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Holographic Superconductors and Superfluids
Lecture 1

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Trieste Spring School Lectures: Holographic Superfluidity and Superconductivity

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Four lectures on holography and the AdS/CFT correspondence applied to condensed matter systems.

I. INTRODUCTION: QUANTUM PHASE TRANSITIONS

Spurred by the concrete proposal of refs. [1–3] for an AdS/CFT correspondence, there are some good reasons why holographic ideas have become so important in high energy theoretical physics over the last ten years. The first and perhaps most fundamental reason is that the AdS/CFT conjecture provides a definition of quantum gravity in a particular curved background space-time. Recall that the original conjecture posits an equivalence between type IIB string theory in the space-time $AdS_5 \times S^5$ and the maximally supersymmetric (SUSY) SU(N) Yang-Mills theory in 3+1 dimensions. (In our notation, AdS_5 is five dimensional anti-de Sitter space and S^5 is a five dimensional sphere.) Yang-Mills theory, at least in principle, can be simulated on a computer as the continuum limit of a lattice theory. Thus, the correspondence gives a definition of type IIB string theory in a fixed ten dimensional background. The low energy limit of type IIB string theory is type IIB supergravity and the correspondence must also yield a quantum theory of gravity. This line of reasoning has led to an improved understanding black hole physics, including a tentative resolution of the black hole information paradox and a better understanding of black hole entropy. But I am not concerned with this line of inquiry in these lecture notes.

Another reason for interest in the AdS/CFT correspondence is that it provides a tool for studying strongly interacting field theories. To see why, recall that the interaction strength of maximally SUSY Yang-Mills theory is described by the 't Hooft coupling $\lambda = g_{\rm YM}^2 N$. Through the AdS/CFT correspondence $\lambda = (L/\ell_s)^4$ where L is the radius of curvature of AdS_5 (and the S^5), and ℓ_s is a length scale that sets the tension of the type IIB strings.[1] Strings are also characterized by a coupling constant, g_s, that describes their likelihood to break. The AdS/CFT dictionary relates the string coupling constant to the gauge theory coupling via $4\pi g_{\rm YM}^2 = g_s$. In the double scaling limit where $N \to \infty$ while λ is kept large and fixed, string theory is well approximated by classical gravity. Keeping ℓ_s small means string theory is well approximated by gravity, while keeping g_s small eliminates quantum effects. Using the AdS/CFT correspondence, enormous progress has been made over the last ten years in understanding the large N, large λ limit of maximally SUSY Yang-Mills theory in 3+1 dimensions and its cousins.

One of the most interesting ideas surrounding this flurry of activity mapping out the properties of strongly interacting SUSY field theories with gravitational duals is that we might learn something

[1] The tension is conventionally defined as $1/(2\pi\ell_s^2)$.

about quantum chromodynamics (QCD). At low energy scales, QCD is a quintessential example of a strongly interacting field theory. Consider temperatures slightly above the deconfinement transition, of the order of 200 MeV, where the baryons and mesons dissolve into a soup of strongly coupled quarks and gluons. Such a non-Abelian soup is probably not so qualitatively different from maximally SUSY Yang-Mills in the double scaling limit. Experiments at the relativistic heavy ion collider (RHIC) combined with hydrodynamic simulations suggest that the viscosity of the quark-gluon plasma is very low (see for example [7]). In contrast, perturbative QCD techniques yield a large viscosity [4], and lattice gauge theory requires a very difficult analytic continuation from the Euclidean theory to extract such a transport coefficient (see for example [5]). AdS/CFT yields, for maximally SUSY Yang-Mills (and indeed for all its cousins in this double scaling limit), the low value $\eta/s = \hbar/4\pi k_B$ for the viscosity to entropy density ratio [6], a number which is compatible, to date, with the RHIC experiment. But I will not be interested in applying holographic techniques to QCD in these lectures.

