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Efficient calculation of scattering amplitudes in supersymmetric gauge and gravity theories
Lecture 3 & 4

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Higher loops in $\mathcal{N} = 4$ SYM and supergravity

Having discussed 1-loop in detail, the logical step is to proceed to higher loops. The goal is still to use only on-shell information to construct these amplitudes. Before we proceed with any actual calculations it is perhaps useful to draw a comparison between 1-loop and higher loops based on what we currently know and on simple counting arguments.

1-loop	higher loops
technology: quadruple cuts freeze integral	no general analog, though recent suggestion exists; there exist integrals with too few propagators
very efficient complete basis	over-complete or undercomplete basis; naive guess insufficient
same basis for any number of external legs	limited experience; 4pt→ 5pt new integrals new structures: $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^2)$ in 1-loop amplitude are relevant current best guess: conformal integrals
cuts easy to disentangle (even without quadruple cuts)	(very) nontrivial zeros; many ways to reorganize cuts; hard to choose; some more useful than others
$4d$ algebra suffices	unclear; safety requires $D-$dim. except perhaps for 4-points

Higher loop strategy: use everything available:

– Begin by doing calculations in $d = 4$; while this is likely to be corrected most of the times, it provides a good starting point, with only a handful of terms left to be determined by d-dimensional calculations.

– regular cuts (not too efficient, though occasionally useful for checks)

– generalized cuts

– test the result using its infrared poles

– test using soft/collinear properties; there is some overlap between these two points, as the IR poles are of soft/collinear type in N=4 SYM. One may however test the integrand.

A quite successful approach continues to be generalized unitarity. One may cut sufficiently many propagators such that one needs to evaluate only a product of tree amplitudes. One needs to be careful however; cutting/fixing too many propagators may lead to missing contact terms.

A set of cuts that come pretty close to a complete localization similar to that of quadruple cuts at 1-loop goes by the name of maximal cuts – in the sense of Bern/Carrasco/Johanson/Kosower. The idea is to cut as many propagators as possible provided one obeys the restriction that there should not be more than 4 cut conditions determining one loop momentum (e.g. hexabox don't cut all 6 legs). At L-loops the maximal number is 4L; this would lead to a complete localization. The topologies available at some fixed number of loops are many, varied and have not been classified in general; however, it is quite clear that some of them have fewer that $4L$ propagators.

At 2-loops, using integral reduction one may argue that at most double-pentagon integrals may exist. Higher ones can be reduced.

figures with 2-loop integrals

Due to integrals with more than maximal number of propagators, maximal cuts may contain some ambiguities. For example, at 2-loops, by cutting 8 propagators leads in some sense to a complete localization, but one can no longer distinguish between say a double-pentagon and a box-pentagon (this issue appears only at 6-points and higher). This is because the integrals are completely localized and there is no parameter to identify the propagators.

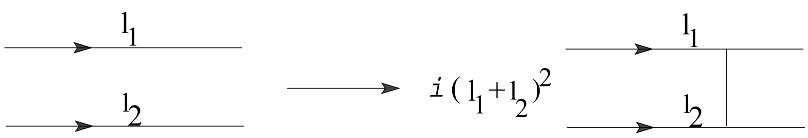
In general a strategy would be: **(a)** if no ambiguity can appear, start with cutting max nr of propagators and extract coefficients of certain integrals; then decrease the number and construct "contact terms". **(b)** if ambiguities exist, then start by cutting the max number that does not exhibit an ambiguity. For example, at 2-loops one may consider cutting 7 propagators; this will leave one component unfixed and thus the possibility of identifying any left-over propagators. One can then proceed further and cut fewer and fewer number of propagators. This will identify more and more of the integrals that appear in the amplitude under discussion and their coefficients.

Still, the cuts containing the most information are those with the fewest number of cut propagators such that the amplitude still falls apart into a product of trees. In this category fall the iterated 2-particle cuts; as the name says, one just does repeated 2-particle cuts. They turn out to be a fairly powerful tool for all amplitudes; for MHV amplitudes they are very special because of a special property of MHV trees. It turns out that 2PC iterate in the the sense that

$$A_{MHV} \times A_{MHV} \propto A_{MHV}$$

The reason to be happy with this observation is that, while the use of unitarity-based method is in principle straightforward, it is nevertheless tedious at times. Finding general features in its application avoids repeating the same steps in the calculation over and over again.

For a 4-particle amplitude the iteration of 2-particle cuts can be explicitly solved and yields the so-called rung rule. It states that the L-loop integrals which follow from iterated 2-particle cuts can be obtained from the $(L-1)$-loop amplitudes by adding a rung in all possible (planar) ways and in the process multiply the numerator by i times the invariant constructed from the momenta of the lines connected by the rung. This rule is illustrated in the figure.



For higher multiplicity amplitudes the rung rule is less effective and it is necessary to explicitly evaluate the relevant iterated cuts.

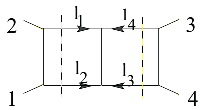
To date there are only a handful of higher loop calculations in N=4SYM that have been carried out: 2-loop 4g, 5g, 6g, 7p-even MHV; 3-loop 4g, 5g; 4-loop 4g and 5-loop 4g and some of them are not yet integrated; 2-loop collinear splitting amplitude.

As an illustration of how the calculation goes for some of them, let me sketch 2-loop 4g:

$$\begin{aligned}&A_4^{\rm tree}(l_2, k_1^-, k_2^-, l_1) A_4^{\rm tree}(-l_1, -l_4, -l_3, -l_2) A_4^{\rm tree}(l_4, k_3^+, k_4^+, l_3)\\ &\quad = i s_{12}(k_2 - l_4)^2 \frac{1}{(l_2-k_1)^2 (l_2+l_3)^2} A(-l_3, 1^-, 2^-, -l_4) A_4^{\rm tree}(l_4, k_3^+, k_4^+, l_3)\end{aligned}$$

$$= \left[i s_{12}(k_2 - l_4)^2 \frac{1}{(l_2 - k_1)^2 (l_2 + l_3)^2} \right] \left[i s_{12} s_{23} \frac{1}{(l_3 - k_1)^2 (l_3 + k_4)^2} \right]$$

$$= -s_{12}^2 s_{23} A_4^{\text{tree}}(k_1^-, k_2^-, k_3^+, k_4^+) \tag{1}$$



Other cuts (the t-channel) may be computed in a similar way and leads to the correct answer.

