The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA International Atomic Energy Agency

2024-10

Spring School on Superstring Theory and Related Topics

23 - 31 March 2009

Efficient calculation of scattering amplitudes in supersymmetric gauge and gravity theories
Lecture 2

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1-loop

To review a little some of the comments from last time, the use of Feynman rules at loop level is complicated (though of course possible) for a variety of reasons. The main one is the fact that symmetries in general and gauge invariance in particular are not enforces diagram by diagram. As one sums all diagrams one finds sometimes quite massive cancellations which lead to a restoration of gauge invariance. This relates to the fact that Feynman rules know about off-shell fields and thus of the unphysical degres of freedom. In most covariant gauges ghosts are necessary cancel them. Attempting to bypass this by going to a physical gauge complicates matters by breaking Lorentz invariance and introducing peculiar looking vertices and propagators. The other way – by using amplitudes as building blocks, is the story for today. In all fairness, Lorentz-violating gauges – such as lc gauge and space-cone gauge – have a distinguished role in the amplitude computing world.

We saw that MHV vertices may be used to construct non-MHV amplitudes as if they were Feynman rules. I mentioned, but I did not prove to you, that this may be understood in the sense that a certain canonical transfomation of the YM Lagrangian in lc gauge yields a Lagrangian in which amplitudes are vertices/terms in the Lagramgian. It is therefore reasonable to wonder whether the same MHV vertices may be used at loop level in the naivest possible way to construct loop amplitudes – that is, as if they were Feynman rules.

Even before trying , there are of course o course arguments why this cannot work:
– MHV vertices are nonlocal; unitarity will be messed up
– construction relies on being in $d = 4$; not clear how to regularize
– off-shell spinors were "invented"; potential problems?
– $i\epsilon$ prescription?

The correct rules have been guessed before the Lagrangian argument for MHV expansion existed.
– $i\epsilon$ taken the same as for the standard Feynman propagator;
– the off-shellness was dealt with in the spirit of dispersion relations:

$$P = P^\flat + z\zeta \qquad z = \frac{P^2}{2P\cdot\zeta} \tag{1}$$

where ζ is a fixed null vector and so if $P^\flat$. This decomposition is unique, up to choice of ζ; one may trade the integration over P for a phase space integral and an integral over z:

$$A = \int \tfrac{d^4L_1}{L_1^2}\tfrac{d^4L_2}{L_2^2}\delta^4(jL_1{+}L_2{+}p_{i+1,\dots,j})A_L(i{+}1,\dots,j,L_1,L_2)A_R(j{+}1,\dots,i,-L_2,-L_1)$$

$$\frac{d^4P}{P^2} = \frac{dz}{z+i\epsilon}(\langle P^\flat dP^\flat\rangle d^2\tilde{P}^\flat - [\tilde{P}^\flat d\tilde{P}^\flat]d^2P^\flat) \tag{2}$$

The z integral is similar to a dispersion integral. Dimensionally regularized the phase space integral.
– one sums over all possible states propagating in the loop

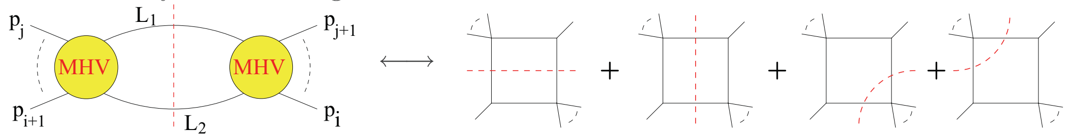
– one sums over all 1-loop diagrams with MHV vertices consistent with the helicity ordering

– this prescription yields vanishing all-+ and one-– amplitudes; it needs to be reanalyzed in absence of susy

Known answer: sum of 2me box integrals:

$$A^1/A^0 = \sum I^{2me}(i,r)$$

The MHV diagrams expose a different organization of the cuts of these these box integrals from the standard one – in the sense that one MHV diagram contributes to pieces of several box integrals. Each term of this in this expansion gives some combination of functions related to the discontinuity of box integrals in various channels:



• one feature of loop amplitudes in N=4 SYM and indeed in all theories with massless particles, IR divergences are contained in integrals. In this particular organization of contributions to amplitudes it is not hard to see that only diagrams with 4-point verices are divergent in the IR. Indeed, in the presence of such 4-point vertices, of one of the momenta of internal legs becomes collinear, then momentum conservation implies that the same is true for the other internal line momentum. Only these diagrams need regularization.