In these lectures, I will try to describe progress in applying holography and AdS/CFT to condensed matter systems. QCD is not the only useful strongly interacting field theory. Field theory has for a long time been a standard tool in a condensed matter theorist's toolbox. For example, near phase transitions, coherence lengths become long enough to allow a continuum description of a crystal lattice or otherwise discretized system of atoms and molecules. While finding a holographic dual for QCD may at best be like finding a needle in the haystack of generalized AdS/CFT correspondences and at worst impossible because of some fundamental mismatch in the requirements necessary for a field theory to have a gravity dual, the odds of a finding a gravity dual to a condensed matter system appear, at least superficially, to be better. There are hundreds of thousands of pre-existing materials to consider. Moreover, using nano-lithography, optical lattices, and other experimental techniques, we may be able to engineer a material with a gravity dual. This last possibility raises the tantalizing prospect of better understanding quantum gravity through material science or atomic physics.

An outline for the rest of these lectures is as follows:

- Using the notion of a quantum phase transition, in the rest of the first lecture I will frame the connection between condensed matter systems and holography in a useful and hopeful way.

- The second lecture is a discussion of old and doubtless well known results in field theory. I have devoted a whole lecture to these results for a few reasons. The first is that it is much easier to understand what extra information AdS/CFT is giving us if we first understand the limitations of field theory. The second is that while the first, third and fourth lectures may not stand the test of time, the contents of this second lecture are true and probably very useful in other contexts.

- In the third lecture, I will holographically compute field theory transport coefficients using a very simple gravitational action consisting of an Einstein-Hilbert and Maxwell term:

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g}(R - 2\Lambda) - \frac{1}{4g^2} \int d^4x \sqrt{-g} F_{\mu\nu} F^{\mu\nu} \ . \tag{1}$$

I will connect these holographic results, in a qualitative way, to measurements of transport coefficients in graphene and high temperature superconductors.

- In the last lecture, justifying the title of this lecture series, I will modify the gravitational action by adding an order parameter that will produce a superconducting or superfluid phase transition. I will focus on the case where the order parameter is a scalar, but one could introduce a vector order parameter as well by promoting the Abelian $F_{\mu\nu}$ in the action above to an SU(2) gauge field.

A. Quantum Phase Transitions

The notion of a quantum phase transition in condensed matter systems provides our motivation for using AdS/CFT. A quantum phase transition is a phase transition between different phases of matter at $T = 0$. Such transitions can only be accessed by varying a physical parameter, such as a magnetic field or pressure, at $T = 0$. They are driven by quantum fluctuations associated with the Heisenberg uncertainty principle rather than by thermal fluctuations. We will be concerned with second order quantum phase transitions in this lecture. Much of the discussion here is drawn from ref. [8].

At $T = 0$ but away from a quantum critical point, a system typically has an energy scale, Δ, perhaps associated with the energy difference between the ground and first excited state. Another important quantity is a coherence length, ξ, characterizing the length scale over which correlations in the system are lost. At the quantum critical point, we expect Δ to vanish and ξ to diverge, but not necessarily in the same way:

$$\Delta \sim (g - g_c)^{\nu z} , \tag{2}$$

$$\xi \sim (g - g_c)^{-\nu} . \tag{3}$$

The quantity z, relating the behavior $\Delta \sim \xi^{-z}$, is usually called the dynamical scaling exponent. At the quantum critical point, the system becomes invariant under the rescaling of time and distance, $t \to \lambda^z t$ and $x \to \lambda x$. Different z occur in different condensed matter systems. For example, $z = 1$ is common for spin systems, and we will see an example of such a system shortly. The case $z = 1$ is special because the quantum critical system typically has a Lorentz symmetry and the scaling is enhanced to a full conformal symmetry group. These lectures will focus mostly on the $z = 1$ case because it is here that the AdS/CFT dictionary is most powerful and well developed. Another common and familiar value is $z = 2$. The free Schrödinger equation is invariant under $z = 2$ scalings, but there are other examples as well, e.g. Lifshitz theories. Generic, non-integer z are possible.

Figure 1 shows a prototypical phase diagram for a system that undergoes a quantum phase transition. Here the physical parameter is a coupling g, and the quantum phase transition occurs at $g = g_c$ and $T = 0$. At low temperatures, we imagine the system is in one of two phases well characterized by some order parameter(s). The blue lines in the phase diagram could be classical thermal phase transitions or softer cross-overs, depending on the dimensionality and nature of the system. The region between the solid black lines is the quantum critical region (QCR).