The 3-loop 4g calculation can be done in a similar way; at 4-loops 4g the iterated 2-particle cuts do not generate the complete integrand.

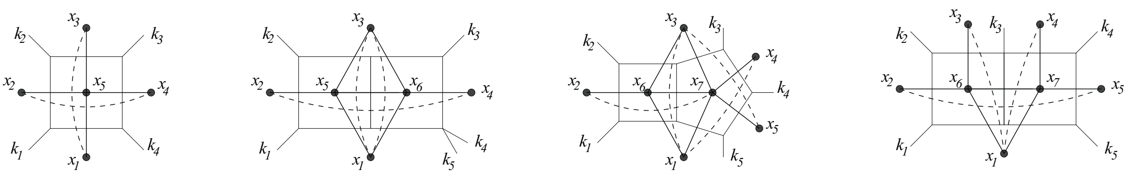
As one increases the number of legs new structures appear: some involving epsilon tensors. With them also complications appear because epsilon tensors provide ingredients for the construction of nontrivial integrands that integrate to 0 and also are nonzero when cut conditions are imposed. Similar objects could have appeared at 1-loop already there however, the fact that an integral basis exists makes their appearance harmless.

A basis of independent integrals covering both the even and the odd part of amplitudes seems hard to construct. For the even part a conjecture has been made, based on inspecting the integrands available in 2006 – i.e. 2, 3-loop 4g and 2-loop 5g. As it turns out, they are given by what came to be known as conformal (or pseudo-conformal) integrals. These are integrals which, when treated in D=4 and with their external lines regularized by a mass, are invariant under an inversion transformation:

$$k_i = x_i - x_{i+1} \quad l_j = x_j - x_j^* \tag{2}$$

where x_i are cyclic variables and x_j^* is the coordinate of a point inside a loop. For the loop momentum one introduces more x-s such that momentum conservation is solved at vertices. Then, the inversion transformation is $x_i \mapsto x_i/|x_i|^2$.

As it turns out, each propagator that appears in a planar color-stripped amplitude is the square of some difference of x-s, so transforms homogeneously – $k_{ij}^2 \mapsto k_{ij}^2/(x_i^2 x_{j+1}^2)$. The same is true for factors of external momentum invariants or rung-rule factors. Likewise, in D=4, the loop integration measure transforms homogeneously. Thus all one has to do is count.



(a) (b) (c) (d)

An interesting property, first observed for the 4g L=2 and then for the 4g L=3 amplitude and then tested for 5-points as well, is that the amplitudes iterates in a certain way. In particular:

$$\exp\left[\sum_n \lambda^n f_n(\epsilon) M_1(n\epsilon; s, t) + C\right] \qquad f_n(\epsilon) = \gamma_n^K + \epsilon G_{0n} + \epsilon^2 C_n \tag{3}$$

This is in line with the factorization of the soft and collinear singularities:

$$\mathcal{M}_n = \left[\prod_i M^{[gg->1]}(s_{i,i+1}/\mu^2, \epsilon) \right]^{1/2} \times h_n \tag{4}$$

and also with the exponentiation of the 2-loop splitting amplitude.

Soft/Collinear factorization

◇ Rescaled amplitude factorizes in three parts:

$$\mathcal{M}_n = S(k, \frac{Q}{\mu}, \alpha_s(\mu), \epsilon) \times \left[\prod_{i=1}^{n} J_i(\frac{Q}{\mu}, \alpha_s(\mu), \epsilon) \right] \times h_n(k, \frac{Q}{\mu}, \alpha_s(\mu), \epsilon)$$

$S(k, \mu, \alpha_s(\mu), \epsilon)$ soft function; captures the soft gluon radiation; defined up to overall function

– $J_i(k, \mu, \alpha_s(\mu), \epsilon)$ independent of color flow; all collinear dynamics

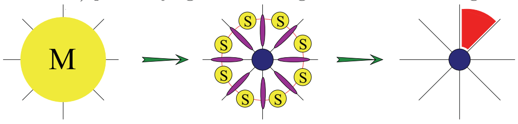
– $h_n(\mu, \alpha_s(\mu), \epsilon)$ is finite as $\epsilon \to 0$

◇ Independence of Q: factorization vs. evolution

Consequences of the large N limit:

1) trivial color structure: S can be absorbed in J

2) planarity: gluon exchange is confined to neighboring legs



$$\mathcal{M}_n = \times \left[\prod_{i=1}^{n} \mathcal{M}^{[gg\to 1]} \left(\frac{s_{i,i+1}}{\mu}, \lambda, \epsilon \right) \right]^{1/2} \times h_n(k, \lambda, \epsilon)$$

Sudakov form factor: decay of a scalar into 2 gluons

Factorization $\mapsto$ differential (RG) equation for $\mathcal{M}^{[gg\to 1]}$

$$\frac{d}{d\ln Q^2} \mathcal{M}^{[gg\to 1]}\left(\tfrac{Q^2}{\mu^2}, \lambda, \epsilon \right) = \frac{1}{2} \left[K(\epsilon, \lambda) + G(\frac{Q^2}{\mu^2}, \lambda, \epsilon) \right] \mathcal{M}^{[gg\to 1]}\left(\tfrac{Q^2}{\mu^2}, \lambda, \epsilon \right)$$

$$\left(\frac{d}{d\ln\mu} + \beta(\lambda) \frac{d}{dg} \right) (K + G) = 0 \qquad \left(\frac{d}{d\ln\mu} + \beta(\lambda) \frac{d}{dg} \right) K(\epsilon, \lambda) = -\gamma_K(\lambda)$$

Exact solution for $\mathcal{N} = 4$ SYM

$$\mathcal{M}_n = \exp\left[-\frac{1}{8} \sum_l a^l \left(\frac{\gamma_K^{(l)}}{(l\epsilon)^2} + \frac{2\mathcal{G}_0^{(l)}}{l\epsilon} \right) \sum_i \left(\frac{\mu^2}{-s_{i,i+1}} \right)^{l\epsilon} \right] \times \; h_n$$

$$f(\lambda) = \sum_l a^l \, \gamma_K^{(l)} \qquad \text{universal scaling function}$$

The exponentiation of the 2-loop splitting amplitude led to the conjecture that at 2-loops all MHV amplitudes have an exponential structure (ABDK). Together with the 3-loop 4g calculation and the structure of IR singularities led to the conjecture that this exponentiation holds to all loops for all MHV amplitudes. We now know that this conjecture may be explained at 4- and 5-points on the basis of dual conformal invariance – if one requires that that it is present to all loop orders (BDS). With 4- and 5-point kinematics it turns out to be impossible to construct invariants under dual inversion. Therefore everything is determined by the IR divergences.

