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Fluid Turbulence

S. Sridhar
Raman Research Institute
Bangalore
India

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S. Sridhar

Raman Research Institute, Bangalore 560080, India

ssridhar@rri.res.in

1. Scaling in the Navier–Stokes equations

The NS equation for an incompressible fluid:

$$\begin{aligned} \frac{\partial \boldsymbol{v}}{\partial t} \,+\, (\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{v} \;&=\; -\boldsymbol{\nabla}\left(\frac{p}{\rho}\right) \,+\, \nu\boldsymbol{\nabla}^2\boldsymbol{v} \\ \boldsymbol{\nabla}\cdot\boldsymbol{v} \;&=\; 0 \\ \text{On solid boundaries,} \qquad \boldsymbol{v} \;&=\; \text{velocity of the boundary} \end{aligned} \tag{1}$$

- Consider two geometrically similar bodies moving in fluids with different densities, and viscosities. Under what conditions are the flows related to each other through some simple scaling? To answer this question, let us begin by casting the NS equations (1) in a dimensionless form.

- Let U and L be typical scales of speed and length of the flow in any one of the two cases considered above. Let

$$\boldsymbol{x} \;=\; L\boldsymbol{x}'\,, \qquad\qquad t \;=\; \frac{L}{U}\,t'\,, \qquad\qquad \boldsymbol{v} \;=\; U\boldsymbol{v}' \tag{2}$$

  where $\boldsymbol{x}'$, t' and $\boldsymbol{v}'$ are dimensionless. The quantity, (p/ρ), has dimensions of velocity-squared. So we write $(p/\rho) = U^2(p/\rho)'$, where $(p/\rho)'$ is dimensionless. Then the NS equation can be written in dimensionless form as,

$$\frac{\partial \boldsymbol{v}'}{\partial t'} \,+\, (\boldsymbol{v}'\cdot\boldsymbol{\nabla}')\,\boldsymbol{v}' \;=\; -\boldsymbol{\nabla}'\left(\frac{p}{\rho}\right)' \,+\, \frac{1}{\mathrm{Re}}\boldsymbol{\nabla}'^2\boldsymbol{v}' \tag{3}$$

  where

$$\mathrm{Re} \;=\; \frac{UL}{\nu}\,; \qquad\qquad \text{Reynolds number} \tag{4}$$

  Note that the Reynolds number can be thought of as the ratio of *inertial* to *viscous* forces acting on a typical fluid element. The velocity field must be of the form,

$$\boldsymbol{v}' \;=\; \boldsymbol{a}(\boldsymbol{x}', t'; \mathrm{Re})\,, \qquad \text{i.e.} \qquad \boldsymbol{v} \;=\; U\,\boldsymbol{a}\left(\frac{\boldsymbol{x}}{L}\,, \frac{Ut}{L}\,; \frac{UL}{\nu}\right) \tag{5}$$

where $\boldsymbol{a}$ is some divergence–free vector function. All other physical quantities, like pressure, can now be constructed. Auto/Aircraft manufacturers use the scaling property of the NS equations by conducting tests on scale–models in wind tunnels.

- **Problems**

  1. Estimate Re for various flows.
  2. Scaling used in wind-tunnel simulations.

2. Low Reynolds number flow past a sphere: Stokes flow

- Consider flow past a solid sphere of radius a. If the fluid velocity far upstream is $\hat{x}U_0$, the Reynolds number for the flow may be defined as $\mathrm{Re} = U_0 a/\nu$. The flow is called *Stokes flow* when $\mathrm{Re} \ll 1$. The inertial term in the NS equation is much smaller than the viscous term, so force balance gives

$$\boldsymbol{\nabla} p \;=\; \eta \boldsymbol{\nabla}^2 \boldsymbol{v} \tag{6}$$

The problem is to find a $\boldsymbol{v}(\boldsymbol{x})$, such that $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$, with $\boldsymbol{v} = 0$ on the surface of the sphere, and $\boldsymbol{v} \to \hat{x}U_0$ for $|\boldsymbol{x}| \to \infty$. Stokes solved this problem and determined that the *drag force* acting on the sphere is

$$\mathbf{F} \;=\; \hat{x}\, 6\pi\eta a U_0 \tag{7}$$

where 2/3 of the force is due to viscous stress and 1/3 is due to pressure. The difference in fluid pressure between the front and back of the sphere is $\Delta p \sim \eta U_0/a$. The pressure acts on a surface area $\sim a^2$, so we expect a drag force $\sim \eta a U_0$. Viscous stresses acting on the surface of the sphere contribute similarly.

