Basic Fluid Dynamics: Solutions

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1 Ideal Fluids

1. *Hydrostatic equilibrium*: A plane-parallel atmosphere, composed of a perfect gas, is in hydrostatic equilibrium in an external gravitational field, $-\hat{z}g$. (a) Derive an expression for the entropy gradient if the atmosphere is isothermal.

Solution

Hydrostatic condition implies $\mathbf{v} = 0$, therefore the Euler equations take the form

$$\frac{1}{\rho}\boldsymbol{\nabla} p + \boldsymbol{\nabla}\Phi = 0 \Rightarrow \frac{dp}{dz} = -\rho g \tag{1}$$

From the first law of thermodynamics we have $TdS = dU + pdV$, and upon dividing by the mass of the gas, it takes the from

$$Tds = du + pd\left(\frac{1}{\rho}\right) \tag{2}$$

or

$$Tds = d\left(u + \frac{p}{\rho}\right) - \frac{1}{\rho}dp = dh - \frac{1}{\rho}dp \tag{3}$$

where we have used the definition of specific enthalpy in the last step. Note that for an ideal gas the specific internal energy u depends only on the temperature T. Since the equation of state for an ideal gas is

$$p = \frac{\rho k_B T}{\mu m_p}\ , \tag{4}$$

it is clear that enthalpy too is only a function of temperature of the gas. Thus, for an isothermal atmosphere we obtain

$$T ds = -\frac{1}{\rho} dp \tag{5}$$

Now using the equation of hydrostatic equilibrium we obtain the gradient of entropy as

$$\frac{ds}{dz} = \frac{g}{T} \tag{6}$$

(c) *Earth's atmosphere*: In the lower stratosphere, the air is isothermal. Use the condition of hydrostatic equilibrium to show that:

$$p(z) \propto \exp(-z/H), \tag{7}$$

where the scale height, $H = kT/(\mu m_p g)$. Estimate the scale height. (Use mean molecular weight $\mu = 29$ and $T = 300\,\mathrm{K}$).

Solution

From the equation of hydrostatic equilibrium

$$\frac{dp}{dz} = -\rho g \tag{8}$$

Using the equation of state and the fact that atmosphere is isothermal we obtain

$$\frac{1}{\rho}\frac{d\rho}{dz} = -\frac{\mu m_p g}{k_B T}, \tag{9}$$

which can be solved to obtain

$$\rho = \rho_0 \exp\left(-\frac{\mu m_p g}{k_B T} z\right) = \rho_0 \exp\left(-z/H\right) \tag{10}$$

Since pressure is proportional to the density, temperature being a constant, pressure too obeys a similar equation

$$p = p_0 \exp\left(-z/H\right) \tag{11}$$

The scale height can be calculated to be $H = 8.7\,\mathrm{km}$

(b) Assuming that the air is isentropic, show that:

$$\frac{dT}{dz} = -\left(\frac{\gamma - 1}{\gamma}\right)\frac{g\mu m_p}{k} \tag{12}$$

Here $\gamma \simeq 1.4$ is the ratio of specific heats for gases like Nitrogen and Oxygen. Why does the above expression vanish for $\gamma = 1$? Obtain expressions for $p(z)$ and $\rho(z)$ if the atmosphere is isentropic.

Solution

From the definition of enthalpy we have

$$dh = Tds + \frac{1}{\rho}dp \tag{13}$$

For an isentropic atmosphere in hydrostatic equilibrium we obtain

$$\frac{dh}{dz} = -g \tag{14}$$

As mentioned earlier, for an ideal gas, enthalpy is only a function of temperature

$$dh = du + d(p/\rho) = \left(c_v + \frac{k_B}{\mu m_p}\right)dT = c_p dT \tag{15}$$

Using this and the expression for enthalpy gradient we obtain

$$\frac{dT}{dz} = -\frac{g}{c_p} \tag{16}$$

Using the fact that $c_p - c_v = k_B/\mu m_p$ and $c_p/c_v = \gamma$, we obtain

$$c_p = \frac{\gamma}{\gamma - 1}k_B/\mu m_p, \tag{17}$$

which gives

$$\frac{dT}{dz} = -g/c_p = -\left(\frac{\gamma - 1}{\gamma}\right)\frac{g\mu m_p}{k_B} \tag{18}$$

