The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA International Atomic Energy Agency

2024-9

Spring School on Superstring Theory and Related Topics

23 - 31 March 2009

Efficient calculation of scattering amplitudes in supersymmetric gauge and gravity theories
Lecture 1

R.S. Roiban

Pennsylvania State University
U.S.A.

Introduction and trees

• Some motivation

The perturbative approach to QFT has been around since the invention of QFTs, some 75 years ago. The more or less standard approach to doing such calculations has been through the use of Feynman rules for off-shell fields. While conceptually algorithmic, it is not extremely efficient for several reasons:

0) they are very general; as such, they cannot take advantage of special features of some specific theory

1) Symmetries of the theory, in particular local symmetries, are not manifest. They are recovered after all Feynman diagrams contributing to some process are summed up.

2) one repeats the same calculation many times; e.g. part of each Feynman diagram contributing to a 6-point amplitude also contributes to a 5-point amplitude. However, as one computes the 6-point amplitude one rarely makes use of simplifications already carried out for the 5-point one. One just does them al over again.

3) simplifications which appear only if external fields are on-shell – especially those that appear in the example above if all 5 particles are on-shell – are not always apparent

Evidence gathered over the years that pretty much all questions that are usually answered in terms of Feynman diagrams, with the exception of the calculation of off-shell Green's functions of fundamental fields which are usually not needed by themselves anyway, can be found also by dealing only with on-shell physical states at all times. The purpose of these lectures is to describe some of the methods that have been developed for the purpose of taking advantage of the on-shell simplifications. On-shell methods are in some sense many and relatively varied. Not all of them apply in all situations; in fact, this restriction is in some sense a source of strength, as when they apply they tend to be quite efficient. As always, there is quite likely room for improvement.

The plan would be: 1) notation and tree level; 2) 1-loop 3) higher loops The theory I will have in mind throughout this discussion is $\mathcal{N} = 4$ SYM. However, these techniques can be and have been extended to theories with less supersymmetry (not much to do at tree-level), theories with tree-level masses, YM theories with various kinds of matter.

In the 4th lecture I will describe how the same technology extends to perturbative calculations in supergravity theories, in particular $\mathcal{N} = 8$ supergravity.

To take it from the top:

As with the calculation of amplitudes through Feynman diagrams, having a good notation and a good organization of amplitudes is crucial. Here, spinor helicity and color ordering still rule the playground. The latter provides a way to separate the color part from the momentum part of amplitudes thus organizing the amplitude in terms of a smaller number of functions and permutations of their arguments. The former provides a way of capturing the physical polarizations of particles in a Lorentz invariant way.

The philosophy is always to look for the smallest object that has a meaning: for FD approach they are vertices, their meaning being related to Lagrangian. Here the smalest objects will be amplitudes with fixed helicities for external lines and also fixed color order.

Let us begin by reviewing these two very important ingredients.

◇ Spinor helicity (massless particles); organization of amplitudes; color ordering

The polarization vector of a gluon should obey the following constraints:

$$k_\mu \epsilon^\mu(k) = 0 \qquad \epsilon \sim \epsilon + \alpha k \quad ; \tag{1}$$

both constraints are consequences of gauge invariance. Thus, only two of the 4 components of ϵ describe physical transverse polarizations. The issue is how to extract them in a Lorentz invariant way.

In 4 dimensions, the masslessness of the fields implies that the momentum is not really the most basic quantity. Indeed, $0 = p^2$ may be written also as $\det p_\mu \bar\sigma$ which implies that $p_\mu \bar\sigma$ has rank 1 so it must be a direct product of two vectors

$$(p_\mu \bar\sigma^\mu)_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha} \quad . \tag{2}$$

This decomposition is clearly not unique: $\lambda_\alpha \sim S\lambda_\alpha \; \tilde\lambda_{\dot\alpha} \sim 1/S\tilde\lambda_{\dot\alpha}$

A meaning of λ_α and $\tilde\lambda_{\dot\alpha}$ may be identified by recalling a well-known identity that which is typically used when constructing cross-sections from scattering amplitudes:

$$(\not p + m) u_s(p) = 0 \qquad \rightarrow \qquad \sum_{s=\pm} u(p)_s \bar u_s(p) = -\not p + m \tag{3}$$

