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GNSS Navigation Solution and Differential GNSS

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GNSS Navigation Solution and Differential GNSS

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Overview

- GPS Measurements (review)
- Determining the GPS navigation solution using least-squares
- Kalman filtering overview
- Inertial navigation systems and integration with GPS
- Differential GPS concepts and techniques
- Carrier-phase ambiguity resolution

GPS Receiver Measurements

What does the receiver measure?

GPS Measurements (Overview)

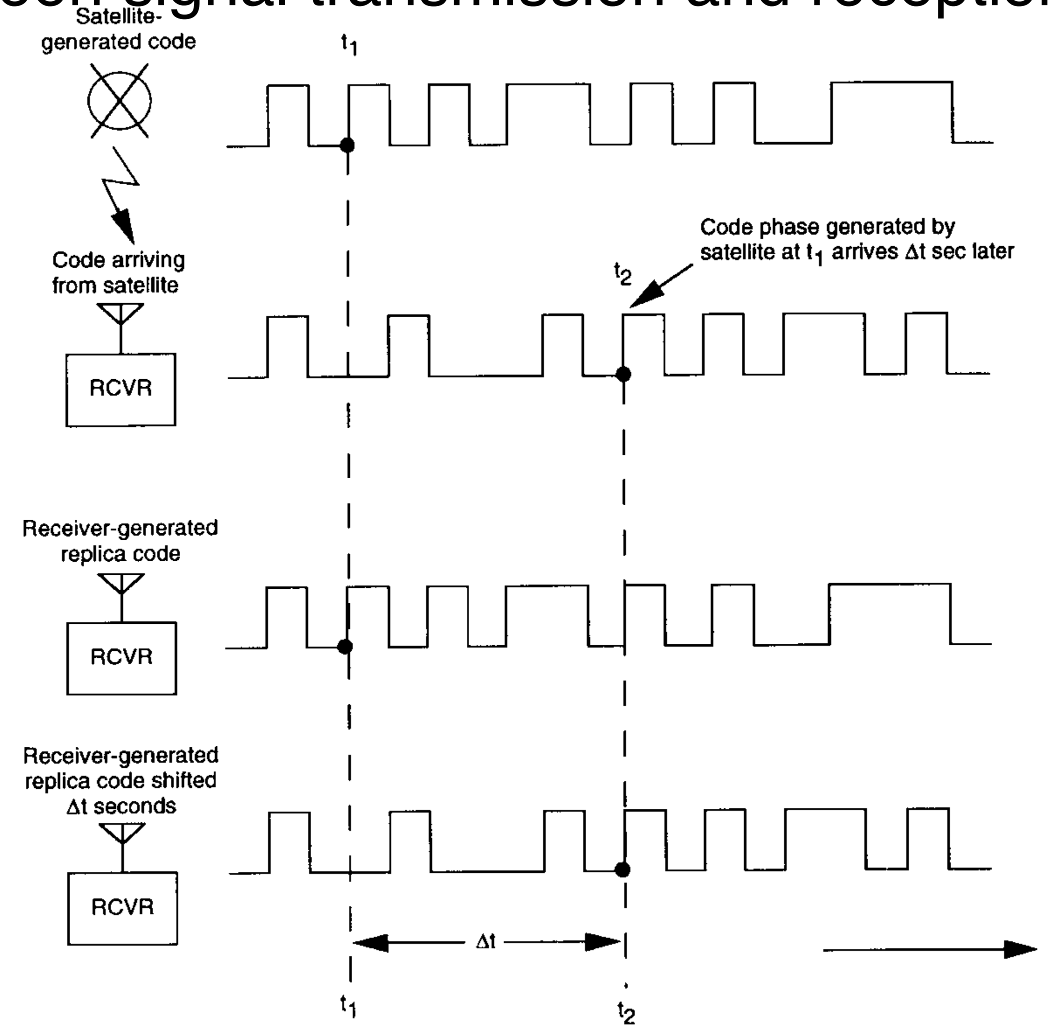
- Each separate tracking loop typically can give 4 different measurement outputs
  - Pseudorange measurement
  - Carrier-phase measurement (sometimes called integrated Doppler)
  - Doppler measurement
  - Carrier-to-noise density C/N_0
- Actual output varies depending upon receiver
  - Ashtech Z-surveyor (or Z-12) gives them all
  - RCVR-3A gives just C/N_0
- Note: We're talking here about *raw measurements*
  - Almost all receivers generate navigation processor outputs (position, velocity, heading, etc.)

Measurement Rates and Timing

- Most receivers take measurements on all channels/tracking loops simultaneously
  - Measurements time-tagged with the receiver clock (receiver time)
  - The time at which a set of measurements is made is called a data epoch.
- The data rate varies depending upon receiver/application. Typical data rates:
  - Static surveying: One measurement every 30 seconds (120 measurements per hour)
  - Typical air, land, and marine navigation: 0.5-2 measurement per second (most common)
  - Specialized high-dynamic applications: Up to 50 measurements per second (recent development)

GPS Pseudorange Measurement

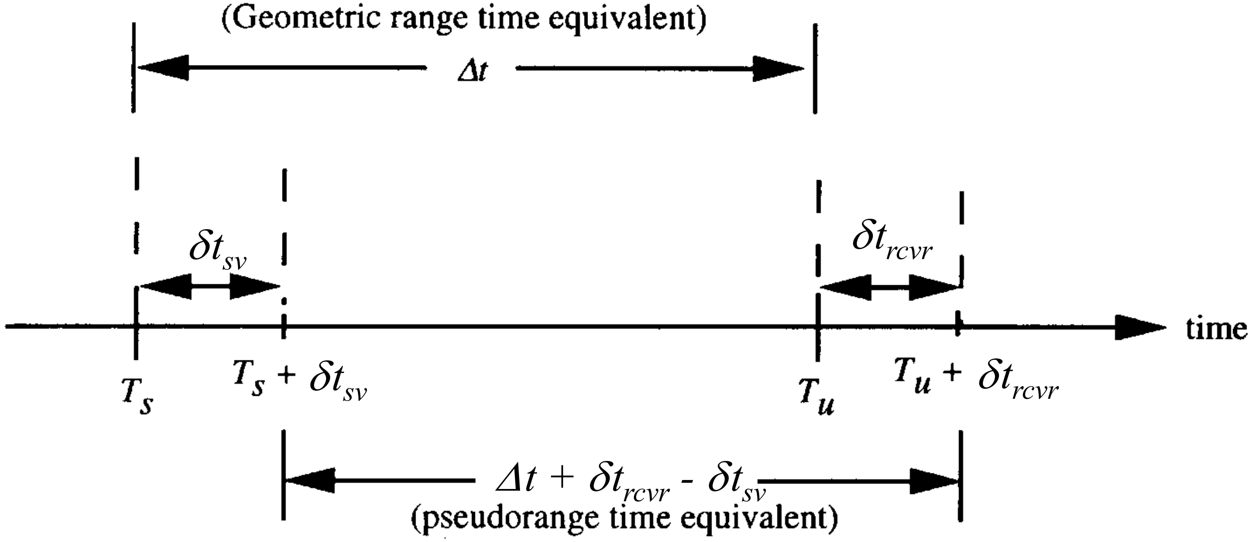
- Pseudorange is a measure of the difference in time between signal transmission and reception



Kaplan (ed.), *Understanding GPS: Principles and Applications*, Artech House, 1996

Effect of Clock Errors on Pseudorange

- Since pseudorange is based on time difference, any clock errors will fold directly into pseudorange



- Small clock errors can result in large pseudorange errors (since clock errors are multiplied by speed of light)
- Satellite clock errors (δt_{sv}) are very small
  - Satellites have atomic time standards
  - Satellite clock corrections transmitted in navigation message
- Receiver clock (δt_{rcvr}) is dominant error

Kaplan (ed.), *Understanding GPS: Principles and Applications*, Artech House, 1996

Doppler Shift

- For electromagnetic waves (which travel at the speed of light), the received frequency f_R is approximated using the standard Doppler equation

$$f_R = f_T\left(1 - \frac{(\boldsymbol{v}_r \cdot \boldsymbol{a})}{c}\right)$$

$f_R =$ received frequency (Hz)

$f_T =$ transmitted frequency (Hz)

$\boldsymbol{v}_r =$ satellite-to-user relative velocity vector (m/s)

$\boldsymbol{a} =$ unit vector pointing along line-of-sight from user to SV

$c =$ speed of light (m/s)

  - Note that $\boldsymbol{v}_r$ is the (vector) velocity difference

$$\boldsymbol{v}_r = \boldsymbol{v} - \dot{\boldsymbol{u}}$$

$\boldsymbol{v} =$ velocity vector for satellite (m/s)

$\dot{\boldsymbol{u}} =$ velocity vector for user (m/s)

- The Doppler shift Δf is then

$$\Delta f = f_R - f_T \quad \text{(Hz)}$$

Doppler Measurement

- The GPS receiver locks onto the carrier of the GPS signal and measures the received signal frequency
  - Relationship between true and measured received signal frequency: $f_{R_{meas}}$

$$f_R = f_{R_{meas}}(1 + \delta\dot{t}_{rcvr})$$

$f_R =$ true received signal frequency (Hz)

$f_{R_{meas}} =$ measured received signal frequency (Hz)

$\delta\dot{t}_{rcvr} =$ receiver clock drift rate (sec/sec)

  - Doppler measurement formed by differencing the measured received frequency and the transmit frequency:

$$\Delta f_{meas} = f_{R_{meas}} - f_T$$

  - Note: transmit frequency is calculated using information about SV clock drift rate given in navigation message

Doppler Measurement Sign Convention

- Sign convention based on Doppler definition
  - A satellite moving away from the receiver (neglecting clock errors) will have a *negative* Doppler shift

$$f_{R_{meas}} < f_T$$

$$\Delta f_{meas} = f_{R_{meas}} - f_T < 0$$

  - Sign convention used for NovAtel (and possibly other) receivers
- Sign convention based on relationship between Doppler and pseudorange
  - Doppler is essentially a measurement of the rate of change of the pseudorange
  - A satellite moving away from the receiver (neglecting clock errors) will have a *positive* Doppler measurement value
  - More common sign convention for GPS receivers (Ashtech, Trimble, and others)
- Carrier-phase measurement follows same convention as Doppler measurement (normally)

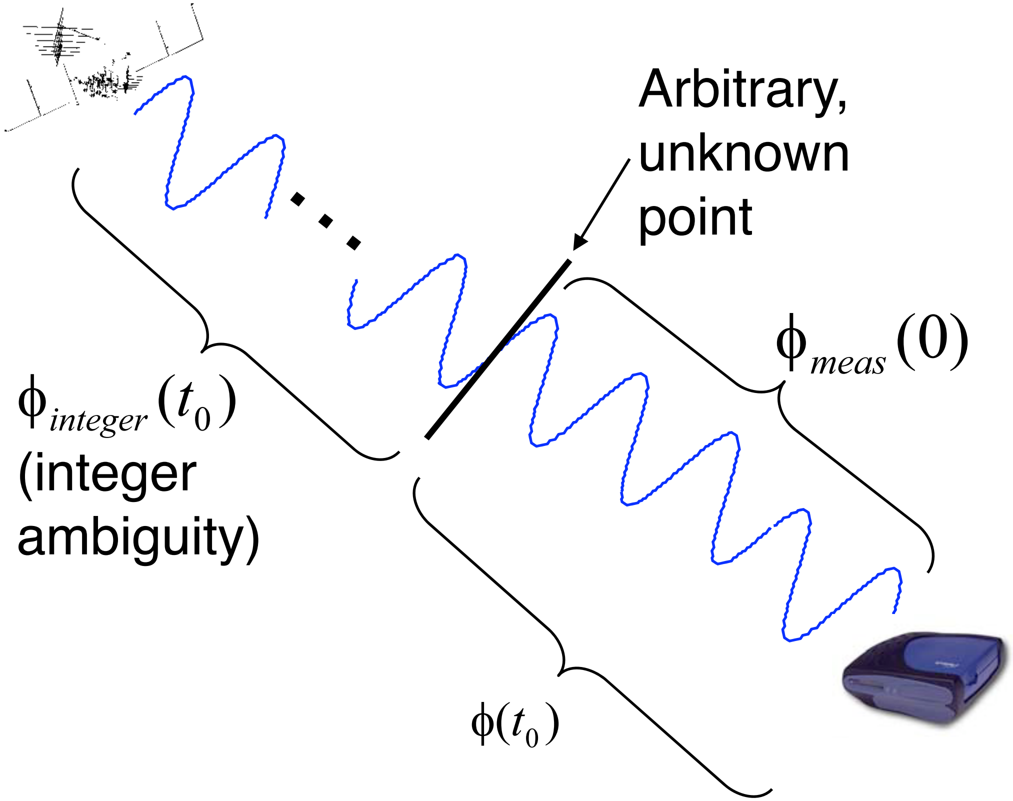
Carrier-Phase (Integrated Doppler) Measurement

- The carrier-phase measurement $\phi_{meas}(t)$ is calculated by integrating the Doppler measurements

$$\text{range}(t) = \underbrace{\int_{t_o}^{t} \Delta f_{meas}(t)dt + \phi(t_0)}_{\substack{\phi_{meas}(t) \\ \text{(can be measured by receiver)}}} + \phi_{integer}(t_0) + \text{clock error} + \text{other errors}$$

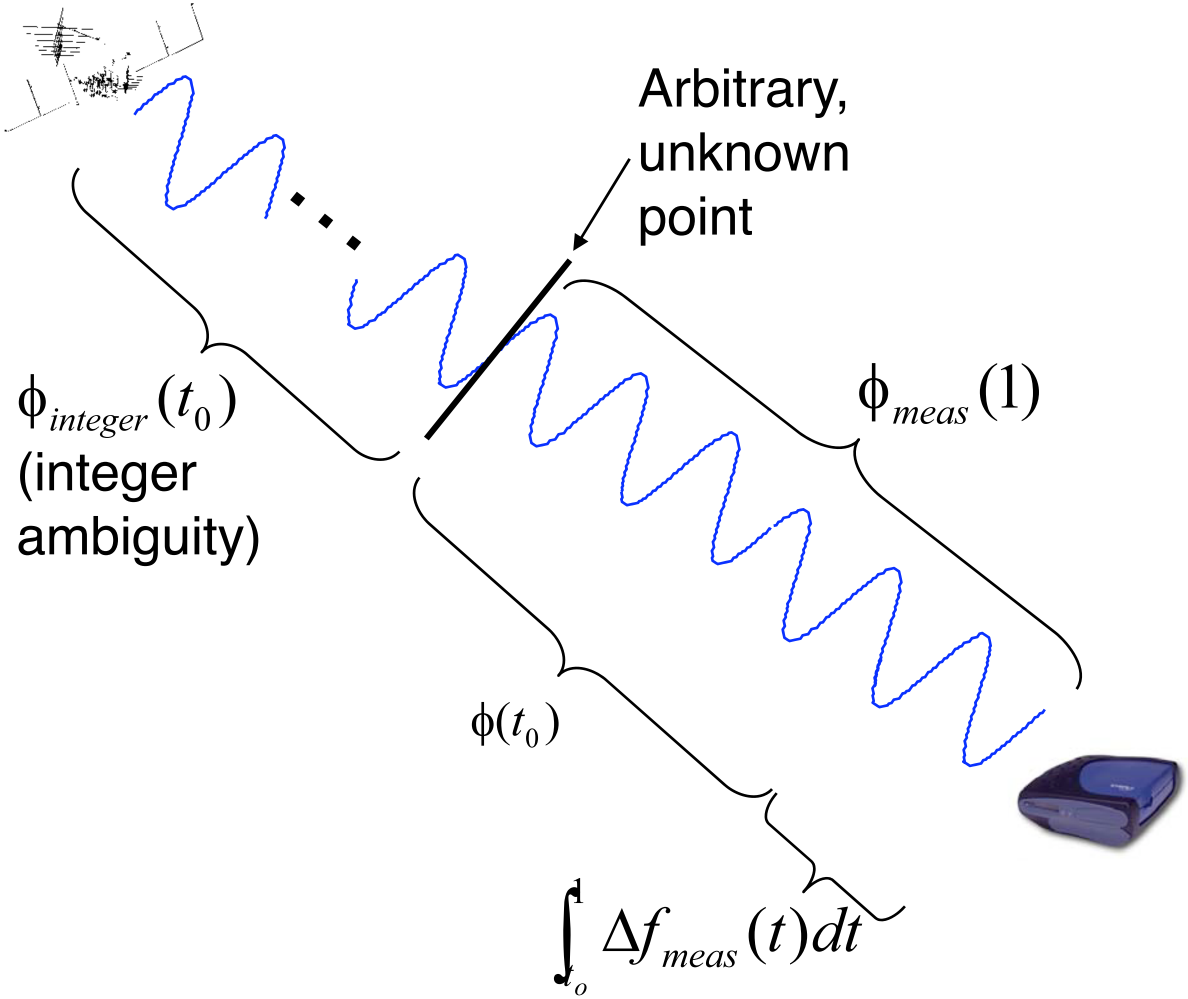
- The integer portion of the initial carrier-phase at the start of the integration ($\phi_{integer}(t_0)$) is known as the "carrier-phase integer ambiguity"
  - Because of this ambiguity, the carrier-phase measurement is not an absolute measurement of position
  - Advanced processing techniques can be used to resolve these carrier-phase ambiguites (carrier-phase ambiguity resolution)
- Alternative way of thinking: carrier-phase measurement is the "beat frequency" between the incoming carrier signal and receiver generated carrier.

Phase Tracking Example
At Start of Phase Lock (Time = 0 seconds)



Ignoring clock and other errors

Phase Tracking Example
After Movement (for 1 Second)



Arbitrary, unknown point

$\phi_{integer}(t_0)$ (integer ambiguity)

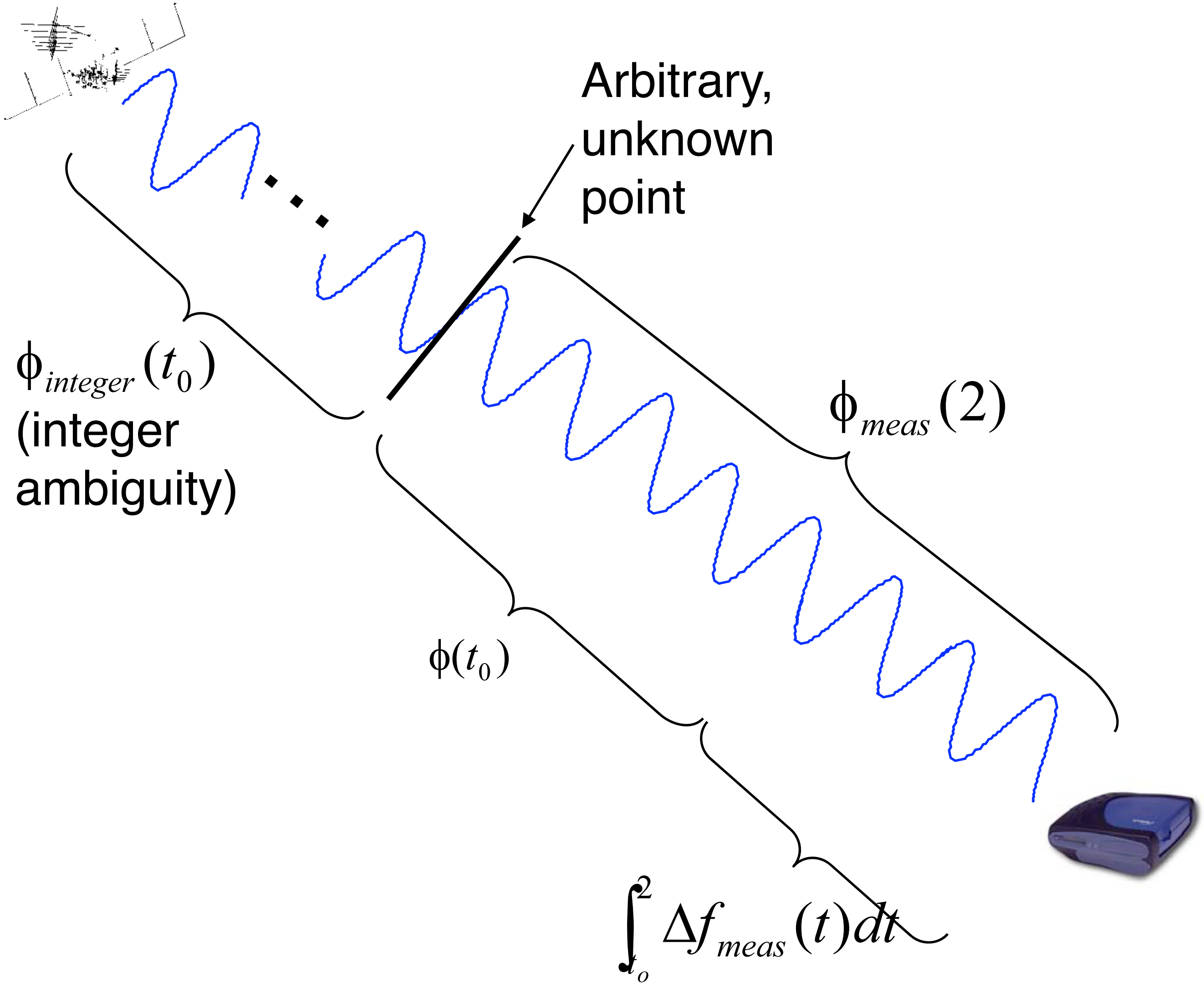
$\phi_{meas}(1)$

$\phi(t_0)$

$\int_{t_o}^{1} \Delta f_{meas}(t)dt$

Ignoring clock and other errors

Phase Tracking Example
After Movement (for 2 Seconds)



Ignoring clock and other errors

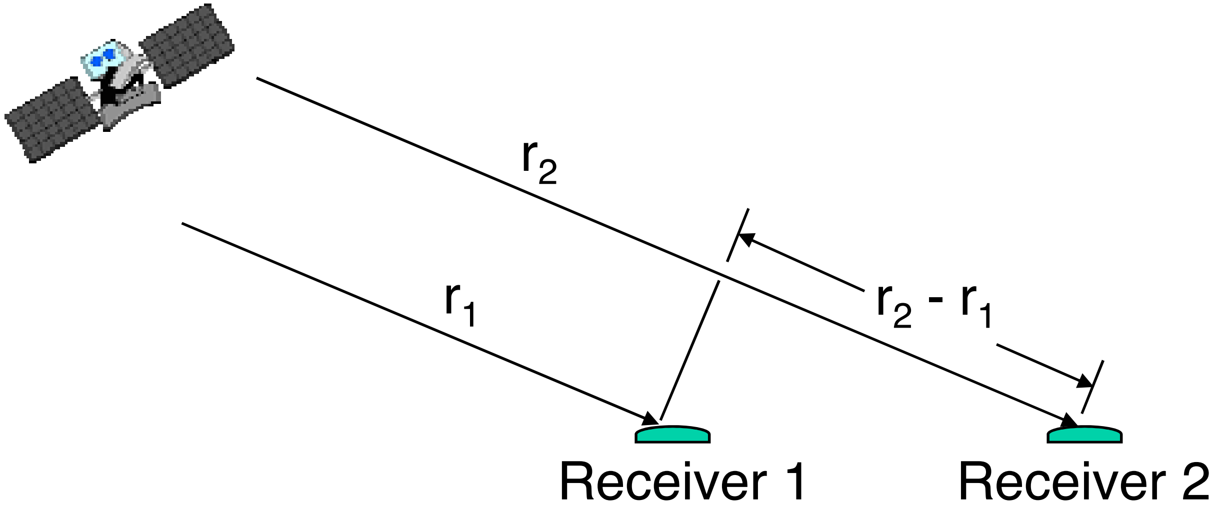
Comparison Between Pseudorange and Carrier-Phase Measurements

	Pseudorange	Carrier-Phase
Type of measurement	Range (absolute)	Range (ambiguous)
Measurement precision	~1 m	~0.01 m
Robustness	More robust	Less robust (cycle slips possible)

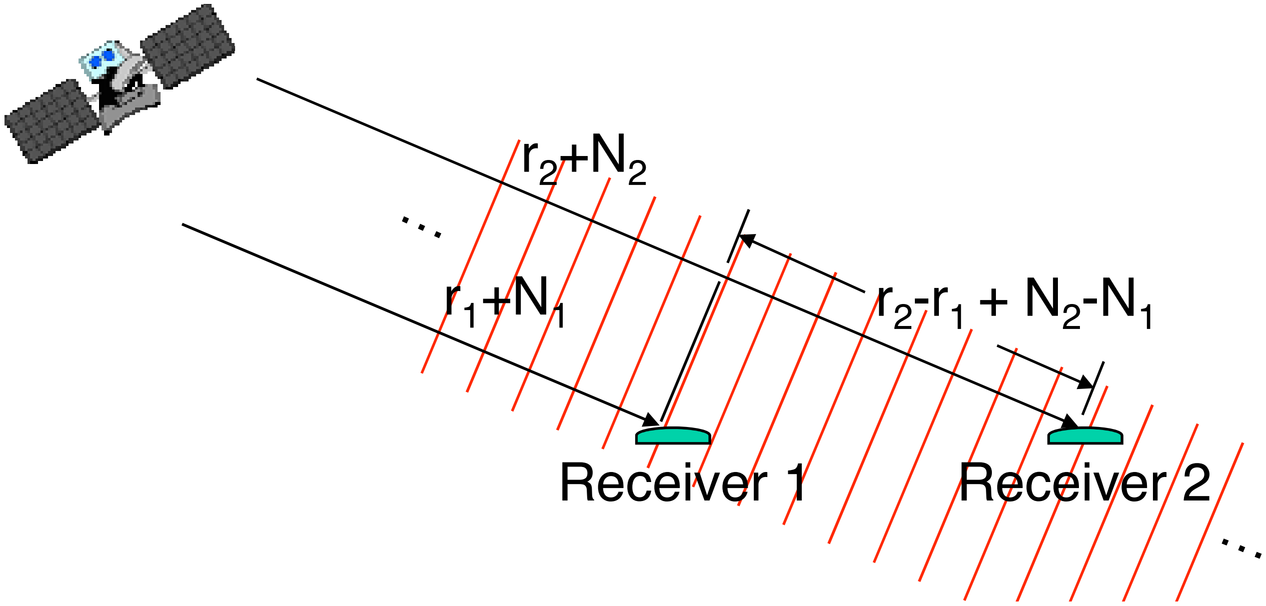
Necessary for
high precision
GPS

What Does a DGPS Measurement Tell You?

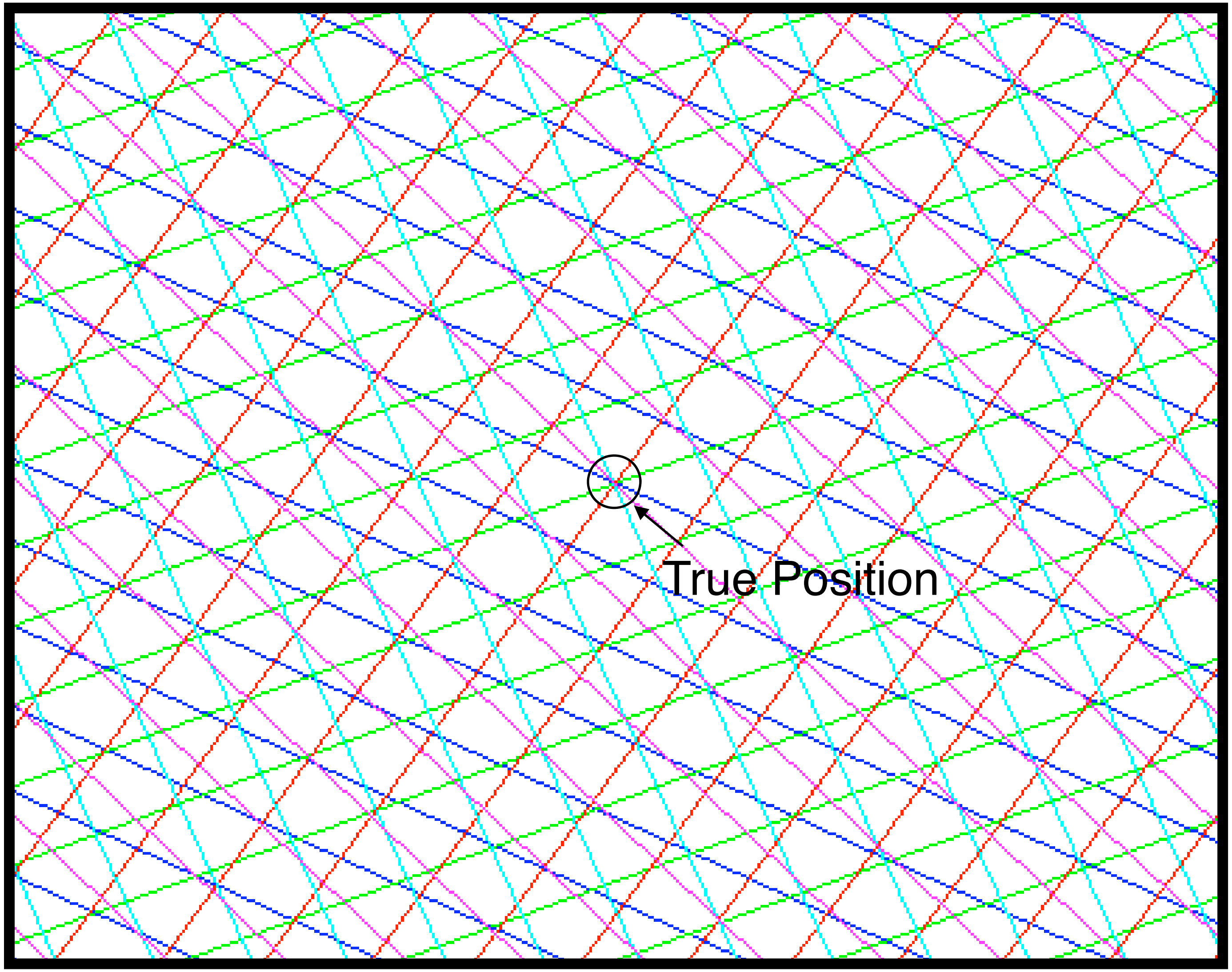
DGPS using pseudorange measurements



DGPS using carrier-phase measurements



Five Satellite Carrier-Phase DGPS Example



Carrier-to-Noise Density (C/N_0)

- The carrier-to-noise density is a measure of signal strength
  - The higher the C/N_0, the stronger the signal (and the better the measurements)
  - Units are dB-Hz
  - General rules-of-thumb:
    - $C/N_0 > 40$: Very strong signal
    - $32 < C/N_0 < 40$: Marginal signal
    - $C/N_0 < 32$: Probably losing lock
- C/N_0 tends to be receiver-dependent
  - Can be calculated many different ways
  - Absolute comparisons between receivers not very meaningful
  - Relative comparisons between measurements in a single receiver are very meaningful

GPS Navigation Solution

"OK, so I have all of these pseudorange measurements. Where in the world are we?"