5-loop result

We also know the conjecture needs to be corrected at 6-points and beyond; this is also the place where dual conformal invariance – if imposed – no longer fully constrains the amplitude. Starting at 6-points it is possible to construct invariants under dual inversion:

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12} s_{45}}{s_{123} s_{345}} \qquad u_2 = \frac{x_{24}^2 x_{51}^2}{x_{25}^2 x_{41}^2} = \frac{s_{23} s_{56}}{s_{234} s_{456}} \qquad u_3 = \frac{x_{35}^2 x_{62}^2}{x_{36}^2 x_{25}^2} = \frac{s_{34} s_{61}}{s_{345} s_{234}} \; . \tag{5}$$

The story becomes similar to 2d CFT; the result is constrained up to arbitrary functions of these cross-ratios. This does happen.

Using the technology I've described it is possible to compute the 6-point amplitude

6-loop result – or not depending on time

- made out of conformal integrals
- subtract BDS ⟶ obtain a dual conformal invariant

The technology for higher-loop and higher-point calculations are still very much under development. Perhaps one of the most pressing things to be figured out are efficient reduction procedures, perhaps similar to those we have at 1-loop. To this end, having an integral basis would be useful.

In retrospect, the various methods we discussed rely to a very large extent on field theory considerations. Recently, based on AdS/CFT, Alday and Maldacena suggested that, at strong coupling, amplitudes are essentially given by the expectation value of certain null polygonal Wilson loops.

Wilson loop figure

J. Maldacena will probably discuss this in detail in his lectures later in the week. However, this string coupling analysis led to the suggestion (DHKS/BHT) that such a relation might hold at weak coupling as well, at least for some amplitudes. This turns out to be correct – at least for MHV amplitudes for the time being and for the examples that have been considered. This is a very interesting suggestion, as it relates amplitudes to something that has little or no right to be related to amplitudes by field theory standards.

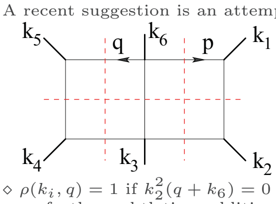
Discussing this relation in some detail is a full lecture in itself, so let me not do it. If true:

– WL is manifestly dual conformal invariant → dual conformal invariance is indeed a symmetry of MHV amplitudes to all loop orders; justifies symmetry argument for $n = 4$ and $n = 5$ amplitudes

– it provides an alternative presentation of MHV amplitudes in terms of simpler albeit non-standard integrals

– an important open problem is whether it can be extended beyond MHV amplitudes

A recent suggestion is an attempt at a complete localization, similar to the one realized by quadruple cuts at 1-loop.



if $\begin{array}{l} k_2^2(q + k_6) = 0 \\ k_1^2(k_{345} - q)^2 = 0 \end{array}$ Proof: $\int d^4q d^4p \prod_i \delta^{(+)}(q_i^2) = \frac{2}{s_{12}(k_{16} + q)^2 \, \rho(k_i, q)}$

◇ $\rho(k_i, q) = 1$ if $k_2^2(q + k_6) = 0 = k_1^2(k_{345} - q)^2$ allowing an additional, "effective" 8th cut!

– further subtleties; additional pole may be missing.

This observation led to the so-called leading singularity method: basically one assumes that only integrals exhibiting this additional singularity appear in the amplitude. One then constructs and ansatz and compares the "generalized cuts" of the ansatz to those of the amplitude

$$c_{\rm doublebox} = -\frac{i}{S} \sum_{h,S,J_1,J_2} \left(n_{J_1} n_{J_2} A_1^{\rm tree} A_2^{\rm tree} A_3^{\rm tree} A_4^{\rm tree} A_5^{\rm tree} A_6^{\rm tree} \right)_h$$

where c is the sum of all coefficients of integrals with at least double-box topology together with the Jacobians from the integral over delta functions. No one such generalized cut determine a specific integral coefficient. However, there are many such cuts. One can count and find that typically there are more cuts than coefficients. If a solution exists it seems likely to be correct. While, no explicit proof of this assertion was given, examples of 5- and 6-point amplitudes at 2-loops seem to support it. There are currently 2 results derived only through this methods which await confirmation by other means: the 3-loop 5-point amplitude and the odd part of the 6-point 2-loop amplitude.

One thing I have not said anything is subleading color. At 1-loop that is algebraically determined by the planar part. At higher loops however this is no longer true. The unitarity-based method works as well for nonplanar amplitudes. 2-particle cuts still iterate. 4g 2- and 3-loop amplitudes have been evaluated. No higher-point calculation is available.

Besides being interesting in their own right, knowing such amplitudes is interesting for a different reason: it is possible to use such data to construct quite easily scattering amplitudes in another theory of some interest – $\mathcal{N}=8$ supergravity. The reason for such calculations is to answer the age-old question when do UV divergences occur in this theory – if ever. The proposed answers by now vary from 3 to 9 loops depending on various assumptions. Basically, N-extended supersymmetry implies that counterterms are possible at $L = N - 1$. Additional symmetries, such as higher dimensional Lorentz symmetry improve the situation to $L = 9$. More recently it was suggested, both on field theory/unitarity grounds as well as using string dualities, that the theory might be finite – though the string theory statement was somewhat softened.