The massless version projected onto the chiral part reads:

$$(p_\mu \bar\sigma^\mu)_{\alpha\dot\alpha} = u(p)_\alpha \bar u(p)_{\dot\alpha} \tag{4}$$

leads to the identifications

$$\bar u(p)_{\dot\alpha} \equiv |p] \equiv \tilde\lambda_{\dot\alpha} \qquad u(p)_\alpha \equiv |p\rangle \equiv \lambda_\alpha \tag{5}$$

All Lorents invariants may be expressed in terms of Lorentz invariant product of spinors:

$$\langle pq\rangle = \epsilon^{ba} u(p)_a \, u(q)_b \qquad [pq] = \langle qp\rangle^* = \epsilon^{\dot a\dot b} \bar u(p)_{\dot a} \, \bar u(q)_{\dot b} \tag{6}$$

Here ϵ is the 2d Levi-Civita tensor.

$$2k_1 \cdot k_2 = [k_1 k_2]\langle k_2 k_1\rangle \quad . \tag{7}$$

Polarization vectors may be expressed in terms of spinors and the spinors associated to a null arbitrary vector

$$\epsilon^+_\mu(k,\xi) = \frac{\langle \xi|\gamma_\mu|k]}{\sqrt{2}\langle \xi k\rangle} \qquad \epsilon^+_{\alpha\dot\alpha}(k,\xi) = \sqrt{2}\,\frac{\xi_\alpha \tilde\lambda_{\dot\alpha}}{\langle \xi k\rangle}$$
$$\epsilon^-_\mu(k,\xi) = -\frac{[\xi|\gamma_\mu|k\rangle}{\sqrt{2}[\xi k]} \qquad \epsilon^-_{\alpha\dot\alpha}(k,\xi) = -\sqrt{2}\,\frac{\lambda_\alpha \tilde\xi_{\dot\alpha}}{[\xi k]}$$

where ξ is an arbitrary null vector $\xi_\mu \sigma^\mu_{\alpha\dot\alpha} = \xi_\alpha \tilde\xi_{\dot\alpha}$. This vector may be chosen independently for each of the external legs. This freedom allows one to find easily that certain amplitudes vanish at tree level.

1) all plus: each vertex brings at most one factor of momentum and tehre are at most $n-2$ vertices for n external legs. So 2 polarization vectors are necessarily multiplies; choose all ξ-s for $+$-helicity proportional

2) one-minus all-plus: same as above except that there are also terms with $\epsilon_+ \cdot \epsilon_- \propto \langle q_+ k_- \rangle [q_- k_+]$; so choose $q_+ = k_-$.

In non-susy theories this argument breaks down at loop level as more momenta become available. In susy theories however, susy Ward identities imply that these amplitudes continue to vanish:

Susy argument: act with susy on $\langle 0|\Lambda^+ g^\pm g^+ \dots g^+|0\rangle$; this matrix element vanishes on Lorentz invariance grounds.

$$\begin{aligned} 0 &= \langle 0|[Q(\eta(q), \Lambda^+ g^+ g^+ \dots g^+]|0\rangle \\ &= -\Gamma^-(q,p_1) A(g^+ g^+ \dots g^+) + \sum_i \Gamma^-(q,p_i) A(\Lambda^+ g^+ \dots \Lambda_i^+ g^+) \end{aligned} \tag{8}$$

$(\Gamma(p,q)^+ = \theta[q,p]$, $\Gamma(p,q)^- = \theta\langle q,p\rangle)$ Fermions have only helicity-conserving interactions, so all but first amplitudes vanish. The first must too.

$$\begin{aligned} 0 &= \langle 0|[Q(\eta(q), \Lambda^+ g^- g^+ \dots g^+]|0\rangle \\ &= -\Gamma^-(q,p_1) A(g^+ g^- \dots g^+) + \sum_i \Gamma^-(q,p_2) A(\Lambda^+ \Lambda^- g^+ \dots g^+) + \sum_i \Gamma^-(q,p_2) A(\Lambda^+ g^- g^+ \dots \Lambda^+ g^+) \end{aligned} \tag{9}$$

the terms under the sum vanish; then choose $q = p_1$.