Pseudorange Equation

- The pseudorange is the sum of the true range plus the receiver clock error
  - We're assuming (for now) that the receiver clock error is the only remaining error
    - SV clock error has been corrected for
    - All other errors are deemed negligible (or have been corrected)

$$\begin{aligned}\rho_j &= \sqrt{\left(x_j - x_u\right)^2 + \left(y_j - y_u\right)^2 + \left(z_j - z_u\right)^2} + c\delta t_u \\ &= f(x_u, y_u, z_u, \delta t_u)\end{aligned}$$

ρ_j = pseudorange measurement from satellite j (m)

x_j, y_j, z_j = ECEF position of satellite j (m)

x_u, y_u, z_u = ECEF position of user (m)

δt_u = receiver clock error (sec)

- For now, only use one type of pseudorange (L1 C/A, L1 P, or L2 P)

Statement of the Problem

- At a given measurement epoch, the GPS receiver generates n pseudorange measurements (from n different satellites)

$$
\begin{aligned}
\rho_1 &= \sqrt{(x_1 - x_u)^2 + (y_1 - y_u)^2 + (z_1 - z_u)^2} + c\delta t_u \\
\rho_2 &= \sqrt{(x_2 - x_u)^2 + (y_2 - y_u)^2 + (z_2 - z_u)^2} + c\delta t_u \\
\rho_3 &= \sqrt{(x_3 - x_u)^2 + (y_3 - y_u)^2 + (z_3 - z_u)^2} + c\delta t_u \\
&\vdots \\
\rho_n &= \sqrt{(x_n - x_u)^2 + (y_n - y_u)^2 + (z_n - z_u)^2} + c\delta t_u
\end{aligned}
$$

- Goal: Determine user position and clock error, expressed in state-vector form as

$$
\mathbf{x} = \begin{bmatrix} x_u \\ y_u \\ z_u \\ c\delta t_u \end{bmatrix}
$$

Solving the Pseudorange Equations

- The n pseudorange equations are non-linear (so no easy solution)
- Ways to solve
  - Closed form solutions
    - Complicated
    - May not give as much insight
  - Iterative techniques based on linearization
    - Often using least-squares estimation
    - Arguably the simplest approach
    - Approach covered in this course
  - Kalman filtering
    - Similar to least-squares approach, except with additional ability to handle measurements over a period of time
    - Will discuss briefly
- What is linearization?
  - Pick a nominal (or approximate) solution
  - Linearize about that point, resulting in a set of linear equations
  - Solve the linear equations
  - Will use Taylor series expansion for linearization

Taylor Series Expansion (1/2)

- Taylor series expansion (1 variable)

$$f(a+\Delta a)=f(a)+\Delta a\frac{df}{da}+\frac{(\Delta a)^2}{2!}\frac{d^2f}{da^2}+\frac{(\Delta a)^3}{3!}\frac{d^3f}{da^3}+\dots$$

- This can be used to linearize about a certain value of the independent variable a.
  - Example: the function $f(t)=2+3t-6t^2$ is a non-linear function in t
  - Suppose we want to linearize about the point $\hat{t}=2$
  - The complete Taylor series expression is

$$f(\hat{t}+\Delta t)=f(\hat{t})+\Delta t\frac{df}{dt}+\frac{(\Delta t)^2}{2!}\frac{d^2f}{dt^2}$$

$$=2+3\hat{t}-6\hat{t}^2+\Delta t(3-12\hat{t})+\frac{(\Delta t)^2}{2}(-12)$$

  - To linearize, we set $\hat{t}=2$ and neglect higher order (non-linear) terms of Δt
    - Valid for perturbations (i.e., small values of Δt)

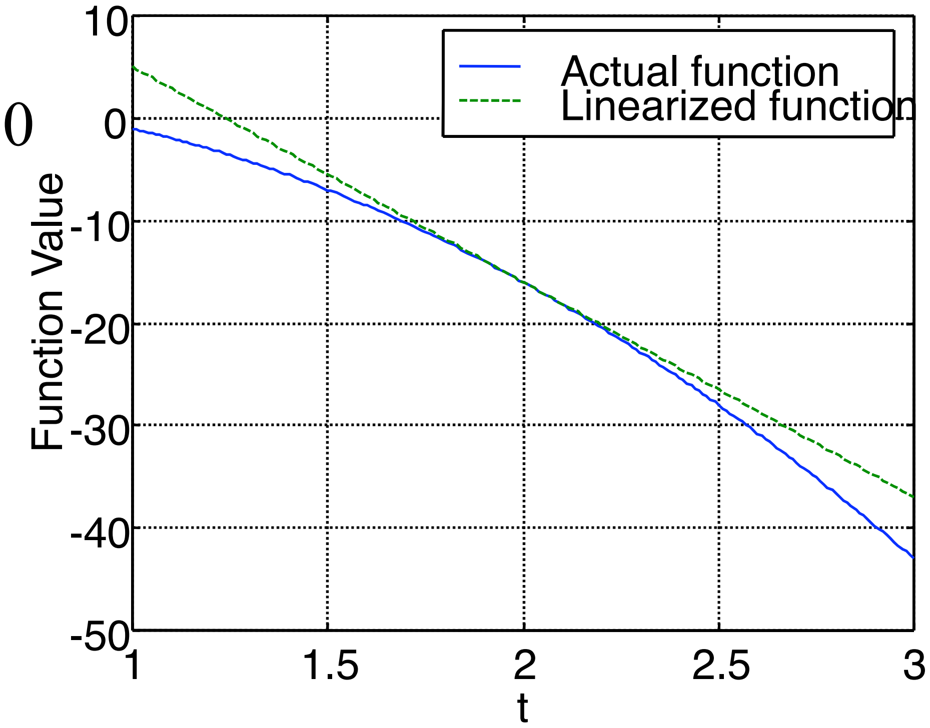
Taylor Series Expansion (2/2)

– (Continued example) Linearized form

$$f(\hat{t}+\Delta t)\Big|_{\hat{t}=2} = f(\hat{t}=2)+\Delta t\frac{df}{dt}\Big|_{\hat{t}=2} + \cancelto{0}{\frac{(\Delta t)^2}{2!}\frac{d^2 f}{dt^2}\Big|_{\hat{t}=2}}$$

$$= 2+3(2)-6(2^2)+\Delta t(3-12(2))$$

$$= -16-21\Delta t$$



- First order Taylor series for function in two variables:

$$f(\hat{a}+\Delta a,\hat{b}+\Delta b)=f(\hat{a},\hat{b})+\Delta a\frac{\partial f}{\partial a}\Big|_{\hat{a},\hat{b}}+\Delta b\frac{\partial f}{\partial b}\Big|_{\hat{a},\hat{b}}+\text{h.o.t.}$$

Linearization of Pseudorange Equations (1/5)

- First, define a nominal state (position and clock error) as

$$\hat{\mathbf{x}} = \begin{bmatrix} \hat{x}_u \\ \hat{y}_u \\ \hat{z}_u \\ c\delta\hat{t}_u \end{bmatrix} = \text{nominal (approximate) state}$$

- An approximate (or expected) pseudorange can then be calculated for satellite j

$$\begin{aligned} \hat{\rho}_j &= \sqrt{(x_j - \hat{x}_u)^2 + (y_j - \hat{y}_u)^2 + (z_j - \hat{z}_u)^2} + c\delta\hat{t}_u \\ &= f(\hat{x}_u, \hat{y}_u, \hat{z}_u, c\delta\hat{t}_u) \end{aligned}$$

  - This approximate (expected) pseudorange is the pseudorange that we would expect to have if our position and clock error were actually $\hat{x}_u, \hat{y}_u, \hat{z}_u,$ and $c\delta\hat{t}_u$.

Linearization of Pseudorange Equations (2/5)

- Relationship between true and approximate position and time

$$x_u = \hat{x}_u + \Delta x_u$$

$$y_u = \hat{y}_u + \Delta y_u$$

$$z_u = \hat{z}_u + \Delta z_u$$

$$c\delta t_u = c\delta \hat{t}_u + \Delta c\delta t_u$$

  - Vector form:

$$\mathbf{x}_u = \hat{\mathbf{x}}_u + \Delta \mathbf{x}_u$$

- Based on these relations, we can write

$$f(x_u, y_u, z_u, c\delta t_u) = f(\hat{x}_u + \Delta x_u, \hat{y}_u + \Delta y_u, \hat{z}_u + \Delta z_u, c\delta \hat{t}_u + \Delta c\delta t_u)$$

- To linearize, right-hand side of equation can be evaluated using a first order Taylor series expansion

Linearization of Pseudorange Equations (3/5)

- First order Taylor series expansion of pseudorange function:

$$
\begin{aligned}
f(\hat{x}_u+\Delta x_u,\hat{y}_u+\Delta y_u,\hat{z}_u+\Delta z_u,c\delta\hat{t}_u+\Delta c\delta t_u) &= f(\hat{x}_u,\hat{y}_u,\hat{z}_u,c\delta\hat{t}_u) \\
&+\frac{\partial f(\hat{x}_u,\hat{y}_u,\hat{z}_u,c\delta\hat{t}_u)}{\partial\hat{x}_u}\Delta x_u+\frac{\partial f(\hat{x}_u,\hat{y}_u,\hat{z}_u,c\delta\hat{t}_u)}{\partial\hat{y}_u}\Delta y_u \\
&+\frac{\partial f(\hat{x}_u,\hat{y}_u,\hat{z}_u,c\delta\hat{t}_u)}{\partial\hat{z}_u}\Delta z_u+\frac{\partial f(\hat{x}_u,\hat{y}_u,\hat{z}_u,c\delta\hat{t}_u)}{\partial c\delta\hat{t}_u}\Delta c\delta t_u \\
&+\text{h.o.t.}
\end{aligned}
$$

- The partial derivatives are

$$
\frac{\partial f(\hat{x}_u,\hat{y}_u,\hat{z}_u,\delta\hat{t}_u)}{\partial\hat{x}_u}=-\frac{x_j-\hat{x}_u}{\hat{r}_j}\qquad\frac{\partial f(\hat{x}_u,\hat{y}_u,\hat{z}_u,\delta\hat{t}_u)}{\partial\hat{y}_u}=-\frac{y_j-\hat{y}_u}{\hat{r}_j}
$$

$$
\frac{\partial f(\hat{x}_u,\hat{y}_u,\hat{z}_u,\delta\hat{t}_u)}{\partial\hat{z}_u}=-\frac{z_j-\hat{z}_u}{\hat{r}_j}\qquad\frac{\partial f(\hat{x}_u,\hat{y}_u,\hat{z}_u,\delta\hat{t}_u)}{\partial c\delta\hat{t}_u}=1
$$

$$
\hat{r}_j=\sqrt{\left(x_j-\hat{x}_u\right)^2+\left(y_j-\hat{y}_u\right)^2+\left(z_j-\hat{z}_u\right)^2}
$$

Linearization of Pseudorange Equations (4/5)

- Using above results, linearized pseudorange equation is

$$\rho_j = \hat{\rho}_j - \frac{x_j - \hat{x}_u}{\hat{r}_j}\Delta x_u - \frac{y_j - \hat{y}_u}{\hat{r}_j}\Delta y_u - \frac{z_j - \hat{z}_u}{\hat{r}_j}\Delta z_u + \Delta c\delta t_u$$

- This can be simplified to $\Delta\rho_j = a_{xj}\Delta x_u + a_{yj}\Delta y_u + a_{zj}\Delta z_u - \Delta c\delta t_u$ where

$$\Delta\rho_j = \hat{\rho}_j - \rho_j$$

$$a_{xj} = \frac{x_j - \hat{x}_u}{\hat{r}_j}, \qquad a_{yj} = \frac{y_j - \hat{y}_u}{\hat{r}_j}, \qquad a_{zj} = \frac{z_j - \hat{z}_u}{\hat{r}_j}$$

Linearization of Pseudorange Equations (5/5)

- Original (nonlinear) equations for n measurements

$$\rho_1 = \sqrt{(x_1 - x_u)^2 + (y_1 - y_u)^2 + (z_1 - z_u)^2} + c\delta\, t_u$$

$$\rho_2 = \sqrt{(x_2 - x_u)^2 + (y_1 - y_u)^2 + (z_1 - z_u)^2} + c\delta\, t_u$$

$$\rho_3 = \sqrt{(x_3 - x_u)^2 + (y_3 - y_u)^2 + (z_3 - z_u)^2} + c\delta\, t_u$$

$$\vdots$$

$$\rho_n = \sqrt{(x_n - x_u)^2 + (y_n - y_u)^2 + (z_n - z_u)^2} + c\delta\, t_u$$

- Linearized (error) equations for the same n measurements

$$\Delta\rho_1 = a_{x1}\Delta x_u + a_{y1}\Delta y_u + a_{z1}\Delta z_u - \Delta c\delta t_u$$

$$\Delta\rho_2 = a_{x2}\Delta x_u + a_{y2}\Delta y_u + a_{z2}\Delta z_u - \Delta c\delta t_u$$

$$\Delta\rho_3 = a_{x3}\Delta x_u + a_{y3}\Delta y_u + a_{z3}\Delta z_u - \Delta c\delta t_u$$

$$\vdots$$

$$\Delta\rho_n = a_{xn}\Delta x_u + a_{yn}\Delta y_u + a_{zn}\Delta z_u - \Delta c\delta t_u$$

Solving the Linearized Pseudorange Equations Using Least-Squares (1/2)

- We can express the set of pseudorange equations in matrix form

$$\Delta\boldsymbol{\rho} = \mathbf{H}\Delta\mathbf{x}$$

$$\Delta\boldsymbol{\rho} = \begin{bmatrix} \Delta\rho_1 \\ \Delta\rho_2 \\ \Delta\rho_3 \\ \vdots \\ \Delta\rho_n \end{bmatrix} \qquad \mathbf{H} = \begin{bmatrix} a_{x1} & a_{y1} & a_{z1} & -1 \\ a_{x2} & a_{y2} & a_{z2} & -1 \\ a_{x3} & a_{y3} & a_{z3} & -1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{xn} & a_{yn} & a_{zn} & -1 \end{bmatrix} \qquad \Delta\mathbf{x} = \begin{bmatrix} \Delta x_u \\ \Delta y_u \\ \Delta z_u \\ \Delta c\delta t_u \end{bmatrix}$$

- Three possible cases
  - $n < 4$: Underdetermined case
    - Cannot solve for $\Delta\mathbf{x}$
    - Is there still useable information?
  - $n = 4$: Uniquely determined case
    - One valid solution for $\Delta\mathbf{x}$ (generally)
    - Solved by calculating $\mathbf{H}^{-1}$ $(\Delta\mathbf{x} = \mathbf{H}^{-1}\Delta\boldsymbol{\rho})$
  - $n > 4$: Overdetermined case
    - No solution that perfectly solves equation (generally)
    - Can use least-squares techniques (which pick solution that minimizes the square of the error)

Solving the Linearized Pseudorange Equations Using Least-Squares (2/2)

- Basic least-squares solution (no measurement weighting)

$$\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\Delta\boldsymbol{\rho}$$

  - Reasonable approach for single-point positioning in presence of SA
- Solution with measurement weighting (weighted least-squares)
  - Useful when
    - Measurements have different error statistics
    - Measurement errors are correlated
  - Measurement error covariance matrix $\mathbf{C}_\rho$
    - Diagonal terms are measurement error variances
    - Off-diagonal terms show cross-correlation between measurement errors

$$\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{C}_\rho^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{C}_\rho^{-1}\Delta\boldsymbol{\rho}$$

  - Note that this is identical to unweighted case if $\mathbf{C}_\rho = \mathbf{I}$ (identity matrix)

Measurement Residuals

- For overdetermined system, generally no valid solution for $\Delta\mathbf{x}$ that solves measurement equation, so

$$\Delta\rho \neq \mathbf{H}\Delta\mathbf{x}$$

- Measurement residuals ($\mathbf{v}$)
  - Corrections that, when applied to measurements, would result in solution of above equation
  - Least-squares minimizes the sum of squares of these residuals

$$\mathbf{v} = \Delta\rho - \mathbf{H}\Delta\mathbf{x}$$

$$\Delta\rho = \mathbf{H}\Delta\mathbf{x} + \mathbf{v}$$

Iterating the Nominal State

- Linearized equations (and resulting $\mathbf{H}$ matrix) calculated using nominal state $\hat{\mathbf{x}}_u$
- Linearization valid when
  - Nominal state is close to true state
  - $\Delta\mathbf{x}$ is "small"
- If $\hat{\mathbf{x}}_u$ is not very accurate (i.e., $\Delta\mathbf{x}$ is large), iteration is required
  - For each iteration, a new value of $\hat{\mathbf{x}}_u$ is calculated based upon the old value and the corrections $\Delta\mathbf{x}$

$$\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$$

  - This new value of $\hat{\mathbf{x}}_u$ is then used to recalculate the corrections $\Delta\mathbf{x}$ (which should be smaller this time)
- Solution must converge
  - For standard GPS positioning, not much of a problem (will generally converge with an initial guess at the center of the Earth)
  - For more non-linear situations (e.g., using pseudolites), this can be more of a problem

Correcting for Satellite Clock Error

- Single point positioning only estimates receiver clock error
  - Assumes all other errors are negligible
  - Requires correction of satellite clock error
- Clock correction (from ICD-GPS-200C)

$$\rho_{corr} = \rho + c\Delta t_{sv}$$

$$\Delta t_{sv} = a_{f_0} + a_{f_1}(t - t_{0_c}) + a_{f_2}(t - t_{0_c})^2 + \Delta t_r$$

$$\Delta t_r = Fe\sqrt{a}\sin E_k$$

ρ_{corr} = pseudorange corrected for SV clock error

ρ = original (raw) pseudorange measurement

Δt_{sv} = SV clock correction

$a_{f_0}, a_{f_1}, a_{f_2}, t_{0_c}$ = SV clock correction parameters from nav message

Δt_r = relativity correction (since not circular orbit)

F = constant = $-4.442807633 \times 10^{-10}$ sec/(meter)$^{1/2}$

e = eccentricity from nav message

$\sqrt{a}$ = square root of semi-major axis from nav message

E_k = Eccentric anomaly (from SV position calculation)

Determining Signal Transmit Time (1/2)

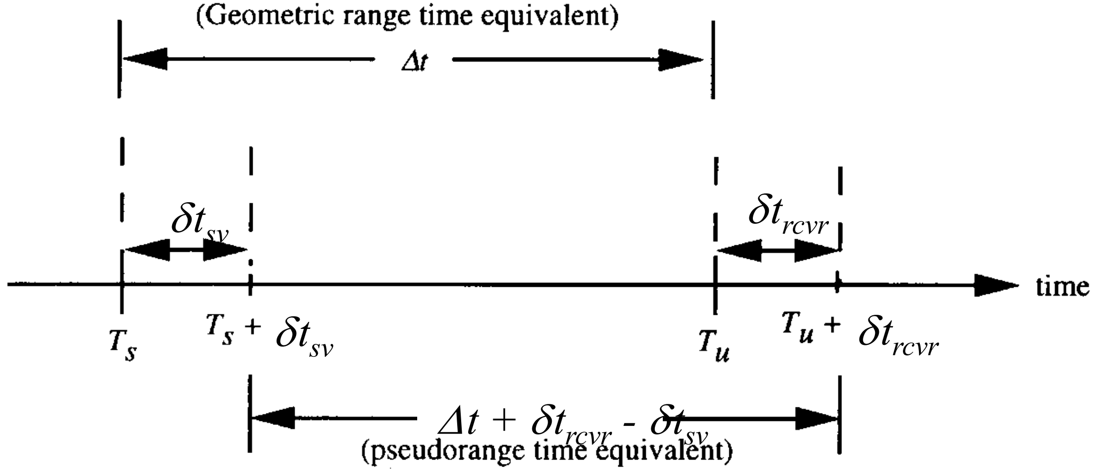
- For satellite position calculation, need true GPS transmit time of the signal (T_s)
  - Receiver provides time of reception according to the receiver clock ($T_u + \delta t_{rcvr}$)
  - From diagram below, if the pseudorange time equivalent is subtracted from the receive time, then the result is the true transmit time plus the satellite clock error

$$\underbrace{T_u + \delta t_{rcvr}}_{\text{receive time}} - \frac{PR}{c} = T_s + \delta t_{sv}$$

$PR =$ pseudorange measurement (m)

$$\boxed{\underbrace{T_u + \delta t_{rcvr}}_{\text{receive time}} - \frac{PR}{c} - \underbrace{\delta t_{sv}}_{} = T_s}$$

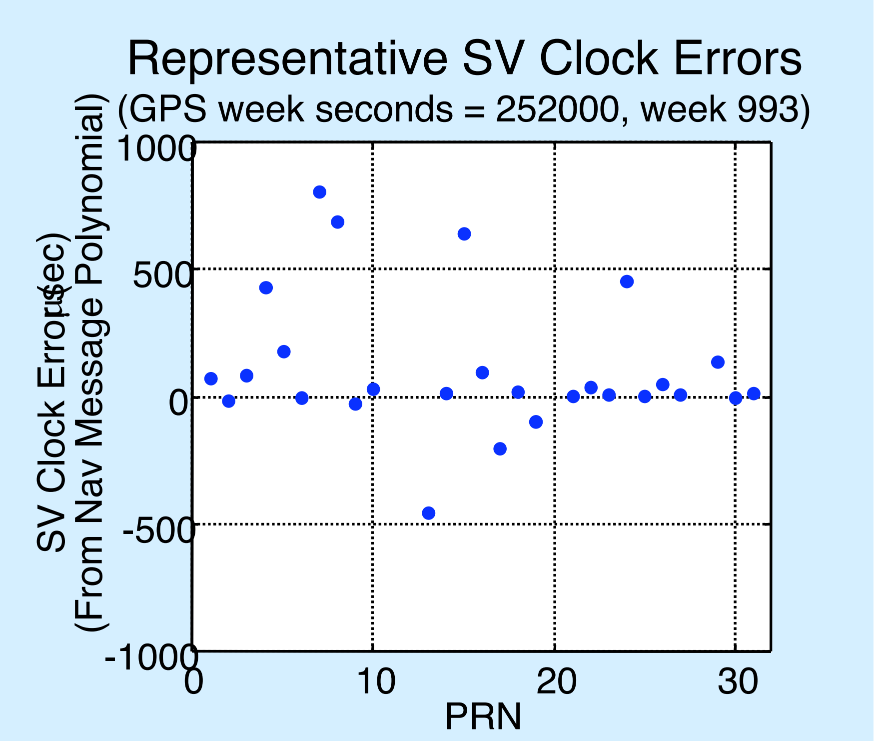
same as Δt_{sv} from the previous slide



Determining Signal Transmit Time (2/2)

- Effect of neglecting δt_{sv} for SV positioning[1]
  - Satellite clock error can grow to up to ~1 msec:
  - Typical satellite velocity is 3900 m/s
  - Worst-case position error from neglecting δt_{sv}

    3900 m/s x 0.001 s = **3.9 m**
  - Effect of neglecting δt_{sv}
    - Single point positioning: Can be significant (but not with SA)
    - Differential positioning: effectively cancelled out (acts like 3.9 m satellite position error)



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[1] The SV clock error δt_{sv} will have a significant effect on the actual pseudorange measurement. This page only describes the impact of δt_{sv} on determining the position of the satellite.

Correcting for Satellite Group Delay

- Each satellite has a slight time bias between the L1 and the L2 signals
  - Not desired, but it's there nonetheless
  - Will affect dual-frequency users, unless it's accounted for
  - Can be measured and/or calibrated out
  - This calibration is accounted for when the control segment generates the satellite clock correction terms from broadcast nav message: $a_{f_0}, a_{f_1}, a_{f_2}$, and t_{0_c}
  - However, this is all designed for the dual-frequency user! Single frequency users need to remove the effect of this dual-frequency correction on their Δt_{sv} value
- Single frequency users must apply the group delay term (TGD) from the nav message to their SV clock correction term (from p. 90 of ICD-GPS-200C)

$$\left(\Delta t_{sv}\right)_{L1} = \Delta t_{sv} - T_{GD}$$

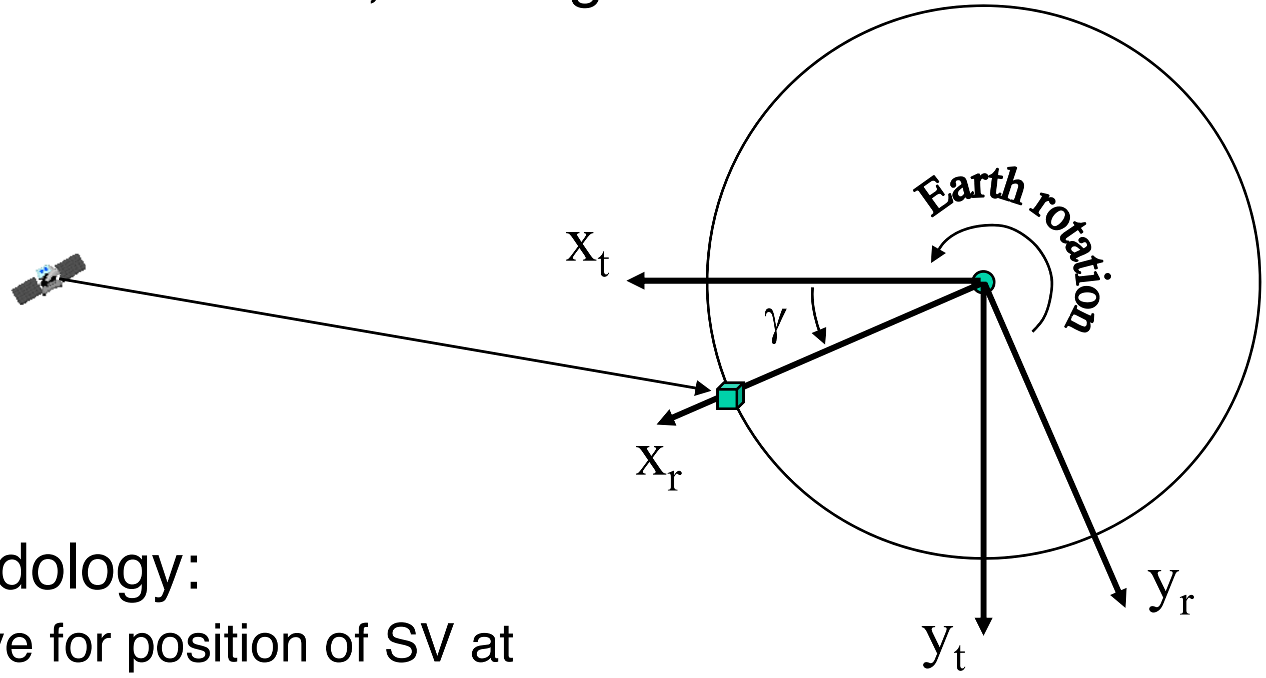
$$\left(\Delta t_{sv}\right)_{L2} = \Delta t_{sv} - \left(\frac{77}{60}\right)^2 T_{GD}$$

Accounting for Signal Travel Time (1/3)

- Signal arrives at receiver ***after*** it is transmitted (due to signal travel time)
  - Transmit time: Time the signal was transmitted
  - Receive time: Time the signal was received
- Satellite position should be calculated based upon transmit time
  - When measuring a signal, we don’t really care what happened after that signal was transmitted
  - Transmit time should be GPS system time (or as close to it as possible)
  - Very good approximate value of transmit time obtained by subtracting pseudorange (expressed in seconds) from the receive time as indicated by the receiver clock
    - Why?
- What other considerations do we need to make for signal travel time?

Accounting for Signal Travel Time (2/3)

- Here's the situation, looking down at the North Pole



- Methodology:
  - Solve for position of SV at transmit time, in ECEF coordinates at transmit time (x_t, y_t, and z_t) using ICD-GPS-200 equations
  - Rotate into ECEF reference frame at the time of reception:

$$\begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix}$$

$$\gamma = \dot{\Omega}_e t_{prop}$$

t_{prop} = Signal propagation time

Accounting for Signal Travel Time (3/3)

- Neglecting atmospheric delay, the signal propagation time is calculated by

$$t_{prop} = \frac{\text{geometric range to satellite}}{\text{speed of light}}$$

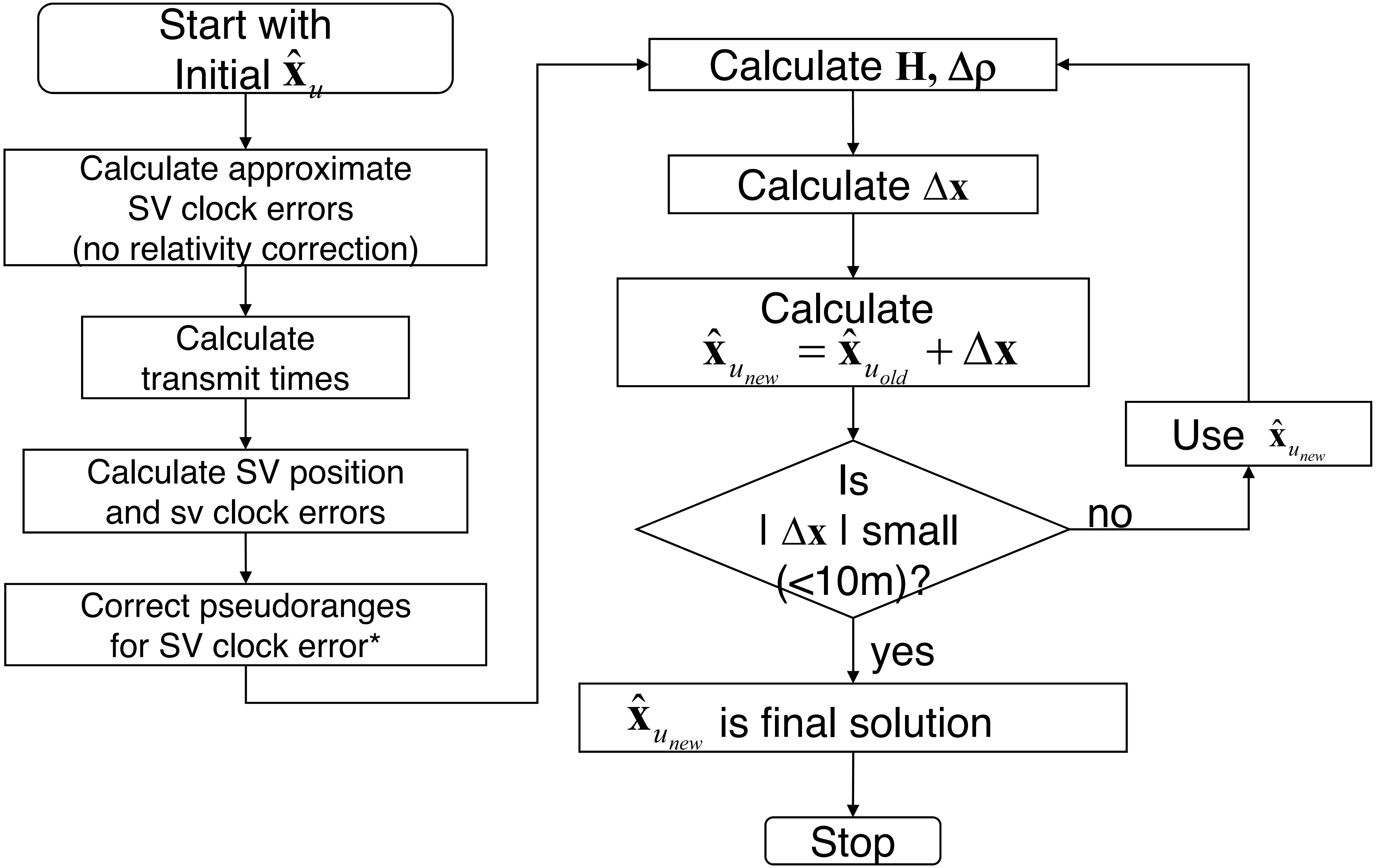
$$= \frac{\left|\boldsymbol{p}_{sv} - \boldsymbol{p}_{rcvr}\right|}{c}$$

$\boldsymbol{p}_{sv}$ = satellite ECEF position vector

$\boldsymbol{p}_{rcvr}$ = receiver ECEF position vector

- Note that the satellite position is needed to calculate t_{prop} (and vice-versa)
  - Satellite position in ECEF coordinates at transmit time is sufficiently accurate (x_t, y_t, and z_t)
  - Note that receiver position must be known
    - Can be approximate

Single Point Positioning Algorithm



Start with Initial $\hat{\mathbf{x}}_u$

Calculate approximate SV clock errors (no relativity correction)

Calculate transmit times

Calculate SV position and sv clock errors

Correct pseudoranges for SV clock error*

Calculate $\mathbf{H}, \Delta\rho$

Calculate $\Delta\mathbf{x}$

Calculate $\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$

Is $|\Delta\mathbf{x}|$ small (<10m)?

no

Use $\hat{\mathbf{x}}_{u_{new}}$

yes

$\hat{\mathbf{x}}_{u_{new}}$ is final solution

Stop

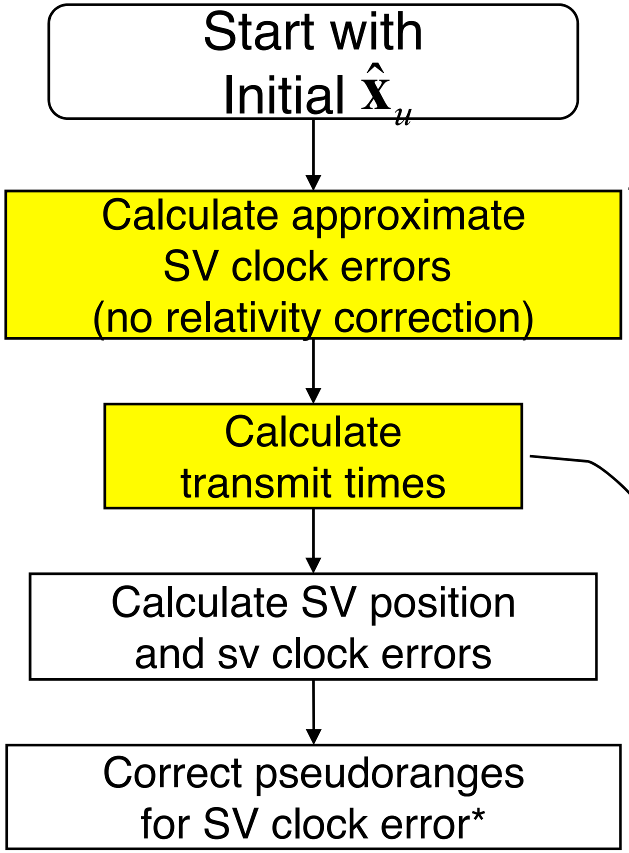
*include group delay correction, if a single-frequency user