3. Transition to turbulence: flow past a cylinder

- **Flow past a cylinder**: Consider flow past a cylindrical obstacle of diameter d, with its axis oriented along the z–axis. If the fluid velocity far upstream is $\hat{x}U_0$, the Reynolds number for the flow may be defined as $\mathrm{Re} = U_0 d/\nu$. If the flow is stationary the velocity field must be of the form

$$\boldsymbol{v} \;=\; U\, \boldsymbol{a}\left(\frac{\boldsymbol{x}}{L}\,;\frac{U_0 d}{\nu}\right) \tag{8}$$

Figures 1 and 2 provide some idea of this steady flow pattern, for quite small values of Re. However, the flow is unstable at higher Re and becomes unsteady.

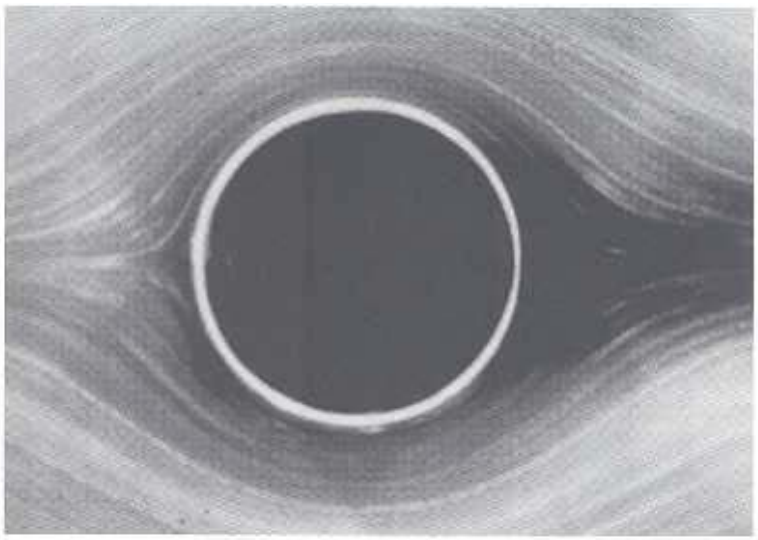

Fig. 1.— Flow at Re = 0.16. When Re $\ll$ 1, there is creeping flow past the cylinder, similar to the case of Stokes flow. The flow is steady, two–dimensional, and has up–down symmetry (left–right symmetry is obtained only when $\nu = 0$).

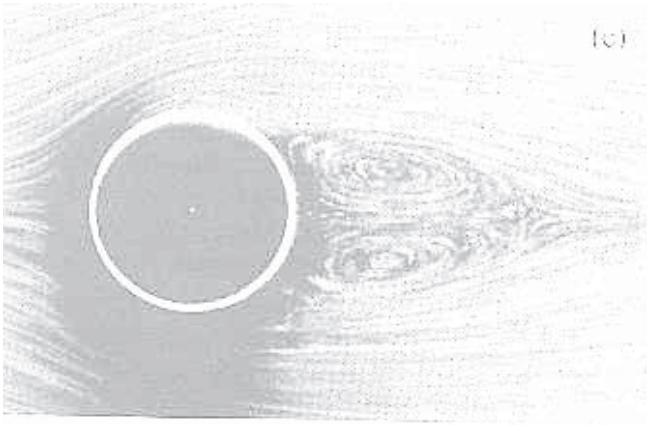

Fig. 2.— Flow at Re = 26. The boundary layer has separated behind the cylinder for Re > 5. Topology of flow changes due to formation of recirculating eddies.

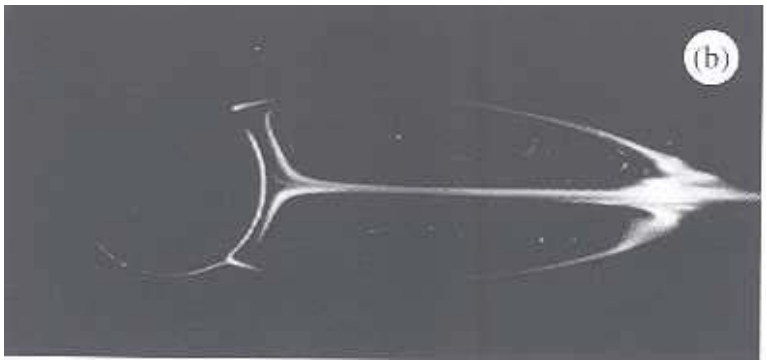

Fig. 3.— Flow at Re = 41 is time–periodic.