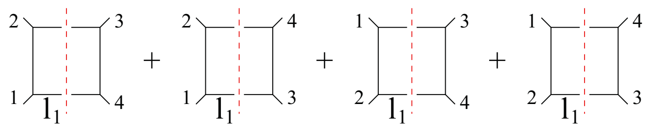
The strategy is to use the Kawai/Lewellen/Tye relations; they are derive based on the fact that the up to 0-mode contributions, the closed string vertex operators are bilinears in open string vertex operators. In the low energy limit, some examples look as follows:

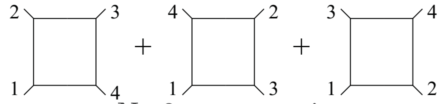
$$\begin{aligned} M_4^{\rm tr}(1,2,3,4) &= -is_{12}A_4^{\rm tr}(1,2,3,4)A_4^{\rm tr}(1,2,4,3) \\ M_5^{\rm tr}(1,2,3,4,5) &= \quad is_{12}s_{34}A_5^{\rm tr}(1,2,3,4,5)A_5^{\rm tr}(2,1,4,3,5) + (2\leftrightarrow 3) \\ M_6^{\rm tr} &= 12 \text{ terms of the type } s^3 A_6 A_6 \end{aligned}$$

It is perhaps clear how this works, but let me describe an example – the calculation of the 4-graviton amplitude at 1-loop:

$$\sum_{\mathcal{N}=8} M_4^{\rm tree}(1,2,l_1,l_2)M_4^{\rm tree}(l_2,l_1,3,4) = (s_{12}s_{23})^2 \frac{s_{12}A_4^{\rm tree}(1,2,3,4)}{(2l_1\cdot k_2)(2l_2\cdot k_4)}\frac{s_{12}A_4^{\rm tree}(2,1,4,3)}{(2l_1\cdot k_1)(2l_2\cdot k_3)}$$

$$\xrightarrow{\text{partial fraction}} \quad s_{12}s_{13}s_{23}M_4^{\rm tree}(1,2,3,4)\left[\frac{1}{2l_1\cdot k_1}+\frac{1}{2l_1\cdot k_2}\right]\left[\frac{1}{2l_2\cdot k_3}+\frac{1}{2l_2\cdot k_4}\right]$$

$$= s_{12}s_{13}s_{23}M_4^{\rm tree}(1,2,3,4)\Big[$$  $$\Big]$$

$$M_4^{1\text{ loop}}(1,2,3,4) = \left[s_{12}s_{23}A_4^{\rm tree}(1,2,3,4)\right]^2\Big[$$  $$\Big]$$

The observation is therefore that, to construct N=8 supergravity cuts and amplitudes, it suffices to know the cuts of leading and subleading color amplitudes of N=4 super-Yang-Mills theory. In this very special example we also see that the integrals which appear are the same as in $\mathcal{N}=4$ SYM. This implies that at 1-loop the UV behavior of N=4 supergravity is the same as that of N=4SYM.

It is of course a good question if this continues.

In connection to this, let me quote a result without arguing for them: “all 1-loop amplitudes in N=8 supergravity are expressible in terms of box integrals with no numerator factors”. This is known as the no-triangle behavior; while correct, it is currently not related to supersymmetry. By unitarity, this implies that to all loop orders, there should be no triangle subamplitudes present. Such a behavior is certainly much better than what one might expect based on the fact that the theory is finite at 1-loop and therefore implies implies a series of all-loop cancellations not expected from a garden-variety supergravity theory.

In the same spirit, good UV behavior at some loop order implies that all subamplitudes of that loop order that appear in higher-loop amplitudes should behave the same way.

Going back to amplitudes – the currently known results are the 2- and the 3-loop 4-graviton amplitudes.

maybe draw the 2-loop answer

slide with the 3-loop answer

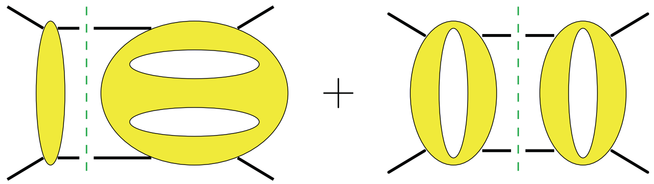
These results support the idea that the UV behavior of N=8 supergravity is the same as that of N=4 SYM. It should me noted that, while there is little room at 2-loops, the 3-loop result could have been finite in a different way – in particular 10 and not 12 factors of loop momenta turning into external momenta. In a sense, these last 2 factors of momenta may be understood as a reflection of the no-triangle behavior. This indeed suggests that something is going on and that the UV behavior of N=8 supergravity is better than implied by supersymetry. Finding out what it is is an important open problem.