$\diamond$ Color decomposition (Berends, Giele; Mangano, Parke, Xu; Bern, Kosower): Any planar amplitude can be written as

$$A = \sum_\rho \mathrm{Tr}[T^a_{\rho(1)} \dots T^a_{\rho(n)}] A(\rho(1) \dots \rho(n))$$

The factors $A(\rho(1)\dots\rho(n))$ are called partial amplitudes, color ordered amplitudes, color-stripped amplitudes. ρ is the set of permutations of $(1,\dots,n)$ which are not cyclic. This is equivalent to fixing 1 leg and summing over all permutations of the remaining legs.

Argument for color ordering: either color-ordered Feynman rules or string diagrams: **draw picture** with Riemann surface with holes and vertex operators inserted on the boundaries; describe multi-trace terms in the decomposition.

• Properties of amplitudes

$*$ cyclicity: this is a consequence of the cyclic symmetry of traces.

$$A(1,\dots n) = A(2,\dots,n,1)$$

$*$ reflection: this is a consequence of the fact that 3-point vertices pick up a sign under such a reflection and that an amplitude with n external legs has n 3-point vertices.

$$A(1,\dots n) = (-)^n A(n \dots 1)$$

$*$ photon decoupling: In a theory with only adjoint fields, the diagonal $U(1)$ does not interact with anyone. Thus, all amplitudes involving this field identically vanish.

$*$ collinear limit: $p_1 \to zp$ and $p_2 \to (1-z)p$ with $p^2 = 0$ (or more precisely: $z = \xi \cdot p_1/(\xi \cdot p)$)

$$A(1,2,3,\dots,n) \mapsto \sum_h \mathrm{Split}_h(z) A(k_{12}^{-h}, 3, \dots, n)$$

with $\mathrm{Split}_h(z)$ being independent of the original amplitude.

• Examples: MHV, NMHV

Due to the vanishing of the all-+ and one-− amplitudes, the simplest tree-level amplitude has 2 negative helicities. In susy theories this continues to be so at loop level as well.

$$A^{\rm tree}_{MHV} = i\frac{\langle ij\rangle^4}{\prod\langle i,i+1\rangle}\delta^{(4)}(\sum_i \lambda_i\tilde\lambda_i) \qquad A^{\rm tree}_{\overline{MHV}} = (-1)^n i\frac{[ij]^4}{\prod[i,i+1]}\delta^{(4)}(\sum_i \lambda_i\tilde\lambda_i)$$

For $\mathcal{N}=4$ SYM all fields may be pakaged into a superfield

$$\Phi = g_- + f_a\eta^a + s_{ab}\eta^a\eta^b + f_{abc}\eta^a\eta^b\eta^c + g_{+abcd}\eta^a\eta^b\eta^c\eta^d \quad ;$$

then, all amplitudes related to the all-gluon amplitude by supersymmetry can be packaged into a single super-expression from which one extracts the component amplitudes by multiplication with the appropriate wave functions and integration over all anticommuting directions:

$$\mathcal{A}^{\rm tree}_{n;MHV} = i\frac{1}{\prod\langle i,i+1\rangle}\delta^{(4)}(\sum_i \lambda_i\tilde\lambda_i)\delta^{(8)}(\sum_i \lambda_i\eta_i^a)$$

For maximal susy YM, at loop level, the MHV ampliudes are proportional to the tree amplitude.

$$A^{\rm L}_{MHV} = A^{\rm tree}_{MHV}\,\mathcal{M}_L(\text{invariants}) \quad .$$

This is a consequence of susy Ward identities. for 4-points one may argue to it based on the properties of the representations of 4d superconformal group $psu(2,2|4)$.

