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Hall MHD

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Hall MHD in a strong magnetic field

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Introduction and motivation

- For a plasma, embedded in a strong external magnetic field, spatial structures tend to elongate along its field lines. This feature allowed a major simplification in the theoretical structure of one fluid MHD, leading to reduced MHD (RMHD).

- A more appropriate theory for fully ionized plasmas is to consider two-fluid effects through a generalized Ohm's law which includes the Hall current. Whenever one deals with phenomena with characteristic length scales comparable or smaller than the ion skin depth c/ω_{pi} , the Hall effect should not be neglected.

- The Hall current causes the magnetic field to become frozen to the electron flow. Another important feature of the ideal Hall MHD description is the self-consistent presence of parallel electric fields able to accelerate particles.

- Here we present what may be called the reduced Hall MHD equations (RHMHD) reflecting two fluid effects like the Hall current and the electron pressure. These physical effects can be relevant in astrophysical environments and also in fusion plasmas.

- Within this approximation we allow for the propagation of circularly polarized normal modes such as whistlers and shear/ion-cyclotron waves. We numerically integrate the RHMHD system of equations to investigate externally driven turbulence.

Hall-MHD equations

- For a fully ionized plasma with ions of mass m_i and massless electrons:

$$m_i n \frac{d\vec{U}}{dt} = en(\vec{E} + \frac{1}{c}\vec{U} \times \vec{B}) - \vec{\nabla} p_i - m_i n \nu_{ie} (\vec{U} - \vec{U}_e)$$

$$0 = -en(\vec{E} + \frac{1}{c}\vec{U}_e \times \vec{B}) - \vec{\nabla} p_e + m_i n \nu_{ie} (\vec{U} - \vec{U}_e)$$

- Mass conservation implies

$$0 = \frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n\vec{U}) \quad , \qquad n_e \cong n_i \cong n$$

- We have Ampere's law

$$\vec{J} = \frac{c}{4\pi} \vec{\nabla} \times \vec{B} = en(\vec{U} - \vec{U}_e)$$

as well as

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Hall-MHD equations

- The dimensionless version, for a length scale L_0, density n_0 and Alfven speed $v_A = B_0 / \sqrt{4\pi m_i n_0}$

$$\frac{dU}{dt} = \frac{1}{\varepsilon}(\vec{E} + \vec{U} \times \vec{B}) - \frac{\beta}{n}\vec{\nabla} p_i - \frac{\eta}{\varepsilon n}\vec{J}$$

$$0 = -\frac{1}{\varepsilon}(\vec{E} + \vec{U}_e \times \vec{B}) - \frac{\beta}{n}\vec{\nabla} p_e + \frac{\eta}{\varepsilon n}\vec{J}$$

where $\vec{J} = \vec{\nabla} \times \vec{B} = \frac{n}{\varepsilon}(\vec{U} - \vec{U}_e)$

- We define the Hall parameter $\varepsilon = \frac{c}{\omega_{pi} L_0}$

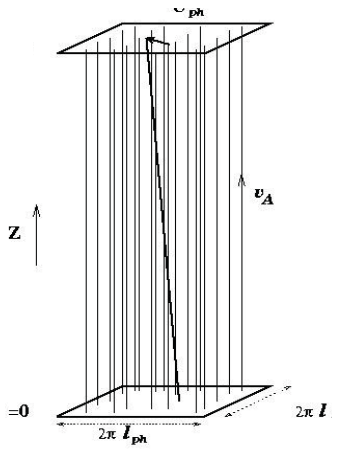
as well as the plasma *beta* $\beta = \frac{p_0}{m_i n_0 v_A^2}$ and the electric resistivity $\eta = \frac{c^2 \nu_{ie}}{\omega_{pi}^2 L_0 v_A}$

- Combining these, we obtain the so called RHMHD equations

$$n\frac{d\vec{U}}{dt} = (\vec{\nabla} \times \vec{B}) \times \vec{B} - \beta \vec{\nabla}(p_i + p_e)$$

$$\frac{\partial \vec{A}}{\partial t} = (\vec{U} - \frac{\varepsilon}{n}\vec{\nabla} \times \vec{B}) \times \vec{B} - \vec{\nabla}\phi + \frac{\varepsilon\beta}{n}\vec{\nabla} p_e - \frac{\eta}{n}\vec{\nabla} \times \vec{B}$$