While this is fine, it also has a series of limitations which make us search for better techniques.

- $d = 4$ is built in because of the use of spinors and the way the action was constructed; while $d = 4$ calculations provide, in general, invaluable guidance, it is not clear how to do a $d \neq 4$ calculation in the same framework. This may affect 2-loop calculations.

- Because of this regularization is implemented at the level of the integrals rather than at the level of the action, which is potentially problematic a higher loops

- integrals which emerge do not look like Feynman integrals without additional work, so one cannot use immediately the fancy technology developed for integral calculations

Old unitarity: A solution to the $d = 4$ issue is provided by what one may call "old unitarity", i.e. the use of the optical theorem. This was used an intrinsic ingredient of the S-matrix theory, which suggested that the S-matrix can be constructed only on general principles of invariances and unitarity. The idea is to use

$$2\Im T = T^\dagger T \tag{3}$$

and relate the discontinuity of the S-matrix elements across cuts in various invariants and lower-loop amplitudes and then use complex analysis (dispersion integrals) to reconstruct the result. The lhs is to be interpreted as a discontinuity because of the $i\epsilon$ in the propagators:

$$\frac{1}{l^2 + i\epsilon} \mapsto -2\pi i \delta^{(+)}(l^2) \equiv -2\pi i \theta(l^0)\delta(l^2) \tag{4}$$

in the mostly $-$ metric!!

Multiplying lower-loop amplitudes and using these propagators leads to a phase space integral. To reconstruct the full amplitude one uses:

$$\Re f(s) = \frac{1}{\pi} P \int_{\infty}^{\infty} dw \frac{\Im f(w)}{w-s} + \Re C_{\infty} \tag{5}$$

The term coming from the contour at infinity vanishes if $f(w) \to 0$ as $w \to \infty$. If it does not, there are subtraction ambiguities related to terms which have no discontinuities.

Obs:

-$d \neq 4$ can be included in this framework; just use d-dimensional T

-however, as in the use of V vertices, evaluating phase space and dispersion integrals does not make use of recent sophisticated techniques for evaluating Feynman integrals: identities, modern reduction techniques, differential equations, reduction to master integrals, etc.

It is however possible to reinterpret the unitarity calculation such that it is no longer necessary to carry out phase space and dispersion integrals; this amonts to making a gedanken experiment using Feynman diagrams. Consider looking at the part of the amplitude which contains some prescribed set of propagators such that if they are cut the amplitude falls apart in at least 2 disconnected pieces. Since the full amplitude is a sum of all Feynman diagrams, the restriction to the part that contains some specified set of propagators isolates in each part one finds by cutting those propagators a sum of all Feynman diagrams with the external and the cut propagators sticking out. Thus each of the parts one finds is itself an amplitude.

example in ϕ^3 theory

This conclusion generalizes the well-known unitarity relation:

$$2\Im T = T^{\dagger} T \tag{6}$$

to cases when cutting some set of propagators does not have a unitarity interpretation. One may give, of course, this interpretation also to a standard unitarity cut. In this way one can identify pieces of a complete amplitude which have some prescribed set of propagators. A simple example is in order: 4-point 1-loop $N = 4$ SYM amplitude. Depending upon using 4 or s dimensional amplitudes one obtains partial or complete information. In practice, valuable guidamce comes from $d = 4$ calculations; additional terms are then constructed by comparing with d-dimensional calculation.