An unexpected symmetry – dual superconformal symmetry – of these amplitudes: First solve momentum conservation:

$$\lambda\otimes\tilde\lambda \equiv k_i = x_i - x_{i+1} \qquad \lambda_{i\alpha}\eta_i^a = (\theta_i)^a_\alpha - (\theta_{i+1})^a_\alpha$$

Then introduce transformations

$$(\lambda_i)_\alpha \mapsto (\lambda_i(x_i)^{-1})_\alpha \qquad (\lambda_i)^\alpha \mapsto ((x_i)^{-1}\lambda_i)^\alpha \qquad (\tilde\lambda_i)_{\dot\alpha} \mapsto ((x_{i+1})^{-1}\tilde\lambda_i)_\alpha \qquad (\tilde\lambda_i)^{\dot\alpha} \mapsto (\tilde\lambda_i(x_i)^{-1})^\alpha$$

$$(\theta_i^a)^\alpha \mapsto ((x_i)^{-1}\theta_i^a)^\alpha \qquad (\theta_i^a)_\alpha \mapsto (\theta_i^a(x_i)^{-1})_\alpha$$

$$\mathcal{A} = i\frac{1}{\prod\langle i,i+1\rangle}\,\frac{1}{n}\sum_{p=1}^{n}\delta^{(4)}(x_p - x_{p+n})\delta^{(8)}(\theta_p - \theta_{p+n})$$

$$\langle i i+1\rangle \mapsto \langle i x_i^{-1}x_{i+1}^{-1} i+1\rangle = \frac{1}{x_i^2 x_{i+1}^2}\langle i x_i x_{i+1} i+1\rangle = \frac{1}{x_i^2 x_{i+1}^2}\langle i i+1\rangle$$

where we used that $(x_{i,i+1})\lambda_i = 0$. Then, the delta functions have opposite weights; can be seen by integrating them and requiring that the result is invariant under the change of coordinates above.

• MHV rules and super-rules

It turns out that one may think of MHV amplitudes and superamplitudes as building blocks of other amplitudes. The idea is to sew MHV amplitudes together as if they were usual Feynman rules.

The rules are:

1) one uses $n-1$ vertices for an amplitude with n negative helicity gluons. Internal lines are not on-shell. One defines the corresponding spinors by introduting some fixed null direction and projecting on it the off-shell momentum:

$$\langle Pk_i\rangle = [\zeta|P|k_i\rangle = \zeta^{\dot{a}} P_{a\dot{a}} k_i^a \qquad |P\rangle[P| = P - \frac{P^2}{2\zeta\cdot P}\zeta \tag{10}$$

2) vertices are connected by a standard scalar Feynman propagator

$$\Delta(p) = \frac{i}{p^2} \quad ;$$

Not surprising since these are physical excitations

3) One sums over all possible diagrams with the prescribed number of vertices and prescribed order of external legs. The same diagram may in principle have different helicities assigned to internal legs.

(Examples: $(- - - + ... +)$**,** $(+ - + - + -)$

◇ independence of η; Lorentz invariance is restores when all diagrams are added up: similar to Feynman diagrams excet that there are fewer diagrams

There is some justification for this: the initial justification of CSW involved the twistor string, which I won't get into. Later is was justified based on YM theory in lc gauge. In that gauge there are 3- and 4-point vertices: $++-$, $--+$, and $++--$ By performing a nonlocal canonical transformation that kills $++-$ one generates all the MHV amplitudes as terms in the Lagrangian.

For superamplitudes the story is essentially the same. The modification is that the propagator picks up a factor which identifies the anticommuting coordinates corresponding to the internal leg in the 2 vertices – i.e. $\delta^{(4)}(\eta^a - \eta^{a\prime})$ – and besides multiplication one also integrates over the internal anticommuting coordinates. All in all:

$$\frac{i}{P^2}\int d^4\eta_P \mathcal{A}_L(\ldots, P^\flat, \eta_P)\mathcal{A}_L(-P^\flat, \eta_P, \ldots)$$

• **On-shell recursion relations:**

◇ **massless**

Key observation: from the standpoint of scattering amplitudes momenta are just parameters. The fact that they are real is only a consequence of the fact that eventually they are interpreted as momenta of particles. Thus, from the perspective of constructing a function which has the properties of amplitudes – which we discussed before – it is of course legal to treat momenta as complex; this may be interpreted as analytic continuation. The result is then analytically continued back to real momenta.

$$\begin{matrix} p_i \to p_i(z) = p_i + z\eta \\ p_j \to p_j(z) = p_j - z\eta \end{matrix} \quad \text{such that} \quad \begin{matrix} p_i + p_j = p_i(z) + p_j(z) \\ p_i(z)^2 = 0 = p_j(z)^2 \end{matrix} \qquad \eta = \lambda_i\tilde{\lambda}_j$$