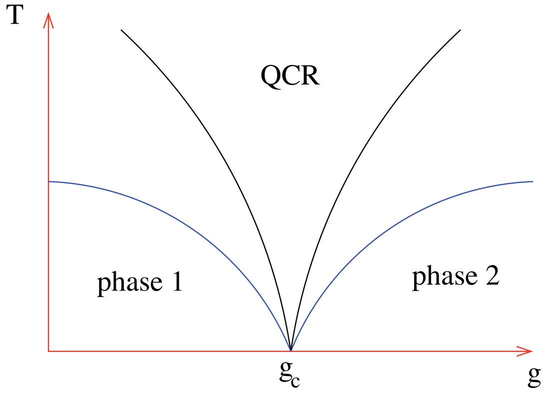

FIG. 1: A typical phase diagram involving a second order quantum critical point.

The usefulness of the notion of a quantum phase transition lies in a wished for ability to understand the system in the QCR. The QCR is characterized by the requirement that T be large compared to the dimensionally appropriate power of $(g-g_c)$. It seems reasonable to expect that the effective scale invariant field theory valid at the critical point, now generalized to nonzero T, can be used to predict the behavior of the system in the QCR. (We can generalize this discussion, replacing T with some other external parameter or set of parameters — chemical potential, magnetic field, etc.)

B. The Quantum Rotor

My next job is to convince you that the set of condensed matter systems with quantum critical points is not empty. Consider the quantum rotor with Hamiltonian:

$$H = gJ\sum_i \hat{L}_i^2 - J\sum_{\langle ij\rangle} \hat{n}_i \cdot \hat{n}_j \ , \tag{4}$$

where we are summing over a lattice indexed by i and where $\langle ij\rangle$ indicates a pair of nearest neighbor sites. Let $\hat{n}_i$ be an N component vector such that $\hat{n}_i^2 = 1$. The operator $\hat{L}_i$ is an angular momentum, and $\hat{L}_i^2$ is thus the kinetic energy term for this vector $\hat{n}_i$ which lives on an $N-1$ dimensional sphere. Taking $J > 0$, the interaction term in $\hat{H}$ will prefer to align the $\hat{n}_j$. The kinetic energy, in contrast, is minimized by randomizing the $\hat{n}_i$. (For more details about this model, the reader is referred to ref. [8] from which the following discussion is drawn.)

The quantum rotor is somewhat more than a toy model that exhibits a quantum phase transition as we tune the value of g. In the limit $g \gg 1$, the sites on the lattice decouple from one another, and the system can be solved exactly. In the ground state, the kinetic energy is minimized by taking $\langle \hat{n}_i \rangle = 0$ such that $\langle \hat{L}_i^2 \rangle = 0$. Correlations between different lattice sites die off exponentially with distance,

$$\langle 0|\hat{n}_i \cdot \hat{n}_j|0\rangle \sim e^{-|x_i - x_j|/\xi} , \tag{5}$$

where ξ is the correlation length. The lowest energy excitation is a particle where a single lattice site has a nonzero $\langle \hat{L}_i^2 \rangle$, and this particle hops from site to site. There is an energy gap $\Delta_+ \sim gJ$ associated with this particle. Because an external field will tend to align the $\hat{n}_i$, the ground state in this limit is a quantum paramagnet.

In contrast, in the opposite limit $g \ll 1$, the system becomes magnetically ordered. It is energetically favorable that $\langle \hat{n}_i \rangle \neq 0$ and for all of the vectors to align:

$$\lim_{|x_i - x_j| \to \infty} \langle 0|\hat{n}_i \cdot \hat{n}_j|0\rangle = N_0^2 . \tag{6}$$

There is a gapless continuum of excited states associated with spin waves, i.e. slow rotations in the direction of $\langle \hat{n}_i \rangle$. One can define an energy scale Δ_- associated with the kinetic term of these excitations — a spin stiffness.

