Galaxies and the Universe — Problems 1

Question 1. Galactic rotation

(a) Given the mass of a disc galaxy, $M = 10^{11} M_\odot$, derive a rough estimate of its rotational velocity V at a distance $r = 8.5\,\mathrm{kpc}$, assuming a point-like mass distribution (i.e., derive *Kepler's rotation law*). Compare your result with the rotational velocity of 220 km/s observed near the Sun; decide if a more realistic model with the gravitating mass distributed within an extended disc, bulge and halo would result in a smaller or larger rotational velocity near the Sun.

(b) Find the rotation velocity within a spherically symmetric dark matter halo whose density is given by $\rho(r) = \rho_s (r/R_s)^n$ with certain constants n, ρ_s and R_s. Find n such that the rotation curve is flat, as observed, i.e., V is independent of r.

Hint: the gravity field at a distance r from the centre of a gravitating sphere depends only on the mass inside the radius r, and is the same as that of a point-like mass equal to the mass within r.

Note: more recent dark halo models have $\rho(r) = \dfrac{\rho_s}{(r/R_s)(1 + r/R_s)^2}$ (Navarro, Frenk & White, ApJ, 490, 493, 1997).

Solution

(a) For a point-like mass M at the origin, the gravitational potential is given by $\Phi = -GM/r$, so the centrifugal equilibrium equation $V^2/r = \partial\Phi/\partial r$ immediately leads to

$$V = \sqrt{GM/r},$$

where V is the rotational velocity, r is the spherical radius, and G is Newton's gravitalm constant. With $M = 10^{11}\,M_\odot$, $r = 8.5\,\mathrm{kpc}$ and $G \approx (1/150{,}000{,}000)\,\mathrm{cm}^{-3}\,\mathrm{g}^{-1}\,\mathrm{s}^{-2} \approx 6.67 \times 10^{-8}\,\mathrm{cm}^{-3}\,\mathrm{g}^{-1}\,\mathrm{s}^{-2}$, we obtain $V \approx 230\,\mathrm{km/s}$.

If the gravitating mass is distributed more uniformly and some of it is at $r > 8.5\,\mathrm{kpc}$, less mass will be confined within $r = 8.5\,\mathrm{kpc}$ and (r) will be smaller.

(b) With $\rho(r) = \rho_s (r/R_s)^n$, the mass within a sphere of a radius r is

$$M(r) = 4\pi \int_0^r \tilde{r}^2 \rho(\tilde{r})\, d\tilde{r} = \frac{M_s}{1 + n/3} \left(\frac{r}{\rho_s}\right)^{n+3},$$

where $M_s = (4\pi/3)\rho_s R_s^3$. Now, the centrifugal balance equation yields

$$V^2 = \frac{GM_s}{R_s(1 + n/3)} \left(\frac{r}{\rho_s}\right)^{n+2},$$

and $V = \mathrm{const}$ for

$$n = -2.$$

This is one of the first models of the galactic dark halos used to explain the flattening of the galactic rotation curves at large galactocentric distance. A difficulty of this specific model of the dark matter distribution is that the total mass diverges at infinity,

$$M(r) \propto r \to \infty \quad \text{as } r \to \infty.$$

Thus, $\rho \propto r^{-2}$ only within some range of galactocentric distances $r \lesssim 50\,\mathrm{kpc}$, and ρ must decrease faster at larger radii.

Question 2. Hydrostatic equilibrium of the interstellar gas

Given that the vertical (along the z-axis, directed across the gas layer) acceleration due to gravity is (Ferrière, ApJ, 497, 759, 1998)

$$g_z = -\left[a_1 \frac{z}{\sqrt{z^2 + H_1^2}} + a_2 \frac{z}{H_2}\right], \tag{1}$$

with $a_1 = 4.4 \times 10^{-9}\,\mathrm{cm\,s^{-2}}$, $a_2 = 1.7 \times 10^{-9}\,\mathrm{cm\,s^{-2}}$, $H_1 = 0.2\,\mathrm{kpc}$ and $H_2 = 1\,\mathrm{kpc}$, find the dependence of the gas density on z from the balance of pressure gradient and the gravitational acceleration. Hence, determine the *scale height* of the gas distribution. (Here the first term in g_z is due to the stellar disc and the second arises from the dark matter halo; H_1 and H_2 are the respective scale heights of the gravitating mass distributions.)

You may assume that the interstellar turbulence is transonic, gas temperature is constant at $T = 10^4\,\mathrm{K}$ (an *isothermal atmosphere*) and the magnetic, cosmic-ray and turbulent pressures are equal to each other.