- Amplitude and propagators: $\begin{cases} A_{1\ldots n} \mapsto A_{1\ldots n}(z) \\ P_{i,\ldots,i+k} \mapsto P_{i,\ldots,i+k}(z) \end{cases}$

– $A(0)$– original amplitude: extract as a contour integral:

$$A_{1\dots n} = \oint_{C_0} \frac{dz}{z} A_{1\dots n}(z) \qquad \textbf{complex plane drawing}$$

Integrate using poles outside the contour; for each pole–

$$\oint \frac{dz}{z} A_L(z) \frac{1}{P_L^2 - 2z\langle i|P_L|j]} A_R(z) = \frac{1}{P_L^2} A_L(z = \frac{P_L^2}{2\langle i|P_L|j]}) A_R(z = \frac{P_L^2}{2\langle i|P_L|j]}) \tag{11}$$

Example: $(- - - + ++)$

When do they work: it is necessary that the amplitudes vanish as $z \to \infty$. In general this needs not be the case. For gauge theories one may argue for this using the CSW rules and picking appropriate shifts. This, together with the fact that there is another derivation of the on-shell recursion, based on the expression of the 1-loop amplitudes, provides a good justification of the CSW rules.

Has been argued through the use of superspace on-shell rec rel that the dual superconformal symmetry observed for MHV exists for all tree amplitudes.

Drawbacks: not as recursive as they seem; when used to evaluate a higher-point amplitude, an amplitude is needed at some shifter momentum – not the one that is actually needed. Thus, it needs to be re-evaluated, unless the recursion is solved analytically. Solutions exist for split helicity, for split-but-one helicity and, more recently, for a supersymmetric version of these recursion relations.

⋄ **massive**

The on-shell recursion relations have been extended to theories with massive particles. The strategy is identical; the details are however different. In paticular, the momentum shifts are not as simple. Nonetheless, one still has modified propagators depending on z and one still pick up their poles:

$$\frac{1}{P^2_{l\dots j\dots l+m} + M^2_{l\dots m}} \mapsto \frac{1}{P^2_{l\dots j\dots l+m}(z) + M^2_{l\dots m}}$$

$$A \mapsto A(z) = \sum_{l,m,h} A_L^h(z) \frac{1}{P^2_{l\dots j\dots l+m}(z) + M^2_{l\dots m}} A_R^{-h}(z)$$

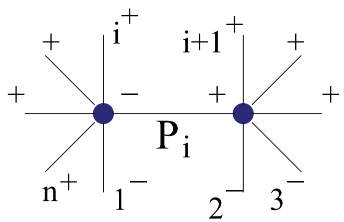
∗ Shifts: $\begin{array}{l} p_i \to p_i(z) = p_i + z\eta \\ p_j \to p_j(z) = p_j - z\eta \end{array} \quad \mapsto \quad \eta \cdot p_i = \eta \cdot p_j = \eta^2 = 0$

∗ simple solution if $m_i = 0$ or $m_j = 0$; otherwise complicated

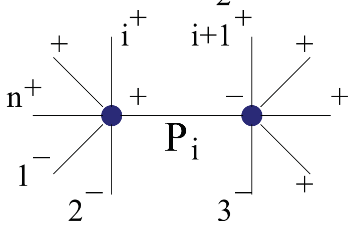
$A = \oint \frac{dz}{z} A(z)$ use instead poles at $z_{l\dots j\dots l+m} = -\frac{P^2_{l\dots j\dots l+m} + M^2_{l\dots m}}{2\eta \cdot P_{l\dots j\dots l+m}}$

Some examples have been worked out by Badger, Dixon, Glover, Khoze

Example: $A_n(---+\cdots+)$

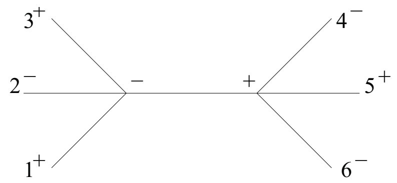


$$\sum_{i=3}^{n-1}\left[\frac{\langle 1P_i\rangle^3}{\langle P_i,i+1\rangle\langle i+1,i+2\rangle\ldots\langle n1\rangle}\right]\frac{1}{P_i^2}\left[\frac{\langle 23\rangle^3}{\langle P_i2\rangle\ldots\langle iP_i\rangle}\right]$$