Let us discuss an example – 1-loop 4-point amplitudes. There are two types of cuts – singlet and nonsinglet (i.e. the complete multiplet of states can cross the cut). In the former type the cut prop's are gluon propagators. In the latter one has the whole multiplet crossing the cut. also, one has cuts in different channels.

$$\begin{aligned}
A(1^-, 2^-, l_1, l_2) A(-l_2, -l_1, 3^+, 4^+) &= -\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 2l_1\rangle\langle l_1 l_2\rangle\langle l_2 1\rangle} \frac{[34]^4}{[34][4l_2][l_2 l_1][l_1 3][34]} \\
= \quad -\frac{\langle 12\rangle[34]^2}{[21]} \frac{\mathrm{Tr}_+[l_1 2 1 l_2 4 3]}{(2k_2\cdot l_1)(2k_1\cdot l_2)(2k_4\cdot l_2)(2k_3\cdot l_1)} &= -\frac{\langle 12\rangle[34]^2}{[21]} \frac{\mathrm{Tr}_+[l_1 2 l_1 l_2 4 l_2]}{(2k_2\cdot l_1)(2k_1\cdot l_2)(2k_4\cdot l_2)(2k_3\cdot l_1)} \\
= \quad -\langle 12\rangle^2[34]^2 \frac{1}{(2k_1\cdot l_2)(2k_3\cdot l_1)} &= i s_{12} s_{23} A(1^- 2^- 3^+ 4^+) \frac{1}{(l_2+k_1)^2 (l_2-k_4)^2}
\end{aligned} \tag{7}$$

where in the propagator-like structures one recognizes the cut of a box integral. From here one may guess that the 1-loop amplitude is just $stA_{tree}I_0$; to make sure of this one should evaluate the cut in the other channel. This is a nonsinglet cut; it turns out that it confirms the guess.

The calculation of the non-singlet cut can be done in a variety of ways; the initial calculaiton was done using susy Ward id. to write out all amplitudes that can appear. A more systematic way, which also extends easily to higher loops, makes use of superspace. The observation is that the sum over states may be realized as integration over the anticummuting variables η:

$$\int d^4\eta_{l_1} d^4\eta_{l_2} \mathcal{A}(\ldots, l_1, \eta_{l_1}, l_2, \eta_{l_2}) \mathcal{A}(l_2, \eta_{l_2}, l_1, \eta_{l_1}, \ldots) \tag{8}$$

represents the sum over all possible assignments of fields to legs carrying momenta l_1 and l_2 which are consistent with both of the two amplitude factors being nonvanishing. Carrying out the integrals may be done in a variety of ways. If the two factors are MHV: use $\delta(a)\delta(b) = \delta(a+b)\delta(b)$ to obtain the overall supercharge conservation. For the remaining delta function notice that the integral is just the 4th power of the determinant of the matrix of coefficients of the system of 2 equations – $\langle l_1 l_2\rangle^4$. In the context of the example – it is however better to use the $\overline{\mathrm{MHV}}$ presentation of the 4-point amplitude.

Final answer:

$$A^{1\,\mathrm{loop}} = i s_{12} s_{23} A^{\mathrm{tree}} \mathrm{Box}(1,2,3,4) \quad . \tag{9}$$

As promissed, integral is IR divergent. Value of integral is

$$\frac{1}{s_{12}s_{23}} \mathrm{Box}(1,2,3,4) = -\frac{1}{\epsilon^2}(s_{12}^{-\epsilon} + s_{23}^{-\epsilon}) + \ln^2 \frac{s_{12}}{s_{23}} + \frac{\pi^2}{2}$$

All IR divergences are either of collinear type (loop momentum is parallel to external momentum) or of soft type (low energy). The double-log ($1/\epsilon^2$) singularity here is an overlap of both.

Things are a somewhat less straightforward if one increases the number of legs and especially if one goes beyond MHV amplitudes. This is because N^nMHV amplitudes are not terribly simple – despite the recent advances. One may use for example the MHV vertex expansion; in this case the sum over states – if cuts are non-singlets – reduces again to an overall Jacobian and one is also left with some more fermionic delta functions whose arguments depend only on external momenta. There are however many terms to analyze and it may be good to bypass that; clues in this direction come from the structure of the amplitude, in particular, from the integrals which may appear.