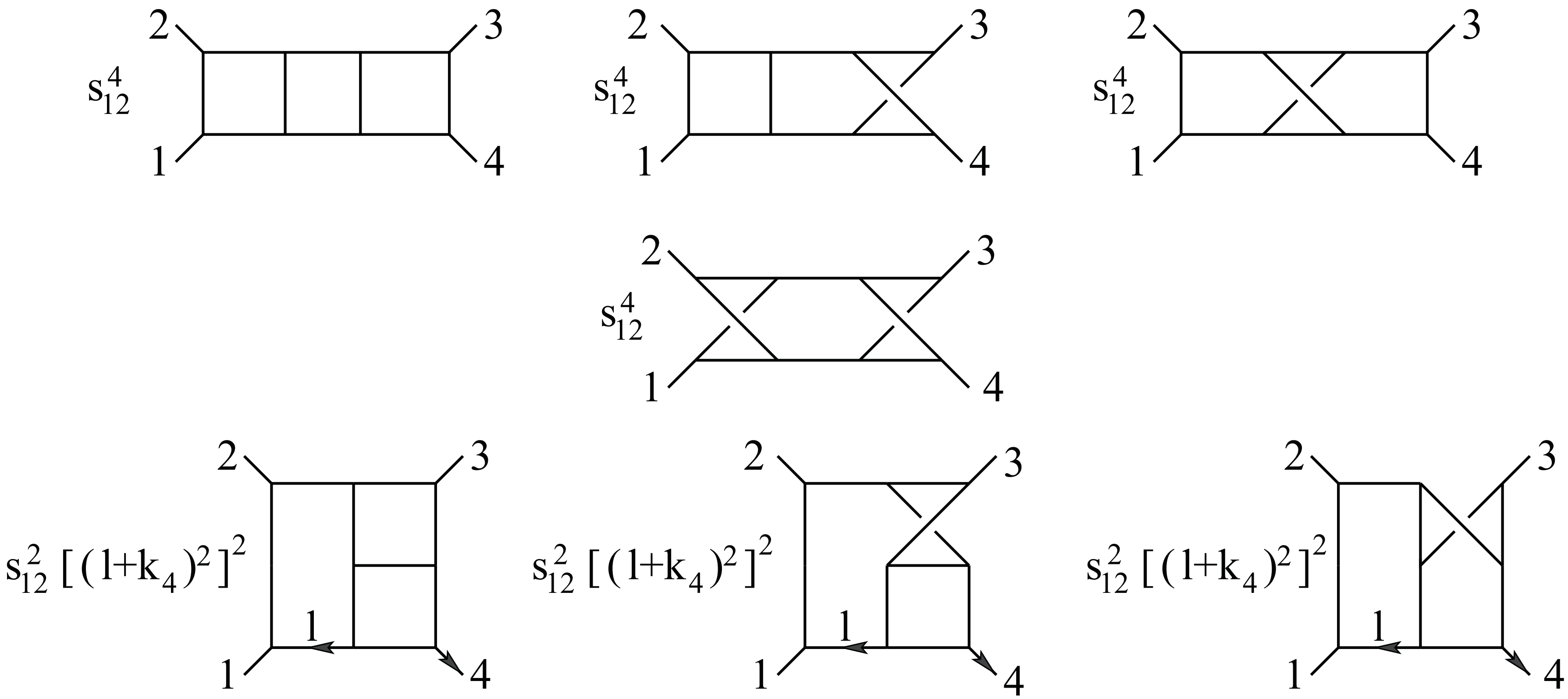
During my lectures I tried to introduce you to the exciting (and technical) world of amplitude calculations. To date these techniques are the most efficient there are; there are further variations which I did not discuss, due to lack of time, some of them will be discussed in J. Maldacena's lectures. I did not discuss the details except for a small number of examples; while other examples can be done in real time, they don't bring additional new ingredients to the story. Those that do are more complicated. As with all techniques there is always room for improvement, perhaps from some of you. Likewise, there are many open problems, some of which I have briefly mentioned during the lectures, which are awaiting a solution.

3-loops 4-graviton amplitude

Bern, Carrasco, Dixon, Johansson, Kosower, RR

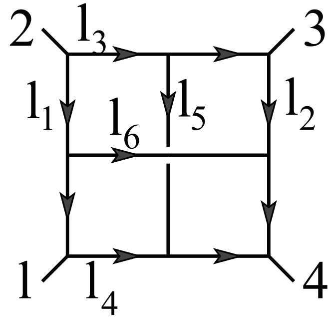
Need to analyze 2-, 3- and 4-particle cuts

- 2-particle cut constructible diagrams 

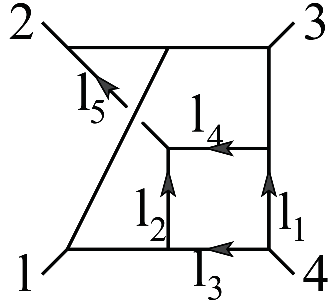


- additional $\mathcal{N}=4$ diagrams and numerator factors

$$s_{12}\left[(l_1+l_2)^2-l_5^2\right] \quad + \quad s_{23}\left[(l_3+l_4)^2-l_6^2\right]$$
$$-s_{12}s_{23}$$

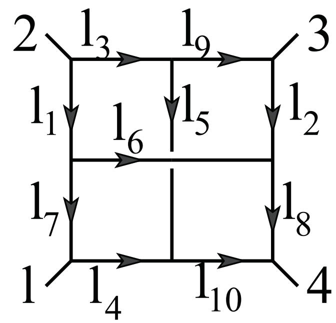


$$s_{12}(l_1+l_2)^2-s_{23}(l_3+l_4)^2+\frac{1}{3}\,(s_{12}-s_{23})\,l_5^2$$

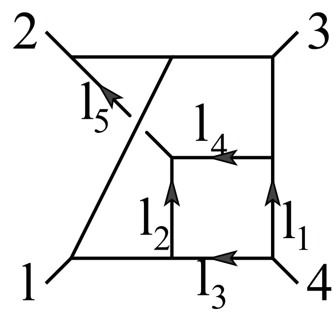


- additional $\mathcal{N}=8$ supergravity diagrams and numerator factors

$$\Big[s_{12}(l_1+l_2)^2+s_{23}(l_3+l_4)^2-s_{12}s_{23}\Big]^2$$
$$-s_{12}^2(2((l_1+l_2)^2-s_{23})+l_5^2)l_5^2$$
$$-s_{23}^2(2((l_3+l_4)^2-s_{12})+l_6^2)l_6^2$$
$$-s_{12}^2(2l_7^2l_2^2+2l_1^2l_8^2+l_2^2l_8^2+l_1^2l_7^2)$$
$$-s_{23}^2(2l_3^2l_{10}^2+2l_9^2l_4^2+l_{10}^2l_4^2+l_3^2l_9^2)$$
$$+2s_{12}s_{23}l_5^2l_6^2$$



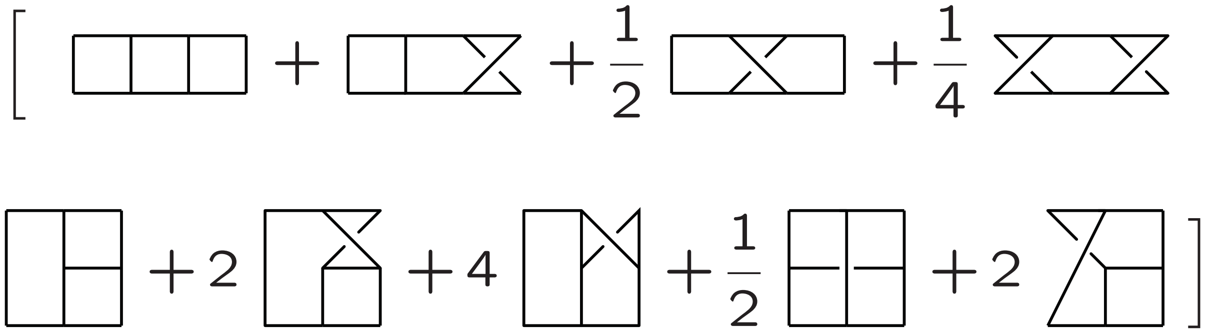
$$\Big(s_{12}(l_1+l_2)^2-s_{23}(l_3+l_4^2)^2\Big)^2$$
$$-\Big(s_{12}^2(l_1+l_2)^2+s_{23}^2(l_3+l_4^2)^2+\tfrac{1}{3}s_{12}s_{23}s_{13}\Big)\,l_5^2$$