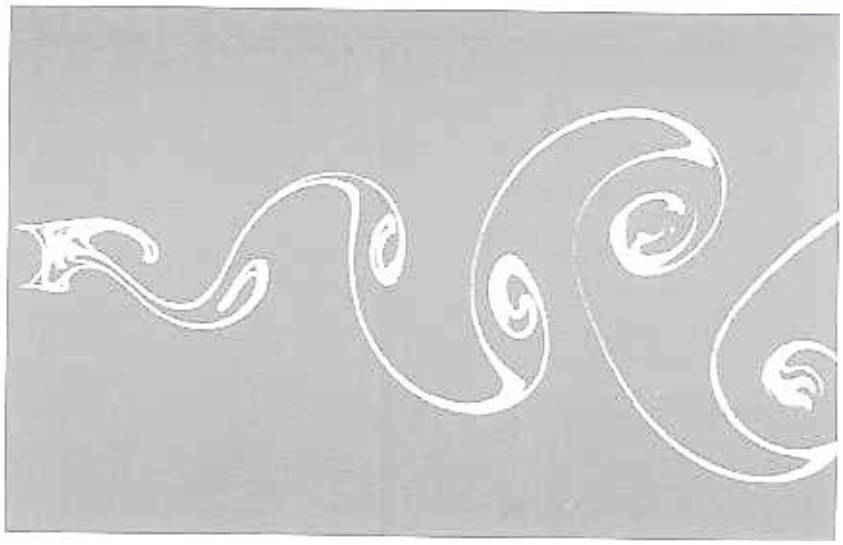

Fig. 4.— Flow for Re = 140. For Re $\sim$ 100, the shedding of the recirculating eddies leads to the formation of alternating vortices, known as a *von Karman street*.

- When Re > 1000, eddies are no longer visible, and the velocity field is irregular on all scales, and the flow is called *turbulent*. At very high Re the flow has *statistically* regular properties, and is called *fully developed turbulence*. A remarkable feature of fully developed turbulence is that its statistical properties (on suitable length and time scales) seem to be independent of the particular manner of its generation. For example, far enough downstream, the flow behind a cylinder is is statistically similar to the flow behind a grid; see Figure 5 below.

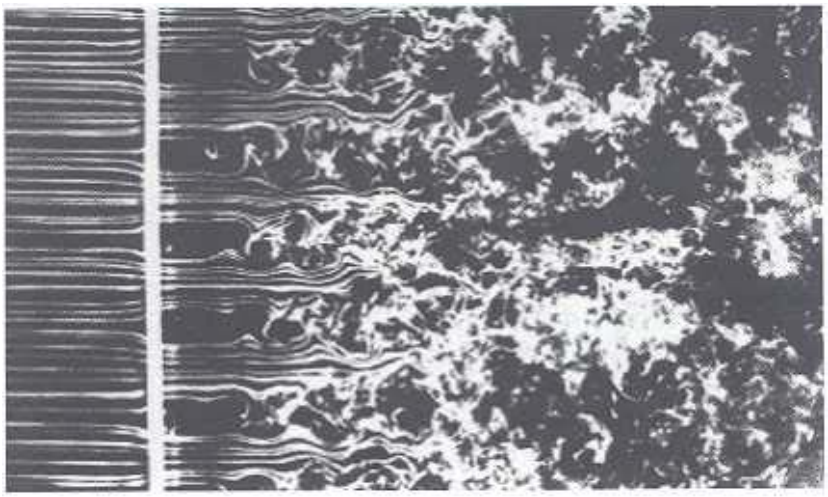

Fig. 5.— Grid generated turbulence.

4. Fully developed turbulence

The flow can be thought of as having a mean velocity, $\boldsymbol{U}(\boldsymbol{x}, t)$, in addition to fluctuations, $\boldsymbol{u}(\boldsymbol{x}, t)$. We write

$$\boldsymbol{v} \;=\; \boldsymbol{U} \;+\; \boldsymbol{u} \tag{9}$$

where the fluctuations have zero mean, $\langle \boldsymbol{u} \rangle = \boldsymbol{0}$. As we shall see later, the relationship between $\boldsymbol{U}$ and $\boldsymbol{u}$ is of great interest and remains one of the great unsolved problems in this area. For now, we imagine that the fluid is forced externally such that the mean velocity $\boldsymbol{U} = \boldsymbol{0}$, and ask what may be said about the statistical properties of the fluctuations. These properties may be stated in terms of *velocity correlation functions*. For simplicity of presentation, we focus on two–point velocity correlators, although higher order correlators play very important roles in the theory. When the forcing is statistically uniform in space, we expect the correlators to be *homogeneous*. This means that quantitites such as $\langle u_i(\boldsymbol{x}, t) u_j(\boldsymbol{x}', t') \rangle$

will be functions of $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}'$. If the forcing is also *isotropic*, then the correlator will depend only on $r = |\boldsymbol{r}|$.

- **Problem**

  3. Two-point velocity correlator for homogeneous isotropic turbulence.

Under conditions of steady driving, the fluctuations may be expected to be *stationary* in time. This means that the two–point correlator depends only on $t - t'$.