(a) Neglect the z-dependence of g_z and adopt for g_z a constant value, equal to $g_z(0.2\,\mathrm{kpc})$ in Eq. (1).

(b) Now, consider $z \ll H_1$ and derive, from Eq. (1), an approximate expression for g_z such that $g \propto z$.

(c) (optional) Solve the hydrostatic equilibrium equation with the full form (1) and find the gas scale heights at $z \to 0$ and $z \to \infty$.

(d) Now, estimate the scale height of an isothermal *hot* gas, $T = 10^6\,\mathrm{K}$ under the same assumptions and assuming $g_z = g_z(H) = \mathrm{const}$ for the sake of simplicity. Can the hot has be confined to a thin disc in a spiral galaxy? What value of H it is reasonable to adopt to obtain a realistic estimate?

Hint: the speed of sound is $10\,\mathrm{km/s}$ at $T = 10^4\,\mathrm{K}$, independent of density and proportional to $T^{1/2}$.

Solution

We want to calculate the scale height of the gas under the different assumptions: (a), (b), (c) & (d).

The equation for hydrostatic support of the disc is

$$\frac{\mathrm{d}P}{\mathrm{d}z} = \rho g_z \tag{2}$$

The assumption that the pressures due to the magnetic field P_{mag}, cosmic rays P_{cr} and turbulence P_{turb} are all equal to the thermal pressure P_{th} of the $10^4\,\mathrm{K}$ gas gives us and that the turbulence is transonic (i.e. the average turbulent gas velocity is around the sound speed $\bar{v} \sim c_s = 10\,\mathrm{km\,s^{-1}}$ and is assumed to be independent of z) gives us

$$P_{\mathrm{th}} = P_{\mathrm{mag}} = P_{\mathrm{cr}} = P_{\mathrm{turb}} = \frac{1}{3}\rho\bar{v}^2. \tag{3}$$

The factor $1/3$ arises as we only require the pressure in the vertical z direction and $\bar{v}$ is the <u>three-dimensional</u> velocity and we have assumed that the turbulence is isotropic, so $\bar{v}_x = \bar{v}_y = \bar{v}_z$. Then Eqs. (2) & (3) give

$$\frac{4\bar{v}^2}{3}\frac{\mathrm{d}\rho}{\mathrm{d}z} = \rho g_z \tag{4}$$

To answer (c) we will use Eq. (1) for the right hand side, but for the rest let us write this equation in a more convenient form as follows:

$$g_z = -\frac{z}{H_2}\left[a_2 + \frac{a_1 H_2}{\sqrt{z^2 + H_1^2}}\right],$$

then

$$g_z = -\frac{z}{H_2}\left[a_2 + \frac{a_1 H_2}{H_1}\frac{1}{\sqrt{1+(z/H_1)^2}}\right], \tag{5}$$

which gives us the convenient equation

$$g_z = -1.7\times 10^{-9}\left(\frac{z}{1\,\mathrm{kpc}}\right)\left[1 + \frac{13}{\sqrt{1+(z/0.2\,\mathrm{kpc})^2}}\right]\,\mathrm{cm\,s^{-2}}. \tag{6}$$

(a): We take g_z as a constant, $g_z(0.2\,\mathrm{kpc}) \simeq -3\times 10^{-9}\ \mathrm{cm/\,s^2}$, and then for an exponential gas distribution in z, $\rho = \rho_0 \exp(-z/h)$, Eq. (4) gives

$$h = \frac{4}{3}\frac{\bar{v}^2}{|g_z|} \simeq 140\,\mathrm{pc}$$

(b): For $z \ll 0.2\,\mathrm{kpc}$ Eq. (6) simplifies to

$$g_z \simeq -1.7\times 10^{-9}\left(\frac{z}{1\,\mathrm{kpc}}\right)(14)\ \mathrm{cm\,s^{-2}} = -2.4\times 10^{-8}\left(\frac{z}{1\,\mathrm{kpc}}\right)\ \mathrm{cm\,s^{-2}} = -g_0\left(\frac{z}{1\,\mathrm{kpc}}\right)\ \mathrm{cm\,s^{-2}}.$$

Then from Eq. (2) we have

$$\frac{\mathrm{d}\ln\rho}{\mathrm{d}\hat{z}} = -\frac{3}{8}\frac{g_0 H_2}{\bar{v}^2}\hat{z},$$

where $\hat{z} = z/H_2$ and $H_2 = 1\,\mathrm{kpc}$. Integration then gives

$$\rho = \rho_0 \exp\left(-\frac{\hat{z}^2}{A^2}\right)$$

with $A^2 = 8\bar{v}^2/(3g_0H_2)$. Compare this to the Gaussian distribution of $\rho(z) = \rho(0)\exp\left[-z^2/2h^2\right]$ and we see that the scale height that we require is given by

$$h = \sqrt{\frac{H_2^2 A^2}{2}} = \sqrt{\frac{8\bar{v}^2 H_2}{3g_0}} \simeq 140\,\mathrm{pc}.$$

[Note: often the factor of two in the argument of the Gaussian ($\rho(z) = \rho(0)\exp\left[-z^2/2h^2\right]$) is omitted and then the quoted Gaussian scale height would be $h = 190\,\mathrm{pc}$.]