GPS Positioning Example

- We'll look at a single case to give an example
- Situation
  - Receiver measurement time (GPS week seconds): 220937
  - Initial $\hat{\mathbf{x}}_u$: $[\underbrace{506071.529 \quad -4882278.667 \quad 4109624.557}_{\text{Initial guess of position (in error by ~50 km)}} \quad \underset{\substack{\uparrow \\ \text{Initial clock error} \\ \text{expressed in m}}}{15.807}]$
  - Measurements:

PRN	**Pseudorange**
12	25022759.323
2	22075351.532
26	21929350.580
15	22677087.545
29	21039894.608
21	24757444.127
30	24032696.422

Example: Calculation of Transmit Time



Start with Initial $\hat{\mathbf{x}}_u$

Calculate approximate SV clock errors (no relativity correction)

Calculate transmit times

Calculate SV position and sv clock errors

Correct pseudoranges for SV clock error*

$$\Delta t_{sv_{approx}} = a_{f_0} + a_{f_1}(t - t_{0_c}) + a_{f_2}(t - t_{0_c})^2$$

(Receive time = 220937)

		Approx SV	Transmit
PRN	**PR/c (s)**	**Clock Error (s)**	**Time (s)**
12	0.083466941	-0.000359172	220936.916892231
2	0.073635447	0.000199964	220936.926164589
26	0.073148440	0.000247391	220936.926604169
15	0.075642622	-0.000107130	220936.924464508
29	0.070181534	-0.000043329	220936.929861795
21	0.082581944	0.000065081	220936.917352974
30	0.080164446	0.000075192	220936.919760362

$$\underbrace{T_u + \delta t_{rcvr}}_{\text{receive time}} - \frac{PR}{c} - \underbrace{\delta t_{sv}}_{} = T_s$$

Example: SV Position and Clock Error and Pseudorange Correction

Start with Initial $\hat{\mathbf{x}}_u$

Calculate approximate SV clock errors (no relativity correction)

Calculate transmit times

Calculate SV position and sv clock errors

Correct pseudoranges for SV clock error*

PRN	SV ECEF X (m)	SV ECEF Y (m)	SV ECEF Z (m)	SV Clock Err (s)
12	-9924909.896	-22418412.217	-10238600.462	-0.000359164
2	19519446.654	-12864169.870	12106498.214	0.000199964
26	9518973.908	-24465469.002	-347289.566	0.000247426
15	6420995.137	-25601178.700	-2907089.329	-0.000107131
29	-8265550.815	-16497554.935	19234406.500	-0.000043326
21	-22299549.612	-6458590.524	13615554.839	0.000065090
30	-18044425.181	-19566072.431	-289953.964	0.000075168

Orbital calculations + time-of-transit rotation correction

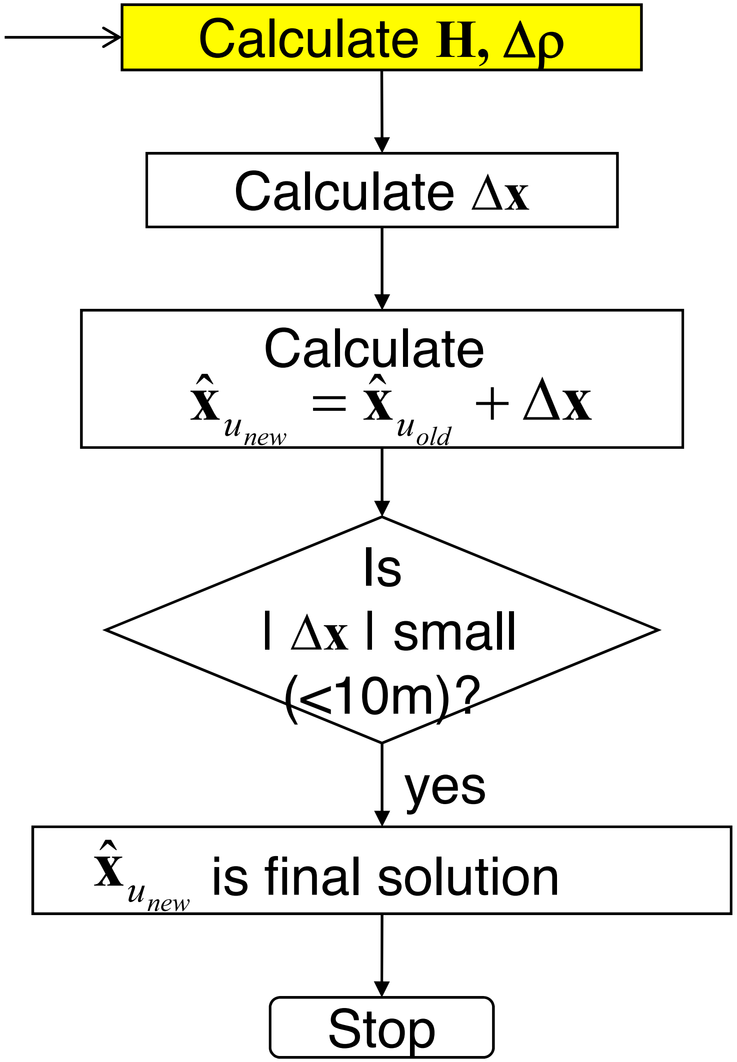
$$\Delta t_{sv} = a_{f_0} + a_{f_1}(t - t_{0_c}) + a_{f_2}(t - t_{0_c})^2 + \Delta t_r$$

			Group Delay
PRN	Original PR (m)	Corrected PR (m)	Tgd (ns)
12	25022759.323	24915088.264	-12.107193470
2	22075351.532	22135304.471	-17.229467630
26	21929350.580	22003529.125	-6.519258022
15	22677087.545	22644973.532	-10.244548320
29	21039894.608	21026908.513	-8.847564459
21	24757444.127	24776961.101	-11.641532180
30	24032696.422	24055233.876	-8.381903172

$$\rho_{corr} = \rho + c\left(\Delta t_{sv} - T_{GD}\right)$$

*include group delay correction, if a single-frequency user

Example: H Matrix (Iteration 1)



Calculate $\mathbf{H}, \Delta\rho$

Calculate $\Delta\mathbf{x}$

Calculate $\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$

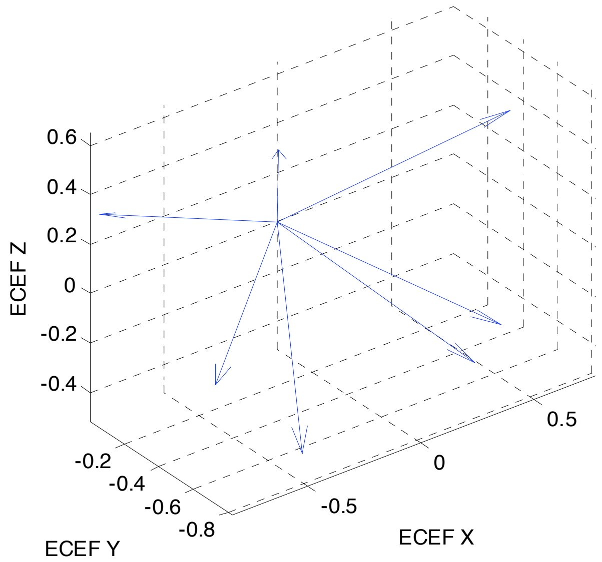
Is $|\Delta\mathbf{x}|$ small (<10m)?

yes

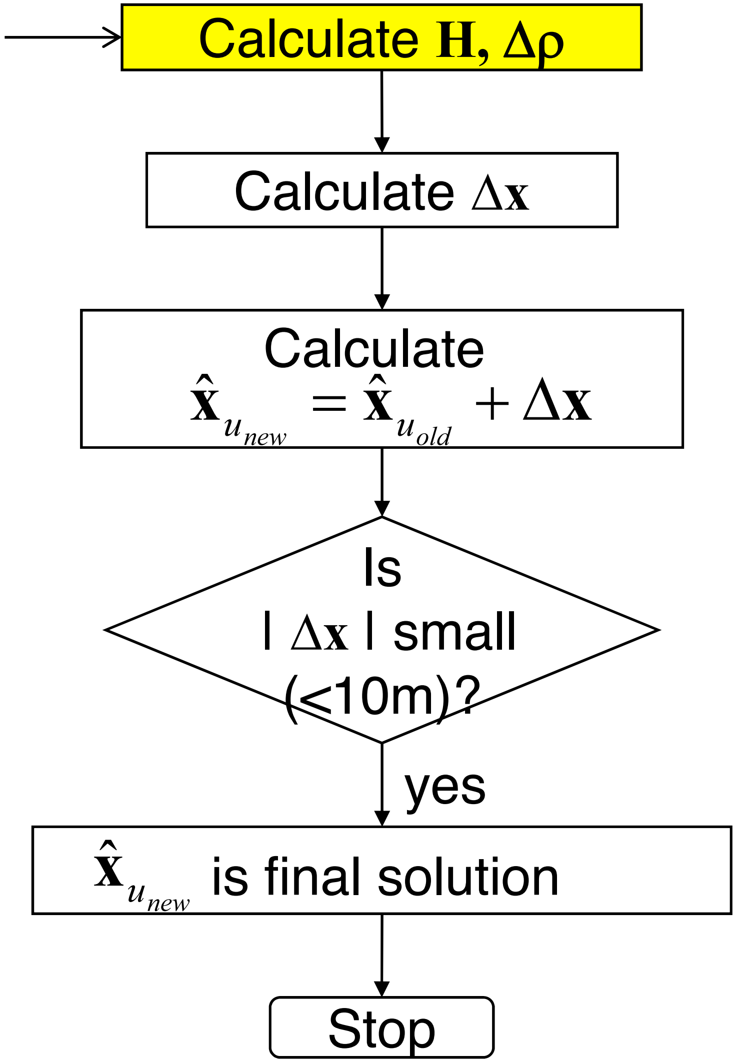
$\hat{\mathbf{x}}_{u_{new}}$ is final solution

Stop

$$\mathbf{H} = \begin{bmatrix} -0.4182 & -0.7030 & -0.5752 & -1 \\ 0.8597 & -0.3609 & 0.3616 & -1 \\ 0.4094 & -0.8896 & -0.2025 & -1 \\ 0.2610 & -0.9143 & -0.3096 & -1 \\ -0.4179 & -0.5533 & 0.7205 & -1 \\ -0.9212 & -0.0637 & 0.3840 & -1 \\ -0.7709 & -0.6102 & -0.1828 & -1 \end{bmatrix}$$



Example: Δρ (Iteration 1)



Calculate **H**, Δρ

Calculate $\Delta\mathbf{x}$

Calculate $\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$

Is I $\Delta\mathbf{x}$ I small (<10m)?

yes

$\hat{\mathbf{x}}_{u_{new}}$ is final solution

Stop

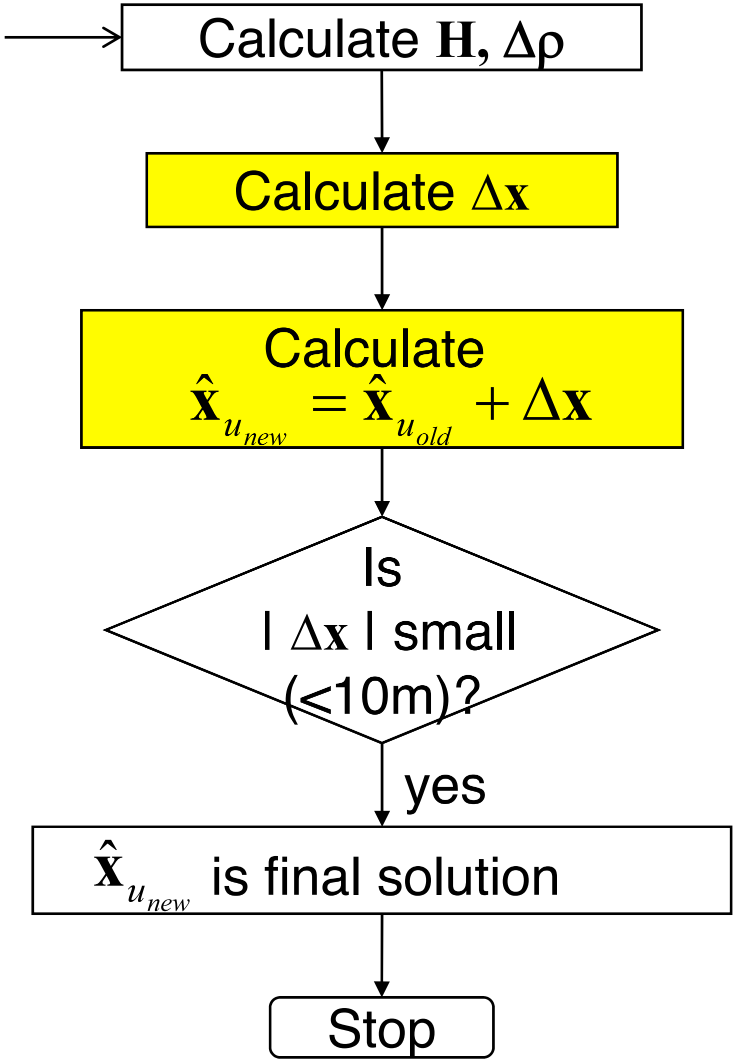
Calculated

Measured (corrected)

$$\Delta\rho = \hat{\rho} - \rho_{corr}$$

PRN	Calculated PR	Measured PR	Delta-Rho
12	24943810.919	24915088.264	28722.655
2	22117181.292	22135304.471	-18123.179
26	22013598.807	22003529.125	10069.682
15	22660408.867	22644973.532	15435.335
29	20990847.857	21026908.513	-36060.657
21	24757718.148	24776961.101	-19242.953
30	24064325.866	24055233.876	9091.990

Example: Solution and Residuals (Iteration 1)



Calculate $\mathbf{H}, \Delta\rho$

Calculate $\Delta\mathbf{x}$

Calculate $\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$

Is $|\Delta\mathbf{x}|$ small (<10m)?

yes

$\hat{\mathbf{x}}_{u_{new}}$ is final solution

Stop

$$\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\Delta\rho$$

$$\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$$

$\hat{\mathbf{x}}_{u_{new}}$	$\hat{\mathbf{x}}_{u_{old}}$	$\Delta\mathbf{x}$
506068.143	506071.529	-3.386
-4882283.665	-4882278.667	-4.998
4059632.252	4109624.557	-49992.305
63.927	15.807	48.120

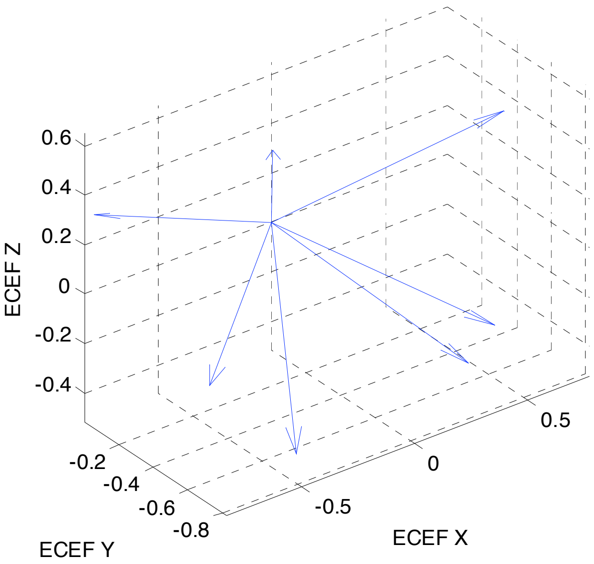
Residuals: $\mathbf{v} = \Delta\rho - \mathbf{H}\Delta\mathbf{x}$

PRN	$\mathbf{v}$	$\Delta\rho$	$\mathbf{H}\Delta\mathbf{x}$
12	9.162	28722.655	28713.493
2	1.699	-18123.179	-18124.878
26	-6.800	10069.682	10076.482
15	-0.178	15435.335	15435.513
29	4.853	-36060.657	-36065.510
21	-3.299	-19242.953	-19239.654
30	-5.436	9091.990	9097.426

Example: H Matrix (Iterations 1 and 2)

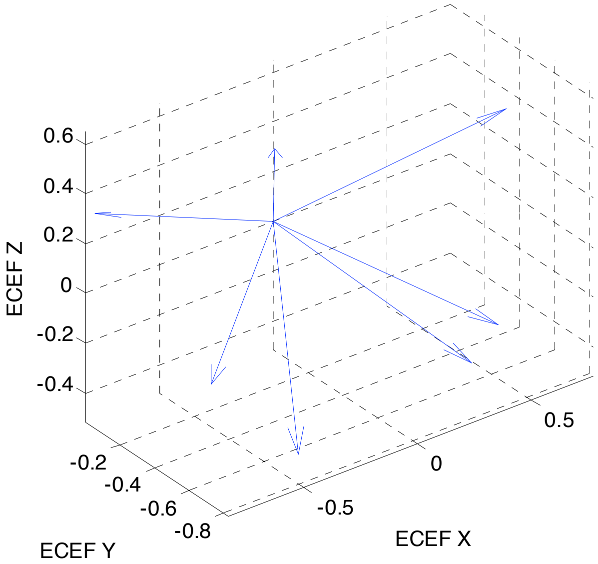
Iteration 1

$$\mathbf{H} = \begin{bmatrix} -0.4182 & -0.7030 & -0.5752 & -1 \\ 0.8597 & -0.3609 & 0.3616 & -1 \\ 0.4094 & -0.8896 & -0.2025 & -1 \\ 0.2610 & -0.9143 & -0.3096 & -1 \\ -0.4179 & -0.5533 & 0.7205 & -1 \\ -0.9212 & -0.0637 & 0.3840 & -1 \\ -0.7709 & -0.6102 & -0.1828 & -1 \end{bmatrix}$$



Iteration 2

$$\mathbf{H} = \begin{bmatrix} -0.4187 & -0.7038 & -0.5739 & -1 \\ 0.8590 & -0.3606 & 0.3635 & -1 \\ 0.4096 & -0.8900 & -0.2003 & -1 \\ 0.2612 & -0.9149 & -0.3076 & -1 \\ -0.4172 & -0.5524 & 0.7217 & -1 \\ -0.9204 & -0.0636 & 0.3857 & -1 \\ -0.7712 & -0.6104 & -0.1808 & -1 \end{bmatrix}$$



Example: $\Delta\rho$ (Iterations 1 and 2)

Iteration 1

Calculated

Measured (corrected)

$$\Delta\rho = \hat{\rho} - \rho_{corr}$$

PRN	Calculated PR	Measured PR	Delta-Rho
12	24943810.919	24915088.264	28722.655
2	22117181.292	22135304.471	-18123.179
26	22013598.807	22003529.125	10069.682
15	22660408.867	22644973.532	15435.335
29	20990847.857	21026908.513	-36060.657
21	24757718.148	24776961.101	-19242.953
30	24064325.866	24055233.876	9091.990

Iteration 2

Calculated

Measured (corrected)

$$\Delta\rho = \hat{\rho} - \rho_{corr}$$

PRN	Calculated PR	Measured PR	Delta-Rho
12	24915130.980	24915088.264	42.716
2	22135355.242	22135304.471	50.771
26	22003576.788	22003529.125	47.662
15	22645023.243	22644973.532	49.711
29	21026941.948	21026908.513	33.435
21	24777000.804	24776961.101	39.703
30	24055278.650	24055233.876	44.773

Example: Solution and Residuals (Iterations 1 and 2)

Iteration 1 $\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\Delta\boldsymbol{\rho}$

$$\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$$

$\hat{\mathbf{x}}_{u_{new}}$	$\hat{\mathbf{x}}_{u_{old}}$	$\Delta\mathbf{x}$
506068.143	506071.529	-3.386
-4882283.665	-4882278.667	-4.998
4059632.252	4109624.557	-49992.305
63.927	15.807	48.120

Residuals: $\mathbf{v} = \Delta\boldsymbol{\rho} - \mathbf{H}\Delta\mathbf{x}$

PRN	$\mathbf{v}$	$\Delta\boldsymbol{\rho}$	$\mathbf{H}\Delta\mathbf{x}$
12	9.162	28722.655	28713.493
2	1.699	-18123.179	-18124.878
26	-6.800	10069.682	10076.482
15	-0.178	15435.335	15435.513
29	4.853	-36060.657	-36065.510
21	-3.299	-19242.953	-19239.654
30	-5.436	9091.990	9097.426

Iteration 2 $\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\Delta\boldsymbol{\rho}$

$$\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$$

$\hat{\mathbf{x}}_{u_{new}}$	$\hat{\mathbf{x}}_{u_{old}}$	$\Delta\mathbf{x}$
506075.869	506068.143	7.726
-4882274.608	-4882283.665	9.057
4059622.275	4059632.252	-9.977
13.120	63.927	-50.807

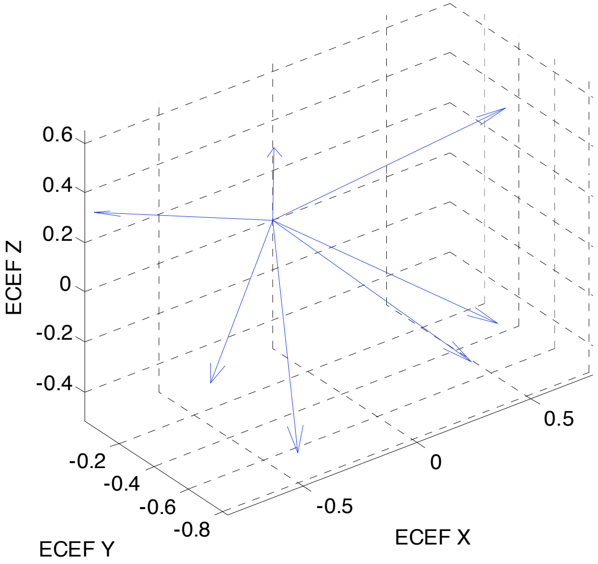
Residuals: $\mathbf{v} = \Delta\boldsymbol{\rho} - \mathbf{H}\Delta\mathbf{x}$

PRN	$\mathbf{v}$	$\Delta\boldsymbol{\rho}$	$\mathbf{H}\Delta\mathbf{x}$
12	-4.208	42.716	46.924
2	0.220	50.771	50.551
26	-0.248	47.662	47.910
15	2.103	49.711	47.609
29	-1.946	33.435	35.381
21	0.431	39.703	39.272
30	3.648	44.773	41.125

Example: H Matrix (Iterations 2 and 3)

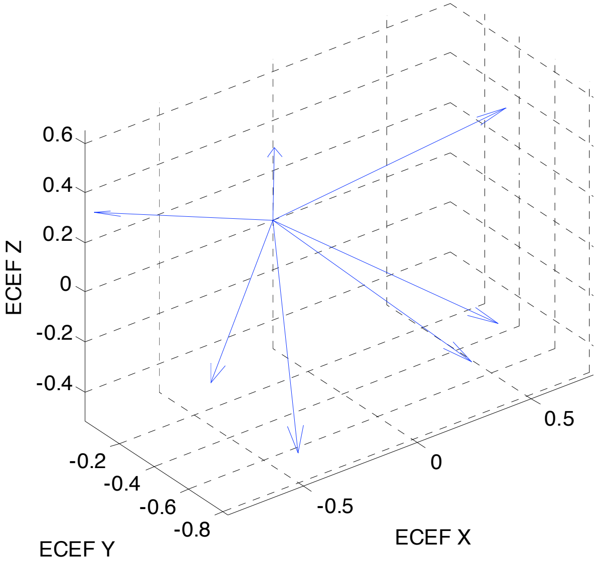
Iteration 2

$$\mathbf{H} = \begin{bmatrix} -0.4187 & -0.7038 & -0.5739 & -1 \\ 0.8590 & -0.3606 & 0.3635 & -1 \\ 0.4096 & -0.8900 & -0.2003 & -1 \\ 0.2612 & -0.9149 & -0.3076 & -1 \\ -0.4172 & -0.5524 & 0.7217 & -1 \\ -0.9204 & -0.0636 & 0.3857 & -1 \\ -0.7712 & -0.6104 & -0.1808 & -1 \end{bmatrix}$$



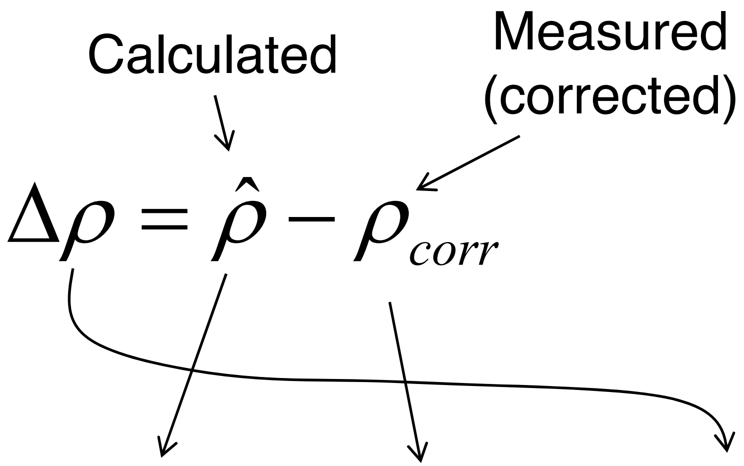
Iteration 3

$$\mathbf{H} = \begin{bmatrix} -0.4187 & -0.7038 & -0.5739 & -1 \\ 0.8590 & -0.3606 & 0.3635 & -1 \\ 0.4096 & -0.8900 & -0.2003 & -1 \\ 0.2612 & -0.9149 & -0.3076 & -1 \\ -0.4172 & -0.5524 & 0.7217 & -1 \\ -0.9204 & -0.0636 & 0.3857 & -1 \\ -0.7712 & -0.6104 & -0.1808 & -1 \end{bmatrix}$$



Example: Δρ (Iterations 2 and 3)

Iteration 2



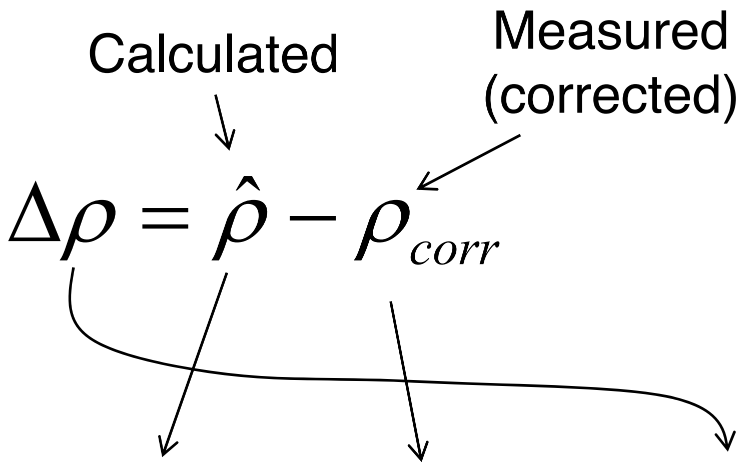
Calculated

Measured (corrected)

$$\Delta\rho = \hat{\rho} - \rho_{corr}$$

PRN	Calculated PR	Measured PR	Delta-Rho
12	24915130.980	24915088.264	42.716
2	22135355.242	22135304.471	50.771
26	22003576.788	22003529.125	47.662
15	22645023.243	22644973.532	49.711
29	21026941.948	21026908.513	33.435
21	24777000.804	24776961.101	39.703
30	24055278.650	24055233.876	44.773

Iteration 3



Calculated

Measured (corrected)

$$\Delta\rho = \hat{\rho} - \rho_{corr}$$

PRN	Calculated PR	Measured PR	Delta-Rho
12	24915084.055	24915088.264	-4.208
2	22135304.691	22135304.471	0.220
26	22003528.878	22003529.125	-0.248
15	22644975.634	22644973.532	2.103
29	21026906.567	21026908.513	-1.946
21	24776961.532	24776961.101	0.431
30	24055237.525	24055233.876	3.648

Example: Solution and Residuals (Iterations 2 and 3)

Iteration 2

$$\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\Delta\boldsymbol{\rho}$$

$$\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$$

$\hat{\mathbf{x}}_{u_{new}}$	$\hat{\mathbf{x}}_{u_{old}}$	$\Delta\mathbf{x}$
506075.869	506068.143	7.726
-4882274.608	-4882283.665	9.057
4059622.275	4059632.252	-9.977
13.120	63.927	-50.807

Residuals: $\mathbf{v} = \Delta\boldsymbol{\rho} - \mathbf{H}\Delta\mathbf{x}$

PRN	$\mathbf{v}$	$\Delta\boldsymbol{\rho}$	$\mathbf{H}\Delta\mathbf{x}$
12	-4.208	42.716	46.924
2	0.220	50.771	50.551
26	-0.248	47.662	47.910
15	2.103	49.711	47.609
29	-1.946	33.435	35.381
21	0.431	39.703	39.272
30	3.648	44.773	41.125

Iteration 3

$$\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\Delta\boldsymbol{\rho}$$

$$\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$$

$\hat{\mathbf{x}}_{u_{new}}$	$\hat{\mathbf{x}}_{u_{old}}$	$\Delta\mathbf{x}$
506075.869	506075.869	0.000
-4882274.608	-4882274.608	0.000
4059622.275	4059622.275	0.000
13.120	13.120	0.000

↑ On order of 10^{-6}

Residuals: $\mathbf{v} = \Delta\boldsymbol{\rho} - \mathbf{H}\Delta\mathbf{x}$

PRN	$\mathbf{v}$	$\Delta\boldsymbol{\rho}$	$\mathbf{H}\Delta\mathbf{x}$
12	-4.208	42.716	46.924
2	0.220	50.771	50.551
26	-0.248	47.662	47.910
15	2.103	49.711	47.609
29	-1.946	33.435	35.381
21	0.431	39.703	39.272
30	3.648	44.773	41.125

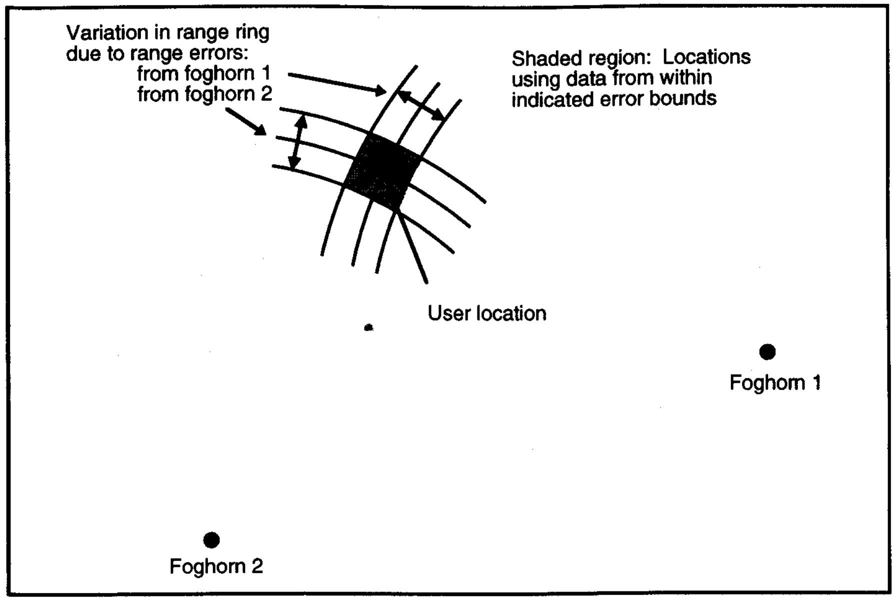
Convergence

- Practically speaking, getting the system to converge with GNSS is easy
  - Example showed case where initial guess was 50 km in error
  - Can start with the center of the Earth as a guess, and it would only add an iteration or two
  - Normally, a receiver will use its last solution as a starting point, so only a single iteration is necessary
- Nonlinearities (which drive the need for iteration) are more severe when dealing with pseudolites
  - Much closer to receiver than satellite
  - H matrix varies more quickly as a function of position