Solving for temperature we obtain

$$T = T_0 - \left(\frac{\gamma - 1}{\gamma}\right) \frac{zg\mu m_p}{k_B} \tag{19}$$

We also have

$$\frac{dp}{dz} = \rho \frac{dh}{dz} = \rho c_p \frac{dT}{dz} = \frac{\mu m_p}{k_B T} c_p \frac{dT}{dz} p \tag{20}$$

which gives

$$p = p_0 \left[1 - \left(\frac{\gamma - 1}{\gamma}\right) \frac{zg\mu m_p}{k_B T_o}\right]^{\gamma/(\gamma-1)} \tag{21}$$

For $\gamma \to 1$ this gives

$$p = p_0 \exp\left(-z/H\right) \tag{22}$$

as it should for an isothermal atmosphere (for an isothermal atmosphere $\gamma = 1$.

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2. *Archimedes' principle* states that, when a solid body is totally or partially immersed in a fluid, the total buoyant upward force of the liquid on the body is equal to the weight of the displaced fluid. Prove the law assuming conditions of hydrostatic equilibrium. Using this result, estimate how much more would one weigh in vacuum.

Solution

The pressure force acts perpendicular to an area element. Let the area element on the submerged surface be $d\mathbf{S}$, pointing out of the submerged volume. The total force on the volume element due to fluid pressure is given by

$$\mathbf{F} = -\int p\, d\mathbf{S} = -\int \boldsymbol{\nabla} p\, d\mathrm{V} \tag{23}$$

Using the equation of hydrostatic equilibrium we obtain

$$\mathbf{F} = \rho g \int d\mathrm{V} = \rho g \mathrm{V}_{\mathrm{sub}} \tag{24}$$

Incidentally, if the object is only partially submerged then a part of the submerged surface lies inside the object. Treating the top part and the bottom part separately, we find that the two contributions cancel and we are left with the above result.

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3. *Fluid equations as conservation laws*: Using the continuity equation, the Euler equation, and the first law of thermodynamics, derive conservation laws for the momentum and energy of an ideal fluid. Hint: proving conservation means writing equations in the form:

$$\frac{\partial}{\partial t}(\text{mom. or energy density}) + \nabla \cdot (\text{mom. or energy current density}) = 0 \tag{25}$$

Solution

Conservation of momentum: We first write down the Euler equations in the component form as follows

$$\frac{\partial}{\partial t} v_i + v_j \partial_j v_i = -\frac{1}{\rho} \partial_i p \,, \tag{26}$$

where summation over repeated indices is assumed. Multiplying both sides by ρ yields

$$\rho \frac{\partial}{\partial t} v_i + \rho v_j \partial_j v_i = -\partial_i p \tag{27}$$

Multiplying the continuity equation by v_i

$$v_i \frac{\partial}{\partial t} \rho + v_i \partial_j (\rho v_j) = 0 \tag{28}$$

Adding the two equation we obtain

$$\frac{\partial}{\partial t}(\rho v_i) + v_i \partial_j (\rho v_j) + \rho v_j \partial_j v_i = -\partial_i p \tag{29}$$

Which can be written as

$$\frac{\partial}{\partial t}(\rho v_i) + \partial_j (\rho v_i v_j + p \delta_{ij}) = 0 \tag{30}$$

Defining the stress tensor

$$T_{ij} = \rho v_i v_j + p\delta_{ij} \tag{31}$$

this can be written as

$$\frac{\partial}{\partial t}(\rho v_i) + \partial_j T_{ij} = 0 \tag{32}$$

<u>Conservation of energy</u>: Total energy per unit volume is the sum of kinetic energy per unit volume, $\frac{1}{2}\rho v^2$, and the internal energy per unit volume, ρu, where u is the specific energy. We shall consider each separately.