5. Turbulent diffusion of a passive scalar

5.1. The advection–diffusion equation

A passive additive, such as perfume in air, is transported and mixed by the flow, but its presence does not affect the flow. In a static homogeneous medium the diffusion of perfume concentration, $n(\boldsymbol{x}, t)$, is determined by the continuity equation,

$$\frac{\partial n}{\partial t} \; + \; \boldsymbol{\nabla} \cdot \boldsymbol{J} \; = \; 0 \tag{10}$$

and *Fick's law* for the flux density, $\boldsymbol{J} = -\kappa \boldsymbol{\nabla} n$, where κ is the molecular diffusivity of perfume in air. Therefore n satisfies the *diffusion equation*:

$$\frac{\partial n}{\partial t} \; = \; \kappa \boldsymbol{\nabla}^2 n \tag{11}$$

Given an initial distribution, $n(\boldsymbol{x}, 0)$, the distribution at any later time is given by

$$n(\boldsymbol{x}, t) \; = \; \frac{1}{(4\pi\kappa t)^{3/2}} \int d^3x' \exp\left[-\frac{|\boldsymbol{x} - \boldsymbol{x}'|^2}{4\kappa t} \right] n(\boldsymbol{x}', 0) \tag{12}$$

- **Problem**

  4. Solve the diffusion equation.

If the initial distribution is concentrated at a point, it will spread over a length scale L in time $\tau \sim L^2/\kappa$. For $\kappa \sim 0.1\,\mathrm{cm}^2\,\mathrm{s}^{-1}$ we may expect perfume to diffuse across a room of size 5 m in time $\tau \sim 600$ hours, a conclusion which contradicts experience! One resolution of this paradox is that air in the room is not static: if the door or a window is open and there is

a breeze, perfume can be advected downwind. What happens if the room is sealed, but air flows randomly within? To address this problem we need to generalise the heat equation to a moving medium.

When the medium is not static, the diffusion equation generalises to the *advection–diffusion equation*:

$$\frac{dn}{dt} \equiv \frac{\partial n}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} n = \kappa \boldsymbol{\nabla}^2 n \tag{13}$$

where $\boldsymbol{u}(\boldsymbol{x}, t)$ is a given incompressible velocity field which could be either deterministic or turbulent. The equation is still linear in n, but the solution cannot be written down as before. However, we can derive some general properties. Integrating the equation over all space, we obtain

$$\frac{d}{dt} \int d^3x \, n(\boldsymbol{x}, t) = 0 \tag{14}$$

where we have assumed that the surface terms vanish at spatial infinity. Multiply equation (13) by n and integrate over all space:

$$\frac{d}{dt} \frac{1}{2} \int d^3x \, n^2(\boldsymbol{x}, t) = -\kappa \int d^3x \, |\boldsymbol{\nabla} n|^2 \leq 0 \tag{15}$$

Therefore we conclude

- The *mean* concentration is conserved.
- The *mean–squared* concentration fluctuations decay over time.

A great deal of work has been done on the advection–diffusion equation, but the general problem for arbitrary velocity fields remains unsolved. Let us now approach the problem using *mean–field* theory.

5.2. Mean–field theory

Let the incompressible velocity field be a random flow with zero mean, $\langle \boldsymbol{u} \rangle = \boldsymbol{0}$. Split the concentration field into mean, $C(\boldsymbol{x}, t)$, plus fluctuations, $c(\boldsymbol{x}, t)$:

$$n = C + c; \qquad \langle c \rangle = 0 \tag{16}$$

When the mean concentration varies over length scales that are much larger than the length scale of variation of the velocity fluctuations, we can interpret the ensemble average, $< >$,

as a spatial average, over scales that are much larger than the length scale of variation of the velocity fluctuations. Then C and c satisfy the following equations:

$$\frac{\partial C}{\partial t} \;=\; -\boldsymbol{\nabla}\cdot\boldsymbol{J}^{\rm t} \;+\; \kappa\boldsymbol{\nabla}^2 C \tag{17}$$

$$\frac{\partial c}{\partial t} \;=\; -\boldsymbol{u}\cdot\boldsymbol{\nabla}C \;-\; \boldsymbol{\nabla}\cdot[c\boldsymbol{u} \;-\; \langle c\boldsymbol{u}\rangle] \;+\; \kappa\boldsymbol{\nabla}^2 c \tag{18}$$

where $\boldsymbol{J}^{\rm t} = \langle c\boldsymbol{u}\rangle$ is the *mean flux density* due to random advection of concentration fluctuations.

We are interested in solving for $C(\boldsymbol{x},t)$, for some known statistical properties of $\boldsymbol{u}(\boldsymbol{x},t)$. It is clear that we can make progress only in certain situations, because equations (17) and (18) are no simpler than the original equation (13). Let us assume that

- The fluctuations are much smaller than the mean: i.e. $|c| \ll |C|$ so we can drop the quadratic terms
- The molecular diffusivity, κ, is so small that the $\kappa\boldsymbol{\nabla}^2 c$ can also be dropped.