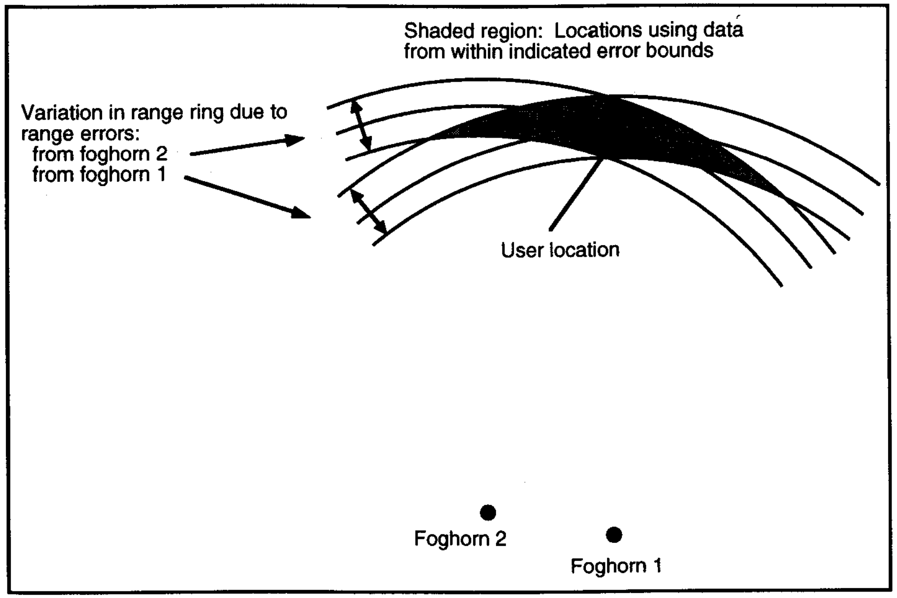
Effect of Geometry on Positioning Accuracy (Foghorn Example)

Consider the foghorn example, except allow for a measurement error

Good Geometry Example



Poor Geometry Example



Obtaining $\mathbf{C}_\mathbf{x}$ from Least-Squares Analysis (2/2)

- According to least-squares theory:

$$\mathbf{C}_\mathbf{x} = \left(\mathbf{H}^T \mathbf{C}_\rho^{-1} \mathbf{H}\right)^{-1}$$

  - Basic assumptions
    - Measurement errors are zero-mean
    - Measurement errors have a Gaussian distribution
- Recall that the least-squares solution with measurement weighting was

$$\begin{aligned}\Delta\mathbf{x} &= \left(\mathbf{H}^T \mathbf{C}_\rho^{-1} \mathbf{H}\right)^{-1} \mathbf{H}^T \mathbf{C}_\rho^{-1} \Delta\rho \\ &= \mathbf{C}_\mathbf{x} \mathbf{H}^T \mathbf{C}_\rho^{-1} \Delta\rho\end{aligned}$$

  - Consider case where the nominal position and clock error (used to calculate $\Delta\rho$) are actually the true position and clock error
    - The $\Delta\rho$ represents the measurement *errors*
    - The $\Delta\mathbf{x}$ represents the position and clock *errors*
    - The $\mathbf{C}_\mathbf{x}$ matrix is a multiplier for the measurement errors ($\Delta\rho$)
      - "Large" $\mathbf{C}_\mathbf{x}$ values → large position errors
      - "Small" $\mathbf{C}_\mathbf{x}$ values → small position errors

Dilution of Precision (DOP)

- In GPS, the concept of Dilution of Precision (DOP) is used
  - Based upon covariance matrix of position and clock errors ($\mathbf{C_x}$)
  - Additional assumptions
    - All measurements have the same variance

$$\sigma_{\rho_1}^2 = \sigma_{\rho_2}^2 = \ldots = \sigma_{\rho_n}^2 = \sigma_{\rho}^2$$

    - Measurement errors are uncorrelated (i.e.,covariance values are zero)

$$\sigma_{\rho_j \rho_k} = 0, \qquad j \neq k$$

  - Using these assumptions

$$\mathbf{C}_{\boldsymbol{\rho}} = \mathbf{I}\sigma_{\rho}^2$$

and

$$\mathbf{C_x} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\sigma_{\rho}^2$$

  - The matrix $\left(\mathbf{H}^T\mathbf{H}\right)^{-1}$ is called the DOP matrix
    - Directly relates measurement errors to position errors

Use of Local-Level Coordinate Frame (1/2)

- Normally, DOPs describe errors in geodetic (local-level) coordinate frame (east, north, up), rather than the ECEF frame.
  - Need to modify the H matrix so that the errors refer to the local-level frame
  - Original H matrix (used to calculate position)

$$\mathbf{H}^E = \begin{bmatrix} \mathbf{a}_1^{E^T} & 1 \\ \mathbf{a}_2^{E^T} & 1 \\ \vdots & \vdots \\ \mathbf{a}_n^{E^T} & 1 \end{bmatrix}$$

    - "$\mathbf{a}$" vectors are unit line-of-sight vectors between user and SV in *ECEF frame*
    - This will give the $\mathbf{C}_\mathbf{x}$ matrix described previously
  - New H matrix for DOP calculations

$$\mathbf{H}^G = \begin{bmatrix} \mathbf{a}_1^{G^T} & 1 \\ \mathbf{a}_2^{G^T} & 1 \\ \vdots & \vdots \\ \mathbf{a}_n^{G^T} & 1 \end{bmatrix}$$

    - "$\mathbf{a}$" vectors are now unit line-of-sight vectors between user and SV in *geodetic (ENU) frame*

Use of Local-Level Coordinate Frame (2/2)

- Local-level "**a**" vectors can be calculated using direction cosine matrix (DCM)

$$\mathbf{a}^{G} = \mathbf{C}_{E}^{G}\mathbf{a}^{E}$$

$$\mathbf{C}_{E}^{G} = \text{DCM that rotates from ECEF to geodetic (E,N,U) frame}$$

$$\mathbf{C}_{E}^{G} = \left(\mathbf{C}_{G}^{E}\right)^{-1} = \left(\mathbf{C}_{G}^{E}\right)^{T}$$

- When $\mathbf{H}^{G}$ is used to calculate the covariance $\mathbf{C}_{\mathbf{x}} = \left(\mathbf{H}^{G^T}\mathbf{H}^{G}\right)^{-1}\sigma_{\rho}^{2}$, then $\mathbf{C}_{\mathbf{x}}$ is defined as

$$\mathbf{C}_{\mathbf{x}} = \begin{bmatrix} \sigma_{e}^{2} & \sigma_{en} & \sigma_{eu} & \sigma_{e\delta t_u} \\ \sigma_{en} & \sigma_{n}^{2} & \sigma_{nu} & \sigma_{n\delta t_u} \\ \sigma_{eu} & \sigma_{nu} & \sigma_{u}^{2} & \sigma_{u\delta t_u} \\ \sigma_{e\delta t_u} & \sigma_{n\delta t_u} & \sigma_{u\delta t_u} & \sigma_{\delta t_u}^{2} \end{bmatrix}$$

  - This is what we desire to describe using DOPs

DOP Values

- Desirable to characterize the $\mathbf{C_x}$ matrix using a single number
  - For DOPs
    - Cross-correlation terms ignored
    - Root-Sum-Square (RSS) value of variables of interest, normalized by σ_{UERE}
    - Example:

$$GDOP = \frac{\sqrt{\sigma_e^2 + \sigma_n^2 + \sigma_u^2 + \sigma_{\delta t_u}^2}}{\sigma_{UERE}}$$

    - GDOP can be calculated directly from DOP matrix

$$\left(\mathbf{H}^{G^T}\mathbf{H}^G\right)^{-1} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{bmatrix} \qquad GDOP = \sqrt{D_{11} + D_{22} + D_{33} + D_{44}}$$

    - Note that GDOP relates UERE with RSS of errors

Key relationship!

$$\sqrt{\sigma_e^2 + \sigma_n^2 + \sigma_u^2 + \sigma_{\delta t_u}^2} = GDOP \times \sigma_{UERE}$$

Types of DOPs

- The "Big Three"
  - GDOP (Geometric DOP)

$$GDOP = \sqrt{D_{11} + D_{22} + D_{33} + D_{44}}$$

$$\sqrt{\sigma_e^2 + \sigma_n^2 + \sigma_u^2 + \sigma_{\delta t_u}^2} = GDOP \times \sigma_{UERE}$$

  - PDOP (Position DOP)

$$PDOP = \sqrt{D_{11} + D_{22} + D_{33}}$$

$$\sqrt{\sigma_e^2 + \sigma_n^2 + \sigma_u^2} = PDOP \times \sigma_{UERE}$$

  - HDOP (Horizontal DOP)

$$HDOP = \sqrt{D_{11} + D_{22}}$$

$$\sqrt{\sigma_e^2 + \sigma_n^2} = HDOP \times \sigma_{UERE}$$

- Less common (for navigators, at least!)
  - VDOP (Vertical DOP)

$$VDOP = \sqrt{D_{33}}$$

$$\sqrt{\sigma_u^2} = VDOP \times \sigma_{UERE}$$

  - TDOP (Time DOP)

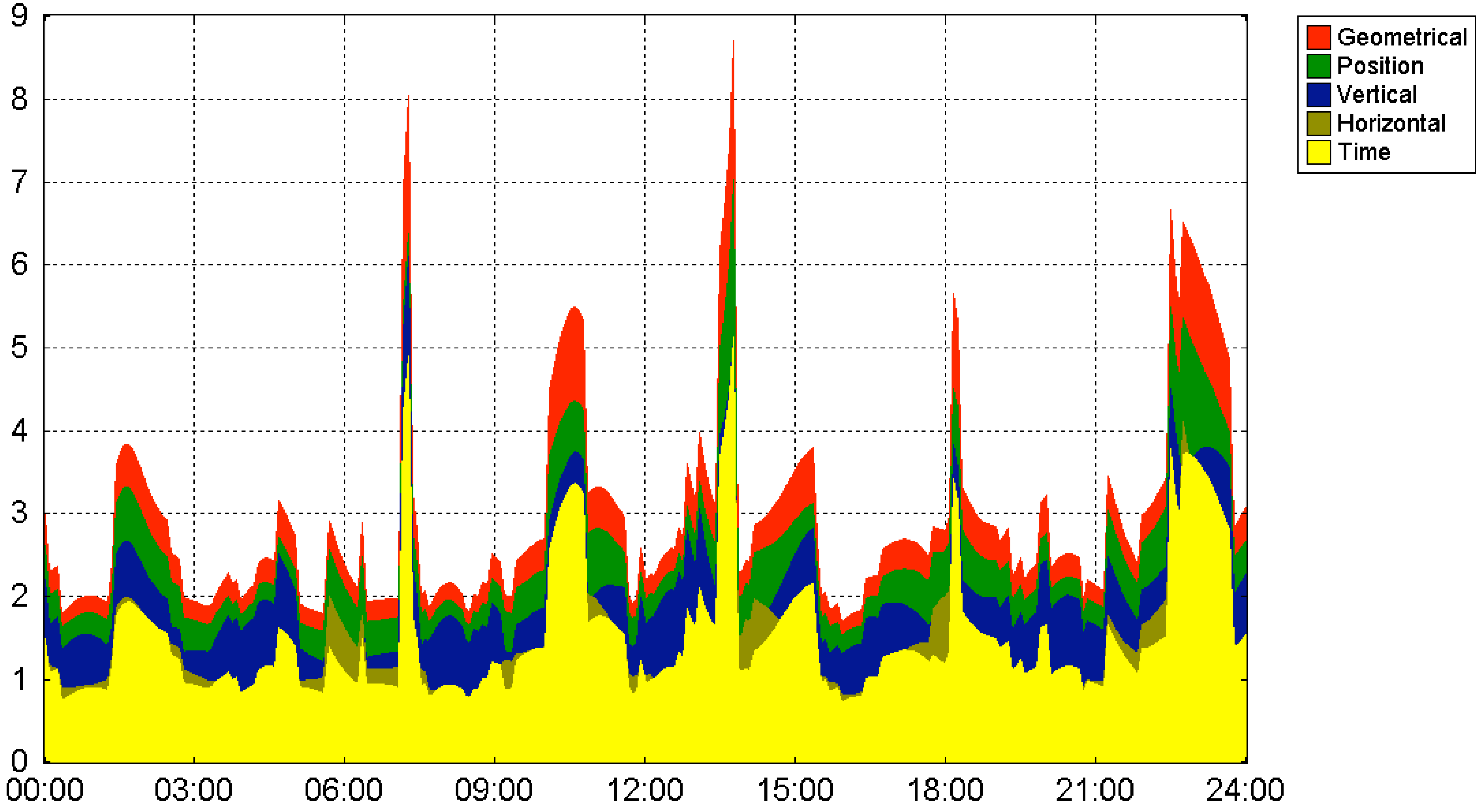
$$TDOP = \sqrt{D_{44}}$$

$$\sqrt{\sigma_{\delta t_u}^2} = TDOP \times \sigma_{UERE}$$

    - Note: time is in units of meters

Typical DOP Plot

Dayton Ohio – 24 Apr 2003 – All Visible SVs (above 10° elevation)

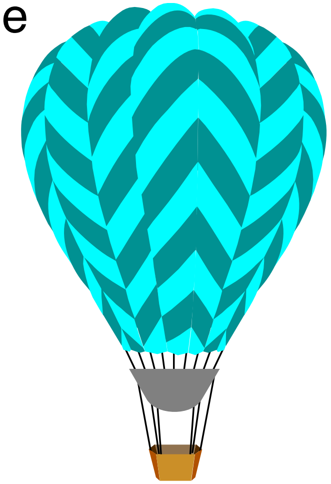


Kalman Filtering Overview

- Kalman filtering is an estimation approach that can be applied to the GPS positioning application
  - Many other application areas
- Concepts
  - Information describing the system
    - State vector
    - Covariance matrix
  - Propagating state and covariance forward in time
  - Using measurements to update the state and covariance
- Will be covered at conceptual level
  - Very few equations
  - Purpose is to describe concept of Kalman filtering as applied to this problem
  - Additional references:
    - Maybeck, *Stochastic Models, Estimation, and Control*, vols. 1-3, Navtech, 1994.
    - Gelb (ed.), *Applied Optimal Estimation*, M.I.T. Press, 1994.
    - Many others

Kalman Filtering: Information Describing the System (1/2)

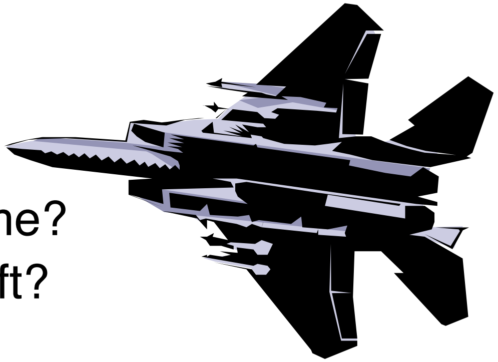
- State vector
  - Set of variables that
    - Describe everything you want to know about the system
    - Include all of the information needed to determine how the system changes over time
  - Example: A reasonable state vector for positioning a hot air balloon would be



$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$$

$x, y, z =$ ECEF position of balloon

$\dot{x}, \dot{y}, \dot{z} =$ ECEF velocity of balloon



  - Does this describe what we want to know?
  - Does this describe how the system changes over time?
  - Would this be a good state vector for a fighter aircraft?

Kalman Filtering:
Information Describing the System (2/2)

- Covariance matrix
  - The covariance matrix basically describes how well the state is known
    - If the system only gives a state output, it's not that useful.
    - If it outputs the state and tells how accurate it is, then you have information that you can confidently act upon.
    - Hot air balloon example: the system state tells me that I'm 300 m above the ground descending at a rate of 10 m/sec.
      - Need to know covariance matrix as well.
        - » Case 1: Position accuracy = 10 m 1- σ, velocity accuracy = 1 m/sec 1-σ → probably not in danger until ~30 seconds
        - » Case 2: Position accuracy = 400 m 1- σ, velocity accuracy = 15 m/sec 1- σ → you could hit the ground any second!
  - How to interpret covariance matrix
    - Diagonal terms are the error variances of the estimated states
    - Off-diagonal terms are cross-covariances, describing the correlations of the errors between the states

Kalman Filtering:
Propagating Covariance and State Forward in Time

- State vector and covariance matrix can be propagated forward in time
  - If you know the current state estimate, you can determine the state estimate at a point in the future
  - If you know the current covariance matrix, you can determine the covariance matrix at a point in the future
  - Information about how the state and covariance changes over time is given in
    - Dynamics matrix **F**: $\dot{\mathbf{x}} = \mathbf{F}\mathbf{x}$
    - State transition matrix $\mathbf{\Phi}$: $\mathbf{x}(t_1) = \mathbf{\Phi}(t_1 - t_0)\mathbf{x}(t_0)$
  - When propagating covariance forward in time, *process noise* is added to account for
    - Unmodeled dynamics
    - Unmodeled system inputs
    - Anything else that decreases the ability to predict the future state using the current state
  - Process noise increases uncertainty (i.e., larger covariance values)

Kalman Filtering: Measurement Updates

- A measurement gives information about the state values
  - Examples: GPS pseudorange (for position or clock bias) or Doppler (for velocity or clock drift)
- Effects of a measurement update
  - State values are adjusted to reflect the measurement
  - Covariance matrix is adjusted to reflect how well the state is known, now that the measurement is available
    - Measurements always decrease uncertainty (i.e., smaller covariance values)
- Measurement noise
  - Description of how precise the measurement is
  - The effect of measurement on state and covariance determined by tradeoff between
    - Measurement noise (how good the measurement is)
    - Covariance matrix (how well the state is known at this point)
- Relationship between measurement and states given by $\mathbf{H}$ matrix (same as least-squares)

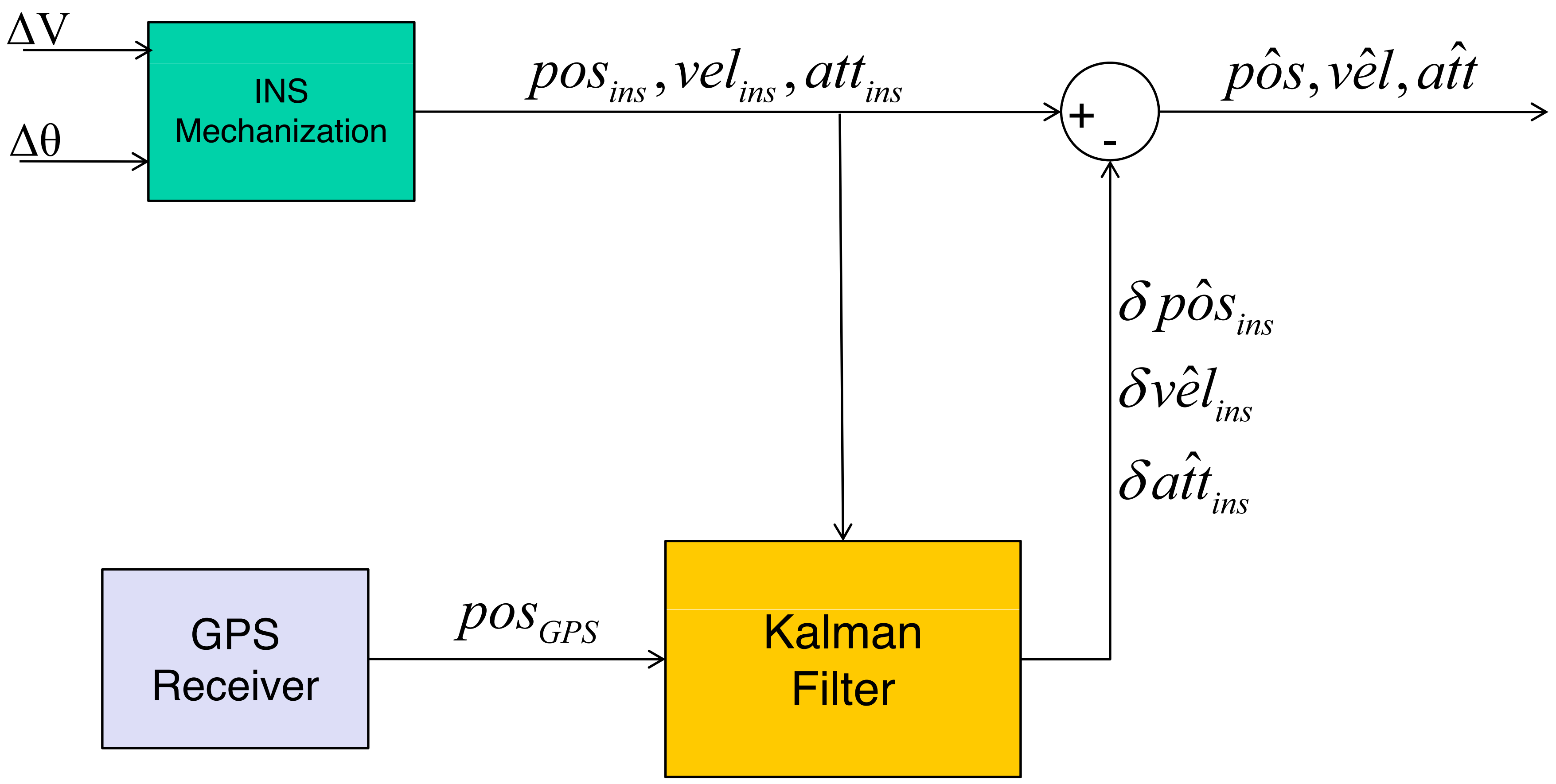
Inertial Navigation Systems

- Sensors
  - Accelerometers
    - Measure specific force f = a + g
  - Gyroscopes
    - Measure rotation about an inertial frame
  - Altitude aiding (required!)
    - Normally a barometric altimeter, but can be other things
- Mechanization equations
  - Attitude computation
  - Resolution of accelerometers into desired frame
  - Subtraction of gravity
  - Double integration
  - Accounting for rotation as vehicle moves around Earth
  - Schuler oscillation

Error Characteristics of Inertial Systems

- Very good high-frequency characteristics
- Long-term drift (poor low-frequency characteristics)
- Categorization of inertial systems
  - Navigation-grade
  - Tactical-grade
  - Commerical-grade
- All inertial systems have errors that grow unbounded unless aided by another sensor
- What would be the ideal sensor?
  - Good low-frequency characteristics (little long term drift)
  - Doesn’t necessarily need to have good high-frequency characteristics
- Good candidate: GPS!

Loosely Coupled, Feed-Forward INS/GPS Integration Approach



ΔV

$\Delta\theta$

INS Mechanization

$pos_{ins}, vel_{ins}, att_{ins}$

$\hat{pos}, \hat{vel}, \hat{att}$

$\delta \hat{pos}_{ins}$

$\delta \hat{vel}_{ins}$

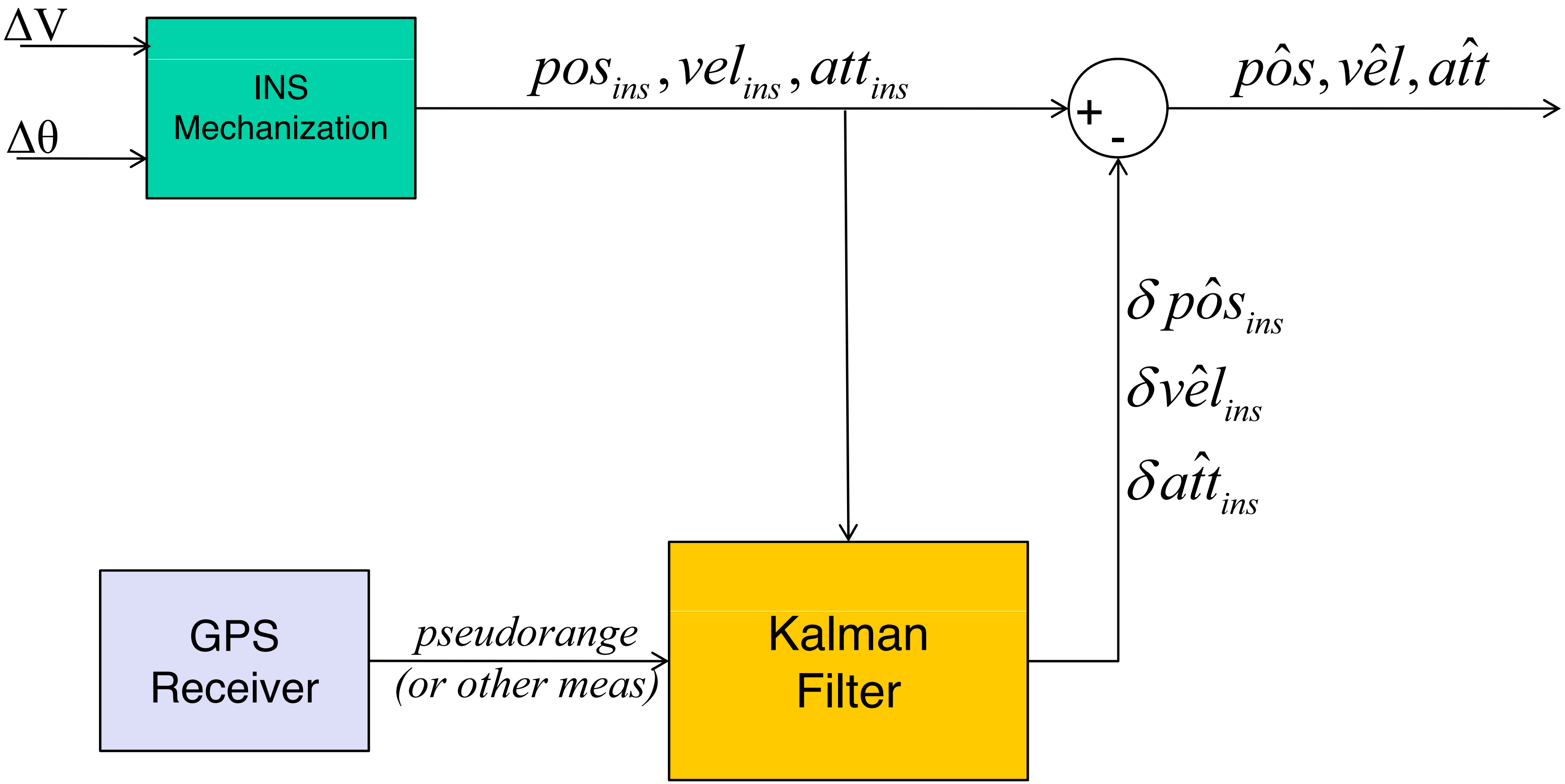
$\delta \hat{att}_{ins}$

GPS Receiver

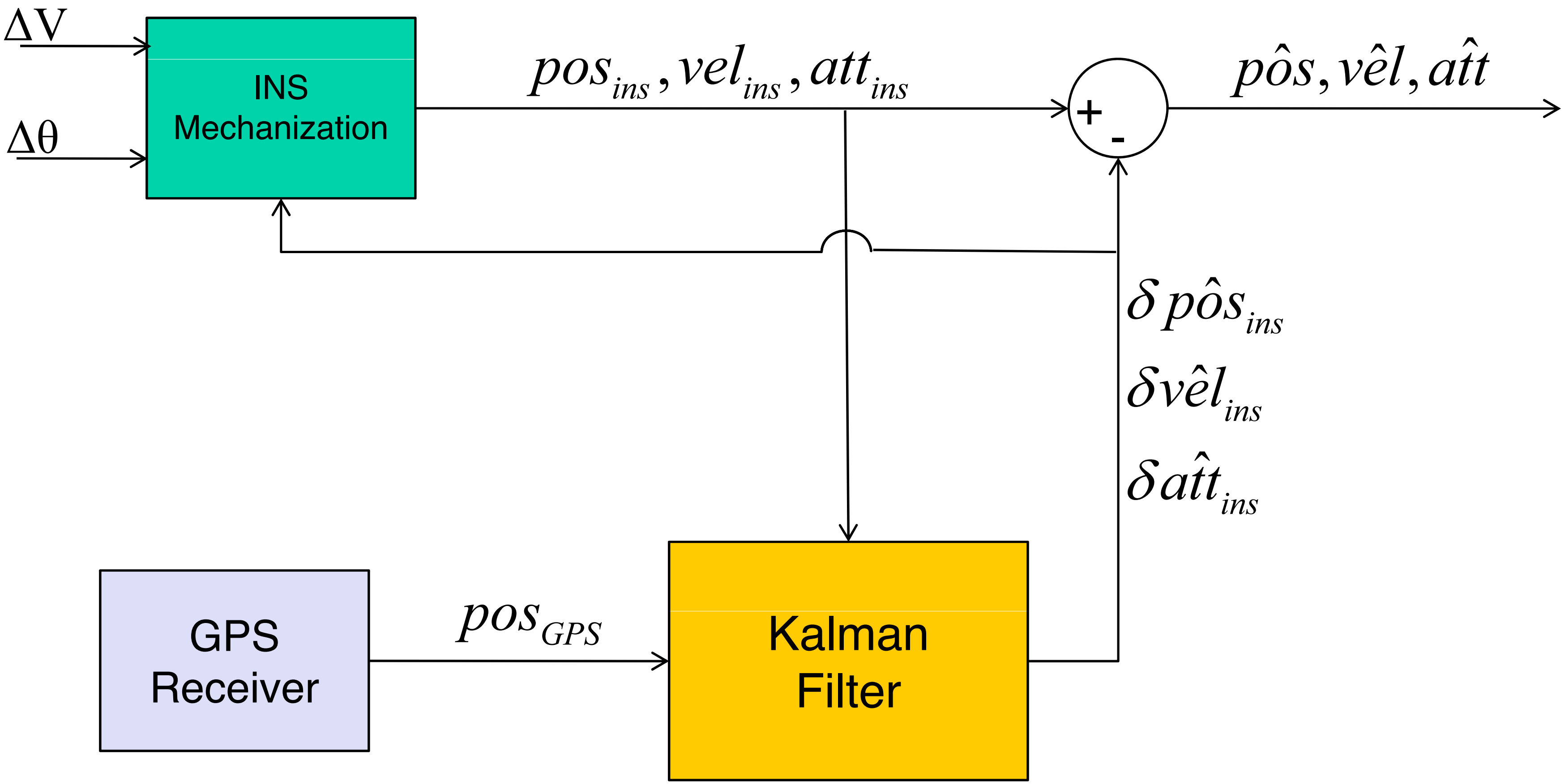
pos_{GPS}

Kalman Filter

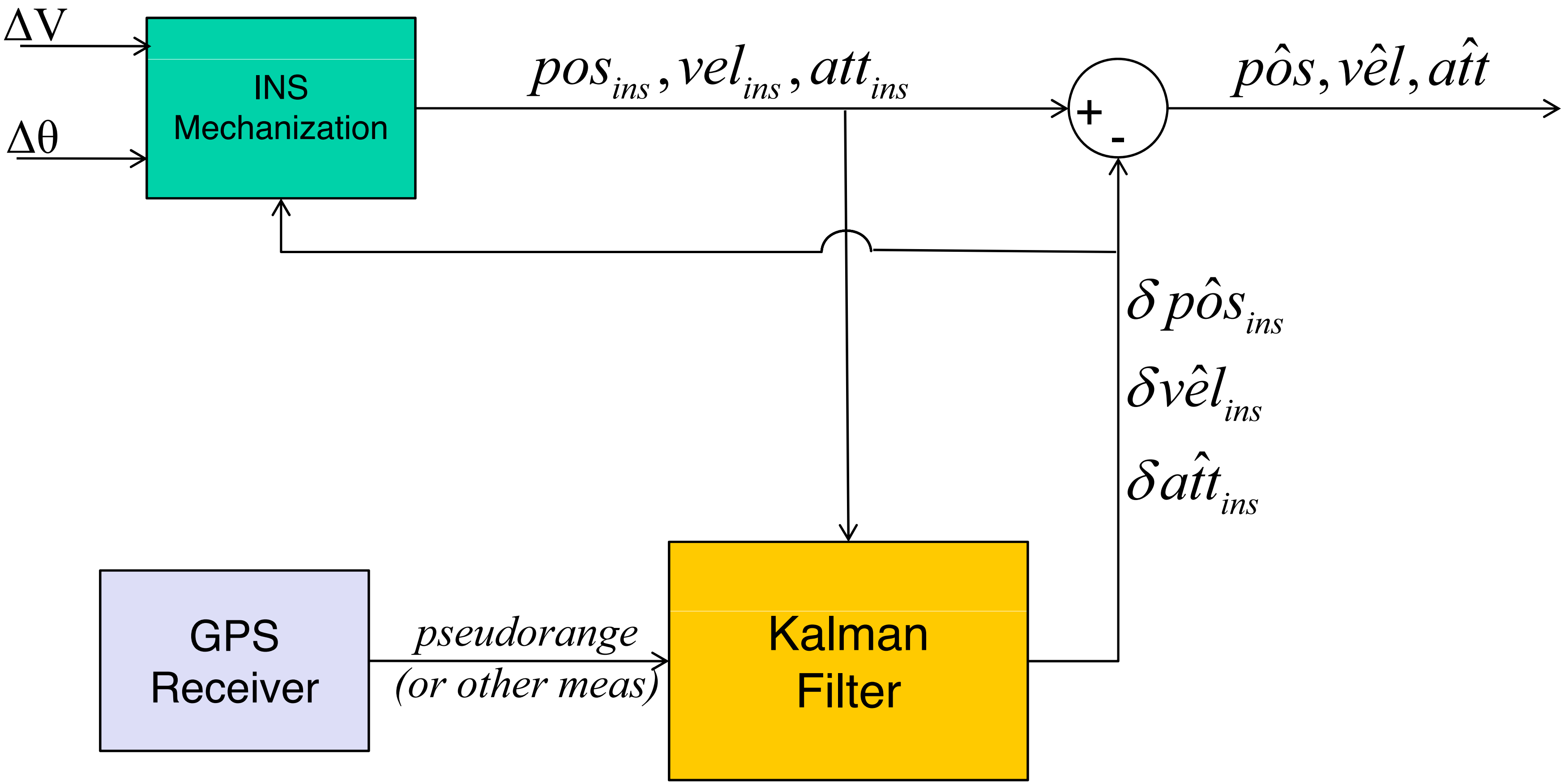
Tightly Coupled, Feed-Forward INS/GPS Integration Approach