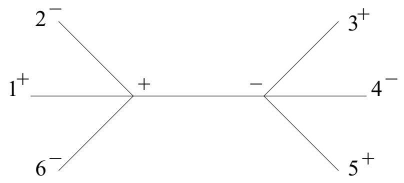


$$+\sum_{i=3}^{n-1}\left[\frac{\langle 12\rangle^3}{\langle 2P_i\rangle\langle P_i,i+1\rangle\ldots\langle n1\rangle}\right]\frac{1}{P_i^2}\left[\frac{\langle 34\rangle^3}{\langle P_i2\rangle\ldots\langle iP_i\rangle}\right]$$

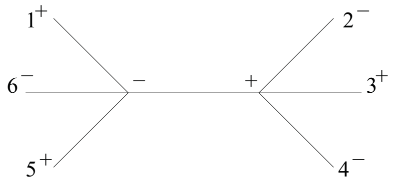
Another example: $A_6(+-+-+-)$ $\langle kP\rangle = \epsilon_{ab}\lambda_k^a P^{a\dot b}\tilde\zeta_{\dot b}$ – arbitrary $\tilde\zeta$



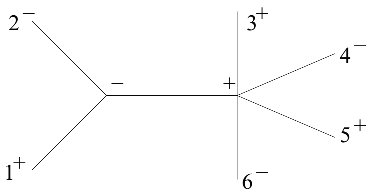
$$\frac{\langle 2p_{123}\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 3p_{123}\rangle\langle p_{123}1\rangle}\frac{1}{p_{123}^2}\frac{\langle 46\rangle^4}{\langle 45\rangle\langle 56\rangle\langle 6p_{123}\rangle\langle p_{123}4\rangle}$$



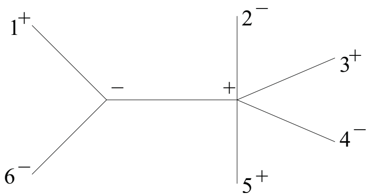
$$\frac{\langle 62\rangle^4}{\langle 61\rangle\langle 12\rangle\langle 2p_{612}\rangle\langle p_{612}6\rangle}\frac{1}{p_{612}^2}\frac{\langle 4p_{612}\rangle^4}{\langle 34\rangle\langle 45\rangle\langle 5p_{612}\rangle\langle p_{612}3\rangle}$$



$$\frac{\langle 6p_{561}\rangle^4}{\langle 56\rangle\langle 61\rangle\langle 1p_{561}\rangle\langle p_{561}5\rangle}\frac{1}{p_{561}^2}\frac{\langle 42\rangle^4}{\langle 23\rangle\langle 34\rangle\langle 4p_{561}\rangle\langle p_{561}2\rangle}$$



$$\frac{\langle 2p_{12}\rangle^3}{\langle p_{12}1\rangle\langle 12\rangle}\frac{1}{p_{12}^2}\frac{\langle 46\rangle^4}{\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 6p_{12}\rangle\langle p_{12}3\rangle} + 2\times(i\to i+2)$$



$$\frac{\langle p_{61}6\rangle^3}{\langle 61\rangle\langle 1p_{61}\rangle}\frac{1}{p_{61}^2}\frac{\langle 24\rangle^4}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 5p_{61}\rangle\langle p_{61}2\rangle} + 2\times(i\to i+2)$$

- Massless fields:

Key observation: momenta may be complex

$$p_i \to p_i(z) = p_i + z\eta \qquad p_j \to p_j(z) = p_j - z\eta$$

such that

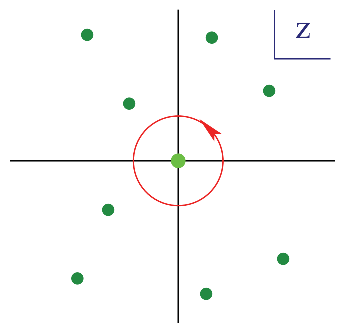
$$p_i + p_j = p_i(z) + p_j(z) \qquad p_i(z)^2 = 0 = p_j(z)^2$$

$$\eta = \lambda_i \tilde{\lambda}_j$$

- Amplitude and propagators: $A_{1\ldots n} \mapsto A_{1\ldots n}(z)$, $P_{i,\ldots,i+k} \mapsto P_{i,\ldots,i+k}(z)$