Hall MHD in a strong field



- Let us assume a strong magnetic field along $\hat{z}$ so that

$$B = \hat{z} + \delta B \quad , \quad |\delta B| \approx \alpha << 1$$

where α represents the typical tilt of field lines with respect to $\hat{z}$. As for RMHD we assume

$$\nabla_\perp \approx 1 \quad , \quad \partial_z \approx \alpha << 1$$

- The magnetic and velocity fields can be expanded in terms of potentials of order α:

$$\vec{B} = \vec{\hat{z}} + \nabla \times (\vec{a\hat{z}} + g\hat{x}) = [a_y, -a_x, 1+b] \quad , \quad b = -g_y$$

$$U = \nabla \psi + \nabla \times (\varphi \hat{z} + f \hat{x}) = [\varphi_y + \psi_x, -\varphi_x + \psi_y, u + \psi_z] \quad , \quad u = -f_y$$

- We want to eliminate the fast scale dynamics, characterized by $\tau_{A\perp} \approx L_0 / v_A$, i.e. $\partial_t \approx 1$

- We obtain the following conditions

$$\nabla_\perp^2 \psi = 0$$

$$\rightarrow \quad \nabla_\perp [b + \beta(p_i + p_e)] = 0$$

$$\rightarrow \quad \nabla_\perp [\phi + \varphi - \varepsilon(b + \beta p_e)] = 0$$

Hall MHD in a strong field

- The relatively slower dynamics, characterized by $\tau_{A//} \approx L_z / v_A$, i.e. $\partial_t \approx \alpha$ is given by the following equations (of order α^2):

$$\partial_t a = \partial_z(\varphi - \varepsilon b) + [\varphi - \varepsilon b, a] + \eta \nabla_\perp^2 a$$
$$\partial_t \omega = \partial_z j + [\varphi, \omega] - [a, j] + \nu \nabla_\perp^2 \omega$$
$$\partial_t b = \partial_z(u - \varepsilon j) + [\varphi, b] + [u - \varepsilon j, a] + \eta \nabla_\perp^2 b$$
$$\partial_t u = \partial_z b + [\varphi, u] - [a, b] + \nu \nabla_\perp^2 u$$

where $j = -\nabla_\perp^2 a$ and $\omega = -\nabla_\perp^2 \varphi$

- These are the RHMHD equations. Their ideal invariants (just as for 3D HMHD) are:

$$E = \frac{1}{2}\int d^3r\,(|\vec{U}|^2 + |\vec{B}|^2) = \frac{1}{2}\int d^3r\,(|\vec{\nabla}_\perp \varphi|^2 + |\vec{\nabla}_\perp a|^2 + u^2 + b^2) \quad \text{energy}$$

$$H_m = \frac{1}{2}\int d^3r\,(\vec{A} \cdot \vec{B}) = \int d^3r\, ab \quad \text{magnetic helicity}$$

$$H_h = \frac{1}{2}\int d^3r\,(\vec{A} + \varepsilon \vec{U}) \cdot (\vec{B} + \varepsilon \vec{\Omega}) = \int d^3r\,(ab + \varepsilon(a\omega + ub) + \varepsilon^2 u\omega) \quad \text{hybrid helicity}$$