Loosely Coupled, Feedback INS/GPS Integration Approach



Tightly Coupled, Feedback INS/GPS Integration Approach



GPS and Time

- Four relevant time standards:
  - UT1: Based on Earth's rotation with respect to sun
  - TAI (International Atomic Time)
    - Fundamental SI unit of time
    - 1 second = duration of 9,192,631,770 periods of the radiation corresponding to the translation between two hyperfine levels of the ground state of the Cesium 133 atom[1]
  - UTC: Atomic-based time standard (tracking TAI), artificially adjusted to stay within 0.9 sec of UT1
    - Occasional leap-second
  - GPS system time: atomic-based time standard based upon UTC (but without leap-seconds)
    - GPS control segment attempts to have GPS system time closely follow UTC
- GPS-UTC time difference is transmitted in navigation message

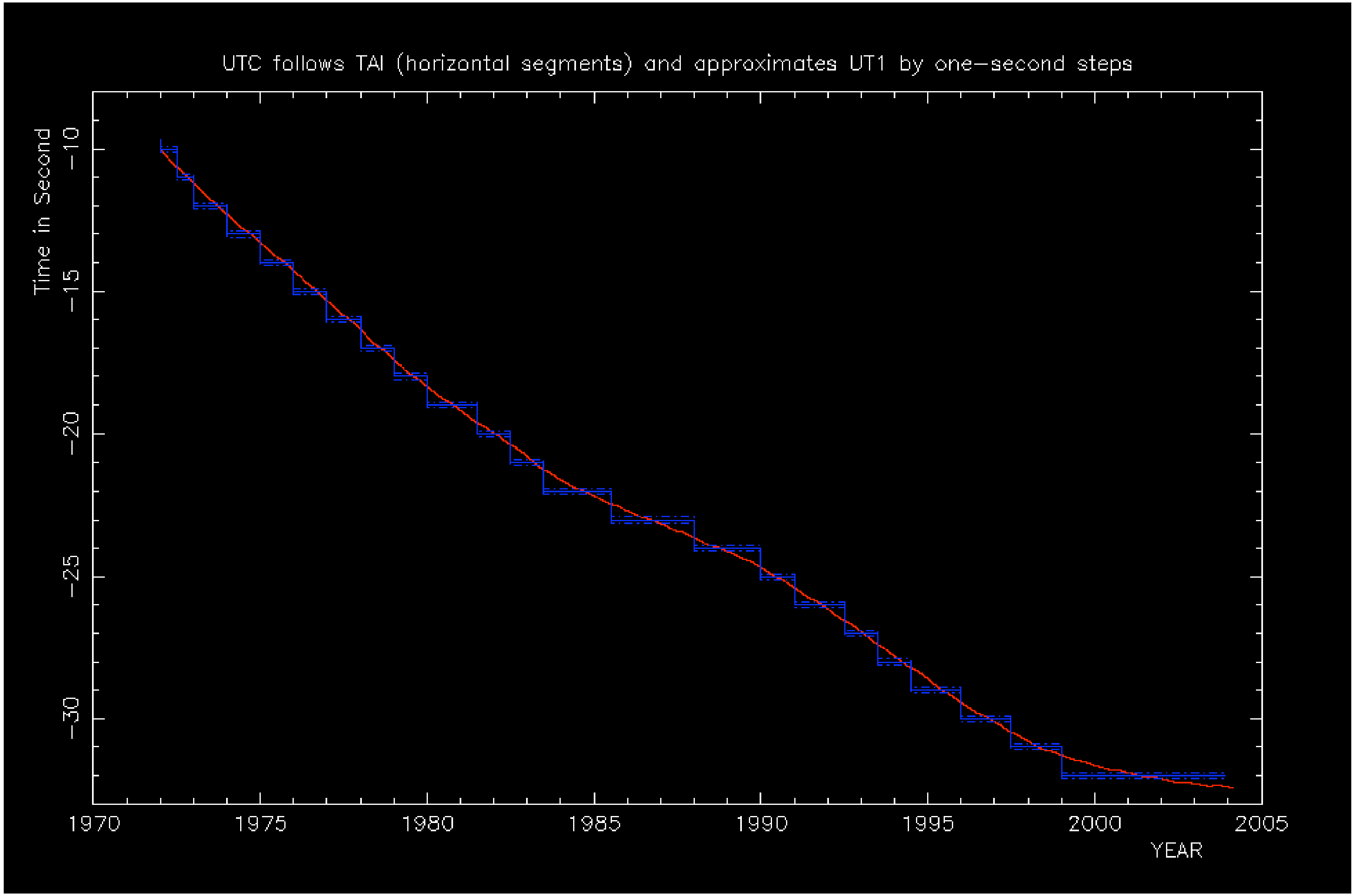
$$\Delta t_{UTC/GPS} = t_{GPS} - t_{UTC} = \underbrace{\Delta t_{LS}}_{\text{Leap Seconds}} + \underbrace{A_0} + \underbrace{A_1}(t_{GPS} - \underbrace{t_{0U}})$$

From Nav Message

  - 1-σ accuracy of $\Delta t_{UTC/GPS}$ is approx. 10 ns

[1] Seeber, G. *Satellite Geodesy: Foundations, Methods, and Applications*, Walter De Gruyter, 1993.

Comparison of UTC and UT1 with TAI



http://hpiers.obspm.fr/eop-pc/earthor/utc/leapsecond.html

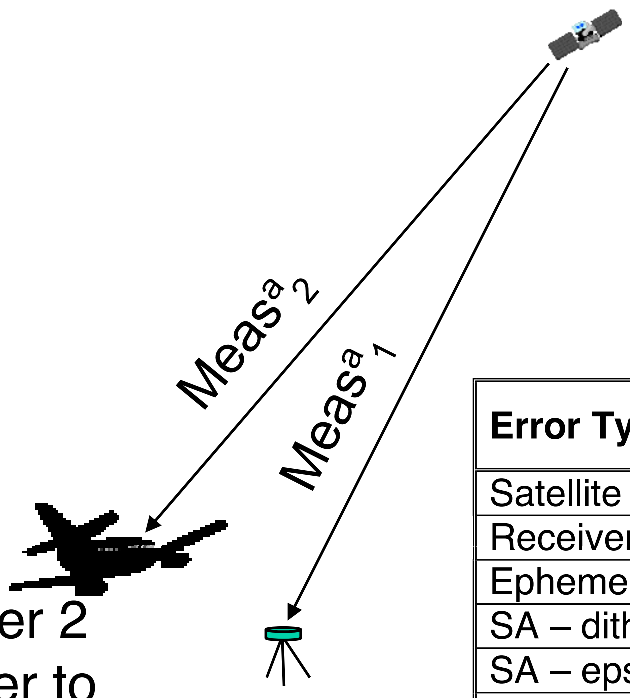
History of UTC-GPS Time Differences

Date	GPS-UTC Time (sec)	
6 Jan 1980	0	(Start of GPS system time)
1 Jul 1981	1	
1 Jul 1982	2	
1 Jul 1983	3	
1 Jul 1985	4	
1 Jan 1988	5	
1 Jan 1990	6	
1 Jan 1991	7	
1 Jul 1992	8	
1 Jul 1993	9	
1 Jul 1994	10	
1 Jan 1996	11	
1 Jul 1997	12	
1 Jan 1999	13	
1 Jan 2006	14	
1 Jan 2009	15	

Differential GPS

“What in the world is differential GPS, and
how does it work?”

Differential GPS Concept



Error Type	**Comparison between $\text{Meas}_1{}^a$ and $\text{Meas}_2{}^a$**	**DGPS Effect on Error**
Satellite clock error	Same	Removed
Receiver clock error	Different (uncorrelated)	Added
Ephemeris error[1]	Very similar[2]	Reduced[2]
SA – dither	Same	Removed
SA – epsilon	Very similar[2]	Reduced[2]
Ionospheric delay	Very similar[2]	Reduced[2]
Tropospheric delay	Very similar[2]	Reduced[2]
Multipath	Different (uncorrelated)	Added (and amplified)
Measurement noise	Different (uncorrelated)	Added (and amplified)

[1] Effect of ephemeris error on positioning (actually only affects the calculated range, not the actual measurement)

[2] Errors grow as the separation distance between receivers 1 and 2 increases. (The errors are the same and are removed for very short baseline distances).

DGPS Variations

- DGPS is a broad term, and there are many different ways DGPS can be applied.
  - Measurements used
    - Code only
    - Carrier-smoothed code
    - Carrier-phase
  - Application type
    - Positioning
    - Attitude
  - Position domain vs. measurement domain
  - Post-processing vs. real-time
  - Type of correction
  - Number of reference receivers
  - Area of coverage
    - LADGPS
    - RADGPS
    - WADGPS
  - Differencing method used
    - Single-differencing
    - Double-differencing

576 possible combinations!

- Each of these will be covered in the slides that follow.

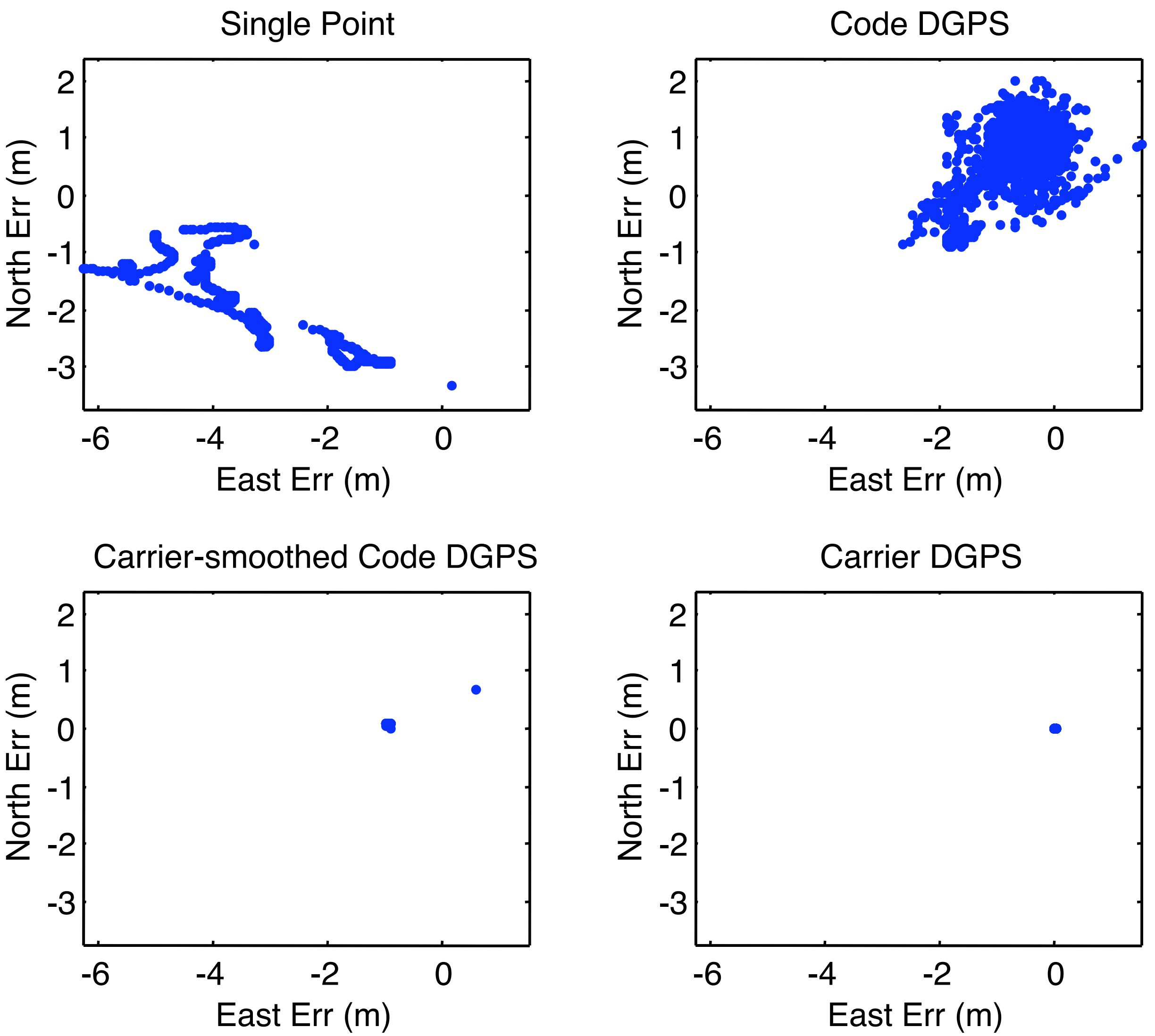
DGPS - Measurements Used (1/5)

- The type of measurements is one of the primary distinguishing factors between different DGPS implementations
  - Code only
    - Simplest to implement
    - Based purely on pseudorange measurements
    - In best case (short baseline), errors include code multipath and noise
    - Typical accuracy: 2-4 m
  - Carrier-smoothed code
    - Carrier-phase measurement is very precise (~1 cm), but it is not an absolute measurement (due to unknown integer ambiguity).
    - Code (pseudorange) measurement is absolute, but it is much less precise (~1-2 m).
    - A filter can be used to combine the carrier-phase and the code measurements to take advantage of their respective strengths.
      - Filter time constant limited by code-carrier ionospheric divergence (due to different signs on ionospheric error term)
    - Carrier-phase smoothing of the code essentially removes most of the code multipath and noise
    - Typical carrier-smoothed code DGPS accuracy: 0.1-0.5 m
    - Relatively easy to implement

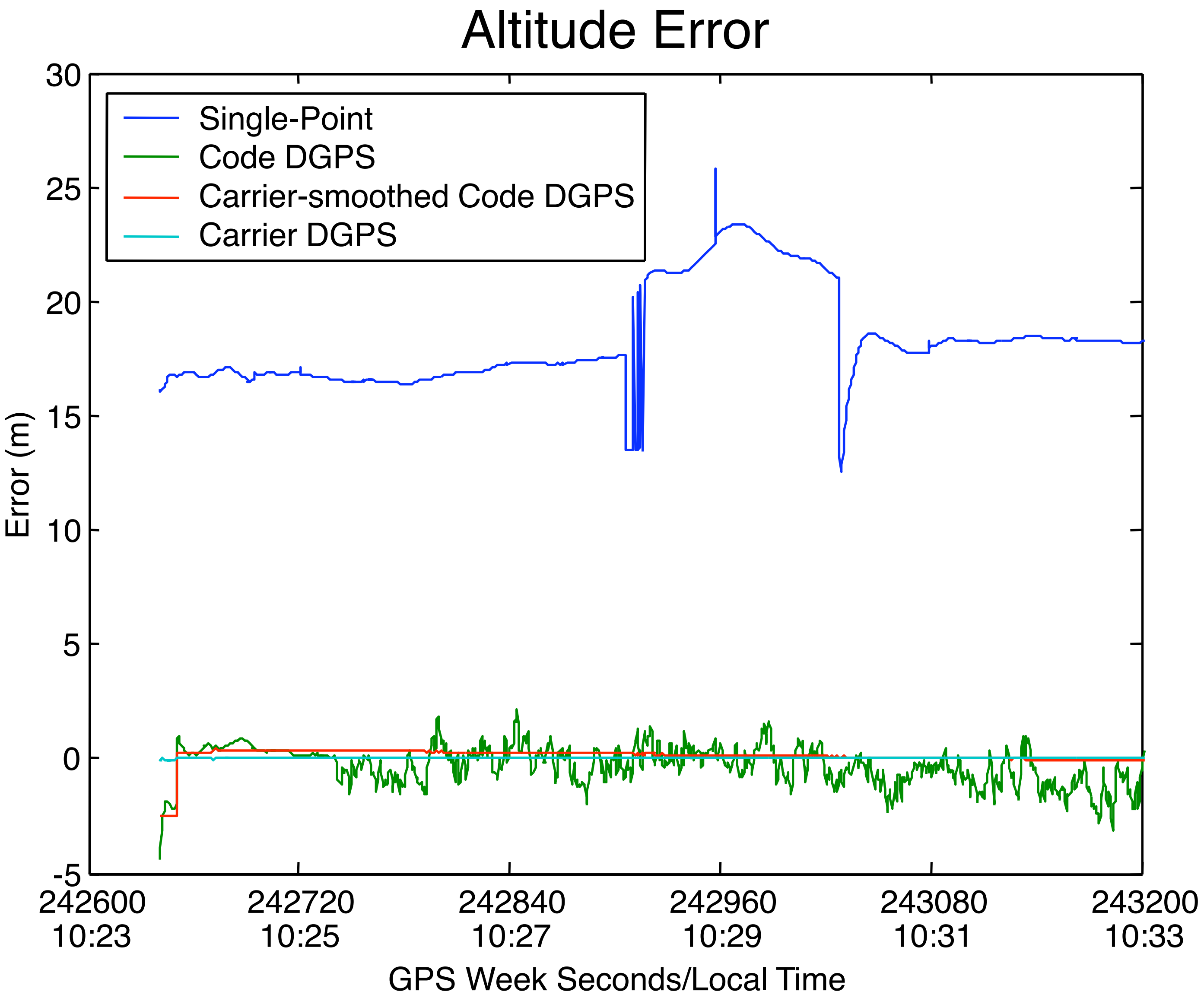
DGPS - Measurements Used (2/5)

- Type of measurements (continued)
  - Carrier-phase
    - GPS receiver can track exact phase of incoming GPS carrier
      - Can determine “where” in the cycle
      - Cannot determine “which” cycle
      - Results in an unknown integer ambiguity
    - If carrier-phase integer ambiguities can be determined, then the carrier-phase measurement will yield the most precise (and accurate) positioning possible
    - Fairly complex to implement
    - Difficult to resolve integer ambiguities over long reference/mobile receiver baselines
    - Normally requires some period of time to resolve ambiguities
      - 1-3 minutes typical
      - Depends upon baseline distance, algorithm
    - Extremely sensitive to loss of carrier-lock (or cycle slips)
    - Often, code measurements will be used to initially aid in determining the integer ambiguities
      - Final solution normally based primarily on carrier-phase measurements
    - Typical accuracy: 0.01-0.05 m

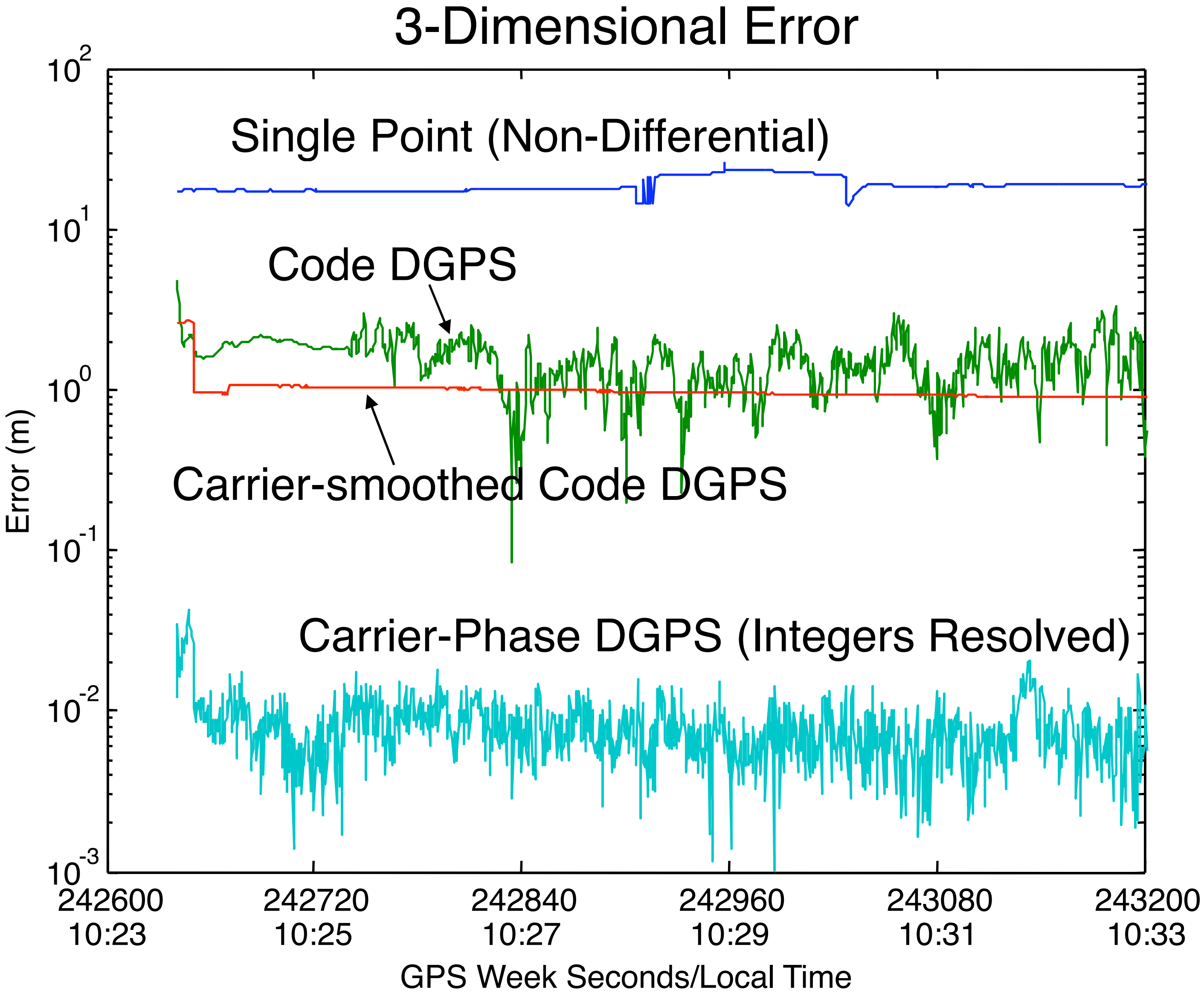
DGPS - Measurements Used (3/5)
Sample Comparison of Horizontal Error



DGPS - Measurements Used (4/5)
Sample Comparison of Altitude Error

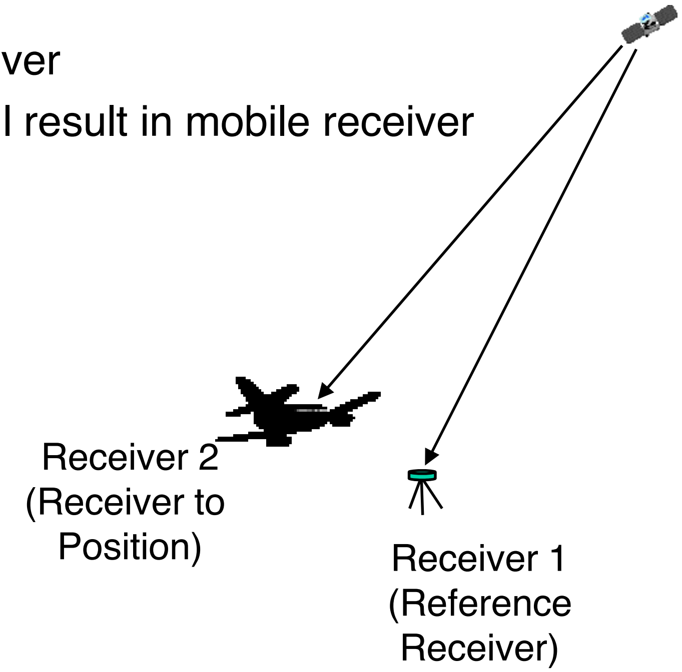


DGPS - Measurements Used (5/5)
Sample Comparison of 3-D Error



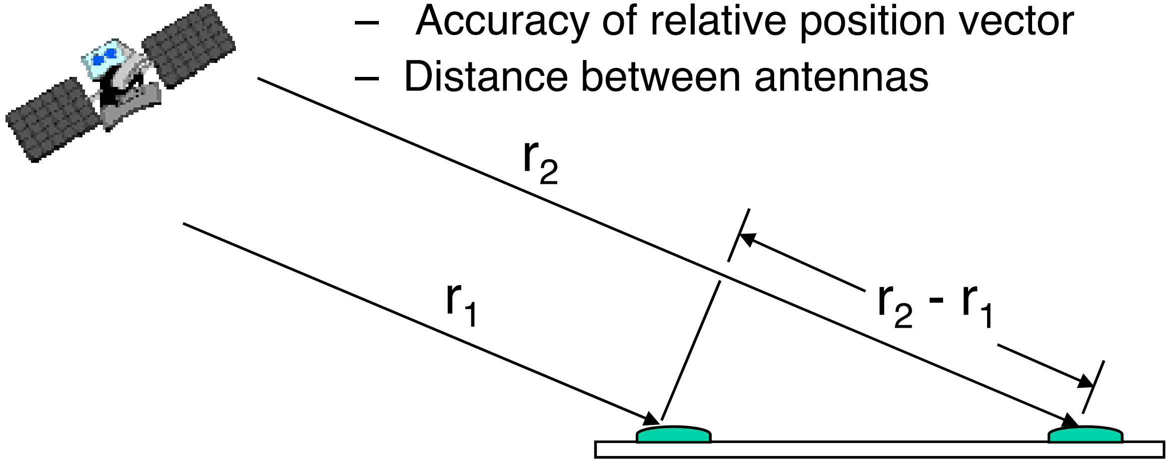
DGPS - Application Type (1/2)

- DGPS gives *relative* position between two receivers
  - Can be expressed as a 3-D vector
- This relative positioning information can be used in two ways
  - Positioning (most common)
    - Know position of reference receiver
    - Can calculate position of “mobile” receiver
    - Errors in reference receiver position will result in mobile receiver positioning errors



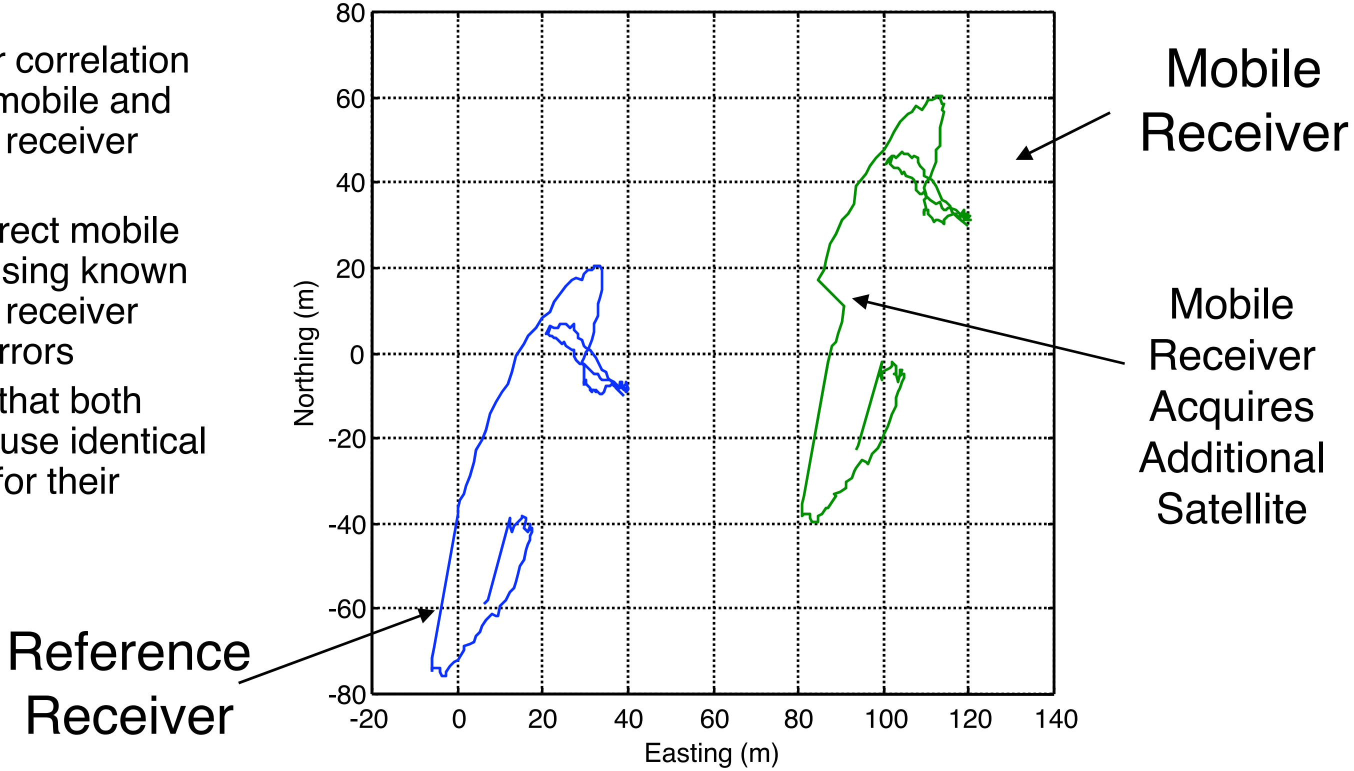
DGPS - Application Type (2/2)

- This relative positioning information can be used in two ways (continued)
  - Attitude determination
    - Antennas are in fixed, known configuration relative to defined “body” axes
    - Relative position vector between antennas is function of attitude of body
    - Can calculate attitude using relative position vector
      - Two antennas → two attitude axes (e.g., yaw and pitch)
      - Three or more antennas → complete attitude
    - Normally based on carrier-phase differential techniques with integer ambiguity resolution for most precise results
      - Relatively easy to resolve integer ambiguities in this case
    - Attitude accuracy depends upon
      - Accuracy of relative position vector
      - Distance between antennas



DGPS - Position vs. Measurement Domain (1/2)

- Position Domain
  - Reference receiver at known point (origin of plot)
  - Mobile receiver located to the northeast
  - Horizontal position of both receivers plotted on local coordinate system
    - Note error correlation between mobile and reference receiver errors
    - Could correct mobile receiver using known reference receiver position errors
    - Requires that both receivers use identical satellites for their solutions!