This has been known for a while: in 4 dimensions, any 1-loop amplitude may be expressed as a combination of bubble, triangle and box integrals with rational coefficients. The strategy is known as "integral reduction" and, in any context, it is a good strategy to simplify the types of integrals that one needs to compute. Here are a few operations which are often used:

-*- a dot product of the loop momentum and an external momentum in the numerator: one may complete squares and produce linear combinations of invese propagators.

-*- a tensor integral: in d dimensions, any vector is a linear combination of some d independent basis vectors. So one may decompose a loop momentum in terms of external momenta and dot products of loop and external momenta.

-*- scalar integral corresponding to more than boxes: at least 6 legs: construct b_i such that $\sum_i p_i^\mu = 0$ and $\sum_i b_i = 0$. Then, construct

$$1 = \frac{\sum_i b_i(p_i^2 + m^2)}{\sum_i b_i(p_i^2 + m^2)} = \frac{\sum_i b_i((l + p_i)^2 + m^2)}{\sum_i b_i(p_i^2 + m^2)}$$

and thus one cancels propagators.

For a pentagon one may use other techniques: find integrals that integrate to 0: e.g. Gram$[l, k1, k2, k3, k4]$ where l is the loop momentum.

-*- integration by parts: given any integral, multiplying the integrand by l^μ and taking d/dl^μ and integrating leads to 0. Then, taking the derivative leads to relations between the integral of interest and other ones – perhaps with fewer distinct propagators

Using such transformations one may reduce any Feynman diagram – and indeed any 1-loop amplitude – to a colection of bubble, triangle and box integrals. For $N = 4$ SYM it turns out that only box integrals appear. One loose argument is that momentum-dependent vertices contain at most 1-loop momentum. Thus, adding one extra leg to the 4-point we alredy computed and then reducing the result produces only boxes again. This is also consistent with the power-counting of the theory. Another argument with the same conclusion is that box integrals are sufficient to account for all factorization properties $N = 4$ amplitudes

Getting back to more practical things – like computing the coefficients of box integrals. for a box integral there are 4 propagators which may be cut.

(draw figure)

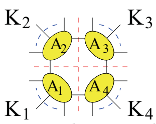
Our discussion before about the relation between cuts and Feynman diagrams implies that in each quadrant one has an on-shell amplitude. A slight subtlety is that sometimes one isolates a 3-point amplitude which vanishes for real momenta; this is so because of momentum conservation implies that the dot product of any 2 momenta vanishes which in real kinematics implies that both the angle and square spinor products vanish as well. At this stage, inspiration from twistor space rescues the day and suggests to go to $+ + --$ signature – i.e. complex momenta. Then, λ and $\tilde{\lambda}$ are no longer related so one may have $p_i \cdot p_j = 0$ and only one of the angle or square spinor products vanishing!

Anyway, cutting 4 props imposes 4 on-shell conditions so the loop momentum is completely determined. Since there is a single box integral which has the cut propagators we have on one side a product of tree amplitude – one for each corner of the box and on the rhs the coefficient of the box with the cut propagators times a Jacobian from doing the delta function integration. So the algorithm is:

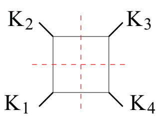
1) start with ansatz for $N = 4$:

$$A_n = \sum c_{i,n_1,n_2,n_3} I_{4m}[i, n_1, n_2, n_3, n_4]$$

2) Isolate one coefficient via the appropriate quadruple cut



$= c_{i,n_{K_1},n_{K_2},n_{K_3},n_{K_4}} \times$



3) compute the coefficient by multiplying the appropriate tree amplitudes (complex momenta are implicitely used if one encounters 3-point tree amplitudes)

– sum over different allowed helicity assignments for internal lines

$$c_{i,n_{K_1},n_{K_2},n_{K_3},n_{K_4}} = \frac{1}{\#\mathrm{sol}} \sum_{\mathrm{hel.s}} (A_1)(A_2)(A_3)(A_4)\Big|_{\mathrm{sol.\,to\,on-shell\,condition}}$$

The sum over helicities/intermediate states is perhaps best carried out in superspace.