- Note contact terms

◇ The complete 4-graviton amplitude at 3-loops

$$\mathcal{M}_4^{(3)} = \frac{\kappa^8}{2^8} stu \mathcal{M}_4^{tree} \sum_{S_3}$$



- It is possible to construct an integrand exhibiting all cancellations: each integral has at most two loop momenta in the numerator

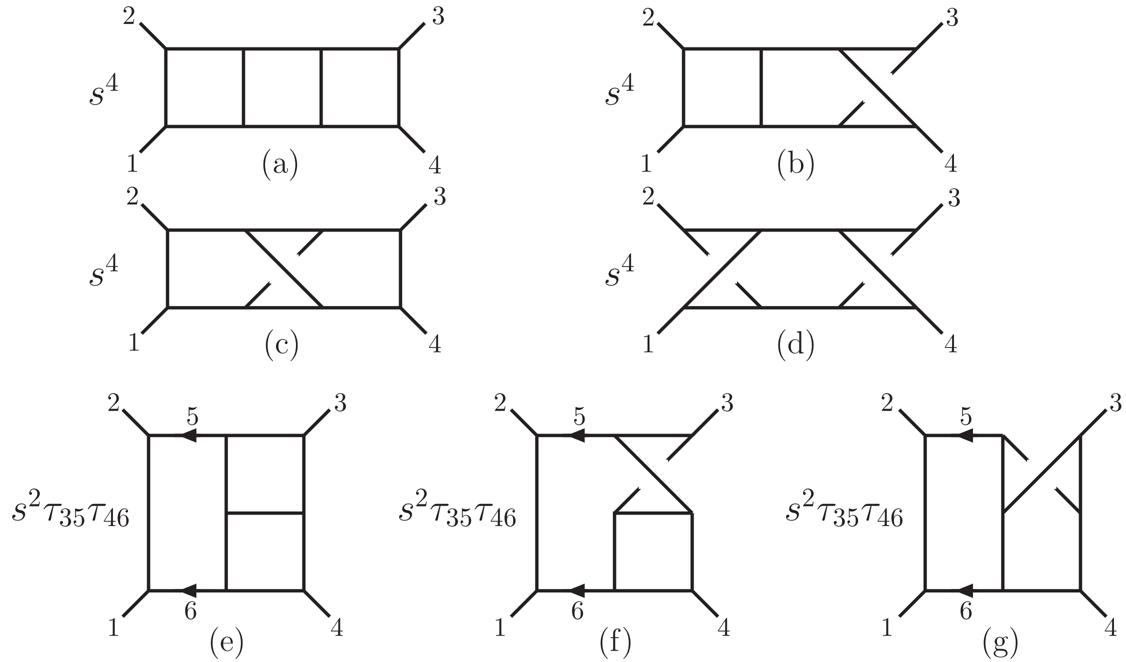
(almost) algebraically related to the previous integrand

fairly large ambiguity

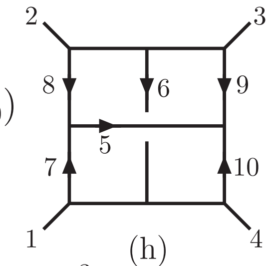
UV div. in $D=6$

definite transcendentality

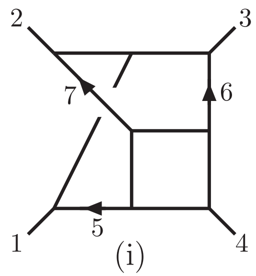
evaluate UV poles in $D=7,9,11$; coef's feed into the duality machine