<u>Kinetic energy</u>

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = \frac{1}{2}v^2\frac{\partial \rho}{\partial t} + \rho\mathbf{v}\cdot\frac{\partial \mathbf{v}}{\partial t} \tag{33}$$

Using equation of continuity and the equations of motion given in the form

$$\frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\omega}\times\mathbf{v} + \boldsymbol{\nabla}\left(\frac{v^2}{2}\right) = -\frac{\boldsymbol{\nabla} p}{\rho} \tag{34}$$

we obtain

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = -\frac{1}{2}v^2\boldsymbol{\nabla}\cdot(\rho\mathbf{v}) - \rho\mathbf{v}\cdot\boldsymbol{\nabla}(\frac{1}{2}v^2) - \rho\mathbf{v}\cdot\frac{\boldsymbol{\nabla} p}{\rho} \tag{35}$$

From the first law of thermodynamics we have

$$\frac{\boldsymbol{\nabla} p}{\rho} = \boldsymbol{\nabla} h - T\boldsymbol{\nabla} s \tag{36}$$

using which the time rate of change of kinetic energy takes the form

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = -\frac{1}{2}v^2\boldsymbol{\nabla}\cdot(\rho\mathbf{v}) - \rho\mathbf{v}\cdot\boldsymbol{\nabla}(\frac{1}{2}v^2) - \rho\mathbf{v}\cdot\boldsymbol{\nabla} h + T\rho\mathbf{v}\cdot\boldsymbol{\nabla} s \tag{37}$$

<u>Internal energy</u>: Consider

$$d(\rho u) = \rho du + u d\rho \tag{38}$$

From first law of thermodynamics

$$du = Tds + \frac{1}{\rho^2}d\rho \tag{39}$$

Combining the above equations

$$d(\rho u) = \rho T ds + h d\rho \tag{40}$$

or

$$\frac{\partial(\rho u)}{\partial t} = \rho T \frac{\partial s}{\partial t} + h \frac{\partial \rho}{\partial t} \tag{41}$$

For an ideal fluid the flow is adiabatic, implying $ds/dt = \partial s/\partial t + \mathbf{v} \cdot \boldsymbol{\nabla} s = 0$. Using this and the continuity equation we get

$$\frac{\partial(\rho u)}{\partial t} = -\rho T \mathbf{v} \cdot \boldsymbol{\nabla} s - h \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) \tag{42}$$

Adding the time rate of change of kinetic and internal energy gives

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho u\right) = -\frac{1}{2} v^2 \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) - \rho \mathbf{v} \cdot \boldsymbol{\nabla}(\frac{1}{2} v^2) - \rho \mathbf{v} \cdot \boldsymbol{\nabla} h - h \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) \tag{43}$$

which can be rearranged to give

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho u\right) + \boldsymbol{\nabla} \cdot \left[(\frac{1}{2}\rho v^2 + \rho h)\mathbf{v}\right] = 0 \tag{44}$$

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4. *Convective instability*: When a fluid is disturbed and it settles back into equilibrium it usually manages to reach mechanical equilibrium faster than thermal equilibrium.

(a) Estimate these time scales for a parcel of air of size 1 m, and 1 km. The coefficient of thermal conductivity, $\kappa = 0.2\,\mathrm{cm}^2\,\mathrm{sec}^{-1}$ and speed of sound$c_s = 350\,\mathrm{m\,sec}^{-1}$.

Solution

The time scale of thermal diffusion is given by

$$\tau_{\mathrm{thermal}} = \frac{l^2}{\kappa} \tag{45}$$

and that of pressure equilibration as

$$\tau_{\mathrm{sound}} = \frac{l}{c_s} \tag{46}$$

Thus for a parcel of size 1 m

$$\tau_{\text{thermal}} = 13.9\,\text{hr} \tag{47}$$
$$\tau_{\text{sound}} = 3\,\text{ms} \tag{48}$$

Thus, for a parcel of size 1 km = 1000 m

$$\tau_{\text{thermal}} = 1.39 \times 10^5\,\text{hr} \tag{49}$$
$$\tau_{\text{sound}} = 73\,\text{s} \tag{50}$$

(b) Earth's atmosphere could be used as an example of this kind: it is in approximate mechanical equilibrium but has temperature gradients in hydrostatic equilibrium. Derive the conditions (called Schwarzchild criterion) under which this equilibrium is stable.