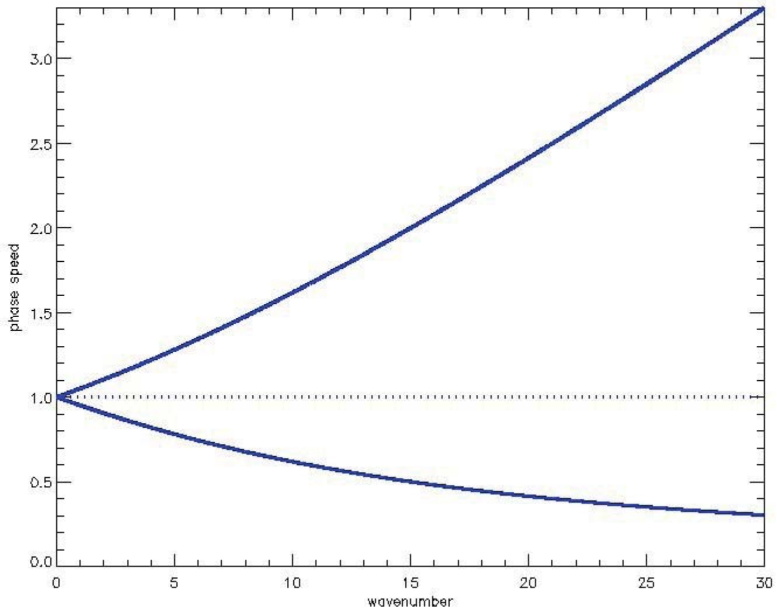
Linear modes of the RHMHD equations

- Retaining the linear terms in the RHMHD eqs we obtain the following dispersion relationship

$$\omega^4 - 2k_z^2(1+\frac{(\varepsilon k_\perp)^2}{2})\omega^2 + k_z^4 = 0$$

which displays the following (dispersive) modes

$$\omega_\pm = \sqrt{k_z^2 + (\frac{\varepsilon k_\perp k_z}{2})^2} \pm \frac{\varepsilon k_\perp k_z}{2}$$



- The positive branch corresponds to (right hand, circularly polarized) *whistlers*, while the negative branch are (left hand polarized) *ion-cyclotron* waves.

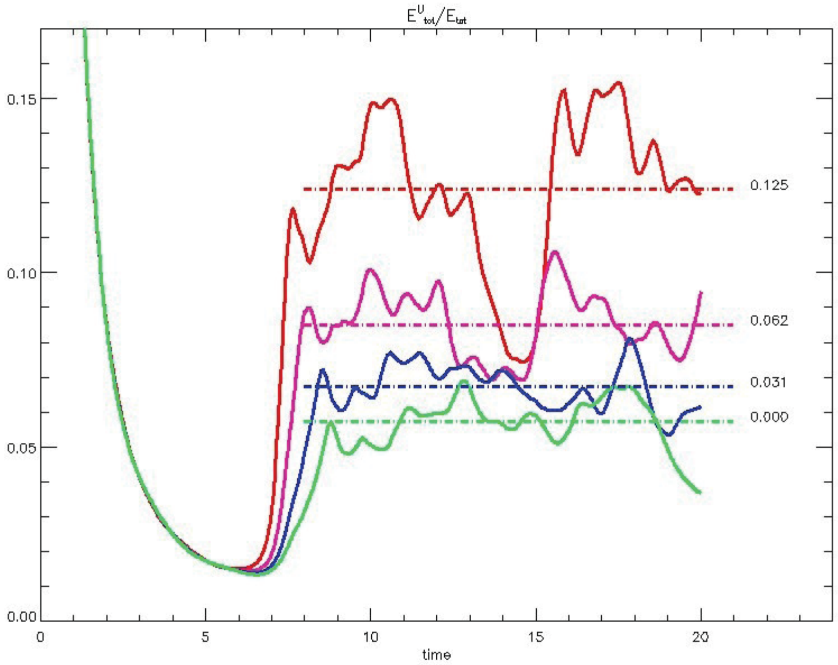
- The phase speed for *whistlers* grows like $c_+ \approx \varepsilon k_\perp$ thus forcing to a very small dt for numerical convergence.

- *Ion-cyclotron* waves, instead, display a decreasing phase speed like $c_- \approx 1/\varepsilon k_\perp$, which makes them good candidates for resonant particle acceleration.

Numerical simulations

- We integrate the RHMHD eqs. numerically, using a spectral scheme in the perpendicular directions and finite differences along the (much smoother) direction z (Gomez, Milano and Dmitruk 2000; also Dmitruk, Gomez & Matthaeus 2003)