Numerical experiments bear out that there is a quantum phase transition between these two different types of order for a critical value $g = g_c$ of the coupling. Referring to Figure 1, phase one for this model would be magnetically ordered while phase two is the quantum paramagnet. As we approach the critical point, it should be energetically easier for the spin waves to rotate more quickly in the magnetically ordered phase or for particle excitations to form in the paramagnetic phase. Thus, as g approaches g_c, we expect the energy scale $\Delta_\pm$ to vanish as a power of $(g - g_c)$. Also, as we move out of the paramagnetic phase, the correlation length ξ should diverge as the vectors $\hat{n}_i$ align.

To see why the quantum rotor is more than a toy, on the experimental side, for two spatial dimensions and $N = 3$, ref. [8] argues that the Hamiltonian models two sheets of La_2CuO_4, the parent compound of a high T_c superconductor I will discuss at greater length in a moment. On the theoretical side, the continuum limit of this model should be very familiar to field theorists. It is the O(N) nonlinear sigma model. The continuum Lagrangian takes the form

$$\mathcal{L} = \frac{1}{2\tilde{g}^2} \left(|\partial_t \vec{n}(x)|^2 - c_{\text{eff}}^2 |\nabla \vec{n}(x)|^2 \right) , \tag{7}$$

subject to the constraint $|\vec{n}(x)|^2 = 1$. It is thus a model with dynamical exponent $z = 1$, as was promised for spin systems. Note that for condensed matter applications, the effective speed of light would typically be much less than the actual speed of light, $c_{\text{eff}} \ll c$. This continuum limit should become a better and better approximation close to the quantum critical point where the correlation lengh ξ diverges and we can coarse grain the $\hat{n}_i$ degrees of freedom.

More loosely, we could soften the constraint and replace it with a scalar potential

$$V(\vec{n}) = \alpha |\vec{n}(x)|^2 + \beta |\vec{n}(x)|^4 , \tag{8}$$

to control the size of the fluctuations, yielding the O(N) vector model. At this point, I refer the reader to a standard field theory textbook such as ref. [12] for a more thorough treatment than I can provide here. In the renormalization group language, the O(N) vector model is known to flow to a strongly interacting Wilson-Fisher fixed point, which in our language is nothing other than a quantum critical point.

C. Quantum Critical Points in the Real World

Quantum phase transitions are believed to be important in describing superconducting-insulator transitions in thin metallic films, as is demonstrated pictorially by rotating Figure 2 ninety degrees counter-clockwise. The rotated diagram is meant to resemble closely Figure 1 where phase one is an insulator, phase two is a superconductor, and g corresponds to the thickness of the film. The insulating transition is a cross-over, while the superconducting transition might be of Kosterlitz-Thouless type. There exists a critical thickness for which the system reaches the quantum critical point at $T = 0$.

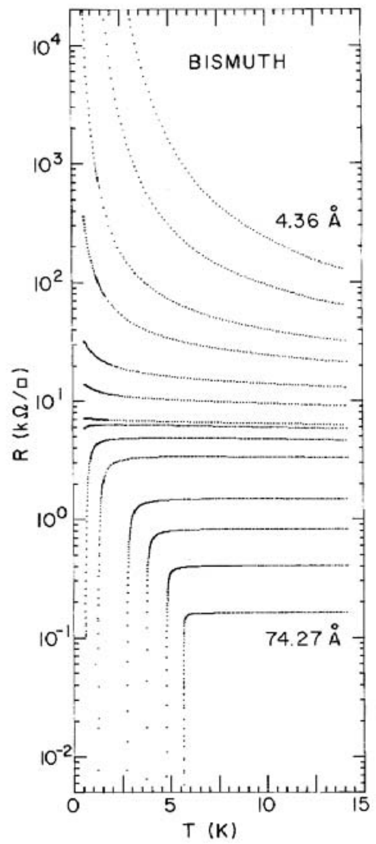

FIG. 2: Resistivity of thin films of bismuth versus temperature. The different curves correspond to different thicknesses, varying from a 4.36 Å film that becomes insulating at low temperatures, to a thicker 74.27 Å film that becomes superconducting. The figure is reproduced from ref. [9].