Solution

If a parcel of air at height z rises by a height η, it would come into pressure equilibrium with its surroundings much faster than it would come into thermal equilibrium. Since the parcel is in equilibrium at height z, where pressure and specific entropy are p and s respectively, then it has a specific volume given by $V(s,p)$. At height $z+\eta$ it will have a specific volume given by $V(s,p(z+\eta))$, since it would have ascended adiabatically. The atmosphere, however, need not have constant entropy at all heights. Therefore, the parcel of air it replaces would have a specific volume $V(s(z+\eta),p(z+\eta))$. For stability we require the parcel to be denser than the air it replaces, that is

$$V(s(z+\eta),p(z+\eta)) - V(s,p(z+\eta)) > 0 \tag{51}$$

Expanding in the small displacement ψ, we obtain

$$\left(\frac{\partial V}{\partial s}\right)_p \frac{ds}{dz} > 0 \tag{52}$$

From the first law of thermodynamics we have

$$Tds = du + pdV = d(u+pV) + Vdp = c_p dT + Vdp \tag{53}$$

from where we deduce that

$$\left(\frac{\partial V}{\partial s}\right)_p = \frac{T}{c_p}\left(\frac{\partial V}{\partial T}\right)_p > 0 \tag{54}$$

Therefore, condition for stability reduces to

$$\frac{ds}{dz} > 0 \tag{55}$$

Expressing the specific entropy in terms of pressure and temperature we obtain

$$\left(\frac{\partial s}{\partial p}\right)_T \frac{dp}{dz} + \left(\frac{\partial s}{\partial T}\right)_p \frac{dT}{dz} > 0 \tag{56}$$

using the first law in the from

$$Tds = c_p dT - V dp \tag{57}$$

the derivatives of entropy be easily evaluated, and using the equation of hydrostatic equilibrium $dp/dz = -\rho g = g/V$ we obtain

$$\left(-\frac{dT}{dz}\right) < \frac{g}{c_p} \tag{58}$$

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5. *Linear theory of sound*: Assume the unperturbed medium is unbounded, static, uniform in its properties: $\rho = \rho_0$, $p = p_0$, and $\mathbf{v}_0 = 0$. The medium is then perturbed:

(i) Write down the linearized continuity and Euler equations satisfied by perturbations ρ_1, p_1, and $\mathbf{v}_1$.

(ii) What is the linearized equation satisfied by the perturbed vorticity?

(iii) Assume that the perturbation gives rise to a pure potential flow, $\mathbf{v}_1 = \nabla\phi_1$. Use this in the linearized Euler equation, and express p_1 in terms of ϕ_1.

(iv) Assume that the flow is barotropic, with sound speed defined by $c_s = \sqrt{dp_0/d\rho_0}$. Derive a wave equation for ϕ_1.

(v) Write down the general solution for ϕ_1, corresponding to plane waves traveling in the $\pm x$ directions.

(vi) What are the corresponding expressions for $\mathbf{v}_1$ and p_1? Are the waves transverse or longitudinal?

Solution

(i) Since $\mathbf{v}_0 = 0$, the nonlinear term in the convective derivative does not contribute anything. The linearized Euler equations are

$$\frac{\partial \mathbf{v}_1}{\partial t} = -\frac{\nabla p_1}{\rho_0}\,, \tag{59}$$

and the linearized continuity equation is

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1 = 0 \tag{60}$$

(ii) Since the fluid is compressible, we first derive the equation of motion for vorticity. The Euler equations are

$$\frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\omega} \times \mathbf{v} = -\frac{\nabla p}{\rho} - \nabla\left(\frac{v^2}{2}\right) \tag{61}$$

Taking the curl on both sides

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \mathbf{v}) = \frac{\nabla \rho \times \nabla p}{\rho^2} \tag{62}$$

The non-linear terms do not contribute in the first order since they are zero for the unperturbed flow (the term on the right hand side is in fact identically zero for a polytropic equation of state), therefore

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = 0 \tag{63}$$

and unless the perturbation has a non-zero vorticity, the vorticity remains zero.