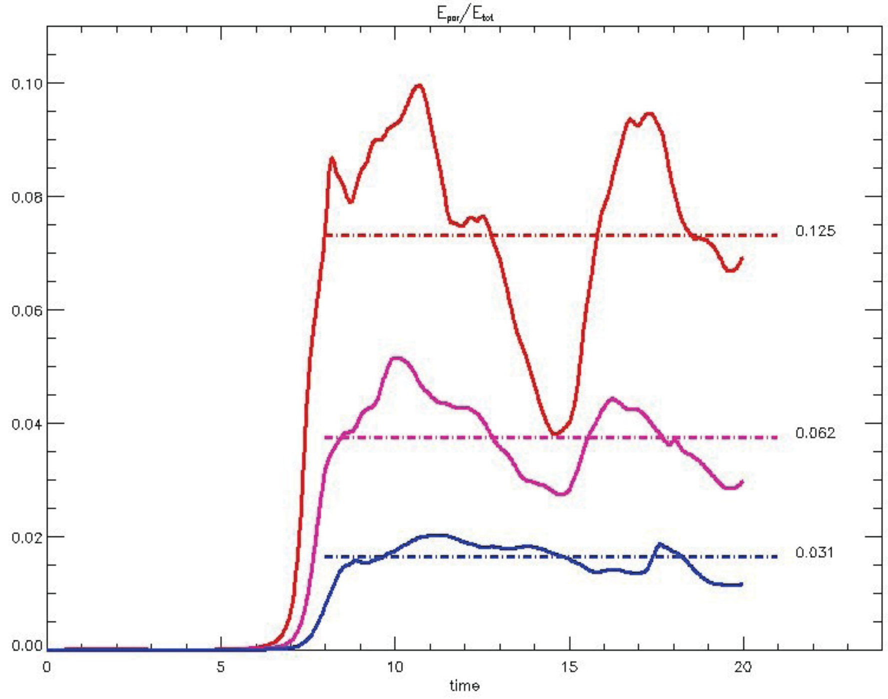
- We show results from 512x512x40 runs performed in (CAPS), our linux cluster with 40 nodes



- Ratio of kinetic to total energy vs. time for different runs corresponding to different values of the Hall parameter (labelled).

CAPS: **C**luster for **A**strophysical **P**lasma **S**imulations

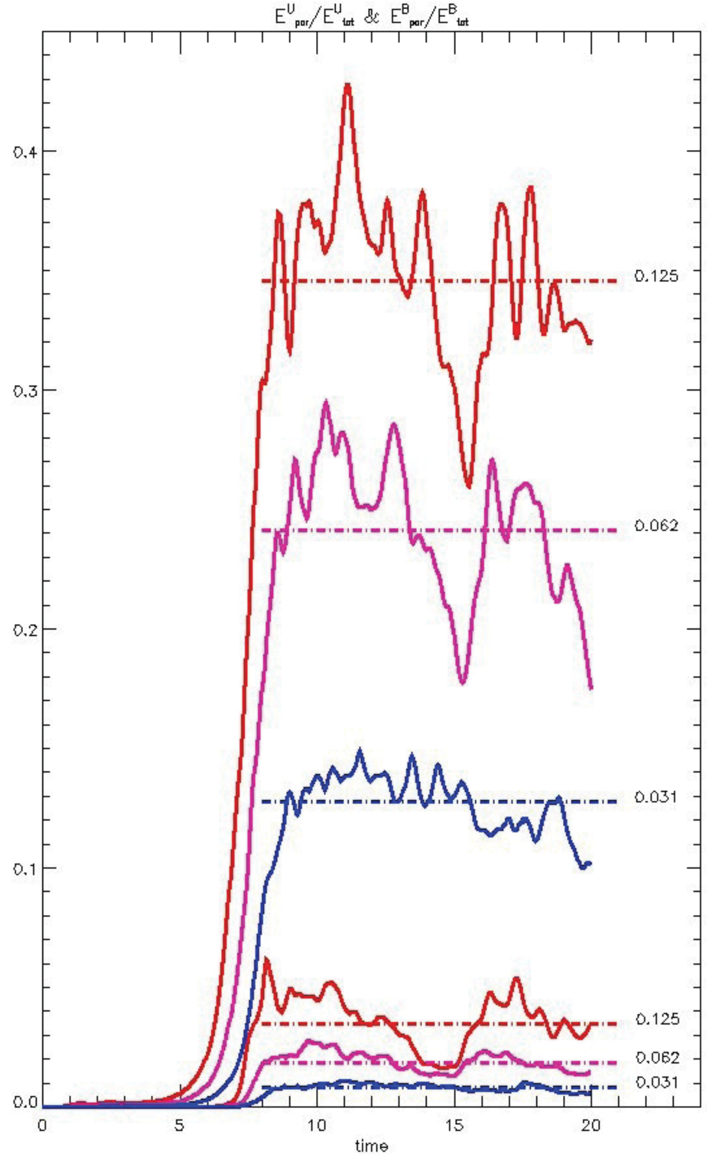
Parallel vs perpendicular energy



- Ratio of parallel to total energy (above) for different values of the Hall parameter.