Then an approximate solution of equation (18) for $c(\boldsymbol{x},t)$, with $c(\boldsymbol{x},0)=0$, is

$$c(\boldsymbol{x},t) \;=\; -\int_0^t dt'\, u_j(\boldsymbol{x},t')\frac{\partial C(\boldsymbol{x},t')}{\partial x_j} \tag{19}$$

This describes the working of velocity fluctuations on the mean concentration to produce concentration fluctuations, and is called the *first order smoothing approximation* (FOSA). We can substitute this solution for c back in equation (17). Let us work out the mean flux density:

$$J_i^{\rm t} \;=\; \langle cu_i\rangle \;=\; -\int_0^t dt'\; \langle u_i(\boldsymbol{x},t)u_j(\boldsymbol{x},t')\rangle\; \frac{\partial C(\boldsymbol{x},t')}{\partial x_j}$$

When C varies over *time scales* that are much larger than the velocity correlation time, $\tau_{\rm c}$,

$$J_i^{\rm t} \;\simeq\; -\kappa_{ij}\,\frac{\partial C(\boldsymbol{x},t)}{\partial x_j} \tag{20}$$

where

$$\begin{aligned}\kappa_{ij} \;&=\; \int_0^\infty dt'\; \langle u_i(\boldsymbol{x},t)u_j(\boldsymbol{x},t')\rangle \\ &=\; \delta_{ij}\,\kappa_{\rm t}, \quad \text{for isotropic random velocities}\end{aligned} \tag{21}$$

Note that equation (20) resembles Fick's law; if the velocity correlator is not isotropic, we get a turbulent contribution to the flux density which is anisotropic. Both κ_{ij} and $\kappa_{\rm t}$ are referred to as *turbulent diffusivities*. We will see later that in a turbulent flow

- Velocity fluctuations occur on a wide range of spatial and temporal scales.
- The dominant contribution to turbulent diffusivity comes from the largest *turbulent eddies*.

Let L be the typical size and $u_{\rm rms}$ the root–mean–squared velocity of the largest eddies (the correlation time $\tau_{\rm c} \sim L/u_{\rm rms}$). From equation (21), we expect that

$$\kappa_{\rm t} \;\approx\; \frac{1}{3} u_{\rm rms}^2 \tau_{\rm c} \;\approx\; \frac{1}{3} u_{\rm rms} \ell \tag{22}$$

Note the analogy with the kinetic theory of gases! Then the mean flux density is given by

$$\langle c\boldsymbol{u} \rangle \;\simeq\; -\kappa_{\rm t}\, \boldsymbol{\nabla} C \tag{23}$$

Substituting this in equation (17), we obtain a diffusion equation for $C(\boldsymbol{x}, t)$:

$$\frac{\partial C}{\partial t} \;=\; (\kappa_{\rm t} + \kappa) \boldsymbol{\nabla}^2 C \tag{24}$$

where we have assumed that the velocity fluctuations are homogeneous, so that $\kappa_{\rm t}$ is independent of space. As before, given an initial distribution, $C(\boldsymbol{x}, 0)$, the solution at a later time is

$$C(\boldsymbol{x}, t) \;=\; \frac{1}{(4\pi\kappa_{\rm T} t)^{3/2}} \int d^3x' \exp\left[-\frac{|\boldsymbol{x} - \boldsymbol{x}'|^2}{4\kappa_{\rm T} t} \right] n(\boldsymbol{x}', 0) \tag{25}$$

where $\kappa_{\rm T} = \kappa_{\rm t} + \kappa$ is the total diffusivity.

Turbulent diffusivity can be much larger than molecular diffusivity. For modest velocity fluctations, $u_{\rm rms} \sim 10\,{\rm cm\,s^{-1}}$ with correlation time $\tau_{\rm c} \sim 30\,{\rm s}$, we have $\kappa_{\rm t} \approx 1000\,{\rm cm^2\,s^{-1}} \approx 10^4\,\kappa$. Therefore we expect concentration to diffusive across a room of size 5 m in about 4 mins, which is a much more reasonable expectation than our earlier estimate suggested.