(iii) Inserting $\mathbf{v}_1 = \nabla \phi_1$ in the Euler equation gives

$$\nabla\left(\frac{\partial \phi}{\partial t} + \frac{p_1}{\rho_0}\right) = 0 \Rightarrow \frac{\partial \phi_1}{\partial t} + \frac{p_1}{\rho_0} = \mathrm{C} \tag{64}$$

(iv) For an adiabatic gas $p = p(\rho)$, which implies $p_1 = c_s^2 \rho_1$. The previous equation then becomes

$$\frac{\partial \phi_1}{\partial t} + c_s^2 \frac{\rho_1}{\rho_0} = 0 \tag{65}$$

Differentiating with respect to time we obtain

$$\frac{\partial^2 \phi_1}{\partial t^2} + \frac{c_s^2}{\rho_0} \frac{\partial \rho_1}{\partial t} = 0 \tag{66}$$

Using the continuity equation we we can write this as

$$\frac{\partial^2 \phi_1}{\partial t^2} - c_s^2 \boldsymbol{\nabla} \cdot \mathbf{v}_1 = 0 \Rightarrow \frac{\partial^2 \phi_1}{\partial t^2} - c_s^2 \nabla^2 \phi_1 = 0 \tag{67}$$

A general solution of this equation is

$$\phi_1 = \phi_0 \mathrm{e}^{i(\mathbf{k}.\mathbf{x} \pm \omega t)} \tag{68}$$

where $\omega / k = c_s^2$. Therefore the solution for pressure is given by

$$p_1 = \boldsymbol{\nabla} \phi_1 = \phi_0 \mathbf{k} \mathrm{e}^{i(\mathbf{k}.\mathbf{x} \pm \omega t)} \tag{69}$$

which is obviously longitudinal.

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5. Describe the flow of water in an idealized bathtub where water enters at a large distance from the drain with a non-zero circulation, and the flow is axisymmetric and stationary. What is the shape that the surface of water acquires? You can neglect the viscosity of water.

Solution

We shall assume that the tub is very large and cylindrically symmetric. We shall further assume that the flux of water is small, therefore, we shall ignore the vertical motion of water while describing the flow. We also assume that the flow is steady and incompressible.

Water enter at a large radius R_0, with a small radial velocity, and circulation Γ_0

$$\Gamma_0 = \oint \mathbf{v} \cdot d\boldsymbol{l} \tag{70}$$

For an ideal fluid, the net circulation remains constant with the flow, therefore, $\Gamma(R) = \Gamma_0$. Thus

$$\oint \mathbf{v} \cdot d\boldsymbol{l} = 2\pi R v_\phi = \Gamma_0 \Rightarrow v_\phi = \frac{\Gamma_0}{2\pi R} \tag{71}$$

Note that this is just a statement of conservation of angular momentum since the angular momentum per unit volume $\rho R v_\phi$ is a constant for this flow.

Continuity equation $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$, in cylindrical polar coordinates is

$$\frac{1}{R}\frac{\partial(Rv_R)}{\partial R} + \frac{1}{R}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} = 0 \Rightarrow v_R \propto \frac{1}{R} \tag{72}$$

Therefore, the speed of the fluid varies inversely with the radius. From Bernoulli's equation

$$\frac{p}{\rho} + \frac{v^2}{2} + gz = 0 \Rightarrow \frac{K}{R^2} + z = z_0 \tag{73}$$

where z_0 is the height at infinity. The constant pressure surface is thus described by

$$(z - z_0) = -\frac{K}{R^2} \tag{74}$$

The vorticity of the flow is zero everywhere except at the origin, where it becomes singular.

$$\boldsymbol{\omega} = \hat{\mathrm{k}}\Gamma_0\delta^2(R) \tag{75}$$

And thus the flow is that of a line vortex.

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2 Viscous Fluids

1. *Molecular origin of shear viscosity*: For most fluids, the shear viscosity coefficient is only determined experimentally. For an ideal gas the shear

viscosity can be determined using kinetic theory of gases. Consider a flow in x direction where the shear .i.e. $\partial v_x/\partial y$ is non-zero. In this case, random motion of the gas molecules moving with typical thermal velocities v_t with a mean free path ℓ ($\ell = 1/(n\sigma)$, n is the number density and σ is the cross-section of collision) deposit different amounts of x-component of momentum across a plane y = const. Show that the (x-component of) momentum deposited per unit time per unit volume of the fluid is given by:

$$F_x \sim \frac{\partial}{\partial y}\left(\eta \frac{\partial u_x}{\partial y}\right), \tag{76}$$

with the coefficient of shear viscosity $\eta = mv_t/\sigma$.