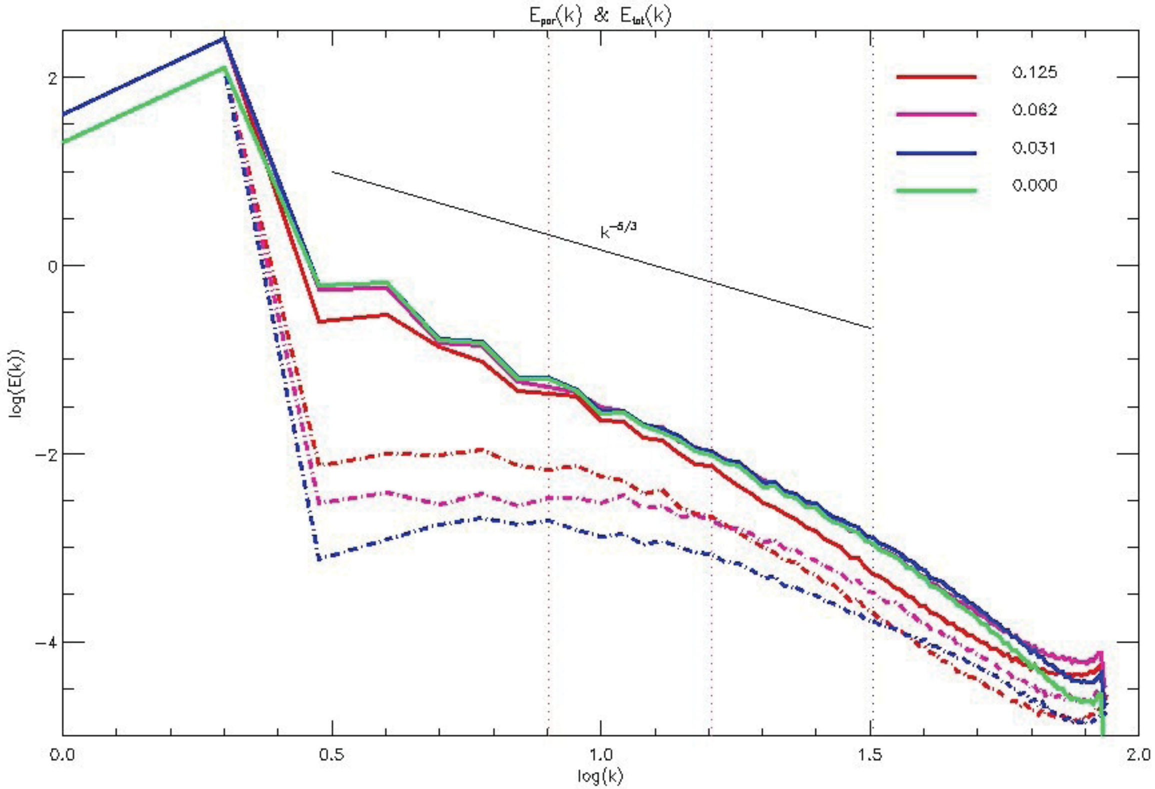
$$E_{\perp} = \frac{1}{2}\int d^3r\,(|\vec{\nabla}_{\perp}\varphi|^2 + |\vec{\nabla}_{\perp}a|^2) \quad , \quad E_{//} = \frac{1}{2}\int d^3r\,(u^2 + b^2)$$

- Ratio of parallel to total kinetic (above) as well as magnetic energy (below).