Question 3. Expanding supernova remnants: the adiabatic phase

1. Find, from dimensional arguments, the Sedov–Taylor solution for the radius of an expanding supernova remnant as a function of time, assuming spherical symmetry, and given that the energy released in the explosion, E_*, is conserved (*adiabatic* expansion), and given the gas density in the ambient medium, ρ_0.

   ### Solution

   We set $R = AE_*^a \rho_0^b t^c$, where $R(t)$ is the radius and A is the constant of proportionality, given in the 'Note' at the end of the question. Dimensionally we have:

   $$L = M^a L^{2a} T^{-2a} M^b L^{-3b} T^c,$$

   giving $a = 1/5$, $b = -1/5$ and $c = 2/5$ so we have the Sedov-Taylor solution

   $$R = A \left(\frac{E_* t^2}{\rho_0} \right)^{1/5}. \tag{7}$$

2. The mass of the supernova remnant consists of the stellar mass ejected in the explosion, $\simeq 4M_\odot$, and the mass of the swept-up interstellar gas. The former is more or less uniformly distributed across the volume of the remnant, whereas the latter is concentrated in a thin shell. Compare the two contributions and decide under what conditions it is reasonable to neglect the mass of the stellar ejecta.

   ### Solution

   We can neglect the mass of the stellar ejecta $M_* = 4M_\odot$ when the mass of the interstellar medium that has been swept up by the expanding shock front into a shell $M_{\rm sh}$ is greater than the mass of the ejecta i.e. $M_{\rm sh} \gg 4M_\odot$. We can estimate the time $t_{\rm ST}$ when the two masses are equal using Eq. (7).

   The shell mass at time t in the Sedov-Taylor phase is given by

   $$M_{\rm sh} = \frac{4\pi}{3} \left(\frac{25}{3\pi} \right)^{3/5} \left(\frac{E_* t^2}{\rho_0} \right)^{3/5} \rho_0 = M_0 t^{6/5}, \tag{8}$$

   giving $t_{\rm ST} \simeq 600\,\text{yr}$.

3. The total energy of the remnant is the sum of kinetic and thermal energies. Can you decide now, whether or not the thermal energy remains constant during the Sedov–Taylor phase?

   ### Solution

   The kinetic energy in the Sedov-Taylor phase is

   $$E_{\rm kin} = \frac{1}{2} M_{\rm sh} \dot{R}_{\rm ST}^2,$$

   with the shell mass given by Eq. (8) and the shell radius by Eq. (7). Thus

   $$E_{\rm kin} \propto R_{\rm ST}^3 \dot{R}_{\rm ST}^2 \propto t^{6/5} t^{-6/5} = \text{constant},$$

   and since the total energy is conserved the thermal energy must remain constant during the Sedov-Taylor phase of the remnant expansion.

Note: by solving the equation of motion, it can be shown that the pre-factor in the dependence of R on t is given by $(25/3\pi)^{1/5}$ (see, e.g., J. E. Dyson & D. A. Williams, *The Physics of the Interstellar Medium,* 2nd edition, IOP, Bristol, 1997, §7.3).

Question 4. A dissipating SN remnant: the turbulent scale in the ISM

Estimate the radius and time, since the start of the expansion, when a supernova remnant merges with the surrounding medium: this happens when the expansion velocity decreases below the local speed of sound, $c_\mathrm{s} = 10\,\mathrm{km\,s^{-1}}$ (and hence the shock disappears).