– cancellation of Jacobian from integral of on-shell condition

– extract one coefficient at a time

– tree-level simplicity translates into 1-loop simplicity

(Not too many) tips for solving the on-shell condition

– if possible, find spinors

– solve conditions at 3-point corners (up to scale freedom)

– choose representation of tree amplitudes; expose loop momenta

– search for inconsistencies implied by these solutions

$\rightarrow$ vanishing contributions (or vanishing coefficients)

– turn holomorphic spinor into antiholomorphic spinor (or vice versa)

$\rightarrow\ \langle lX\rangle = \dfrac{[i|l|X\rangle}{[il]}$ for some external line i

– ratios of the type $\dfrac{\langle lX\rangle}{\langle lY\rangle}$ may sometimes be simplified

...

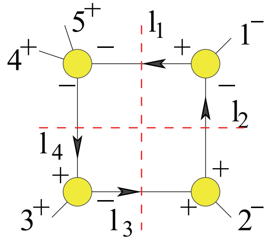
– If nothing works, reconstruct loop momenta; use explicit sol.

Examples

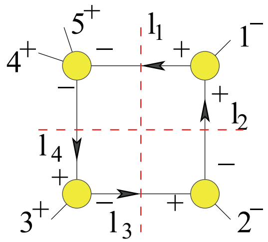
Using this strategy all 1-loop amplitudes may be found algebraically

The strategy can be somewhat improved by using super-amplitudes instead of component amplitudes. An advantage is that the internal state sum is done automatically. Following this it has been shown that dual inversion/conformal invariance/covariance is realized in all 1-loop amplitudes.

Example: 5-points MHV amplitude: five I_{1m} (incoming momenta)

$a)$ 

$$\left(\frac{\langle 1l_2\rangle^3}{\langle l_2l_1\rangle\langle l_11\rangle}\right)\left(\frac{[l_3l_2]^3}{[l_22][2l_3]}\right)\left(\frac{[3l_4]^3}{[l_4l_3][l_33]}\right)\left(\frac{\langle l_1l_4\rangle^3}{\langle l_44\rangle\langle 45\rangle\langle 5l_1\rangle}\right)$$

$b)$ 

$$\left(\frac{[l_2l_1]^3}{[1l_2][l_11]}\right)\left(\frac{\langle 2l_2\rangle^3}{\langle 2l_3\rangle\langle l_3l_2\rangle}\right)\left(\frac{[3l_4]^3}{[l_4l_3][l_33]}\right)\left(\frac{\langle l_1l_4\rangle^3}{\langle l_44\rangle\langle 45\rangle\langle 5l_1\rangle}\right)$$

$a)$ $[l_1l_2]=[l_11]=[l_21]=0 \qquad \tilde\lambda_{l_1}\propto\tilde\lambda_1\ ;\ \tilde\lambda_{l_2}\propto\tilde\lambda_1$

$\langle l_2l_3\rangle=\langle l_22\rangle=\langle 2l_3\rangle=0 \qquad \lambda_{l_2}\propto\lambda_2\ ;\ \lambda_{l_3}\propto\lambda_2$ inconsistent

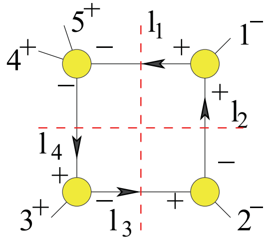
$\langle l_3l_4\rangle=\langle l_33\rangle=\langle 3l_4\rangle=0 \qquad \lambda_{l_3}\propto\lambda_3\ ;\ \lambda_{l_4}\propto\lambda_3$

$b)$ $\langle l_1l_2\rangle=\langle l_11\rangle=\langle l_21\rangle=0 \qquad \lambda_{l_1}\propto\lambda_1\ ;\ \lambda_{l_2}\propto\lambda_1$