6. Reynolds averaging of the NS equations

This is an approach to the NS equations similar in spirit to the mean–field theory of the previous section. However, the velocity field is not a passive quantity; in fact it the

dynamics of the velocity field that is of interest. Let us allow for external stirring of the fluid by including a random acceleration field, $\boldsymbol{f}(\boldsymbol{x},t)$, in the NS equations, which is also assumed to be incompressible with zero mean: $\boldsymbol{\nabla}\cdot\boldsymbol{f}=0$; $\langle\boldsymbol{f}\rangle=\boldsymbol{0}$. Then the forced NS equation is

$$\frac{\partial\boldsymbol{v}}{\partial t} \;+\; (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} \;=\; -\frac{1}{\rho}\boldsymbol{\nabla}p \;+\; \nu\boldsymbol{\nabla}^2\boldsymbol{v} \;+\; \boldsymbol{f}. \tag{26}$$

The pressure, $p(\boldsymbol{x},t)$, is determined by requiring that equation (26) preserve the incompressibility of the flow:

$$\frac{1}{\rho}\boldsymbol{\nabla}^2 p \;=\; -\boldsymbol{\nabla}\cdot[(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}] \tag{27}$$

Following Reynolds, it is conventional to split the flow into *mean* and *fluctuating* components. For the forced situation, we imagine ensemble averages, $<\ >$, being ultimately referred to the statistics of the forcing (but there are many important problems in which there is no forcing and the random fluctuations are due to instabilities amplifying small perturbations). We write

$$\begin{aligned} \boldsymbol{v} \;&=\; \boldsymbol{U} \;+\; \boldsymbol{u}\,, \qquad \langle\boldsymbol{v}\rangle \;=\; \boldsymbol{U}\,, \qquad \langle\boldsymbol{u}\rangle \;=\; \boldsymbol{0} \\ p \;&=\; P \;+\; p'\,, \qquad \langle\, p\,\rangle \;=\; P\,, \qquad \langle\, p'\,\rangle \;=\; 0 \end{aligned} \tag{28}$$

where $(\boldsymbol{U},P)$ and $(\boldsymbol{u},p')$ are the mean and fluctuating velocity and presssure, respectively. We assume that the velocity fields are all incompressible: $\boldsymbol{\nabla}\cdot\boldsymbol{U}=\boldsymbol{\nabla}\cdot\boldsymbol{u}=0$. Following standard procedure, we can derive equations for the mean and fluctuations. The mean-field equations are

$$\frac{\partial\boldsymbol{U}}{\partial t} \;+\; (\boldsymbol{U}\cdot\boldsymbol{\nabla})\boldsymbol{U} \;=\; -\frac{1}{\rho}\boldsymbol{\nabla}P \;+\; \nu\boldsymbol{\nabla}^2\boldsymbol{U} \;+\; \boldsymbol{F} \tag{29}$$

$$\frac{1}{\rho}\boldsymbol{\nabla}^2 P \;=\; -\boldsymbol{\nabla}\cdot[(\boldsymbol{U}\cdot\boldsymbol{\nabla})\boldsymbol{U}] \;+\; \boldsymbol{\nabla}\cdot\boldsymbol{F} \tag{30}$$

where $\boldsymbol{F}$ is given in component form by

$$F_i \;=\; -\frac{1}{\rho}\partial_j R_{ij}\,, \qquad R_{ij} \;=\; \rho\,\langle u_i u_j\rangle \tag{31}$$

where R_{ij} is the (symmetric) *Reynolds stress tensor*. It can be thought of as the turbulent contribution to the total stress tensor which drives the mean–field. When the turbulence is *homogeneous*, R_{ij} will be independent of space and $\boldsymbol{F} = \boldsymbol{0}$, so there is no turbulent driving of the mean–field. The first step in mean–field theory is to determine R_{ij} in terms of the mean–field, $\boldsymbol{U}$, and the forcing, $\boldsymbol{f}$. This means solving for the fluctuations. The equations governing the fluctuations

$$\frac{\partial \boldsymbol{u}}{\partial t} \; + \; (\boldsymbol{U}\cdot\boldsymbol{\nabla})\boldsymbol{u} \; + \; (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{U} \; + \; (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \langle(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\rangle \; = \; -\frac{1}{\rho}\boldsymbol{\nabla}p' \; + \; \nu\boldsymbol{\nabla}^2\boldsymbol{u} \; + \; \boldsymbol{f} \quad (32)$$

$$\frac{1}{\rho}\boldsymbol{\nabla}^2 p' \; = \; -\boldsymbol{\nabla}\cdot[(\boldsymbol{U}\cdot\boldsymbol{\nabla})\boldsymbol{u} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{U} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} - \langle(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\rangle] \quad (33)$$

are, in general, no easier to solve that the original NS equations. We can make progress if the fluctuations are, in some sense, small. But, in practice, they are not small and we need to face up to the problem of *strong turbulence*.