For this purpose, use the expression for the snowplough phase as given in the lectures,

$$R = R_0 \left[1 + 4\frac{V_0}{R_0}(t - t_0)\right]^{1/4}, \quad V = \dot{R},$$

where

$$t_0 = 3.5 \times 10^4 \left(\frac{E_*}{10^{51}\,\mathrm{erg}}\right)^{4/17} n_0^{-9/17}\,\mathrm{yr}, \quad V_0 = 230 \left(\frac{E_*}{10^{51}\,\mathrm{erg}}\right)^{1/17} n_0^{2/17}\,\frac{\mathrm{km}}{\mathrm{s}},$$

and $n_0 \simeq 0.1\,\mathrm{cm^{-3}}$ is the ambient gas number density. Here t_0, R_0 and V_0 are the time, remnant radius and its expansion velocity at the transition from the adiabatic phase to the momentum-conserving phase (i.e., the time when cooling of the shocked interstellar gas in the spherical shell becomes important). You may assume that $t \gg t_0$ (but you should confirm that this is true using your final result!). The resulting maximum SNR radius in fact controls the turbulent scale in the ISM.

Solution

At the transition from Sedov-Taylor to snowplough phases we find $t_0 \approx 10^5\,\mathrm{yr}$ and $V_0 \approx 175\,\mathrm{km\,s^{-1}}$. At the time t_0 the remnant has the radius $R_0 \simeq 50\,\mathrm{pc}$, using Eq. (7). For $t \gg t_0$ we have

$$R \simeq R_0 \left(4\frac{V_0}{R_0}t\right)^{1/4},$$

and

$$\dot{R} \simeq \frac{1}{4}(4V_0R_0^3)^{1/4}t^{-3/4}.$$

When the remnant stops expanding $\dot{R} = c_s = 10\,\mathrm{km\,s^{-1}}$, so we obtain

$$t \simeq \left(\frac{R_0}{4c_s}\right)^{4/3}\left(4\frac{V_0}{R_0}\right)^{1/3} = 3 \times 10^6\,\mathrm{yr},$$

and then the radius when the remnant merges with the ambient ISM is

$$R \simeq R_0 \left(\frac{V_0}{c_s}\right)^{1/3} = 50\,\mathrm{pc}\left(\frac{175}{10}\right)^{1/3} = 130\,\mathrm{pc}.$$

Question 5. Kinetic energy injection by SNe into the ISM

1. Using your results from Question 4, estimate the fraction of the supernova explosion energy that is injected into the ISM as kinetic energy: estimate the kinetic energy of the SNR (given that almost all of its mass is in the dense shell expanding at the speed V). The total explosion energy can be adopted as $E_* = 10^{51}\,\mathrm{erg}$ (after §7.3.6 in Dyson & Williams, *op. cit.*).

   ### Solution

   The kinetic energy at the end of the snowplough phase is

   $$E_\mathrm{kin} = \frac{1}{2}M\dot{R}^2 = \frac{1}{2}\frac{4\pi}{3}\rho_0 R^3\dot{R}^2 \simeq 10^{50}\,\mathrm{erg},$$

so using our results from Q. 4 we find

$$\frac{E_{\rm kin}}{E_*} \simeq \frac{10^{50}}{10^{51}} = 10\%.$$

2. The energy of a turbulent flow is transferred from larger to smaller scales and then lost to heat at a dissipation scale. Therefore, any turbulent flow needs to be continuously supported by an energy source: without that, the turbulence decays.

   Given that the interval between individual supernova explosions in the Milky Way is about 50 yr, calculate the turbulent velocity v_0 of the diffuse gas (number density $n_0 = 0.1\,{\rm cm}^{-3}$) that can be supported by the kinetic energy supply from the SNe. Assume that the turbulent scale is $l_0 = 100\,{\rm pc}$, and the time scale of kinetic energy transfer along the spectrum is l_0/v_0.

   ## Solution

   The total energy available, throughout the Galactic disc, to drive turbulence from supernova explosions per year is given by

$$\dot{E}_g = \frac{E_*}{10\tau_{\rm SN}},$$

   where $\tau_{\rm SN} = 50\,{\rm yr}$ is the interval between explosions and we have used the 10% efficiency estimated above.

   Kinetic energy is transferred down the turbulent cascade at a rate of

$$\dot{e}_{\rm turb} = \frac{1}{2}\rho_0 v_0^2 \frac{v_0}{l_0}$$

   and so the rate at which the turbulent energy cascade will remove energy from the injection scale of l_0, throughout the Galactic disc, is

$$\dot{E}_g = \dot{e}_{\rm turb} V_g = \frac{1}{2}\frac{v_0^3}{l_0}\rho_0\,\pi R_g^2 2h_g,$$

   where $R_g \simeq 15\,{\rm kpc}$ is the radius of the Galaxy, $h_g \simeq 200\,{\rm pc}$ is the scale height of the disc and $\rho_0 \simeq 0.1 m_H\,{\rm cm}^{-3}$. This gives an estimate for the turbulent velocity in the diffuse gas of

$$v_0 \simeq 30\,{\rm km\,s^{-1}}.$$

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