$[l_2l_3]=[l_22]=[2l_3]=0 \qquad \tilde\lambda_{l_2}\propto\tilde\lambda_2\ ;\ \tilde\lambda_{l_3}\propto\tilde\lambda_2$ proceed

$\langle l_3l_4\rangle=\langle l_33\rangle=\langle 3l_4\rangle=0 \qquad \lambda_{l_3}\propto\lambda_3\ ;\ \lambda_{l_4}\propto\lambda_3$

Reorganize factors using momentum conservation:

$b)$ 

$$\left(\frac{[l_2 l_1]^3}{[1 l_2][l_1 1]}\right)\left(\frac{\langle 2 l_2\rangle^3}{\langle 2 l_3\rangle\langle l_3 l_2\rangle}\right)\left(\frac{[3 l_4]^3}{[l_4 l_3][l_3 3]}\right)\left(\frac{\langle l_1 l_4\rangle^3}{\langle l_4 4\rangle\langle 45\rangle\langle 5 l_1\rangle}\right)$$

$$\langle 2 l_2\rangle[l_2 l_1] = \langle 2|l_1+1|l_1] = \langle 21\rangle[1 l_1]$$

$$\langle l_3 l_2\rangle[l_2 1] = \langle l_3|l_3+2|1] = \langle l_3 2\rangle[21]$$

$$c_{123(45)} = \frac{1}{2}\frac{\langle 12\rangle^3[1 l_1]^2\langle l_1|(4+5)|3]^3}{[12]\langle 34\rangle\langle 45\rangle\langle 2 l_3\rangle^2[3 l_3]^2\langle 5 l_1\rangle}$$

$$\langle l_1 l_4\rangle[l_4 3] = \langle l_1|l_1-4-5|3] = -\langle l_1|(4+5)|3]$$

$$\langle 4 l_4\rangle[l_4 l_3] = \langle 4|l_3+3|l_3] = -\langle 43\rangle[3 l_3]$$

Last: leftover spinors $\leftrightarrow$ momentum conservation and constraints

$$\lambda_{l_1} = \alpha_1\lambda_1 \quad \tilde\lambda_{l_3} = \beta_2\tilde\lambda_2$$

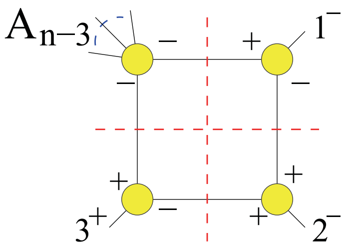
$$\lambda_{l_1}\tilde\lambda_{l_1} = \lambda_1\tilde\lambda_1 + \lambda_{l_2}\tilde\lambda_{l_2} \qquad \alpha_1\tilde\lambda_{l_1} = \tilde\lambda_1 + \beta_1\tilde\lambda_{l_2}$$

$$\lambda_{l_3}\tilde\lambda_{l_3} = \lambda_2\tilde\lambda_2 + \lambda_{l_2}\tilde\lambda_{l_2} \qquad \beta_2\lambda_{l_3} = \lambda_2 + \alpha_2\lambda_{l_2} = \lambda_2 + \alpha_2\beta_1\lambda_1$$

Homogeneity (no α and β): $c_{123(45)} = -\frac{1}{2}s_{12}s_{23}\left[\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\right]$

A general vanishing result:

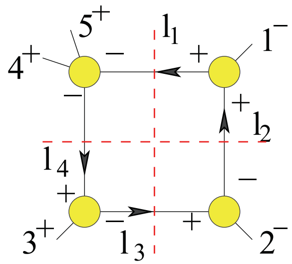
- The coefficient of a 1-mass box integral vanishes if there are two adjacent corners with the same helicity configuration



$$\left(\frac{\langle 1 l_2\rangle^3}{\langle l_2 l_1\rangle\langle l_1 1\rangle}\right)\left(\frac{[l_3 l_2]^3}{[l_2 2][2 l_3]}\right)\left(\frac{[3 l_4]^3}{[l_4 l_3][l_3 3]}\right) A_{n-3}$$