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2. Using Navier-Stokes and continuity equations along with the first law of thermodynamics, show that:

$$\begin{aligned} \frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}v^2 + u\right)\right] &+ \nabla.\left[\rho\mathbf{v}\left(\frac{1}{2}v^2 + h\right) - \zeta\theta\mathbf{v} - 2\eta\sigma.\mathbf{v}\right] \\ &= \rho T\frac{ds}{dt} - \zeta\theta^2 - 2\eta\sigma_{ij}\sigma^{ij} \end{aligned} \tag{77}$$

Solution

We have done part of this problem already while deriving the energy conservation equation. Let us cast the Navier-Stokes equation in the following form

$$\rho\frac{dv_i}{dt} = -\frac{\partial p}{\partial x_i} - \frac{\partial T_{ij}}{\partial x_j} \tag{78}$$

where

$$T_{ij} = -2\eta\sigma_{ij} - \zeta\theta\delta_{ij}, \quad \sigma_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\frac{\partial v_l}{\partial x_l}\delta_{ij}\right), \quad \theta = \frac{\partial v_l}{\partial x_l} \tag{79}$$

Recall the kinetic energy part from the previous problem

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = \frac{1}{2}v^2\frac{\partial}{\partial t}\rho + \rho\mathbf{v}\cdot\frac{\partial}{\partial t}\mathbf{v} \tag{80}$$

It is clear that the second term on the right hand side introduces the following additional terms

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = \ldots + 2\eta v_i \partial_j \sigma_{ij} + \zeta v_i \partial_j \theta \delta_{ij} \tag{81}$$

This can be written as

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = \ldots + 2\eta \partial_j (v_i \partial_j \sigma_{ij}) + \zeta \partial_j (v_i \theta \delta_{ij}) - 2\eta (\partial_j v_i)\sigma_{ij} - \zeta (\partial_j v_i)\theta \delta_{ij} \tag{82}$$

or

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = \ldots + \boldsymbol{\nabla} \cdot (2\eta \mathbf{v} \cdot \boldsymbol{\sigma} + \zeta \mathbf{v} \theta) - 2\eta \sigma_{ij}\sigma_{ij} - \zeta \theta^2 \tag{83}$$

Apart from this we also cannot assume conservation of entropy (since dissipative processes actually produce it), therefore, the ds/dt term that had dropped out in the previous case remains on the right hand side, leading to the result we are required to prove.

Q. Interpret different terms of this equation and show that it means that both the coefficients of viscosity must be positive.

Solution

If there are no sources or sinks of energy, the left hand side and the right hand side of Eq 77 vanish indentically. Note that the flux of energy now contains additional terms dependent on shear. This is not surprising since shear transports momentum through the fluid and transport of momentum involves transport of energy as well.

Putting the right hand side to zero we obtain

$$\rho T \frac{ds}{dt} = \zeta \theta^2 + 2\eta \sigma_{ij}\sigma^{ij} \tag{84}$$

since entropy of the fluid always increases due to the second law of thermodynamics, it is clear that both coefficients of viscosity must be positive.

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3. For an incompressible flow, show that the pressure is determined entirely by the velocity field and derive a Poisson equation for the pressure.

Solution

For an incompressible fluid the Navier-Stokes equation can be written as

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} = -\boldsymbol{\nabla}\left(\frac{p}{\rho}\right) + \nu \nabla^2 \mathbf{v} \tag{85}$$

where $\boldsymbol{\omega}$ is the vorticity. Taking divergence on both sides and using the incompressibility condition $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$, we obtain

$$\nabla^2 p = -\rho \partial_i (v_j \partial_j v_i) \tag{86}$$

which is the required Poisson equation

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4. Determine the flow of a viscous fluid of thickness h on an inclined plane due to the force of gravity.