DGPS - Position vs. Measurement Domain (2/2)

- Measurement domain
  - Differential corrections are given for each *measurement*
  - These corrections are then applied to the mobile receiver measurements
    - Results in corrected measurements
    - Position calculated using corrected measurements
  - Advantages
    - Doesn't require same satellite coverage at mobile and reference receivers
      - Reference receiver can only generate corrections for measurements that it can see
    - Standardized formats are defined
      - RTCM SC-104 messages
    - Makes it possible to detect individual measurement errors
  - Disadvantages
    - Requires that more data be transmitted to mobile user than position domain approach
    - Not generally a large problem with modern radio data modems
    - Insignificant for non-real-time applications

DGPS -
Post-Processing vs. Real-Time

- Post-processing
  - Data is collected separately by each receiver
  - Later, data is combined and processed
  - Advantages
    - No data latency (can correlate times exactly)
    - Does not require real-time data link
    - Easier to implement (both hardware and software)
    - Can study and fix anomalies
    - Allows for use of other data and tools that may not be available real-time
      - Precise orbits
      - Ionospheric grid data
- Real-time
  - Differential corrections are sent to mobile receiver as soon as possible (i.e., near real-time)
    - Hard-wire (close applications)
    - Ground radio data link (10s of km)
    - Satellite data link (large areas)
  - Advantages
    - Many applications require real-time positioning!
    - Reduces data turn-around time, enables field checking

DGPS - Type of Correction

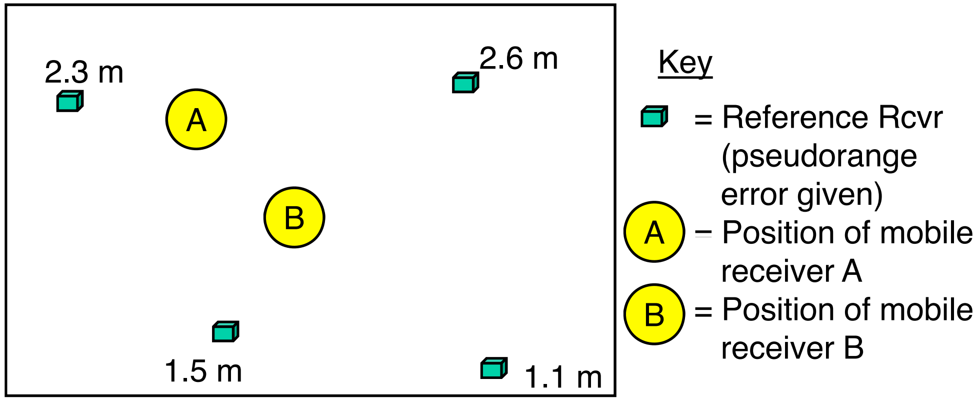
- Two ways to give corrections in measurement domain
  - Corrections to measurements
    - Actual correction values to be applied to each individual measurement
    - Simple, easy to implement
  - Explicit representation of errors
    - DGPS corrections describe all of the errors in a particular measurement
    - Sometimes, error functions or data are transmitted
      - Different error sources can then be combined to generate a correction for a single measurement
      - Example
        - » Precise ephemeris (to remove satellite position error)
        - » Ionospheric grid (to remove ionospheric error)
        - » Tropospheric model parameters (to improve tropospheric model)
    - Advantages
      - Generally valid for wider area of coverage
      - More flexible
    - Disadvantages
      - More complex
      - Requires more differential data to be transmitted

DGPS - Number of Reference Receivers (1/4)

- Single reference receiver is simplest and most common case
  - Errors grow as distance between reference and mobile receivers grows
  - Motivates need for multiple reference stations for some applications
- Multiple reference receivers using code measurements
  - Can involve anywhere from two to hundreds of reference receivers
  - Normally, different error sources are explicitly estimated (satellite position, ionosphere, etc.)
  - Alternatively, individual measurement corrections can be generated for each reference station, and a linear combination of these corrections can be used to generate corrections for a specific point
    - Based upon relative positions between specific point and reference receivers

Example:

What should be the error at
- location A?
- location B?

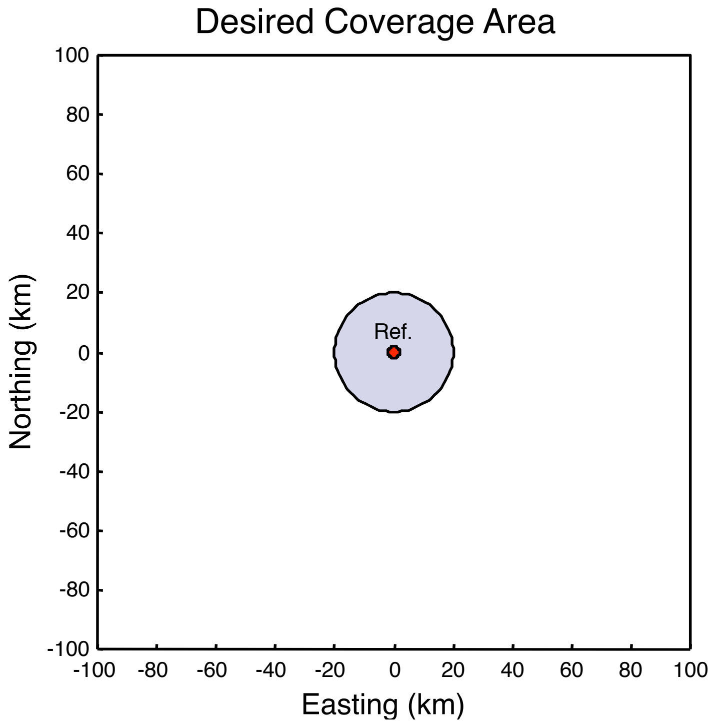


DGPS - Number of Reference Receivers (2/4)

- Multiple reference receivers using carrier-phase measurements
  - More difficult than code approaches because of integer ambiguities
- Motivation

Single Reference Receiver

Not Enough Coverage

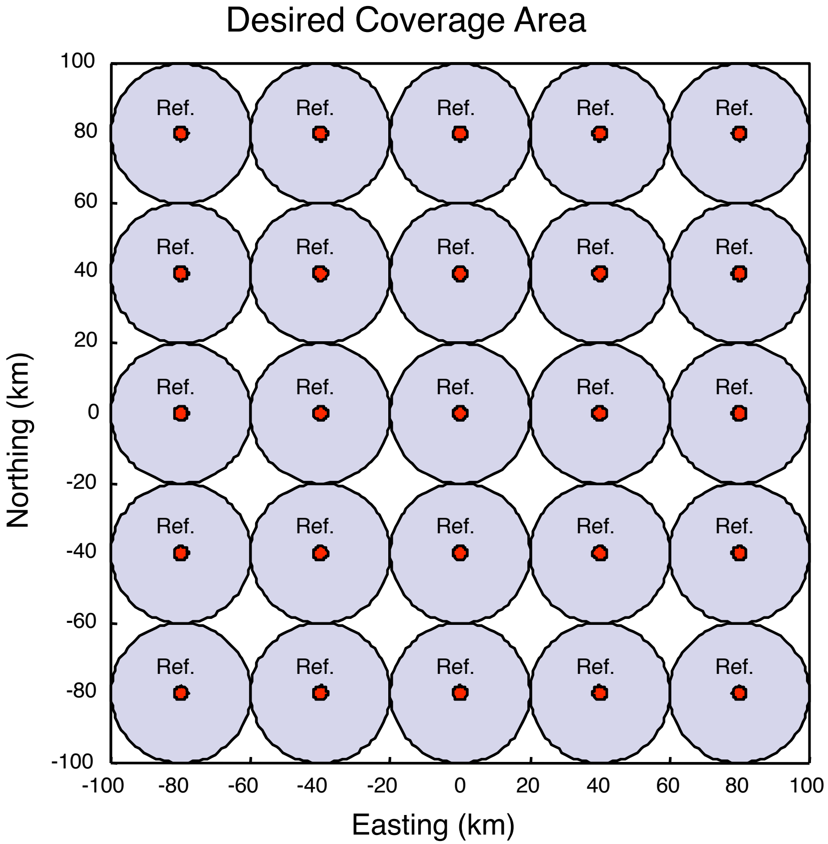


DGPS - Number of Reference Receivers (3/4)

- Motivation (continued)

Independent Ref. Receivers

Not Efficient

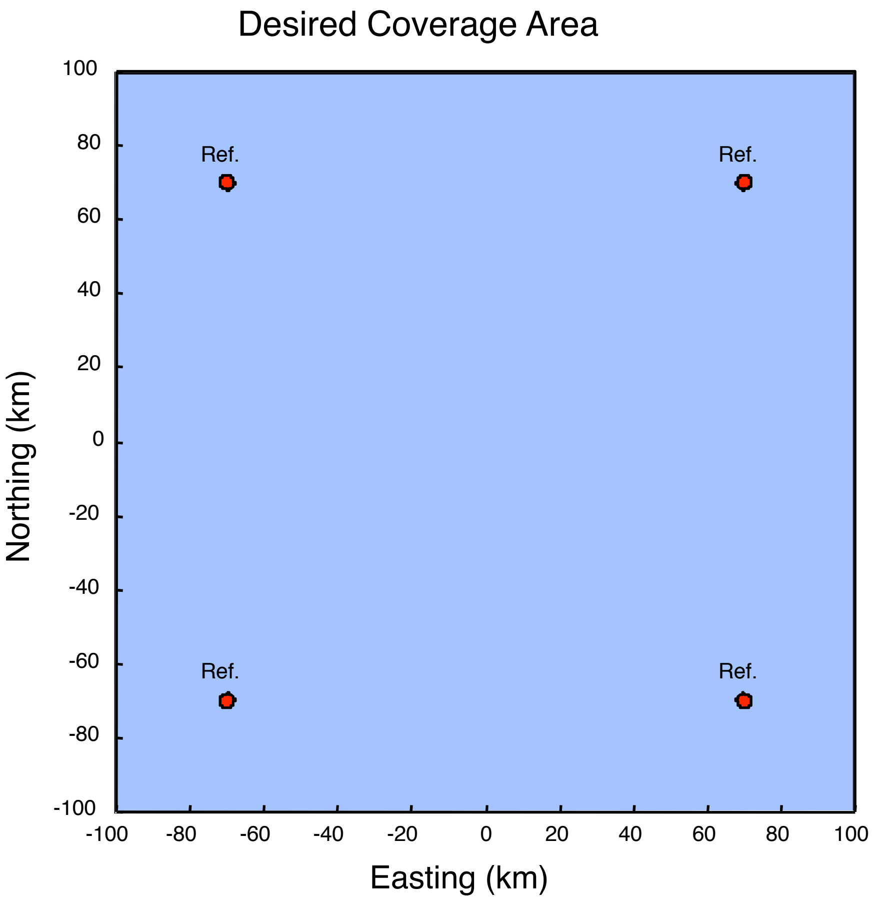


DGPS - Number of Reference Receivers (4/4)

- Motivation (continued)

Reference Receiver Network

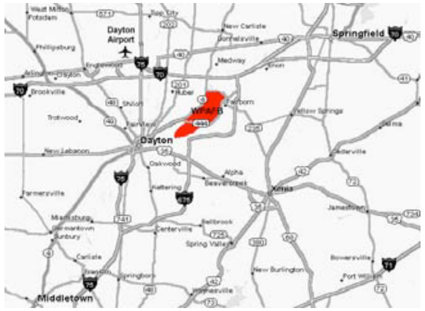
Efficiently Covers Large Area



- One method: NetAdjust

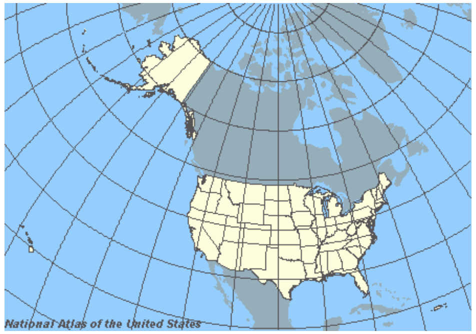
DGPS - Area of Coverage

- DGPS is deployed on three different scales
  - Local Area Differential GPS (LADGPS)
    - Covers tens of km
    - Typically involves single reference station
    - Accuracy varies between ~ 0.01-2 m
    - Example: aircraft landing, surveying



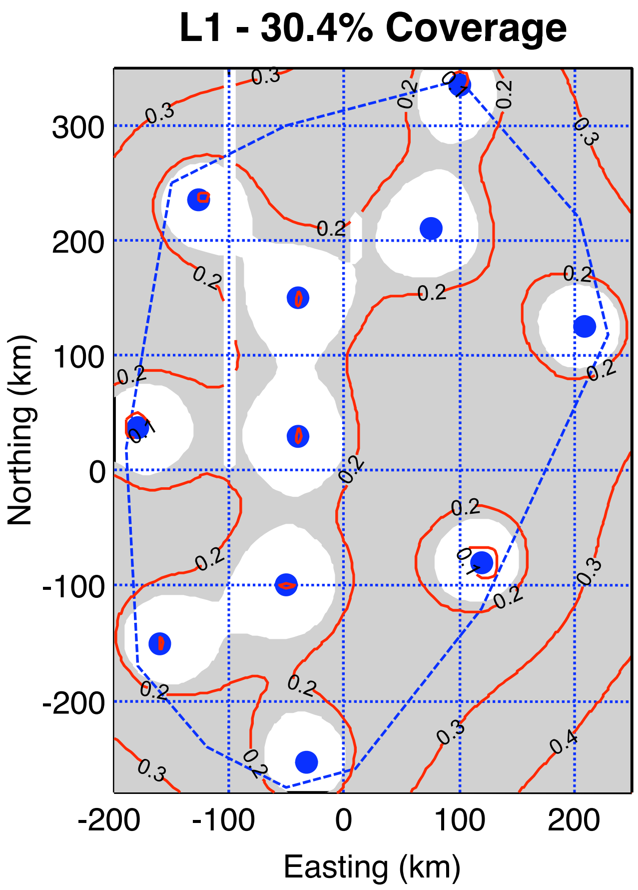
  - Regional Area Differential GPS (RADGPS)
    - Covers hundreds of km
    - Involves multiple reference receivers
    - Can achieve decimeter (or sometimes centimeter) level accuracy
    - Example: Norwegian reference receiver network

  - Wide Area Differential GPS (WADGPS)
    - Covers thousands of km
    - Involves multiple reference receivers
    - Not as accurate as RADGPS or LADGPS (typically 1-2 m accuracy)
    - Example: Wide Area Augmentation System (WAAS) for non-precision (and Cat I) approach

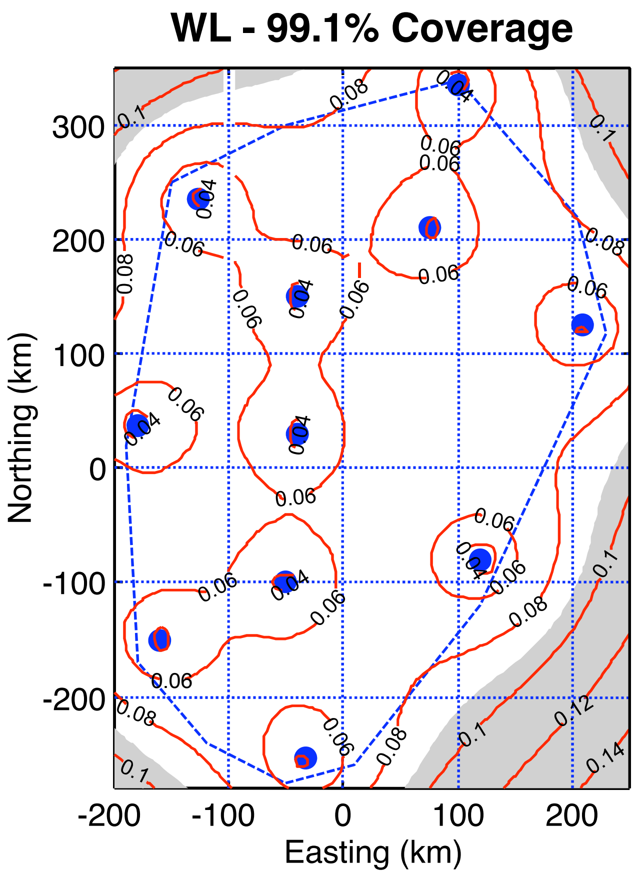


National Atlas of the United States

RADGPS Coverage Example
L1 vs. WL - Conditions of Test



L1 - 30.4% Coverage

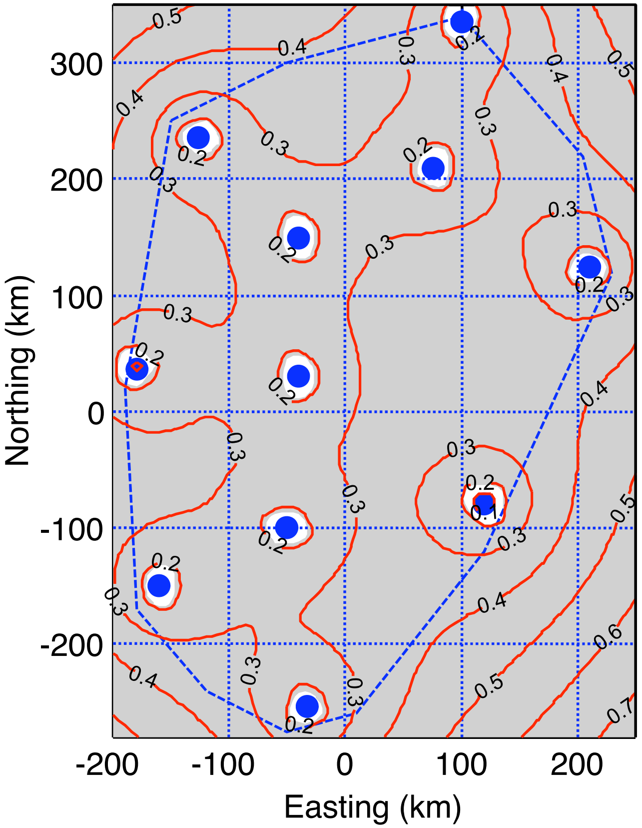


WL - 99.1% Coverage

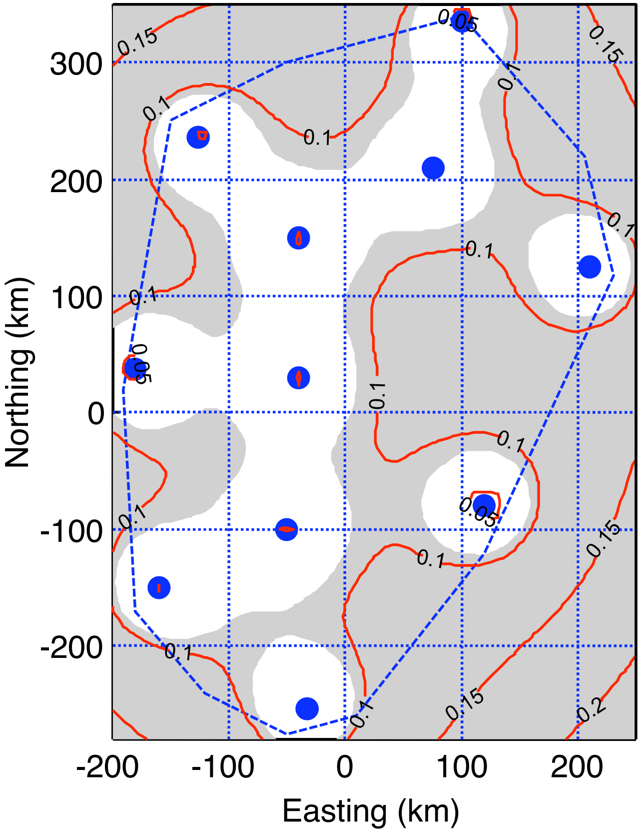
RADGPS Coverage Example

L1 vs. WL - Increased Ionosphere

Increased Ionosphere - L1
4.1% Coverage



Increased Ionosphere – WL
49.4% Coverage



DGPS - Differencing Methods - Pseudorange Measurement Errors

- Two types of differencing methods are common
  - Single differencing
  - Double differencing
- Choice of method depends upon application. Typically
  - Code differential → single differencing
  - Carrier-phase differential → double differencing
- Pseudorange errors
  - Original representation

$$\rho = r + c(\delta t_u - \delta t_{sv} + \delta t_D)$$

neglect

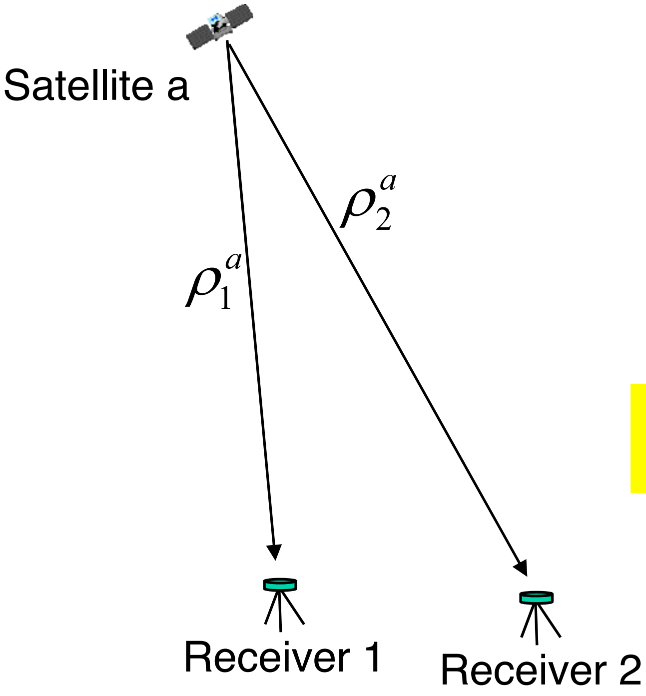
$$\delta t_D = \delta t_{trop} + \delta t_{iono} + \delta t_{noise\&res} + \delta t_{mp} + \cancel{\delta t_{hw}} + \delta t_{SA}$$

  - Simplification

$$\begin{array}{ccccccccccc} \rho = r + c\delta t_u - c\delta t_{sv} & + & c\delta t_{trop} & + & c\delta t_{iono} & + & c\delta t_{noise\&res} & + & c\delta t_{mp} & + & c\delta t_{SA} \\ & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ \rho = r + c\delta t_u - c\delta t_{sv} & + & T & + & I & + & \upsilon & + & m & + & SA \end{array}$$

DGPS - Differencing Methods (1/2)

- Single differencing
  - Difference measurements between one satellite and two receivers



$$\Delta\rho_{12}^a \equiv \rho_1^a - \rho_2^a$$

$$= r_1^a + c\delta t_{u_1} - c\delta t_{sv}^a + T_1^a + I_1^a + \upsilon_1^a + m_1^a + SA^a$$

$$- r_2^a - c\delta t_{u_2} + c\delta t_{sv}^a - T_2^a - I_2^a - \upsilon_2^a - m_2^a - SA^a$$

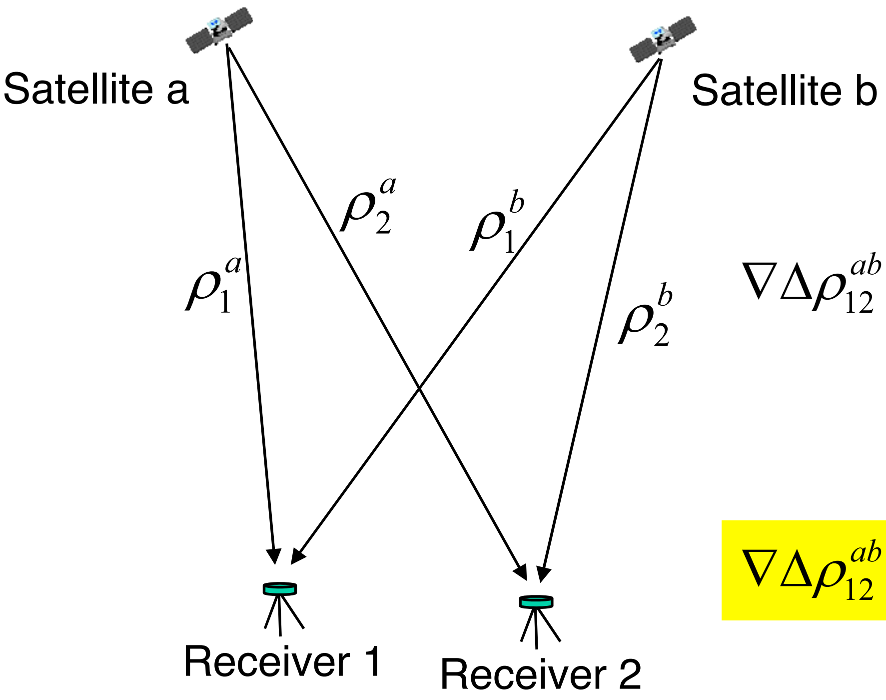
$$\Delta\rho_{12}^a = \Delta r_{12}^a + \Delta c\delta t_{u_{12}} + \Delta T_{12}^a + \Delta I_{12}^a + \Delta\upsilon_{12}^a + \Delta m_{12}^a$$

  - SV clock error and SA *cancelled*[1]
  - Tropospheric, ionospheric errors *reduced*
  - Multipath and noise *amplified* (by factor of $\sqrt{2}$)

[1] Assuming that only the dither portion of SA is utilized (if SA is on at all!)

DGPS - Differencing Methods (2/2)

- Double differencing
  - Difference between two single difference measurements



$$\nabla\Delta\rho_{12}^{ab} \equiv \Delta\rho_{12}^{a} - \Delta\rho_{12}^{b} = \rho_1^a - \rho_2^a - (\rho_1^b - \rho_2^b)$$

$$= \Delta r_{12}^{a} + \Delta c\delta t_{u_{12}} + \Delta T_{12}^{a} + \Delta I_{12}^{a} + \Delta \upsilon_{12}^{a} + \Delta m_{12}^{a}$$

$$- \Delta r_{12}^{b} - \Delta c\delta t_{u_{12}} - \Delta T_{12}^{b} - \Delta I_{12}^{b} - \Delta \upsilon_{12}^{b} - \Delta m_{12}^{b}$$

$$\nabla\Delta\rho_{12}^{ab} = \nabla\Delta r_{12}^{ab} + \nabla\Delta T_{12}^{ab} + \nabla\Delta I_{12}^{ab} + \nabla\Delta \upsilon_{12}^{ab} + \nabla\Delta m_{12}^{ab}$$

  - SV clock error, rcvr clock error, and SA *cancelled*[1]
  - Tropospheric, ionospheric errors *reduced*
  - Multipath and noise *amplified* (by factor of 2)

[1] Assuming that only the dither portion of SA is utilized (if SA is on at all!)

Implementation of LADGPS Using Code Measurements

"OK, so now I know what DGPS is in all of its variations. How do I actually implement it?"

DGPS Implementation

- Will cover the simplest DGPS case (for positioning)
  - Code measurements
  - Measurement domain
  - Post-processing
  - Corrections to measurements
  - Single reference receiver
  - LADGPS
  - Single-differencing
- While the simplest, this is also the most common
  - Understanding this case will give insight into most other cases
  - Algorithmically very similar to single-point positioning algorithm

Single Differencing vs. Measurement Corrections

- Two ways to approach the problem using single differencing
  - Use of single difference observables
    - Generate the single difference observables
    - Re-derive the positioning algorithms in a manner parallel to what was shown for single-point positioning
    - Use the single difference observables to estimate position (using the newly derived algorithm)
  - Use of measurement corrections
    - Use the reference receiver measurements and known position to calculate measurement corrections for each satellite
      - Calculation of measurement corrections does not require knowledge of mobile receiver measurements or position
    - Apply measurement corrections to mobile receiver measurements, *effectively* reducing the errors in those measurements
    - Perform single-point positioning using the corrected pseudorange measurements (rather than the original, uncorrected measurements)
  - Both methods are completely identical!
    - Will yield same results
    - Differ only in conceptual approach

Generating (and Interpreting) Measurement Corrections

- Denote reference receiver as receiver 1
- Reference receiver position is known
- Can calculate satellite position for each measurement
  - Important note: This position will be generally different from the position calculated for the mobile receiver, even though it's the same satellite at the same time epoch. Why?
- Measurement correction for satellite a ($\delta\rho_1^a$) calculated by

$$\begin{aligned}\delta\rho_1^a &= \rho_1^a - r_{calc1}{}^a \\ &= r_1^a - r_{calc1}{}^a + c\delta t_{u1}{}^a - c\delta t_{sv1}{}^a + T_1^a + I_1^a + \upsilon_1^a + m_1^a + SA_1^a \\ &= \delta r_{eph_1}{}^a + c\delta t_{u1}{}^a - c\delta t_{sv1}{}^a + T_1^a + I_1^a + \upsilon_1^a + m_1^a + SA_1^a\end{aligned}$$

  - This measurement-minus-range value is essentially a measurement of the errors that should be removed from the mobile receiver!

Applying Measurement Corrections (1/2)

- Denote mobile receiver as receiver 2
- Pseudorange measurement from satellite a is represented by

$$\rho_2^a = r_2^a + c\delta t_{u2}^{\ a} - c\delta t_{sv2}^{\ a} + T_2^a + I_2^a + \upsilon_2^a + m_2^a + SA_2^a$$

- Apply measurement correction:

$$\begin{aligned}
\rho_{2\,corr}^a &= \rho_2^a - \delta\rho_1^a \\
&= r_2^a + c\delta t_{u2}^{\ a} - c\delta t_{sv2}^{\ a} + T_2^a + I_2^a + \upsilon_2^a + m_2^a + SA_2^a \\
&\quad - \delta r_{eph_1}^{\ a} - c\delta t_{u1}^{\ a} + c\delta t_{sv1}^{\ a} - T_1^a - I_1^a - \upsilon_1^a - m_1^a - SA_1^a \\
&= r_2^a - \delta r_{eph_1}^{\ a} + \left(c\delta t_{u2}^{\ a} - c\delta t_{u1}^{\ a}\right) + \Delta T_{21}^a + \Delta I_{21}^a + \Delta\upsilon_{21}^a + \Delta m_{21}^a
\end{aligned}$$

  - Receiver clock error is now the difference between the reference and mobile receiver clock errors
  - SV clock and SA completely removed
  - Ionospheric and tropospheric errors reduced (removed for short baselines)
  - Multipath and noise are now difference between reference and mobile receiver multipath and noise

Applying Measurement Corrections (2/2)

- Corrected measurement (from last slide)

$$\begin{aligned}\rho_{2\,corr}^{a} &= \rho_2^a - \delta\rho_1^a \\ &= r_2^a - \delta r_{eph_1}{}^{a} + \left(c\delta t_{u2}{}^{a} - c\delta t_{u1}{}^{a}\right) + \Delta T_{21}^a + \Delta I_{21}^a + \Delta\upsilon_{21}^a + \Delta m_{21}^a\end{aligned}$$

  - This is very similar to single difference measurement!
- Note that this corrected measurement includes the difference between the actual and calculated ranges between the satellite and the reference receiver

$$\delta r_{eph_1}{}^{a} = r_1^a - r_{calc1}{}^{a}$$

  - In the process of generating position solution using least-squares (or any other method), the calculated range between mobile receiver and satellite will be subtracted from the measurement
  - Ephemeris prediction errors will nearly cancel as well.

DGPS Errors

"So now I know what DGPS is and how it is applied. What kinds of errors can I expect, and how do those errors grow with baseline distance?"

DGPS Errors

- Errors completely cancelled by DGPS
  - Receiver clock error
  - Satellite clock error
  - SA[1]
- DGPS errors can be grouped into two classes
  - Uncorrelated errors
    - Errors that are not spatially related
    - Do not increase with reference/mobile baseline distance
    - Include multipath and measurement noise
    - DGPS actually increases these errors

Typical Multipath + Noise Error Standard Deviation Values

	Single Meas (non-DGPS)	Single Difference	Double Difference
Code	0.5-1.5 m	0.7-2.1 m	1-3 m
Carrier-Phase	0.2 - 1 cm	0.3 - 1.4 cm	0.4 - 2 cm

  - Correlated errors
    - Are spatially related
    - Increase with baseline distance
    - Include satellite position (ephemeris), ionospheric, and tropospheric errors

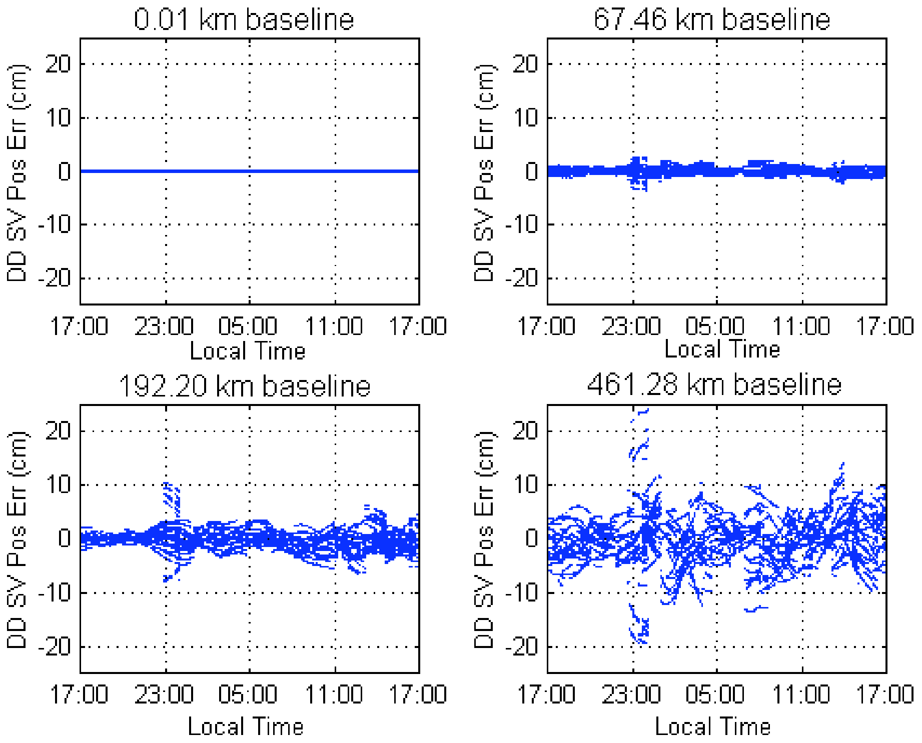
[1] Assuming that only the dither portion of SA is utilized (if SA is on at all!)