$$\begin{array}{rl} [l_1 l_2] = [l_1 1] = [l_2 1] = 0 & \qquad \tilde\lambda_{l_1} \propto \tilde\lambda_1 \ ; \ \tilde\lambda_{l_2} \propto \tilde\lambda_1 \\ \langle l_2 l_3\rangle = \langle l_2 2\rangle = \langle 2 l_3\rangle = 0 & \qquad \lambda_{l_2} \propto \lambda_2 \ ; \ \lambda_{l_3} \propto \lambda_2 \\ \langle l_3 l_4\rangle = \langle l_3 3\rangle = \langle 3 l_4\rangle = 0 & \qquad \lambda_{l_3} \propto \lambda_3 \ ; \ \lambda_{l_4} \propto \lambda_3 \end{array}$$

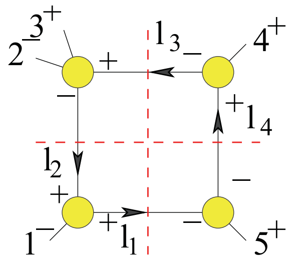
◇ cannot solve on-shell conditions for generic external momenta



$$c_{123(45)} = -\frac{1}{2} s_{12} s_{23} \left[\frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \right]$$

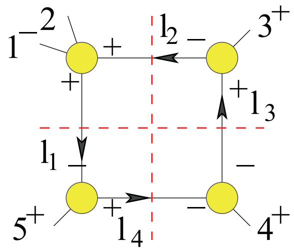
- $(12345) \longrightarrow (51234)$

Remaining nonvanishing coefficients



$$c_{451(23)} = -\frac{1}{2} s_{45} s_{51} \left[\frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \right]$$

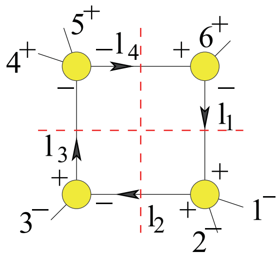
- $(12345) \longrightarrow (45123)$



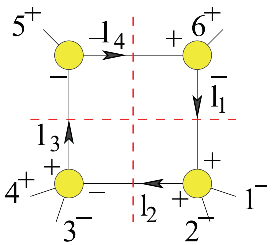
$$c_{345(12)} = -\frac{1}{2} s_{34} s_{45} \left[\frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \right]$$

Another example: (pieces of) 6-points NMHV amplitude

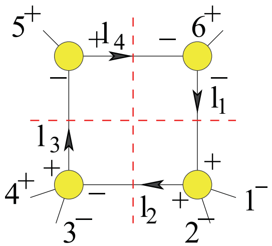
$6 \times I_{1m}$; $3 \times I_{2me}$; $6 \times I_{2mh}$



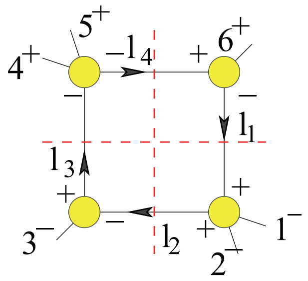
$$\left(\frac{\langle 12\rangle^3}{\langle 2l_2\rangle\langle l_2l_1\rangle\langle l_1 1\rangle}\right)\left(\frac{\langle l_2 3\rangle^3}{\langle 3l_3\rangle\langle l_3l_2\rangle}\right)\left(\frac{\langle l_4l_3\rangle^3}{\langle l_3 4\rangle\langle 45\rangle\langle 5l_4\rangle}\right)\left(\frac{[l_4 6]^3}{[6l_1][l_1l_4]}\right)$$



$$\left(\frac{\langle 12\rangle^3}{\langle 2l_2\rangle\langle l_2l_1\rangle\langle l_1 1\rangle}\right)\begin{bmatrix} \dfrac{\langle l_2 3\rangle^3}{\langle 34\rangle\langle 4l_3\rangle\langle l_3l_2\rangle} \\ \dfrac{[l_3l_2]^3}{[l_2 3][34][4l_3]} \end{bmatrix}\left(\frac{\langle l_4l_3\rangle^3}{\langle l_3 5\rangle\langle 5l_4\rangle}\right)\left(\frac{[l_4 6]^3}{[6l_1][l_1l_4]}\right)$$