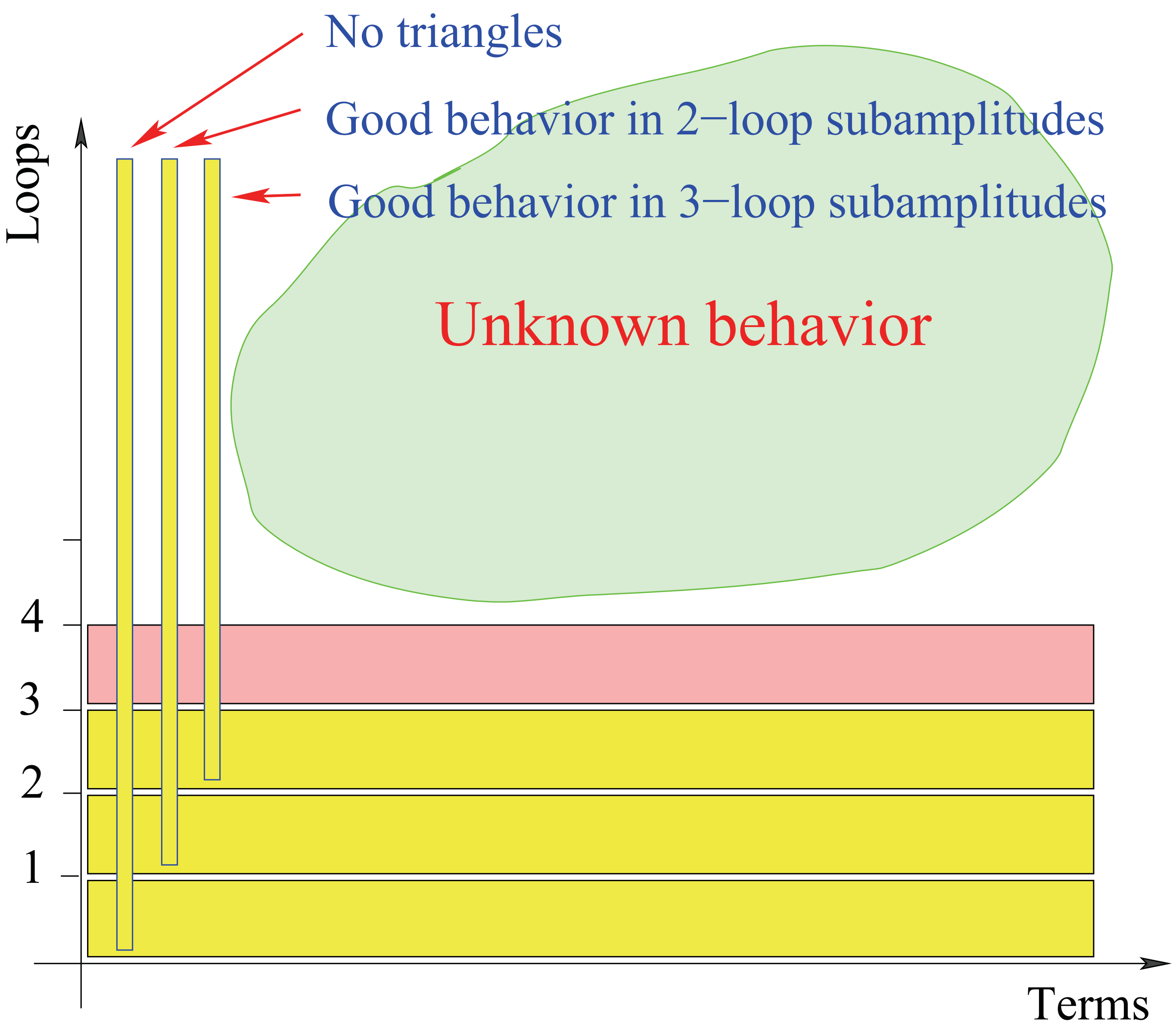


$$(s(\tau_{26}+\tau_{36})+t(\tau_{15}+\tau_{25})+st)^2$$
$$+(s^2(\tau_{26}+\tau_{36})-t^2(\tau_{15}+\tau_{25}))(\tau_{17}+\tau_{28}+\tau_{39}+\tau_{4,10})$$
$$+s^2(\tau_{17}\tau_{28}+\tau_{39}\tau_{4,10})+t^2(\tau_{28}\tau_{39}+\tau_{17}\tau_{4,10})$$
$$+u^2(\tau_{17}\tau_{39}+\tau_{28}\tau_{4,10})$$



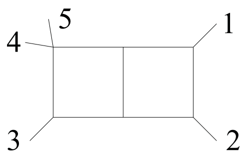
$$(s\tau_{45}-t\tau_{46})^2-\tau_{27}(s^2\tau_{45}+t^2\tau_{46})$$
$$-\tau_{15}(s^2\tau_{47}+u^2\tau_{46})-\tau_{36}(t^2\tau_{47}+u^2\tau_{45})$$
$$+l_5^2s^2t+l_6^2st^2-\tfrac{1}{3}l_7^2stu$$

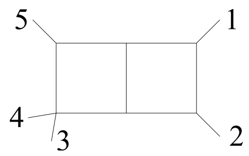


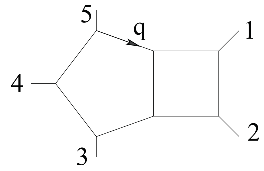

No triangles
Good behavior in 2−loop subamplitudes
Good behavior in 3−loop subamplitudes
Unknown behavior
Loops
4
3
2
1
Terms

5-gluon 2-loop amplitude in $\mathcal{N} = 5$ SYM Bern, Czakon, Kosower, RR, Smirnov

$$A_5^{2\,\text{loops;even}} = -\frac{1}{2} A_5^{tree} \sum_{\text{cyc}}$$

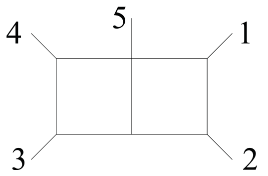
$$\Big\{ s_{12}^2 s_{23}$$ 

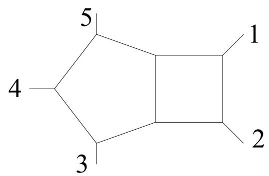
$$+ s_{12}^2 s_{15}$$ 

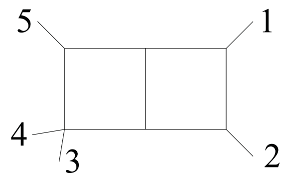
$$+ s_{12} s_{34} s_{45} (q - k_1)^2$$ 

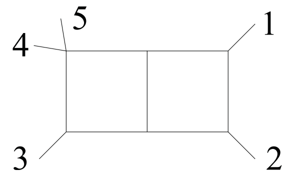
$$\Big\}$$

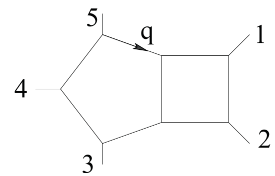
$$A_5^{2\,\text{loops;odd}} = \frac{1}{32} A_5^{\text{tree}} \text{Tr}\,(\gamma_5 \not{k}_1 \not{k}_2 \not{k}_3 \not{k}_4) \frac{s_{12} s_{23} s_{34} s_{45} s_{51}}{G(1,2,3,4)} \sum_{\text{cyc}}$$