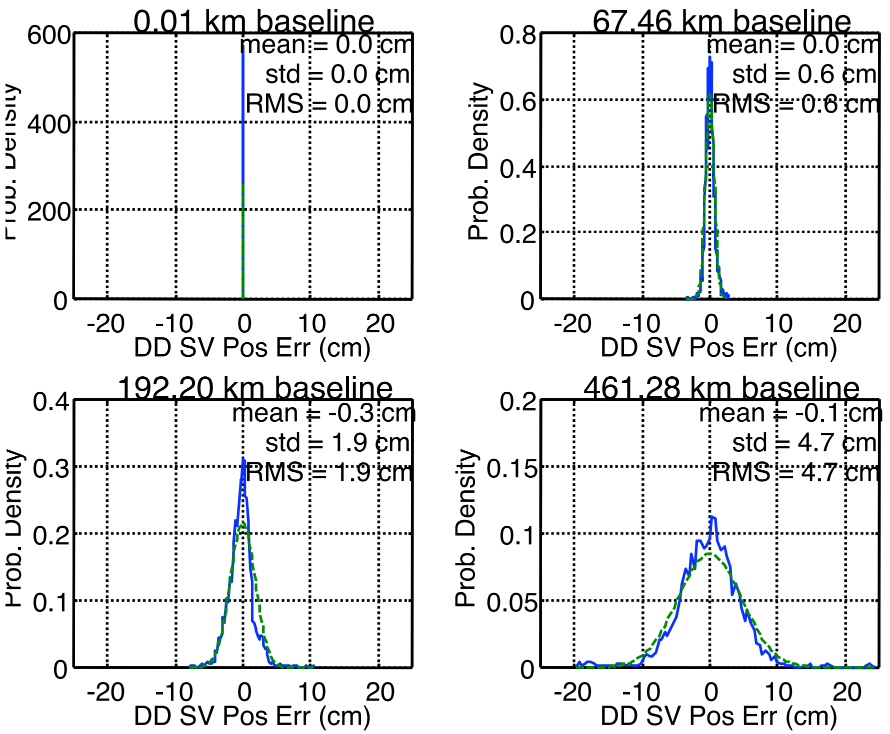
Differential Satellite Position Errors

- Satellite position errors are errors in ephemeris that cause calculated SV position to differ from true SV position
  - Absolute (non-DGPS) error
    - Zenith: ~1 m (1-σ)
    - Non-zenith axes: ~3 m (1-σ)
  - For a given measurement, it is the projection of the 3-D SV position error onto the measurement line-of-sight vector that counts
    - With DGPS, line-of-sight vectors converge as reference/mobile baseline distance goes to zero
  - Satellite position error can be determined using precise ephemeris as truth
    - Precise ephemeris accurate to ~10 cm
  - Differential satellite position errors typically less than 5 cm (1-σ), except for very long baselines (> 500 km)
    - True as long as same set of ephemeris is used for both reference and mobile receivers

Sample SV Position DGPS Error (Double Difference)

Data collected in Norway on Sep 30th, 1998



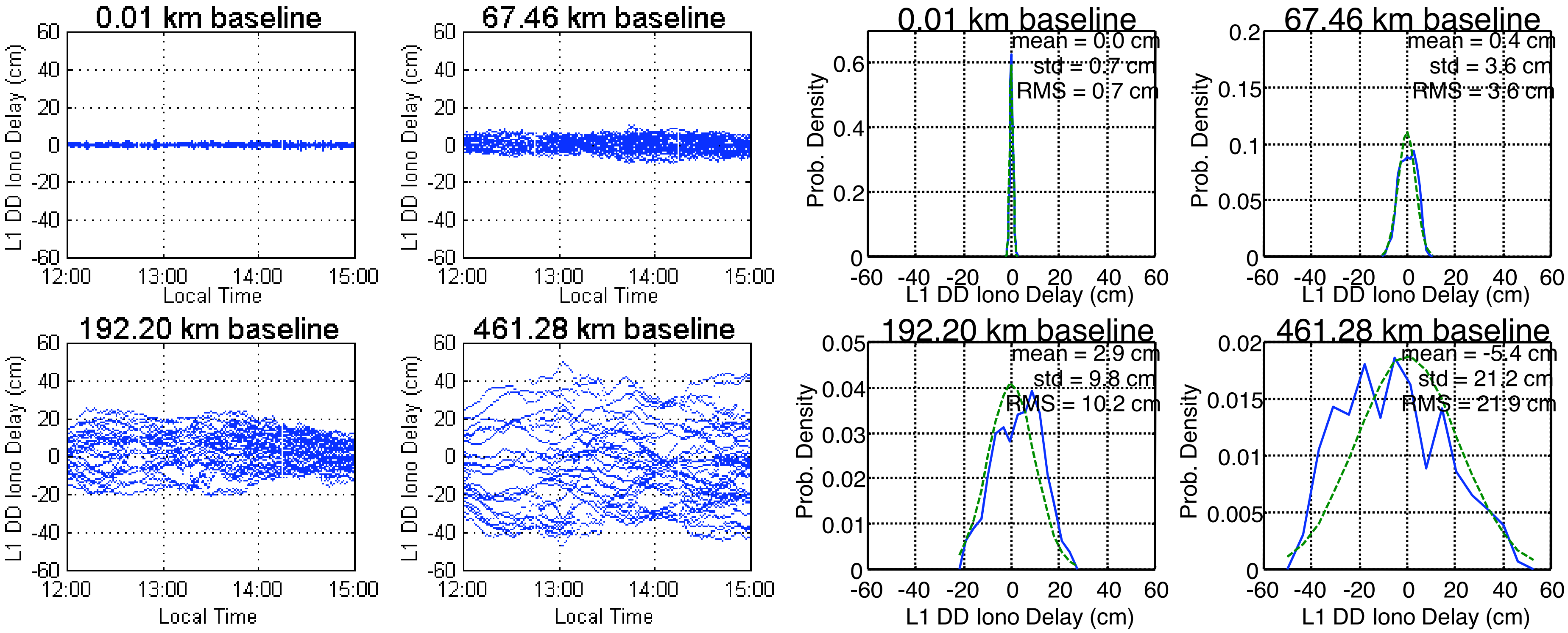


Differential Ionospheric Errors

- Ionospheric errors are spatially correlated
  - Signal from same satellite to two nearby receivers passes through approximately same ionosphere
    - Exception: scintillation
      - Highly local effects
      - Can affect one receiver but not another (unless receivers are collocated)
  - DGPS ionospheric error follows same general trends as overall (non-DGPS) ionospheric error
    - Maximum at ~14:00 local time
    - Minimum at night
    - Varies with solar cycle
  - Ionospheric delay (or phase advance) can be precisely measured using linear combination of phase measurements
    - Requires successful resolution of L1 and L2 carrier-phase ambiguities
    - Accurate to ~1 cm (includes effects of carrier-phase multipath and noise)

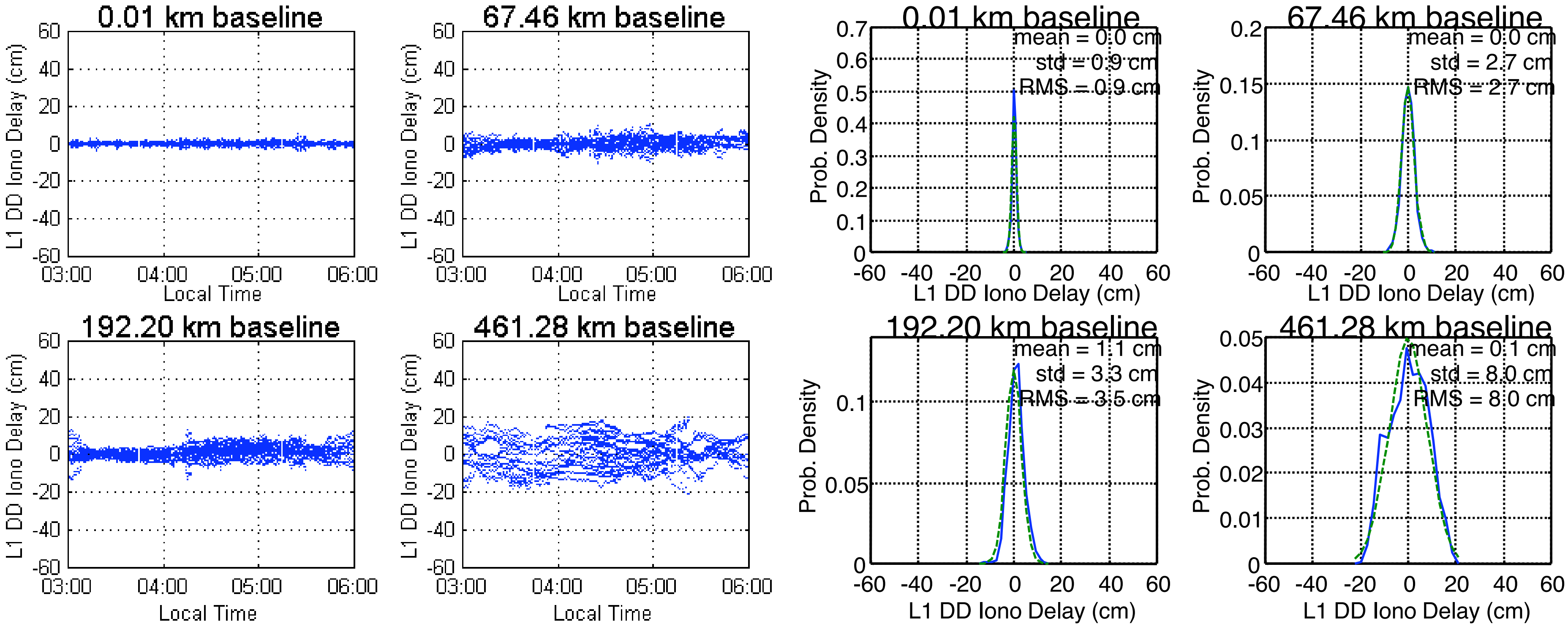
Sample Afternoon Ionospheric DGPS Errors (Double Difference)

Data collected in Norway on Sep 30th, 1998
(between minimum and mid-point of solar cycle)



Sample Nighttime Ionospheric DGPS Errors (Double Difference)

Data collected in Norway on Sep 30th, 1998
(between minimum and mid-point of solar cycle)

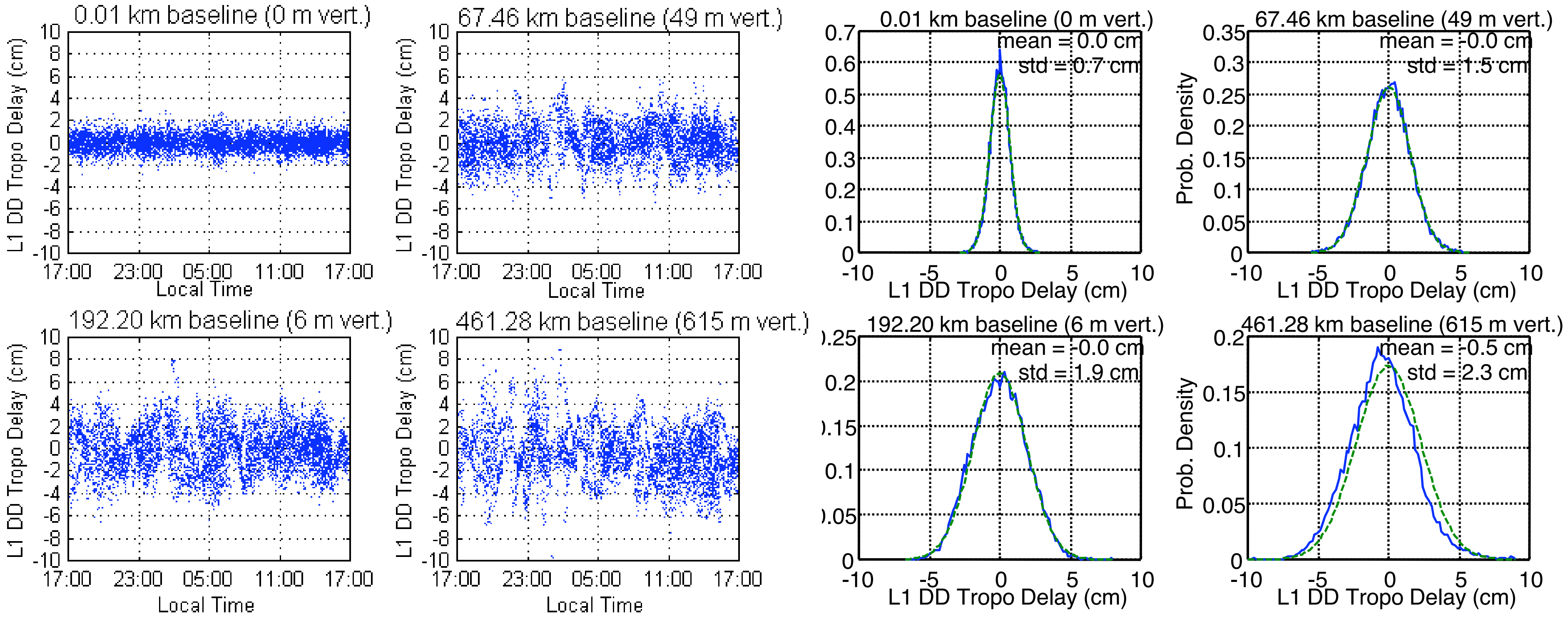


Differential Tropospheric Errors

- Tropospheric errors highly sensitive to altitude of receiver and elevation of satellite
  - Most of the error can be effectively modeled
  - Important to always apply tropospheric model for DGPS
    - If don’t apply, then can introduce differential errors on order of meters for receivers at different altitudes
    - Should use same tropospheric model (if possible)
- With a good model, differential tropospheric errors are relatively small
  - Under normal conditions don’t exceed ~3 cm (1-σ) for baselines < 500 km
  - Can be worse under extreme conditions (e.g., high humidity)
- Differential tropospheric error can be calculated from carrier-phase measurements
  - Use ionospheric-free combination with precise orbits to remove other errors
  - All that remains is tropospheric error (plus multipath and noise)

Sample Tropospheric DGPS Errors (Double Difference)

Data collected in Norway on Sep 30th, 1998
Modified Hopfield Tropospheric Error Model Applied



DGPS - Practical Considerations

- Reference/remote receiver baseline distance
  - Code DGPS
    - Multipath and Noise Dominates
    - Under normal conditions, other errors don’t become significant until baseline reaches 100-200 km.
  - Carrier-phase DGPS (ambiguity resolution)
    - Ambiguity resolution process is based upon tests of measurement residuals
    - Errors (especially biases) in measurements cause significant problems for ambiguity resolution
    - Typical max baseline length to resolve ambiguities effectively (kinematic mode):
      - L1 only: 15-25 km
      - L1 and L2 (widelane): 40-60 km
- Data latency
  - Takes some amount of time for corrections to arrive at mobile receiver for real-time DGPS
  - SA was “fastest-moving” error (when it existed)
    - Max of 19 mm/s^2 acceleration and 2 m/s range rate
    - Data latency of 1 second could cause up to 2 m of DGPS error
  - Sometimes, corrections and time derivatives are transmitted

Carrier-Phase Integer Ambiguity Resolution

Taking the Mystery out of Resolving Ambiguities

Ambiguity Resolution Background

- History and terminology
  - Field started with surveyors (static)
  - Stop-and-go
  - Kinematic
  - On-the-fly
- Ambiguity resolution baseline lengths
  - Static, 12+ hours of dual-frequency data, top-notch software: over 1000 km (with accuracies of a few mm!)
  - Kinematic, 10+ minutes of dual frequency data, widelane ambiguities: ~60 km
  - Kinematic, 10+ minutes of single frequency data, L1 ambiguities: ~20 km
- Reader/listener beware!
  - Algorithms are notoriously data-set dependent
    - Easy to tweak the algorithm to work very well on a particular data set
    - Much more difficult to get it to work well on just about any data set
    - Results, especially those that show surprising improvement, should be taken with a grain of salt

Why We Want to Resolve the Carrier-Phase Integer Ambiguities

- Double-difference phase measurement error equation:

$$\nabla\Delta\phi = \frac{1}{\lambda}\left(\nabla\Delta r + \nabla\Delta\, T + \frac{\nabla\Delta I}{f^2} - \nabla\Delta m + \nabla\Delta\upsilon\right) + \nabla\Delta N$$

  - If we have a "short" baseline (e.g., less than 15 km), then the atmospheric terms can be neglected:

$$\nabla\Delta\phi = \frac{1}{\lambda}\left(\nabla\Delta r + \nabla\Delta m + \nabla\Delta\upsilon\right) + \nabla\Delta N$$

- If we can determine $\nabla\Delta N$, then the double difference phase measurement ($\nabla\Delta\phi$) is a very precise measurement of position
  - What happens to the precision as the baseline length increases?

Basic Components of All Ambiguity Resolution Algorithms

- There are ***many*** different algorithms used to determine integer ambiguities
  - Can be intimidating to study
  - Everyone brings their own slant to the problem
- Most algorithms perform two primary operations
  - Determine the ambiguity search space
    - Come up with the sets of ambiguities that might be correct
  - Selection of correct ambiguity set
    - Pick the correct set of ambiguities out from your search space
- We'll cover basic approaches for both of these operations

Ambiguity Search Space

- Definition of ambiguity set
  - At any given measurement epoch, there are a set of double difference ambiguities
  - We know they are all integers
  - Example of three ambiguity sets

SV Pair	Amb Set 1	Amb Set 2	Amb Set 3
23-3	142093	142092	142093
23-6	-872329	-872329	-872329
23-18	3209874	3209875	3209873
23-27	-243098	-243097	-243098
23-29	49879087	49879087	49879088

- How might we arrive at these different sets of ambiguities?

Determining Ambiguity Search Space

- Position-based approaches
  - Ambiguity function method
    - Search purely over position domain
    - Will not cover in detail
  - Use of knowledge of position to define candidate ambiguity sets
    - Often generates too many candidate ambiguities
    - Will not cover in detail
- Ambiguity-centered approaches (using primarily ambiguity estimates and covariance)
  - Simple rounding
    - Only works when errors are very low and you have very good estimates of ambiguities
  - Least squares AMbiguity Decorrelation Algorithm (LAMBDA)
    - Will cover in following slides
  - Fast Ambiguity Search Filter (FASF)
    - Will cover in following slides

Least squares AMbiguity Decorrelation Algorithm (LAMBDA)[1]

- Developed by Teunissen in 1993
- Often, ambiguity estimates can be highly correlated
  - Low correlation example

$$\hat{X} = \begin{bmatrix} \vdots \\ \vdots \\ \nabla\Delta N_{12} \\ \nabla\Delta N_{13} \\ \nabla\Delta N_{14} \\ \nabla\Delta N_{15} \end{bmatrix} = \begin{bmatrix} \cdots \\ \cdots \\ 10615431.981 \\ 10791937.073 \\ 10921812.941 \\ 10176265.986 \end{bmatrix} \qquad P_{\nabla\Delta N} = \begin{bmatrix} 0.0109 & -0.0056 & 0.0011 & 0.0012 \\ -0.0056 & 0.0159 & -0.0068 & 0.0059 \\ 0.0011 & -0.0068 & 0.0188 & 0.0037 \\ 0.0012 & 0.0059 & 0.0037 & 0.0164 \end{bmatrix}$$

  - High correlation example

$$\hat{X} = \begin{bmatrix} \vdots \\ \vdots \\ \nabla\Delta N_{12} \\ \nabla\Delta N_{13} \\ \nabla\Delta N_{14} \\ \nabla\Delta N_{15} \end{bmatrix} = \begin{bmatrix} \cdots \\ \cdots \\ 10193431.428 \\ 10860011.744 \\ 10853654.623 \\ 10593562.704 \end{bmatrix} \qquad P_{\nabla\Delta N} = \begin{bmatrix} 6.2901 & 2.1378 & 0.5454 & 0.9417 \\ 2.1378 & 4.2692 & 1.3040 & 0.2931 \\ 0.5454 & 1.3040 & 3.2627 & 1.1254 \\ 0.9417 & 0.2931 & 1.1254 & 2.9983 \end{bmatrix}$$

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[1] Slides on LAMBDA method adapted from presentation by Capt Paul Henderson at AFIT, Summer 1999

LAMBDA

- Basic idea
  - Decorrelate highly correlated phase ambiguities from float carrier phase solution by applying an ambiguity transformation (Z-transformation)
    - Not the z-transform used in signal processing
- Find floating point ambiguities and corresponding variance matrix (a, Q_a)
  - These results are highly correlated, especially for short observation periods
  - Causes problems in finding integer solutions
- Transform floating point solution/variance to decorrelated ambiguities/variance
  - Use “Z-transformation”

LAMBDA Z-transform

- Z-Transformation

$$\hat{z} = Z^T \hat{x}, \qquad z = Z^T x, \qquad Q_z = Z^T Q_z Z$$

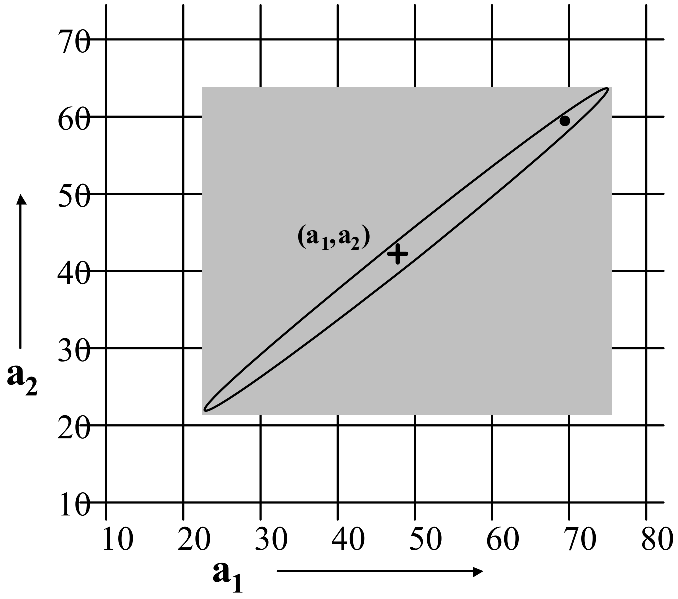
- Transformation must meet certain conditions to preserve the integer nature of ambiguities
  - Must be volume preserving (one-to-one relation)
  - Must reduce the product of ambiguity variances
  - Must have integer elements
- For a two dimensional example

$$\mathbf{Z}_\mathbf{2}^\mathbf{T} = \begin{bmatrix} 1 & 0 \\ z_{21} & 1 \end{bmatrix} \qquad \mathbf{Z}_1^\mathbf{T} = \begin{bmatrix} 1 & z_{12} \\ 0 & 1 \end{bmatrix}$$

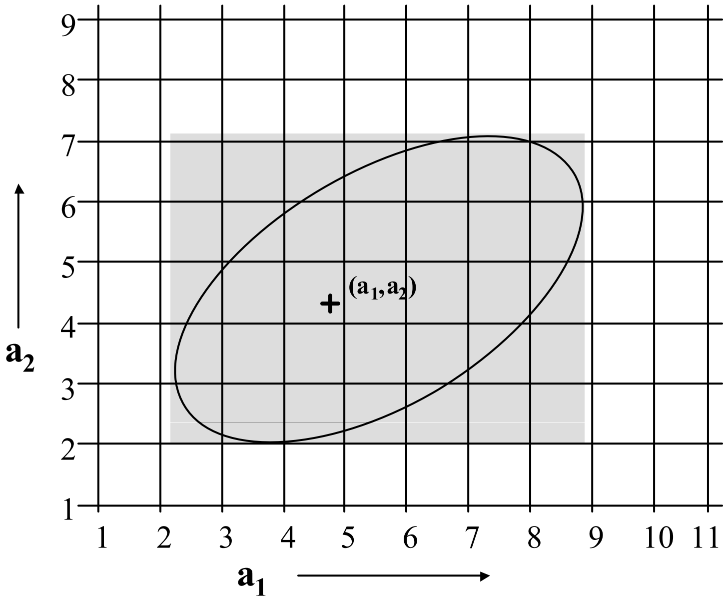
$$z_{21} = \operatorname{int}\left(-\sigma_{21}\sigma_1^{-2}\right) \qquad z_{12} = \operatorname{int}\left(-\sigma_{12}\sigma_2^{-2}\right)$$

LAMBDA Graphical Depiction

2-D Example (before)



2-D Example (after)



LAMBDA Example

- Initial covariance:

$$P_{\nabla\Delta N} = \begin{bmatrix} 6.290 & 5.978 & 0.544 \\ 5.978 & 6.292 & 2.340 \\ 0.544 & 2.340 & 6.288 \end{bmatrix}$$

- Z-transform matrix:

$$Z = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -1 \\ 3 & -3 & 1 \end{bmatrix}$$

- Transformed covariance:

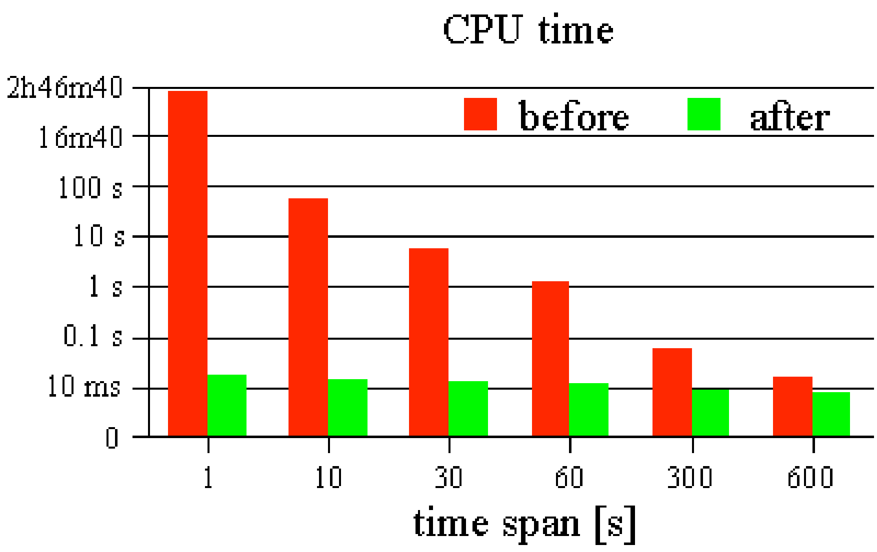
$$P_{\nabla\Delta N} = \begin{bmatrix} 0.626 & 0.230 & 0.082 \\ 0.230 & 4.476 & 0.334 \\ 0.082 & 0.334 & 1.146 \end{bmatrix}$$

LAMBDA (continued)

- The search space still may not contain the solution (correct gridpoint)
  - Scale the search space to guarantee that it contains at least one gridpoint
  - For best results scale the search space to contain at least two gridpoints (for validation) but not many more than two

$$\left(\hat{z}-z\right)^T Q_{\hat{z}}^{-1}\left(\hat{z}-z\right) \leq \chi^2$$

- Results
  - Graphs based on 10.4 km Baseline
  - 7 satellites using dual frequency phase data
  - Run on 486-66MHz PC



Fast Ambiguity Search Filter (FASF)

- Developed by Chen and Lachapelle in 1993
- Principle:
  - Determine search range for best known single ambiguity

$$\hat{x}_n - k\sigma_n \le \nabla\Delta N_{\text{int}} \le \hat{x}_n + k\sigma_n$$

  - For each possible integer value:
    - Assume that the integer is correct
    - Calculate new conditional covariance and ambiguity estimate, conditioned on the fact that the first ambiguity is known
      - Will shrink covariance terms
    - Now, select the best known integer value from the remaining ambiguities, and repeat the process
  - Recursive
    - Will either result in valid ambiguity set, or will be “pruned”, if the conditional covariance doesn’t allow for a valid integer somewhere along the process
    - Very efficient

Calculation of Conditional State and Covariance

- Conditional state estimate and covariance efficiently calculated as follows:

$$\tilde{x} = \hat{\boldsymbol{x}} - \boldsymbol{p}_n(\hat{x}_n - \nabla\Delta N_{\text{int}}) / \sigma_n^2$$

$$P_{\tilde{x}} = P_{\hat{x}} - (\boldsymbol{p}_n \boldsymbol{p}_n^T) / \sigma_n^2$$

$\tilde{x}$ = Estimated parameter vector conditioned upon $\hat{x}_n = \nabla\Delta N_{\text{int}}$

$P_{\tilde{x}}$ = Covariance matrix conditioned upon $\hat{x}_n = \nabla\Delta N_{\text{int}}$

$\boldsymbol{p}_n = n^{th}$ column of the covariance matrix $P_{\hat{x}}$

σ_n^2 = scalar variance of the n^{th} parameter (taken from diagonal of $P_{\hat{x}}$)

FASF Example - Valid Ambiguity Set

Initial state and covariance

$$\hat{x} = \begin{bmatrix} 2434.912 \\ -24987.144 \\ 23421.980 \end{bmatrix} \qquad P_{\hat{x}} = \begin{bmatrix} 3.145 & 3.140 & 2.552 \\ 3.140 & 3.146 & 2.487 \\ 2.552 & 2.487 & 2.644 \end{bmatrix}$$

Determine search range of ambiguity with lowest variance

$$\sigma_{\hat{x}_3} = \sqrt{2.644} = 1.626$$

$$\hat{x}_3 - 3\sigma_{\hat{x}_3} \le N_3 \le \hat{x}_3 + 3\sigma_{\hat{x}_3}$$

$$23417.102 \le N_3 \le 23426.858$$

Select ambiguity

$$N_3 = 23418$$

Conditional state and covariance

$$\tilde{x} = \begin{bmatrix} 2431.071 \\ -24990.888 \\ 23418 \end{bmatrix} \qquad P_{\tilde{x}} = \begin{bmatrix} 0.682 & 0.740 & 0 \\ 0.740 & 0.807 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Determine search range of ambiguity with lowest variance

$$\sigma_{\hat{x}_1} = \sqrt{.682} = 0.826$$

$$\tilde{x}_1 - 3\sigma_{\hat{x}_1} \le N_1 \le \tilde{x}_1 + 3\sigma_{\hat{x}_1}$$

$$2428.593 \le N_1 \le 2433.548$$

Select ambiguity

$$N_1 = 2429$$

Conditional state and covariance

$$\tilde{x} = \begin{bmatrix} 2429 \\ -24993.133 \\ 23418 \end{bmatrix} \qquad P_{\tilde{x}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.0045 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Determine search range of final ambiguity

$$\sigma_{\hat{x}_2} = \sqrt{.0045} = 0.067$$

$$\tilde{x}_2 - 3\sigma_{\hat{x}_2} \le N_2 \le \tilde{x}_2 + 3\sigma_{\hat{x}_2}$$

$$-24993.335 \le N_2 \le -24992.932$$

Select ambiguity

$$N_2 = -24993$$

Valid ambiguity set

$$\hat{x} = \begin{bmatrix} 2429 \\ -24993 \\ 23418 \end{bmatrix}$$

FASF Example - Invalid Ambiguity Set (Pruning)

Initial state and covariance

$$\hat{x} = \begin{bmatrix} 2434.912 \\ -24987.144 \\ 23421.980 \end{bmatrix} \quad P_{\hat{x}} = \begin{bmatrix} 3.145 & 3.140 & 2.552 \\ 3.140 & 3.146 & 2.487 \\ 2.552 & 2.487 & 2.644 \end{bmatrix}$$

Determine search range of ambiguity with lowest variance

$$\sigma_{\hat{x}_3} = \sqrt{2.644} = 1.626$$
$$\hat{x}_3 - 3\sigma_{\hat{x}_3} \le N_3 \le \hat{x}_3 + 3\sigma_{\hat{x}_3}$$
$$23417.102 \le N_3 \le 23426.858$$

Select ambiguity

$$N_3 = 23418$$

Conditional state and covariance

$$\tilde{x} = \begin{bmatrix} 2431.071 \\ -24990.888 \\ 23418 \end{bmatrix} \quad P_{\tilde{x}} = \begin{bmatrix} 0.682 & 0.740 & 0 \\ 0.740 & 0.807 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Determine search range of ambiguity with lowest variance

$$\sigma_{\hat{x}_1} = \sqrt{.682} = 0.826$$
$$\tilde{x}_1 - 3\sigma_{\hat{x}_1} \le N_1 \le \tilde{x}_1 + 3\sigma_{\hat{x}_1}$$
$$2428.593 \le N_1 \le 2433.548$$

Select ambiguity

$$N_1 = 2433$$

Conditional state and covariance

$$\tilde{x} = \begin{bmatrix} 2433 \\ -24988.795 \\ 23418 \end{bmatrix} \quad P_{\tilde{x}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.0045 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Determine search range of final ambiguity

$$\sigma_{\hat{x}_2} = \sqrt{.0045} = 0.067$$
$$\hat{x}_2 - 3\sigma_{\hat{x}_2} \le N_2 \le \hat{x}_2 + 3\sigma_{\hat{x}_2}$$
$$-24988.997 \le N_2 \le -24988.593$$

No valid N_2 ambiguity, so this combination is not a valid ambiguity set!