One of the most exciting (and also controversial) prospects for the experimental relevance of quantum phase transitions is high temperature superconductivity. Consider the parent compound La_2CuO_4 of one of the classic high T_c superconductors, $La_{2-x}Sr_xCuO_4$. La_2CuO_4 is actually not a superconductor at all but an anti-ferromagnetic insulator at low temperatures. The physics of this layered compound is essentially two dimensional. The copper atoms are arranged in a square lattice on separated sheets with effectively one electron per unit cell. The spins of the electrons pair up in an anti-ferromagnetic order.

To turn La_2CuO_4 into a superconductor, the compound can be doped with strontium which has the effect of removing one electron for every lanthanum atom replaced with strontium. Figure 3 is a phase diagram for $La_{2-x}Sr_xCuO_4$. Once the doping x becomes sufficiently large, the compound superconducts at low temperature. Introducing some vocabulary, the doping which yields the highest T_c is called the optimal doping; for this material, $x_o \approx 3/20$ yielding a $T_c \approx 40$ K. When $x > x_o$, the compound is referred to as overdoped, while when $x < x_o$, the compound is called underdoped.

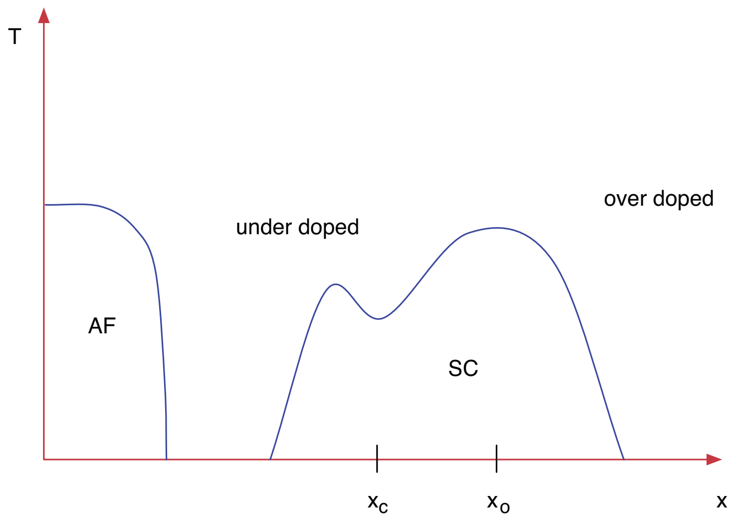

FIG. 3: A cartoon phase diagram for a superconductor such as $La_{2-x}Sr_xCuO_4$. AF stands for anti-ferromagnetic and SC for superconducting.

Overdoped high T_c superconductors are better understood than their underdoped counterparts. For temperatures $T > T_c$, the material behaves like a Fermi liquid where quasiparticle, electron-like degrees of freedom are effectively weakly coupled. Moreover, the phase transition seems to follow the BCS paradigm where the electrons form Cooper pairs as we lower the temperature below T_c. In contrast, in the underdoped region, the effective degrees of freedom are believed to be strongly interacting. The superconducting to normal phase transition is likely to involve disordering the phase of the condensate rather than breaking Cooper pairs, if it indeed makes sense to talk about quasiparticles at all in this regime. Because the electrons may remain in bound states in the normal, under doped region of the phase diagram, this region is sometimes called the pseudogap. For more details about these issues, the reader might try ref. [10].

Speculations about the relevance of a quantum phase transition are related to the dip in T_c at a doping of $x_c = 1/8$ and a possible connection between this dip in T_c and experimental evidence for so-called striped phases where spin and charge density waves break translational invariance [11]. The conjecture is that we can add a third axis to our phase diagram corresponding to an extra control parameter g in some model Lagrangian for the system, as pictured in Figure 4. In this figure, the chemical potential μ plays the role of doping. In the third direction, we may find a quantum critical point where the dip in T_c becomes more pronounced and reaches the $T = 0$ plane. One might hope to gain theoretical control over the pseudogap region using the effective field theory of the quantum critical point.