$$\begin{bmatrix} \dfrac{\langle 12\rangle^3}{\langle 2l_2\rangle\langle l_2l_1\rangle\langle l_1 1\rangle} \\ \dfrac{[l_2l_1]^3}{[l_1 1][12][2l_2]} \end{bmatrix}\left(\frac{\langle l_2 3\rangle^3}{\langle 34\rangle\langle 4l_3\rangle\langle l_3l_2\rangle}\right)\left(\frac{[5l_4]^3}{[l_3 5][l_4l_3]}\right)\left(\frac{\langle l_1l_4\rangle^3}{\langle l_4 6\rangle\langle 6l_1\rangle}\right)$$



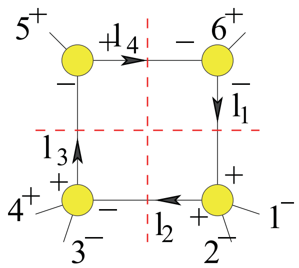
$$[l_2 3] = [3 l_3] = 0 \;\rightarrow\; \tilde\lambda_{l_2} = \alpha \tilde\lambda_3 \;;\quad \tilde\lambda_{l_2} = \beta \tilde\lambda_3$$

$$\langle l_4 6\rangle = \langle 6 l_1\rangle = 0 \;\rightarrow\; \lambda_{l_1} = \gamma \lambda_6 \;;\quad \lambda_{l_4} = \delta \lambda_6$$

$$0 = [3|(l_2 - l_1)|6\rangle = [3|(1+2)|6\rangle !!? - \text{ inconsistent}$$

$$\Longrightarrow c^{2me}_{(12)3(45)6} = 0$$

- Vanishing theorem for $c^{2me}_{(-\cdots-)-(+\cdots+)+}$



$$[l_4 6] = [6 l_1] = 0 \;\rightarrow\; \tilde\lambda_{l_4} = \alpha \tilde\lambda_6 \;;\quad \tilde\lambda_{l_1} = \beta \tilde\lambda_6$$

$$\langle l_3 5\rangle = \langle 5 l_4\rangle = 0 \;\rightarrow\; \lambda_{l_4} = \gamma \lambda_5 \;;\quad \lambda_{l_3} = \delta \lambda_5$$

$$l_4 = \alpha\gamma \lambda_5 \tilde\lambda_6$$

$$c^{2mh\ \mathrm{B}}_{(12)(34)56} \propto \frac{[5|l_4 l_1 l_2|3\rangle^3}{[12]\langle 34\rangle[1|l_1|6\rangle[2|l_2 l_3 l_4|6\rangle[5|l_3|4\rangle}$$

$$= \frac{[5|l_4 6(1+2)|3\rangle^3}{[12]\langle 34\rangle[1|l_4|6\rangle[2|(3+4)5 l_4|6\rangle[5|l_4|4\rangle}$$

$$= \frac{[56]\langle 56\rangle[6|(1+2)|3\rangle^3}{[12]\langle 34\rangle[16][2|(3+4)|5\rangle\langle 54\rangle}$$

Comments

- Algorithmic; yields any 1-loop amplitude in $\mathcal{N}=4$ SYM
    - the 4-mass box remains unpleasant
- Simplicity due to new structures: $[a|b\ldots c|d\rangle$
- IR equations feed this simplicity back to trees $\rightarrow$ rec. rel.
- Existing explicit results:
    - all MHV amplitudes — Bern, Dixon, Dunbar, Kosower
    - all $\leq$ 7-point amplitudes — Britto, Cachazo, Feng; Bern, del Duca, Dixon, Kosower
    - all split-helicity NMHV amplitudes — Bern, Dixon, Kosower
- other fields of $\mathcal{N}=4$ SYM on external lines