Selection of Correct Ambiguity Set

- Recall the basic measurement equation

$$\nabla\Delta\phi = \frac{1}{\lambda}\left(\nabla\Delta r + \nabla\Delta T + \frac{\nabla\Delta I}{f^2} - \nabla\Delta m + \nabla\Delta \upsilon \right) + \nabla\Delta N$$

  - Rearrange to get

$$\underbrace{\nabla\Delta\phi - \nabla\Delta N - \frac{1}{\lambda}\nabla\Delta r}_{\text{measurement residuals}} = \underbrace{\frac{1}{\lambda}\left(\nabla\Delta T + \frac{\nabla\Delta I}{f^2} - \nabla\Delta m + \nabla\Delta \upsilon \right)}_{\text{differential } (\nabla\Delta) \text{ errors}}$$

  - Four different cases:

	Low ∇Δ Errors	High ∇Δ Errors
Correct ∇Δ*N*	Low Residuals	Generally High Residuals
Wrong ∇Δ*N*	Generally High Residuals	Generally High Residuals

  - Can use ratio test

$$\text{ratio} = \frac{vR^{-1}v^T \text{ (2nd best)}}{vR^{-1}v^T \text{ (best)}}$$

$v =$ measurement residuals

$R =$ measurement covariance matrix

    - Correct ambiguity if ratio > constant (typically, 2 or greater)

GPS Measurement Combinations

“OK, so I have an idea of how to use a few measurements (individually) to do something. Is there a way that I can do more by combining measurements?”

Typical Units for GPS Measurements

- Pseudorange
  - Normally expressed in meters
  - Sometimes expressed in seconds (can be converted to meters by multiplying by speed of light)
- Doppler
  - Hertz (cycles per second)
  - Sign convention can vary
- Carrier-phase
  - Cycles (for particular frequency)
  - Sometimes output as integer portion and fractional portion
    - Both are added together to get full cycles
    - This “integer” portion has nothing to do with integer carrier-phase ambiguities

Relationship Between Pseudorange and Doppler Measurements

- Relationships below are approximate, due to errors and dynamics
- Doppler gives information about rate of change of code

$$\Delta f(t_0)\lambda \approx \frac{\rho(t_1)-\rho(t_0)}{t_1-t_0}$$

or

$$\rho(t_1) \approx \rho(t_0)+(t_1-t_0)\Delta f(t_0)\lambda$$

$\Delta f =$ Doppler measurement (Hz)
$\lambda =$ Carrier wavelength (m)
$\rho =$ Code measurement (m)
$t_0, t_1 =$ Two (nearby) time epochs

- This relationship not normally used explicitly
  - Can be used to determine Doppler sign convention/units on Doppler

Relationship Between Carrier-Phase and Doppler Measurements

- Carrier-phase is integration of Doppler measurement
- Can be expressed between epochs as

$$\Delta f(t_0) \approx \frac{\phi(t_1) - \phi(t_0)}{t_1 - t_0}$$

or

$$\phi(t_1) \approx \phi(t_0) + (t_1 - t_0)\Delta f(t_0)$$

Δf = Doppler measurement (Hz)

ϕ = Carrier-phase measurement (cycles)

t_0, t_1 = Two (nearby) time epochs

- Relationship is very useful for detecting phase cycle slips

Relationship Between Pseudorange and Carrier-Phase Measurements

- Note that both pseudorange and carrier-phase measurements have similar relationship with Doppler
  - Difference is wavelength scaling factor (λ)
- Implies that pseudorange and carrier-phase measure essentially the same thing
  - Difference is carrier-phase integer ambiguity (starting point of integration)

$$\frac{\rho(t_0)}{\lambda} \approx \phi(t_0) - N$$

$\lambda =$ Carrier wavelength (m)

$\rho =$ Code measurement (m)

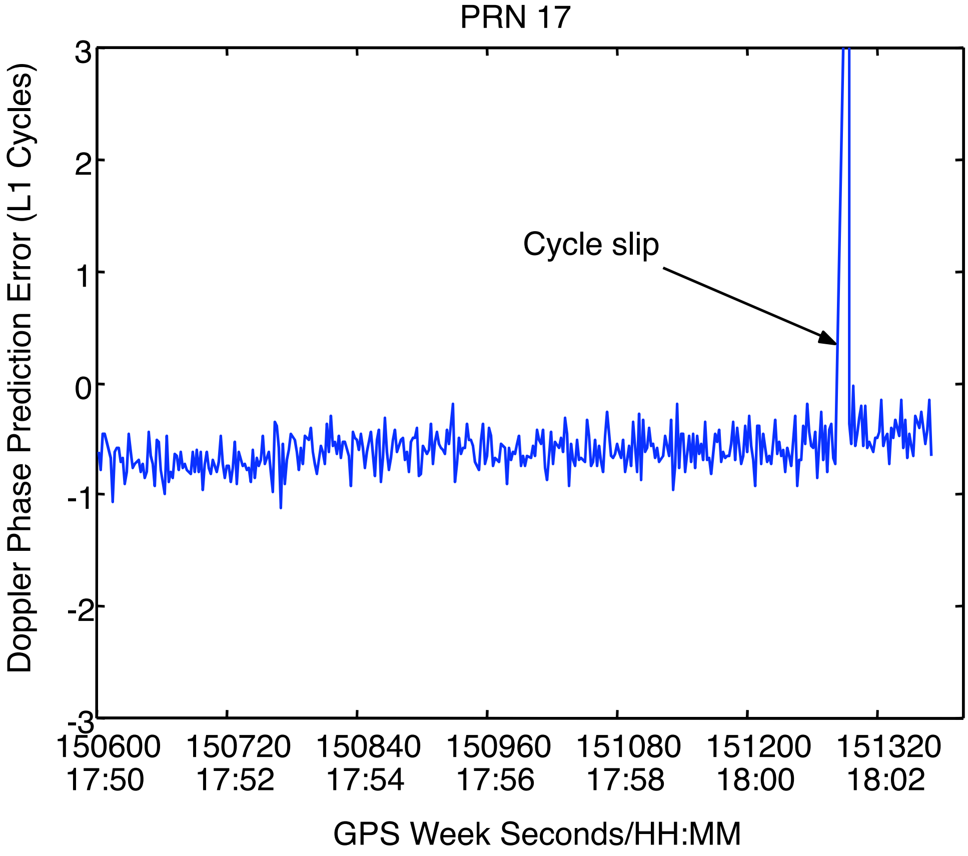
$\phi =$ Carrier-phase measurement (cycles)

$t_0 =$ Particular time epoch

$N =$ Carrier-phase integer ambiguity

  - Very useful relationship
    - If can determine N, can use carrier-phase for positioning
    - Implies that code and phase can be combined (filtered) to remove some errors

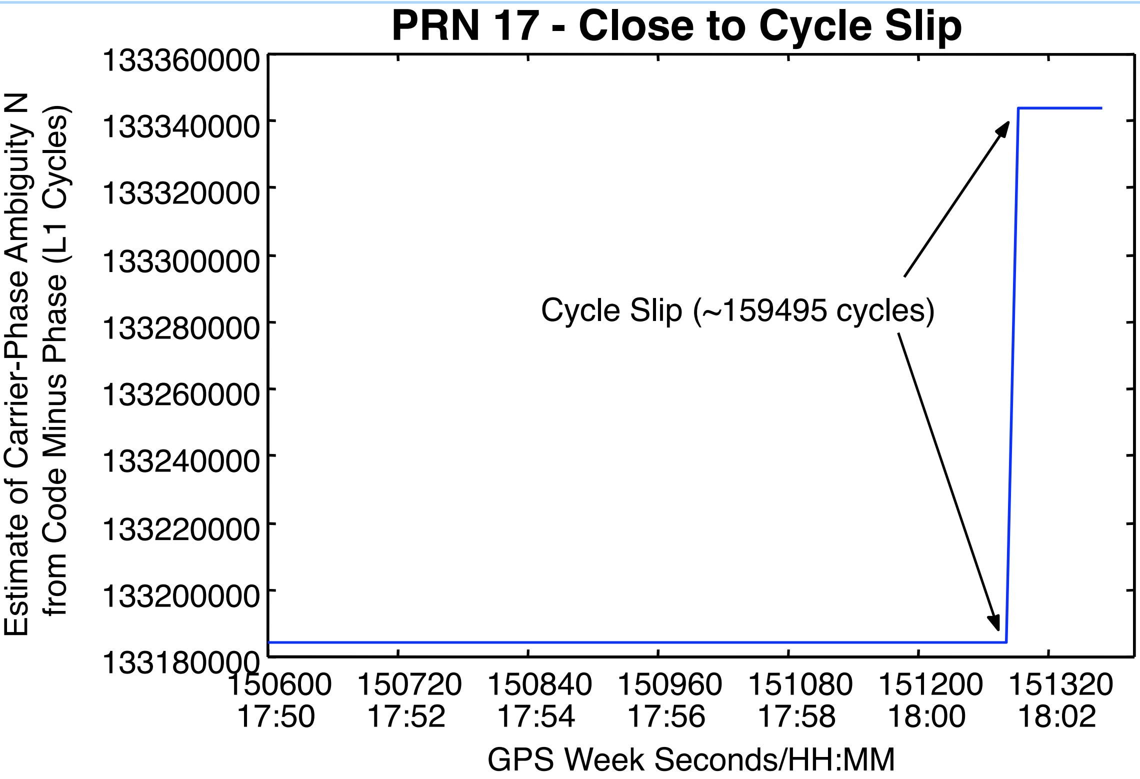
Example: Using Doppler to Predict New Carrier-Phase Measurement

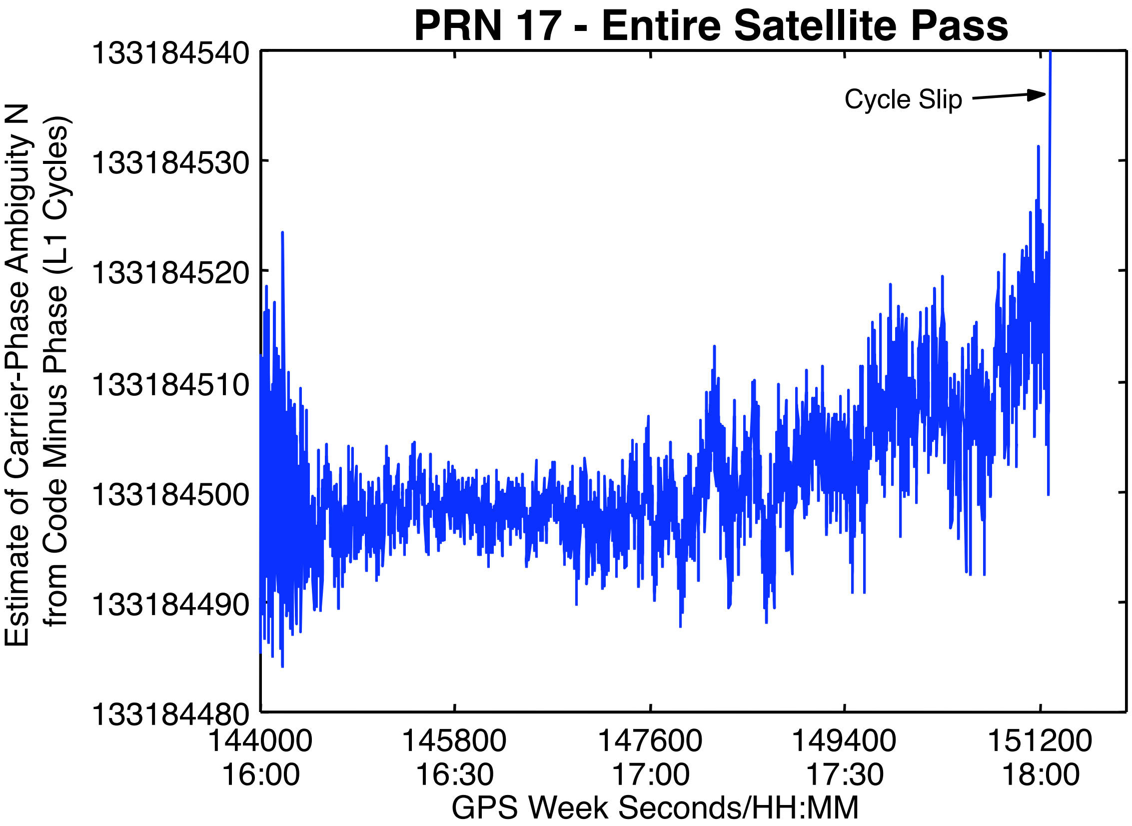


t_k	ϕ_k	$t_{k+1}-t_k$	Δf_k	Predicted Φ_{k+1}	Actual Φ_{k+1}	Prediction Error
151268	20520314.15	2	4781.74	20529877.62	20529877.89	-0.26
151270	20529877.89	2	4782.02	20539441.92	20539442.61	-0.69
151272	20539442.61	2	4782.65	20549007.91	20549008.30	-0.39
151274	20549008.30	2	4783.03	20558574.36	20558575.16	-0.80
151276	20558575.16	2	4783.73	20568142.62	20568143.04	-0.42
151278	20568143.04	2	4784.20	20577711.44	20577711.80	-0.36
151280	20577711.80	2	4784.69	20587281.17	20587281.83	-0.66
151282	20587281.83	2	4785.25	20596852.34	20596853.07	-0.73
151292	20596853.07	10	4785.77	20644710.77	20485242.47	159468.30*
151294	20485242.47	2	4788.44	20494819.35	20494819.72	-0.38
151296	20494819.72	2	4788.80	20504397.33	20504397.88	-0.55

*Cycle slip is indicated here

Example: Code Minus Phase Measurements to Estimate Carrier-Phase Ambiguity





Measurement Combinations

- Generation of new observable by combining measurements from two different frequencies
- Generation of new observable by combining code and carrier-phase measurements
- Using various measurements within an estimation algorithm (e.g., a Kalman filter)

GPS Code and Phase Measurement Representations

- Code measurement (ρ):

$$\boxed{\rho = r + c\delta t_u - c\delta t_{sv} + T + \frac{I}{f^2} + \upsilon + m + SA} \quad (8.1)$$

$r =$ true range to satellite (m)

$c\delta t_u =$ receiver clock error (m)

$c\delta t_{sv} =$ satellite clock error (m)

$T =$ tropospheric delay (m)

$I =$ ionospheric delay factor ($=$ 40.30 TEC) (Hz^{2}m)

$f =$ carrier frequency (Hz)

$\upsilon =$ measurement noise (m)

$m =$ multipath (m)

$SA =$ selective availability (m)

- Phase measurement (ϕ):

$$\boxed{\phi = \frac{1}{\lambda}\left(r + c\delta t_u - c\delta t_{sv} + T - \frac{I}{f^2} + \upsilon + m + SA\right) + N} \quad (8.2)$$

$\lambda =$ carrier wavelength (m)

$N =$ carrier-phase integer ambiguity (cycles)

  - Note 1: Sign change on ionospheric term
  - Note 2: Ionospheric error is frequency dependent
  - Note 3: Multipath and noise are different for every measurment

Use of Dual Frequency Phase Measurements

- Will analyze phase measurements
  - More useful for precise positioning applications
  - Same principles apply to code measurements
- Create a dual frequency observable as a linear combination of the L1 and L2 phase measurements (ϕ_{L1} and ϕ_{L2}, respectively)

$$\phi_{j,k} = j\phi_{L1} + k\phi_{L2}$$

- Applying to equation 8.2 yields

$$\phi_{j,k} = \frac{1}{\lambda_{j,k}}\left(r + c\delta t_u - c\delta t_{sv} + T + SA\right)$$

$$+\frac{j}{\lambda_1}\left(m_{L1} + \upsilon_{L1}\right) + \frac{k}{\lambda_2}\left(m_{L2} + \upsilon_{L2}\right)$$

$$-\frac{I}{c}\left(\frac{jf_2 + kf_1}{f_1 f_2}\right) + jN_1 + kN_2$$

where $\lambda_{j,k} = \left(\dfrac{\lambda_1\lambda_2}{j\lambda_2 + k\lambda_1}\right)$

Typical Carrier-Phase Measurement Combinations

- Widelane (WL) ($j=1$, $k=-1$)

$$\phi_{WL} = \phi_{1,-1} = \phi_{L1} - \phi_{L2}$$

  - Maintains integer nature of carrier-phase ambiguity
  - Has longer wavelength (86 cm) than L1 or L2 (19 cm or 24 cm, respectively)
    - Makes it easier to resolve carrier-phase ambiguities
- Ionospheric free (IF) ($j=1$, $k=-f_2/f_1$)
  - Eliminates ionospheric error
  - Ambiguity is not an integer
    - Can still be used if L1 and L2 (or L1 and WL) ambiguities are known
- Ionospheric ($j=\lambda_1$, $k=-\lambda_2$)
  - Cancels out everything except multipath, noise, and ionospheric error
  - Ambiguity not an integer (similar to IF case)
  - Commonly used for precise (differential) ionospheric measurements

Summary of Carrier-Phase Measurement Combinations

Name	j	k	Wavelength ($\lambda_{j,k}$)	Ambiguity
Widelane (WL)	1	-1	~0.86 m	$N_1 - N_2 = N_{WL}$
Ionospheric Free (IF)	1	$-f_2/f_1$	~0.48 m	$N_1 - (f_2/f_1)N_2 = N_{IF}$
Ionospheric	λ_1	$-\lambda_2$	∞	$\lambda_1 N_1 - \lambda_2 N_2$
L1 Only	1	0	~0.19 m	N_1
L2 Only	0	1	~0.24 m	N_2

Combining Code and Carrier-Phase Measurements

- We noted previously

$$\frac{\rho(t_0)}{\lambda} \approx \phi(t_0) - N$$

- We need to examine this more closely
- Recall:

$$\rho = r + c\delta t_u - c\delta t_{sv} + T + \frac{I}{f^2} + \upsilon + m + SA$$

$$\phi = \frac{1}{\lambda}\left(r + c\delta t_u - c\delta t_{sv} + T - \frac{I}{f^2} + \upsilon + m + SA \right) + N$$

- From this, we can analyze the code-minus-carrier observable

$$\rho - \lambda\phi = 2\frac{I}{f^2} + \upsilon_\rho + m_\rho - \upsilon_\phi - m_\phi - \lambda N$$

υ_ρ = code measurement noise (m)

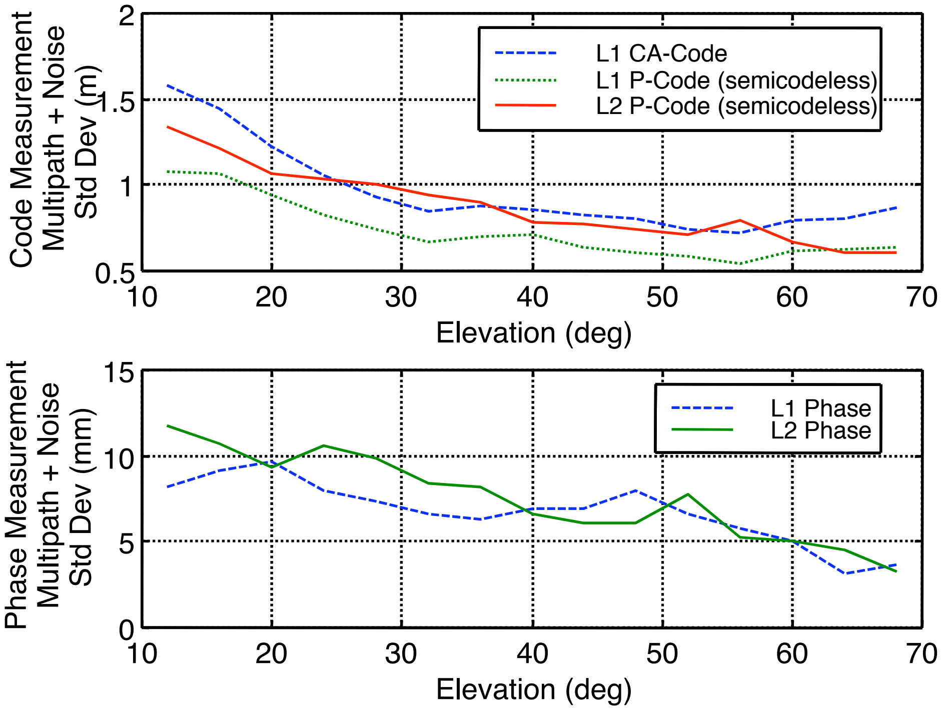
m_ρ = code measurement multipath (m)

υ_ϕ = phase measurement noise (m)

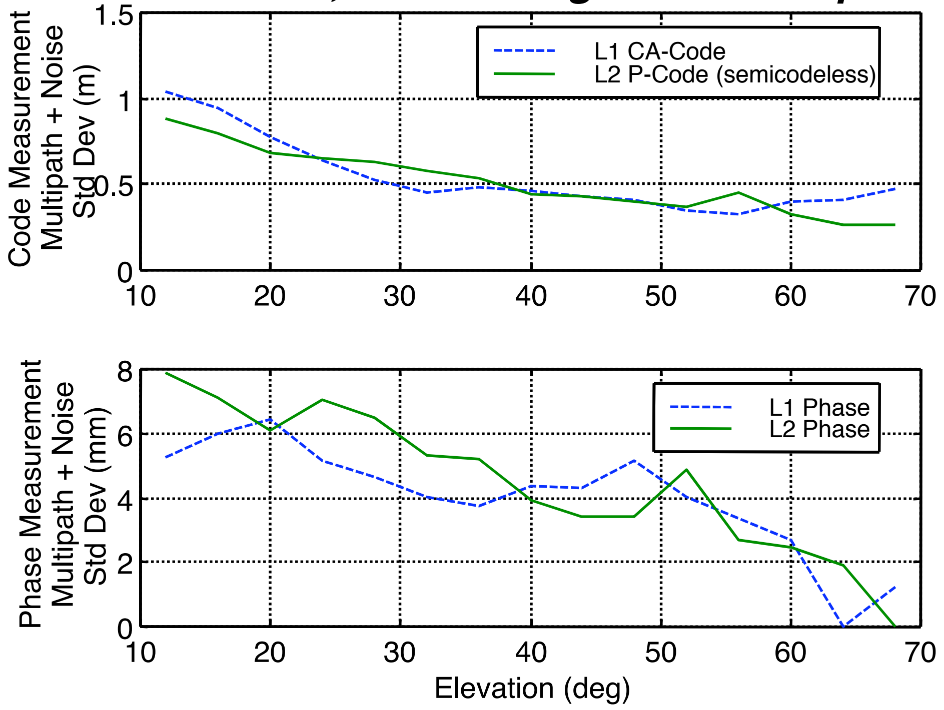
m_ϕ = phase measurement multipath (m)

Examples of Multipath (plus Noise) vs. Satellite Elevation

Ashtech Z-12 Receiver, Dorne-Margolin Groundplane Antenna

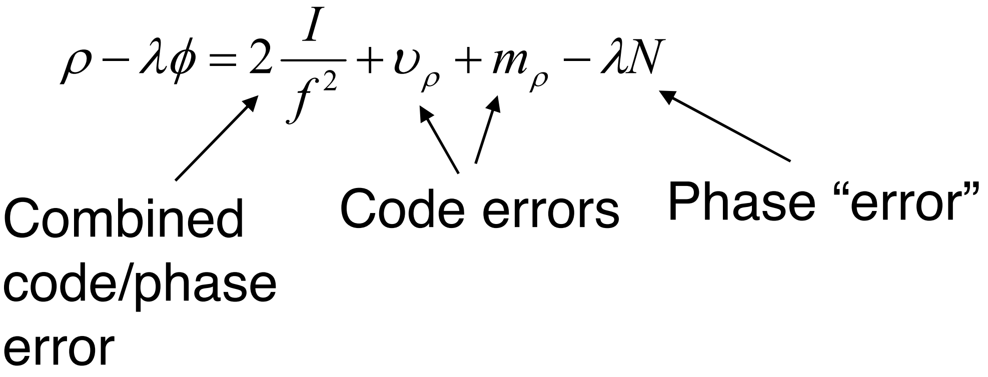


Trimble 4000ssi Receiver, Dorne-Margolin Groundplane Antenna



Errors in Combined Code/Carrier Measurements

- We can neglect the phase noise and multipath, yielding



$$\rho - \lambda\phi = 2\frac{I}{f^2} + \upsilon_\rho + m_\rho - \lambda N$$

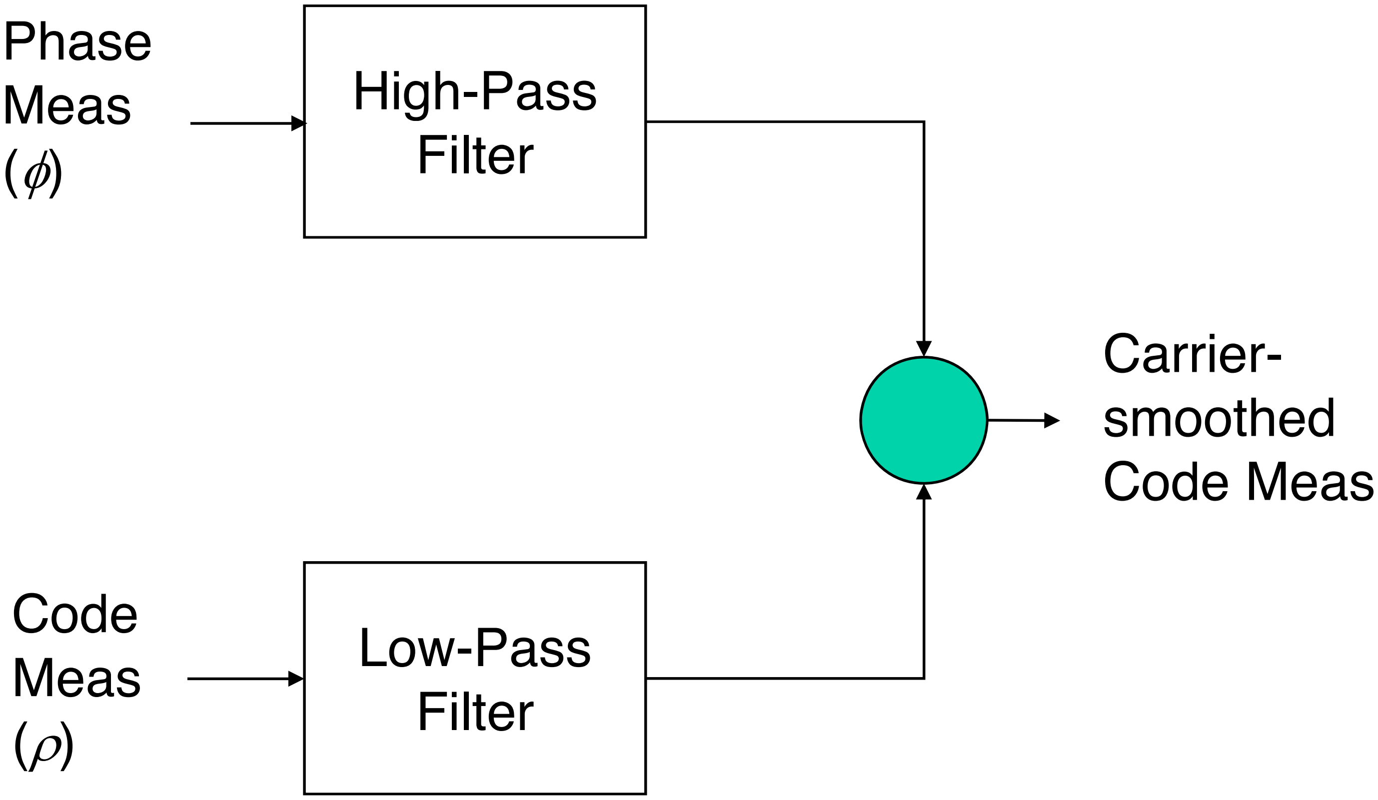
Combined code/phase error

Code errors

Phase “error”

- Time correlation of error sources
  - Ionospheric error:
  - Code measurement noise:
  - Code multipath:
  - Carrier-phase ambiguity:

Conceptual Diagram of Code/Carrier Smoothing



Phase Meas (ϕ)

High-Pass Filter

Code Meas (ρ)

Low-Pass Filter

Carrier-smoothed Code Meas

Algorithm for Code/Carrier Smoothing

- Concept of code/carrier smoothing for GPS first proposed by Hatch[1]
- Progressive weighting algorithm proposed by Lachapelle et al.:[2]
- Recursive filter to progressively increase weight on ϕ while decreasing weight on ρ

$$\underset{\substack{\uparrow \\ \text{Computed} \\ \text{smoothed} \\ \text{pseudorange}}}{\hat{\rho}_k} = W_{\rho_k} \underset{\substack{\uparrow \\ \text{Measured} \\ \text{pseudorange}}}{\rho_k} + W_{\phi_k}\Big[\underset{\substack{\uparrow \\ \text{Previously} \\ \text{smoothed} \\ \text{pseudorange}}}{\hat{\rho}_{k-1}} + \underset{\substack{\uparrow \\ \text{Range difference} \\ \text{from measured} \\ \text{carrier-phase}}}{\lambda(\phi_k - \phi_{k-1})}\Big]$$

  - Weights are incremented over time (within bounds)

$$W_{\rho_k} = W_{\rho_{k-1}} - 0.01 \qquad (0.01 \le W_{\rho_k} \le 1.00)$$

$$W_{\phi_k} = W_{\phi_{k-1}} + 0.01 \qquad (0.00 \le W_{\phi_k} \le 0.99)$$

  - Initialization (at $k=0$)

$$\left.\begin{aligned} W_{\rho_0} &= 1.0 \\ W_{\phi_0} &= 0.0 \end{aligned}\right\} \text{results in } \hat{\rho}_0 = \rho_0$$

---

[1] Hatch, R., "The Synergism of Code and Carrier Measurements," *Proceeding of the Third International Geodetic Symposium on Satellite Doppler Positioning*, DMA/NGS, pp. 1213-1232, Washington, D.C., 1982.

2Lachapelle, Hagglund, Falkenberg, Bellemare, Casey, and Eaton, "GPS Land Kinematic Experiments", *Proceedings of the Fourth International Geodetic Symposium on Satellite Positioning*, Austin, Texas, 1986.

Dual Ramp Code/Carrier Smoothing Technique

- Code/carrier ionospheric divergence causes a drift in the smoothed code measurement over time
  - A short time constant on the smoothing algorithm can help reduce this
  - If the time constant is too short, then carrier smoothing is less effective
- One approach is to calculate smoothed values starting at two different times
  - After a ramp has been used for a specified maximum time, then you switch to a newer ramp and restart the old ramp