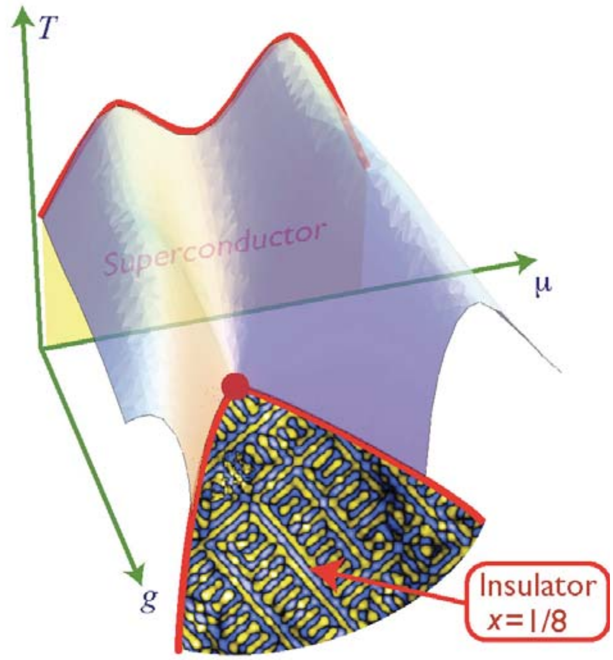

FIG. 4: A third conjectural axis has been added to our phase diagram for a high T_c superconductor. This figure was taken from ref. [11].

The reader may ask why we have invented this extra quantum critical point when, by our definition, there appear to be three perfectly good quantum critical points already present in the phase diagram of Figure 3. The answer is ultimately unsatisfactory and is indicative of the speculative nature of the last few paragraphs. Not all quantum phase transitions are created equal, and this putative fourth quantum critical point promises to be a little simpler and cleaner. The transition from the normal phase to the superconducting phase in the under doped region, for instance, is believed to be disorder driven and thus involves breaking translation invariance.[2]

D. On the Role of AdS/CFT

Thus far, I have tried to argue that quantum phase transitions are important in understanding superconducting insulator transitions in thin films and may be important in the physics of high T_c superconductors. I also exhibited the quantum rotor, a model theoretical system which undergoes such a phase transition.

[2] I would like to thank Markus Mueller for discussion on this point.

In general, it can be difficult to describe a system at a strongly interacting quantum critical point. Weakly coupled effective degrees of freedom may be difficult to identify or not exist, as I sketched for the case of high T_c superconductors. The reader may ask, "Can't we always discretize the system and simulate it on a computer?" To avoid problems with oscillatory numerical integrals, lattice models are almost always formulated for computers in Euclidean time. For questions about equilibrium physics (with no chemical potential), the answer is often, "Yes, the lattice is good enough." However, if we want to ask questions about physics at nonzero density, about real time physics, about transport coefficients and response to perturbations, numerical lattice models require tricky and usually untrustworthy analytic continuations.

Returning to the role of holography in this story, AdS/CFT provides a tool to study a class of strongly interacting field theories with Lorentz symmetry in d space-time dimensions by mapping them to classical gravity in $d+1$ space-time dimensions. The correspondence is a very useful way of working out the equation of state, real time correlation functions and transport properties such as diffusion constants, conductivities, and viscosities. The ambitious program is to find an example of an AdS/CFT correspondence that describes a real world material. Less ambitiously, we may learn universal or semi-universal properties about a class of strongly interacting field theories. These are often field theories and questions for which AdS/CFT is our only calculational tool.

The reader may object that there could well be structural reasons why this program is doomed to fail. AdS/CFT correspondences typically involve some underlying supersymmetry while condensed matter systems do not. In mitigation, I note that introducing chemical potential and temperature breaks supersymmetry, that if we stay away from the $T = 0$ and $\mu = 0$ limits, the physics may not be so different whether the underlying theory is supersymmetric or not.

The reader may also complain that the restriction to $z = 1$ appears to be limiting given the many different types of scaling that appear in condensed matter. In response, I note there has been recent progress in extending AdS/CFT to $z \neq 1$. For example refs. [13, 14] have conjectured a gravity dual for a theory with Schrödinger symmetry, i.e. the symmetry group of the free Schödinger equation. Clearly, this group has $z = 2$. There are serious experimental reasons to understand strongly interacting systems with this symmetry group [15]. Consider a dilute gas of lithium-6 or potassium-40 atoms in an optical trap. The interaction strength between these fermionic atoms can be tuned with an external magnetic field. At a Feshbach resonance, the scattering length becomes larger than the system size, and these so-called fermions at unitarity obey an approximate Schrödinger symmetry. Although I will not address this question here, it is very interesting to ask whether AdS/CFT can say anything useful about these strongly interacting atomic systems.

