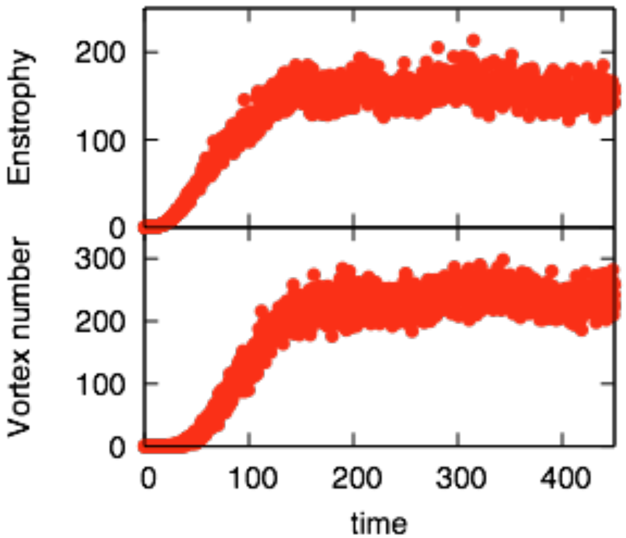


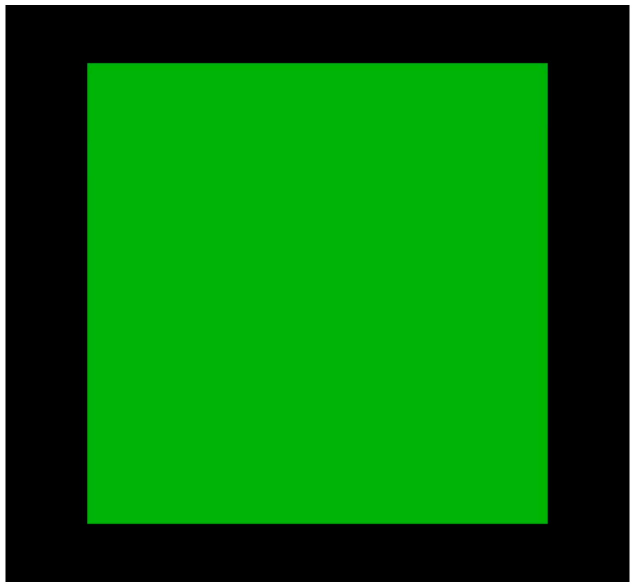



# Two-dimensionla quantum turbulence (the GP model)

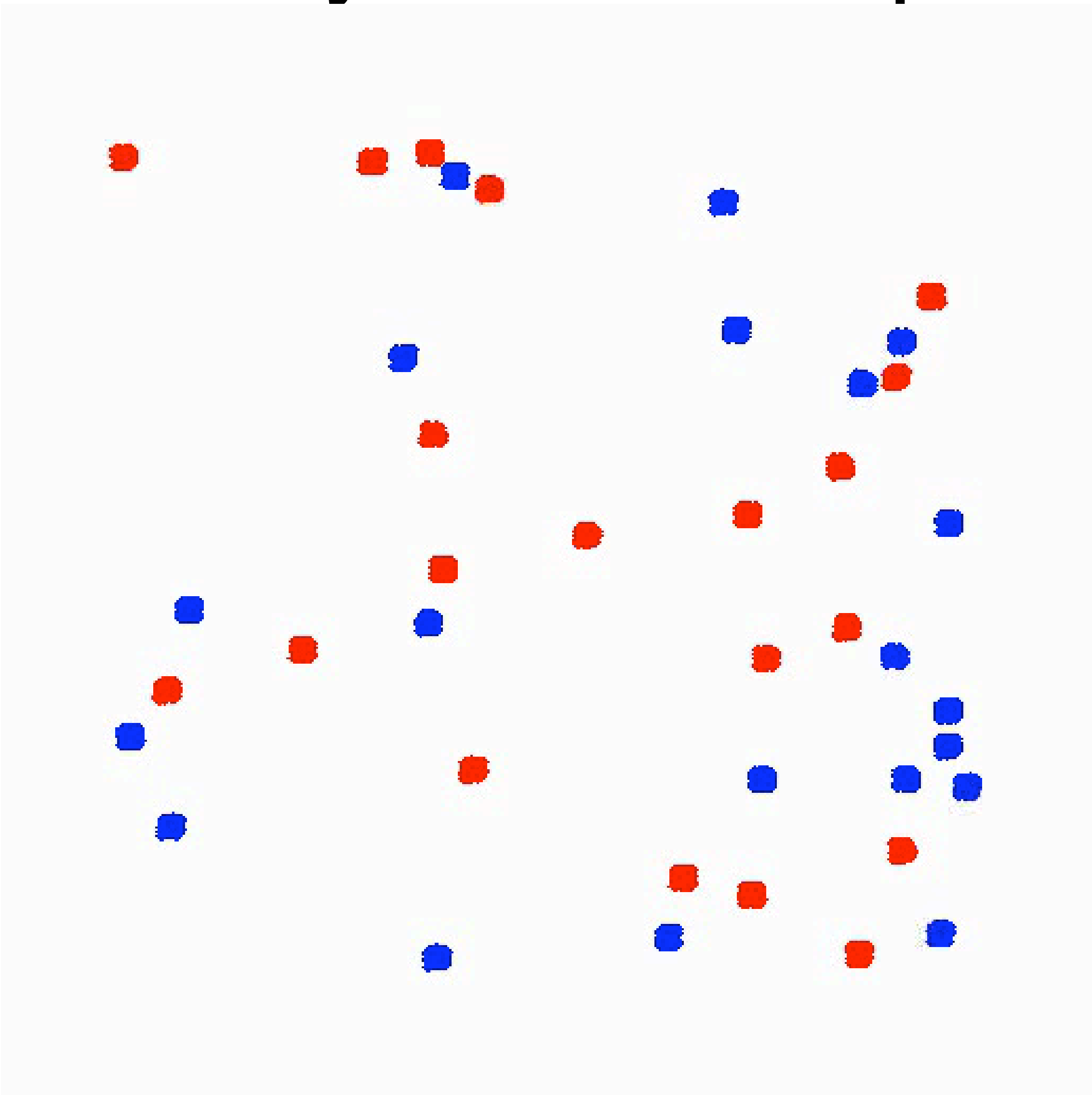


# Simulation by the vortex-point model



# Summary

1. Introduction -Brief history-
2. Quantum turbulence in trapped Bose gas
3. Big simulation towards the classical-quantum crossover
4. Two-dimensional quantum turbulence