$$\times \Big\{$$ 

$$+ 2 s_{12}$$ 

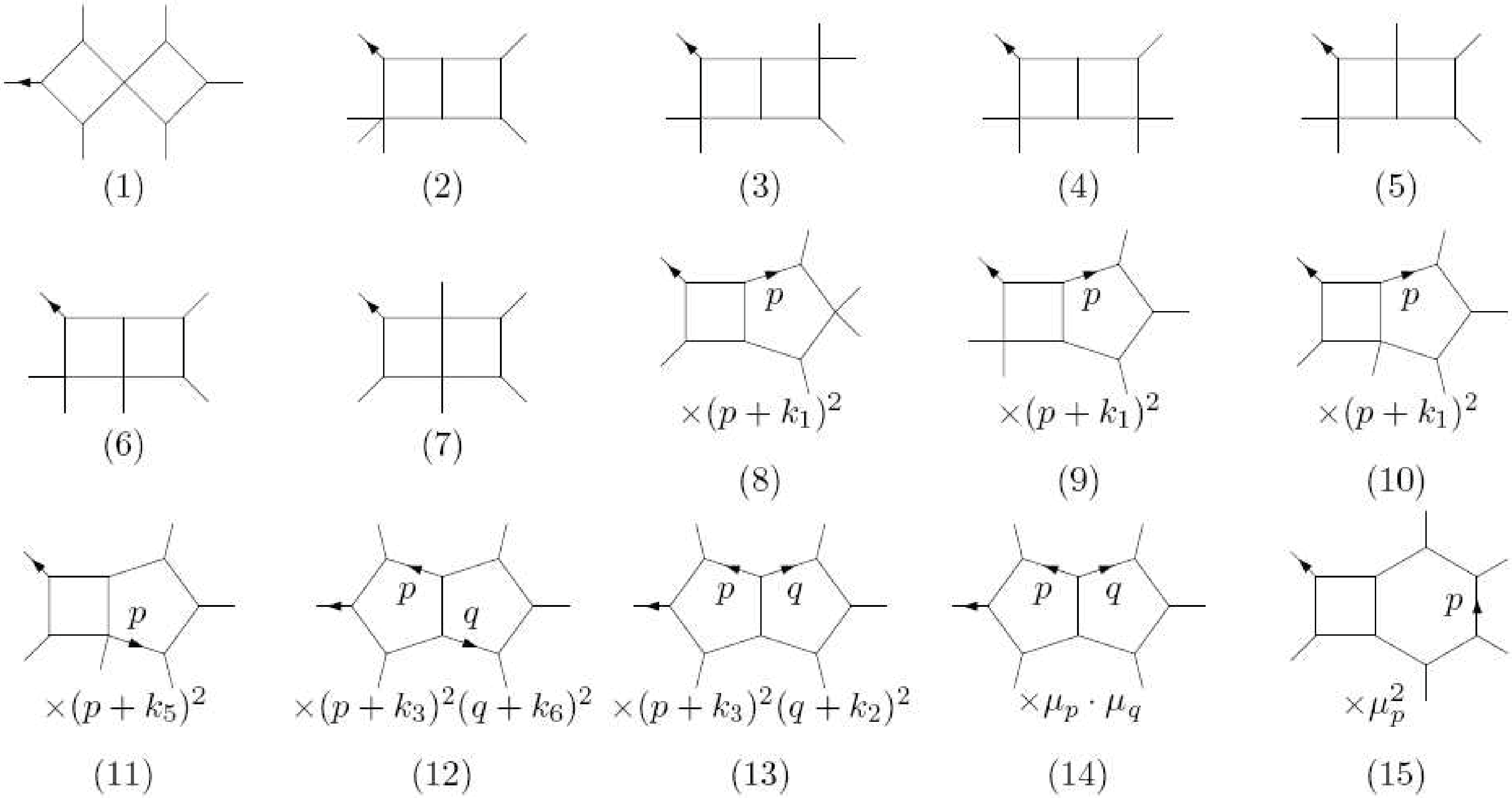
$$- \frac{s_{12}(s_{12}s_{15} - s_{12}s_{23} + s_{23}s_{34} - s_{15}s_{45} + s_{34}s_{45})}{s_{23}s_{34}s_{45}}$$ 

$$+ \frac{s_{12}(-s_{12}s_{51} + s_{12}s_{23} - s_{23}s_{34} + s_{45}s_{51} + s_{34}s_{45})}{s_{34}s_{45}s_{51}}$$ 

$$+ \frac{(s_{12}s_{51} + s_{12}s_{23} - s_{23}s_{34} + s_{45}s_{34} - s_{45}s_{51})}{s_{23}s_{51}} (q + k_1)^2$$ 

$$\Big\}$$

Integrals entering the 6-point amplitude at 2-loops:



$$
\begin{aligned}
M_6^{(2),D=4}(\epsilon) \;=\; \frac{1}{16}\sum_{12\text{ perms.}} \Bigg[& \frac{1}{4}c_1 I^{(1)}(\epsilon) + c_2 I^{(2)}(\epsilon) + \frac{1}{2}c_3 I^{(3)}(\epsilon) + \frac{1}{2}c_4 I^{(4)}(\epsilon) + c_5 I^{(5)}(\epsilon) \\
& + c_6 I^{(6)}(\epsilon) + \frac{1}{4}c_7 I^{(7)}(\epsilon) + \frac{1}{2}c_8 I^{(8)}(\epsilon) + c_9 I^{(9)}(\epsilon) \\
& + c_{10} I^{(10)}(\epsilon) + c_{11} I^{(11)}(\epsilon) + \frac{1}{2}c_{12} I^{(12)}(\epsilon) + \frac{1}{2}c_{13} I^{(13)}(\epsilon) \Bigg]
\end{aligned}
$$

- The coefficients:

$$
\begin{aligned}
c_1 &= \; s_{16}s_{34}s_{123}s_{345} + s_{12}s_{45}s_{234}s_{345} + s_{345}^2(s_{23}s_{56} - s_{123}s_{234}) \\
c_2 &= 2\, s_{12}s_{23}^2 \\
c_3 &= \; s_{234}(s_{123}s_{234} - s_{23}s_{56}) \\
c_4 &= \; s_{12}s_{234}^2 \\
c_5 &= \; s_{34}(s_{123}s_{234} - 2s_{23}s_{56}) \\
c_6 &= -s_{12}s_{23}s_{234} \\
c_7 &= 2\, s_{123}s_{234}s_{345} - 4s_{16}s_{34}s_{123} - s_{12}s_{45}s_{234} - s_{23}s_{56}s_{345} \\
c_8 &= 2\, s_{16}(s_{234}s_{345} - s_{16}s_{34}) \\
c_9 &= \; s_{23}s_{34}s_{234} \\
c_{10} &= \; s_{23}(2s_{16}s_{34} - s_{234}s_{345}) \\
c_{11} &= \; s_{12}s_{23}s_{234} \\
c_{12} &= \; s_{345}(s_{234}s_{345} - s_{16}s_{34}) \\
c_{13} &= -s_{345}^2 s_{56} \\
c_{14} &= 2\, s_{126}(s_{123}s_{234}s_{345} - s_{16}s_{34}s_{123} - s_{12}s_{45}s_{234} - s_{23}s_{56}s_{345}) \\
c_{15} &= 2\, s_{16}\,(s_{123}s_{234}s_{345} - s_{16}s_{34}s_{123} - s_{12}s_{45}s_{234} - s_{23}s_{56}s_{345})
\end{aligned}
$$