Not all condensed matter systems with $z = 2$ have the Schrödinger symmetry. Another possibility is a Lifshitz scaling symmetry, i.e. the symmetry group of a Lagrangian of the form

$$\mathcal{L} = (\partial_t \phi)^2 - \kappa (\nabla^2 \phi)^2 \ . \tag{9}$$

Ref. [16] presents a proposal for a gravity dual for a strongly interacting field theory with such a symmetry. Despite these recent advances, we shall focus henceforth on the $z = 1$ case with Lorentz symmetry. It is this case for which the AdS/CFT dictionary is most detailed and reliable.

[1] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," Phys. Lett. B **428**, 105 (1998) [arXiv:hep-th/9802109].

[3] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253 (1998) [arXiv:hep-th/9802150].

[4] P. Arnold, G. D. Moore and L. G. Yaffe, "Transport coefficients in high temperature gauge theories: (I) Leading-log results," JHEP **0011**, 001 (2000) [arXiv:hep-ph/0010177].

[5] H. B. Meyer, "A calculation of the bulk viscosity in SU(3) gluodynamics," Phys. Rev. Lett. **100**, 162001 (2008) [arXiv:0710.3717 [hep-lat]].

[6] P. Kovtun, D. T. Son and A. O. Starinets, "Viscosity in strongly interacting quantum field theories from black hole physics," Phys. Rev. Lett. **94**, 111601 (2005) [arXiv:hep-th/0405231].

[7] M. Luzum and P. Romatschke, "Conformal Relativistic Viscous Hydrodynamics: Applications to RHIC results at $\sqrt{s_{NN}} = 200$ GeV," Phys. Rev. C **78**, 034915 (2008) [arXiv:0804.4015 [nucl-th]].

[8] S. Sachdev, "Quantum Phase Transitions," Cambridge University Press, Cambridge (1999).

[9] D. B. Haviland, Y. Liu, and A. M. Goldman, "Onset of superconductivity in the two-dimensional limit," Phys. Rev. Lett. **62**, 2180 (1989).

[10] E. W. Carlson, V. J. Emery, S. A. Kivelson, and D. Orgad, "Concepts in High Temperature Superconductivity," arXiv:cond-mat/0206217.

[11] S. Sachdev, "Quantum magnetism and criticality," Nature Physics **4**, 173 (2008), [arXiv:0711.3015 [cond-mat]].

[12] M. E. Peskin and D. V. Schroeder, "An Introduction to quantum field theory," Addison-Wesley (1995).

[13] D. T. Son, "Toward an AdS/cold atoms correspondence: a geometric realization of the Schroedinger symmetry," Phys. Rev. D **78**, 046003 (2008) [arXiv:0804.3972 [hep-th]].

[14] K. Balasubramanian and J. McGreevy, "Gravity duals for non-relativistic CFTs," Phys. Rev. Lett. **101**, 061601 (2008) [arXiv:0804.4053 [hep-th]].

[15] W. Ketterle, M. W. Zwierlein, "Making, probing and understanding ultracold Fermi gases," arXiv:0801.2500 [cond-mat.other].

[16] S. Kachru, X. Liu and M. Mulligan, "Gravity Duals of Lifshitz-like Fixed Points," Phys. Rev. D **78**, 106005 (2008) [arXiv:0808.1725 [hep-th]].

[17] L. Onsager, "Reciprocal Relations in Irreversible Processes, I," Phys. Rev. **37** (1931) 405.

[18] Y. Wang, L. Li, and N. P. Ong, "Nernst effect in high-T_c superconductors," Phys. Rev. B **73** 024510 (2006).

[19] S. A. Hartnoll and C. P. Herzog, "Ohm's Law at strong coupling: S duality and the cyclotron resonance," Phys. Rev. D (2007) [arXiv:0706.3228 [hep-th]].