In applications to engineering problems and astrophysical flows, an intuitive approach has proven very useful. One argues as follows: Galilean invariance implies that the Reynolds stress tensor must be independent of the mean velocity, but will depend on the gradients of the mean–velocity. Then, going by analogy with the kinetic theory of gases, one boldly declares that

$$R_{ij} \; \equiv \; \rho\,\langle u_i u_j\rangle \; \approx \; -\rho\nu_{\rm t}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \quad (34)$$

where

$$\nu_{\rm t} \; \approx \; \frac{1}{3}u_{\rm rms}L \quad (35)$$

is called the *turbulent viscosity*, similar to the turbulent diffusivity we came across in the previous section. The Reynolds stress of equation (34) is then substituted in (29) to get a closed equation for the mean velocity field.

$$\frac{\partial U_i}{\partial t} \; + \; U_j\frac{\partial U_i}{\partial x_j} \; = \; -\frac{1}{\rho}\frac{\partial P}{\partial x_i} \; + \; \nu\frac{\partial^2 U_i}{\partial x_j \partial x_j} \; + \; \frac{\partial}{\partial x_j}\left[\nu_{\rm t}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\right] \quad (36)$$

As before P is determined by requiring that $\boldsymbol{\nabla}\cdot\boldsymbol{U} = 0$. It is important to note that both $u_{\rm rms}$ and L can vary with space, so the turbulent viscosity, $\nu_{\rm t}$, cannot be assumed to be homogeneous; it really depends on the details of the flow one is dealing with.

7. Kolmogorov spectrum for homogeneous isotropic turbulence

- Imagine that an incompressible fluid is stirred *randomly*, with random rms velocity amplitudes u_L, which are correlated on length scale L. The Reynolds number is assumed to be very large:

$$\mathrm{Re} \equiv \frac{L u_L}{\nu} \sim \frac{\text{inertial force}}{\text{viscous force}} \gg 1$$

so that the turbulence may be thought of as fully developed. In a steady state, over length scales much smaller than L, the velocity field will be statistically invariant under translations and rotations. Hence this flow is also known as *homogeneous, isotropic turbulence*.

- The mean energy input rate (per unit mass) is

$$\varepsilon \sim \frac{u_L^3}{L} \tag{37}$$

In a steady state, ε must be equal to the mean rate of viscous dissipation of kinetic energy into heat, $\varepsilon_{vis} \sim \nu \left\langle (\partial u_i/\partial x_j)^2 \right\rangle$. The velocity gradient cannot be set equal to u_L/L. For large Re, the gradient must be taken on some appropriate length scale, that is much smaller than L.

- *Kolmogorov 1941*: Velocity fluctuations are created on small scales through the *nonlinear* interactions provided by the $(\boldsymbol{u} \cdot \boldsymbol{\nabla})\, \boldsymbol{u}$ in the NS equation. The energy transfer rate through scale r is also

$$\varepsilon \sim \frac{u_r^3}{r}\,; \qquad \text{Kolmogorov cascade} \tag{38}$$

Therefore

$$u_r \sim (\varepsilon r)^{1/3} \sim u_L \left(\frac{r}{L}\right)^{1/3} \tag{39}$$

- Eddies of size r turn over in time

$$t_r \sim \frac{r}{u_r} \sim t_L \left(\frac{r}{L}\right)^{2/3} \tag{40}$$

The time for diffusion of momentum over scale r, due to viscosity,

$$t_r^{vis} \sim \frac{r^2}{\nu_{\mathrm{vis}}} \tag{41}$$

decreases more rapidly, as r decreases, than t_r. Hence the *cascade* of kinetic energy is dissipated as heat on the (viscous) scale, ℓ, at which $t_\ell \sim t_\ell^{\mathrm{vis}}$:

$$\ell \sim \frac{L}{\mathrm{Re}^{3/4}} \ll L \tag{42}$$

The range of scales, $\ell \ll r \ll L$ is called the *inertial-range*. The number of degrees of freedom in a fully turbulent flow is of order $(L/\ell)^3 \sim \mathrm{Re}^{9/4} \gg 1$.

- The 3-dim power spectrum of velocity fluctuations on scale $k \sim 1/r$ is

$$E(k) \;\sim\; \frac{u_r^2}{k^3} \;\propto\; \frac{1}{k^{11/3}}\,; \qquad \frac{1}{L} < k < \frac{1}{\ell} \tag{43}$$

which is also known as the *Kolmogorov* spectrum.

- **Problem**:

5. Verify that the rate of energy dissipation rate (per unit mass), $\varepsilon_{vis} \sim \nu \left\langle (\partial u_i/\partial x_j)^2 \right\rangle$, is indeed independent of ν, and equal to the rate of energy input, ε.

6. Among eddies of sizes ranging from L to ℓ, which eddies dominate the contribution to (i) energy, (ii) turbulent diffusivities and viscosities?