  – $A(0)$– original amplitude

$$\longrightarrow A_{1\ldots n} = \oint_{C_0} \frac{dz}{z} A_{1\ldots n}(z)$$

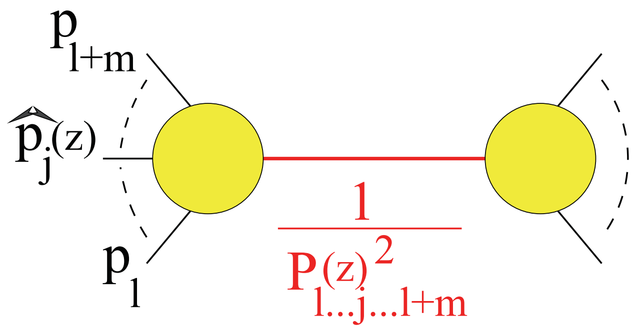


Properties:

- $A(z)$ is a rational function of z
- $A(z)$ has only simple poles in z

  – at $z = z_{lm}$ for which

  $$P_{l,\ldots,j,\ldots,l+m}(z_{lm})^2 = 0$$



- $\lim_{z\to\infty} A(z) = 0$ (nontrivial fact)

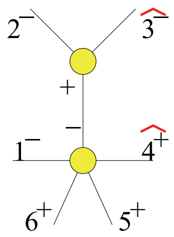
$\Rightarrow$ rotate contour: $A(z) = \sum_{lm} \frac{c_{lm}}{z - z_{lm}}$

• c_{lm} are products of amplitudes evaluated at $z = z_{lm}$

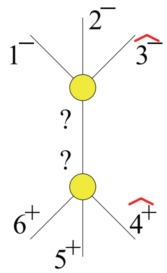
$$A = \sum_{l,m;h} A_L^h(z_{lm}) \frac{i}{P^2_{l\ldots j\ldots m}} A_R^{-h}(z_{lm}) \qquad z_{lm} = \frac{P^2_{l\ldots j\ldots m}}{2[j|P_{l\ldots j\ldots m}|i\rangle}$$

Example: split-helicity $(- - - + + +)$ amplitude

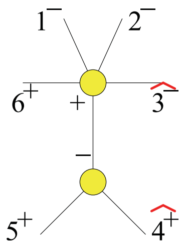
- determine z from on-shell condition of internal leg



$$\frac{\langle 2\hat{3}\rangle^3}{\langle\hat{3}\widehat{p_{23}}\rangle\langle\widehat{p_{23}}2\rangle}\frac{1}{p_{23}^2}\frac{\langle 1\widehat{p_{23}}\rangle^3}{\langle\widehat{p_{23}}\hat{4}\rangle\langle\hat{4}5\rangle\langle 56\rangle\langle 61\rangle} \qquad z=\frac{p_{23}^2}{\langle 4|P_{23}|3]}$$



$$0$$



$$\frac{[\widehat{p_{45}}6]^3}{[\widehat{p_{23}}6][61][12][2\hat{3}][\hat{3}\widehat{p_{23}}]}\frac{1}{p_{45}^2}\frac{[\hat{4}5]^3}{[5\widehat{p_{45}}][\widehat{p_{45}}\hat{4}]} \qquad z=\frac{p_{45}^2}{\langle 4|p_{45}|3]}$$

Just shake...

$$A_{1^-2^-3^-4^+5^+6^+}=\frac{1}{\langle 5|p_{34}|2]}\left(\frac{\langle 1|p_{23}|4]^3}{[23][34]\langle 56\rangle\langle 61\rangle p_{234}^2}+\frac{\langle 3|p_{45}|6]^3}{[61][12]\langle 34\rangle\langle 45\rangle p_{345}^2}\right)$$