Solution

Let the direction of the flow be x and the direction perpendicular to the plane by z, then the gravitational potential is given by $\Phi = gz\cos(\theta) - gx\sin(\theta)$, where θ is the angle of the inclined plane.

The condition that the height of the flow remains a constant, along with the continuity equation implies that v_x is a constant. Moreover, the condition that at the top of the fluid the pressure is given by the atmospheric pressure, which is a constant along the direction of the flow, implies that the pressure is only a function of z. The x component of the Navier-Stokes equation is given by

$$\frac{\partial}{\partial x}\left(\frac{p}{\rho} + gz\cos\theta - gx\sin\theta\right) = \nu \nabla^2 v_x \tag{87}$$

which gives

$$\frac{d^2 v_x}{dz^2} = -\frac{g\sin\theta}{\nu} \tag{88}$$

The general solution of this equation is given by

$$v_x(z) = -\frac{1}{2\nu} g \sin\theta\, z^2 + Bz + C \tag{89}$$

The flow satisfies the boundary condition $v_x(0) = 0$. Furthermore, if we ignore the viscosity of air, it is clear that the condition that $\eta\sigma_{ij} = 0$ at $z = h$, where h is the height of the flow, and that $\eta\sigma_{ij}$ be continuous, implies that $\sigma_{ij} = 0$. Note that continuity of $\eta\sigma_{ij}$ is a necessary condition since its divergence occurs in the Navier-Stokes equations, and discontinuity would lead to singular terms. The two constants can be determined from these two boundary conditions, $v_x(0) = 0$ and $dv_x/dz(h) = 0$, to give the solution

$$v_x(z) = \frac{g \sin\theta}{2\nu} z(2h - z) \tag{90}$$

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5. *Poiseuille flow*: Consider fluid flow in a pipe of radius a. The flow is such that the only component of velocity that is non-zero is along the pipe and its variation is only along the cross-section .i.e. the flow can be determined by a velocity field: $v_z(r)$.

a) Show that for a given pressure gradient along the tube, dp/dz, the velocity field is given by:

$$v_z(r) = \frac{dp}{dz}\frac{1}{4\eta}(a^2 - r^2). \tag{91}$$

Solution

Since the flow is steady $\partial_t = 0$. For a long pipe the flow is invariant along the length of the tube, therefore, the gradient of the flow along the length vanishes. Thus, the left hand side of the Navier-Stokes equation is zero. Ignoring any gravitational field we obtain

$$\boldsymbol{\nabla}\left(\frac{p}{\rho} + \frac{v^2}{2}\right) = \nu\nabla^2\mathbf{v} \tag{92}$$

Taking projection along the z-axis we obtain

$$\frac{dp}{dz} = \eta \nabla^2 v_z \tag{93}$$

Note that dp/dz is a constant. Expressing the Laplacian in the cylindrical polar coordinates and noting that the only variation is along r, we obtain

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) v_z(r) = \frac{1}{\eta}\frac{dp}{dz} \tag{94}$$

Integrating twice we obtain

$$v_z(R) = \frac{1}{\eta}\frac{dp}{dz}\frac{r^2}{4} + C \log r + D \tag{95}$$

where, C and D are constants. Since the flow is non singular at the origin, $C = 0$. The other boundary condition is that $v_z(z) = 0$, imposing which we obtain

$$v_z(r) = \left|\frac{dp}{dz}\right| \frac{1}{4\eta}(a^2 - r^2). \tag{96}$$

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b) Compute the tangential force (per unit area) on the walls of the pipe.

Solution

The force per unit area is given by ηS_{rz}, all other components of the stress tensor vanish.

$$S_{rz} = \left|\frac{dp}{dz}\right| \frac{a}{2\eta} \tag{97}$$

The total force on a piece of length l is given by

$$F = 2\pi a l \eta \, S_{rz} = \left|\frac{dp}{dz}\right| \pi a^2 l \tag{98}$$

which is not a surprising result since πa^2 is the crossectional area of the tube and the pressure gradient times the length is the pressure difference across the segment of the tube. Clearly, the pressure forces are required to balance the total viscous forces on the tube.