8. Formulation of energy transfer through scales

Let us restrict the forced NS equation

$$\frac{\partial \boldsymbol{v}}{\partial t} \;+\; (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} \;=\; -\frac{1}{\rho}\boldsymbol{\nabla} p \;+\; \nu\boldsymbol{\nabla}^2\boldsymbol{v} \;+\; \boldsymbol{f}. \tag{44}$$

to a cubical box of side L. The mean flow is assumed to be zero, $\langle \boldsymbol{v} \rangle = \boldsymbol{0}$ and $\langle \boldsymbol{f} \rangle = \boldsymbol{0}$. The pressure is determined by requiring incompressibility. We define the averages $\langle\ \rangle$, as volume averages over the box. For any quantity $Q(\boldsymbol{x}, t)$,

$$\langle Q \rangle \;=\; \frac{1}{L^3}\int_{\mathrm{box}} d^3x\, Q(\boldsymbol{x}, t) \tag{45}$$

Energy conservation: Take dot product of NS equations with $\boldsymbol{v}$ and average.

$$\frac{1}{2}\frac{d}{dt}\left\langle v^2 \right\rangle \;=\; -\nu \left\langle \omega^2 \right\rangle \;+\; \langle \boldsymbol{v}\cdot\boldsymbol{f} \rangle \tag{46}$$

Scale–by–scale energy budget: Expand all quantities in Fourier series. For example, the velocity field is expanded as

$$\boldsymbol{v}(\boldsymbol{x}, t) \;=\; \sum_{\boldsymbol{k}} \tilde{\boldsymbol{v}}(\boldsymbol{k}, t) \exp\left[i\boldsymbol{k}\cdot\boldsymbol{x}\right] \tag{47}$$

where $\boldsymbol{k} = 2\pi\boldsymbol{n}/L$ with $\boldsymbol{n} = (n_1, n_2, n_3)$ is an ordered triple of integers.

For any wavenumber $K > 0$, the velocity field can be decomposed into low and high wavenumber components (similarly for the forcing):

$$\begin{aligned} \boldsymbol{v}(\boldsymbol{x},t) &= \boldsymbol{v}_K^<(\boldsymbol{x},t) \;+\; \boldsymbol{v}_K^>(\boldsymbol{x},t) \\ \boldsymbol{v}_K^<(\boldsymbol{x},t) &= \sum_{|\boldsymbol{k}|\le K} \tilde{\boldsymbol{v}}(\boldsymbol{k},t)\exp\left[i\boldsymbol{k}\cdot\boldsymbol{x}\right] \\ \boldsymbol{v}_K^>(\boldsymbol{x},t) &= \sum_{|\boldsymbol{k}|> K} \tilde{\boldsymbol{v}}(\boldsymbol{k},t)\exp\left[i\boldsymbol{k}\cdot\boldsymbol{x}\right] \end{aligned} \tag{48}$$

Energy conservation for the low wavenumber component:

$$\begin{aligned} \frac{dE_K}{dt} &= S_K \;-\; D_K \;-\; \Pi_K \\ E_K &= \frac{1}{2}\left\langle |\boldsymbol{v}_K^<|^2 \right\rangle ; \qquad \text{average energy in low wavenumbers} \\ S_K &= \left\langle \boldsymbol{v}_K^< \cdot \boldsymbol{f}_K^< \right\rangle ; \qquad \text{average rate of energy injection in low wavenumbers} \\ D_K &= \nu\left\langle |\boldsymbol{\omega}_K^<|^2 \right\rangle ; \qquad \text{average energy dissipation rate in low wavenumbers} \\ \Pi_K &= \left\langle \boldsymbol{v}_K^< \cdot [(\boldsymbol{v}_K^< \cdot \boldsymbol{\nabla})\,\boldsymbol{v}_K^> + (\boldsymbol{v}_K^> \cdot \boldsymbol{\nabla})\,\boldsymbol{v}_K^>] \right\rangle ; \qquad \text{energy transfer rate} \end{aligned} \tag{49}$$

The average energy in low wavenumbers, E_K , is quadratic in the velocity. Its rate of change with time depends on the energy transfer rate from low to high wavenumbers, Π_K , which is third order in the velocities. If we write down an equation for the time evolution of Π_K , it will involve fourth order velocity correlators and so on. The hierarchy of equations is never ends and this is the reason why turbulence remains an unsolved problem (however, this does not prevent people from taking bold steps, such as "closing" the hierarchy in some way). Despite this fundamental difficulty, Kolmogorov was able to derive a remarkable formula, which remains as one of the few exact results in the theory of turbulence.

Kolmogorov's four–fifth's law: In homogeneous isotropic turbulence, at large Re,

$$\left\langle [\delta v_\ell(\boldsymbol{r})]^3 \right\rangle \;=\; -\frac{4}{5}\,\varepsilon\, r \tag{50}$$

where

- The length scale r lies in the inertial range.
- $\delta v_\ell(\boldsymbol{r})$ is the component of the velocity fluctuation, $\delta v(\boldsymbol{r}) = \boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{v}(\boldsymbol{x})$, in the direction of the displacement $\boldsymbol{r}$.
- ε is the average energy dissipation rate per unit mass.

References

Blandford and Thorne: Applications of classical physics, chapter 14

Landau and Lifshitz: Fluid mechanics

Frisch: Turbulence