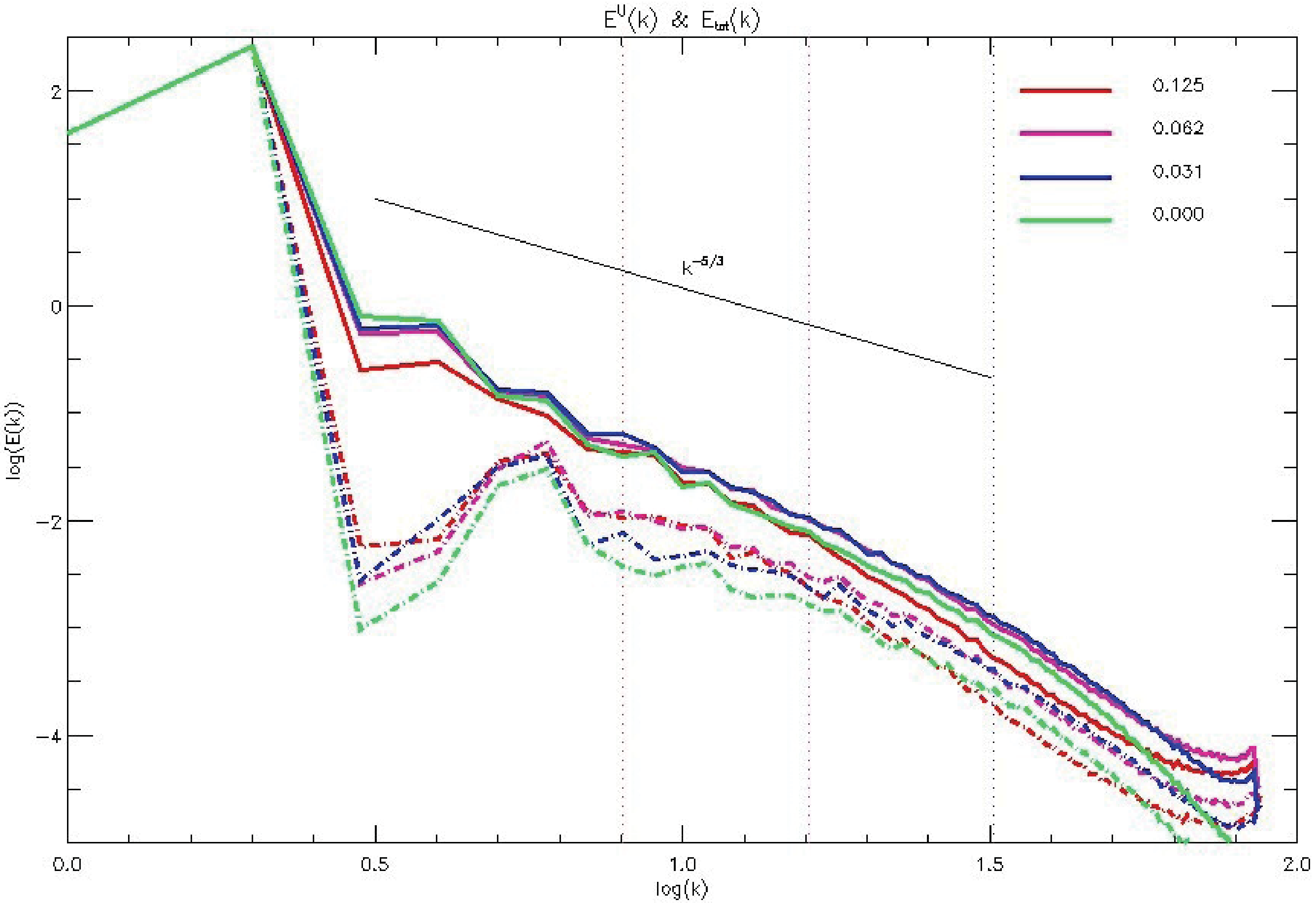
Energy spectra

- We also computed energy power spectra for different values of the Hall parameter ε .
- The Kolmogorov slope $k^{-5/3}$ is also displayed for reference.
- The dotted curves correspond to the parallel energy spectra.
- The vertical dotted lines indicate the location of the Hall scale $k_\varepsilon \cong 1/\varepsilon$ for each run.



Energy spectra

- Energy power spectra for different values of ε.
- The dotted curves are the spectra for kinetic energy.



Helicities

- Magnetic and hybrid helicities vs. time for different values of the Hall parameter.

- For the particular forcing applied in these runs (applied boundary motions at large scales) there is no net injection of magnetic helicity.

- For $\varepsilon \neq 0$ there is a net injection of hybrid helicity, which is larger for larger values of ε

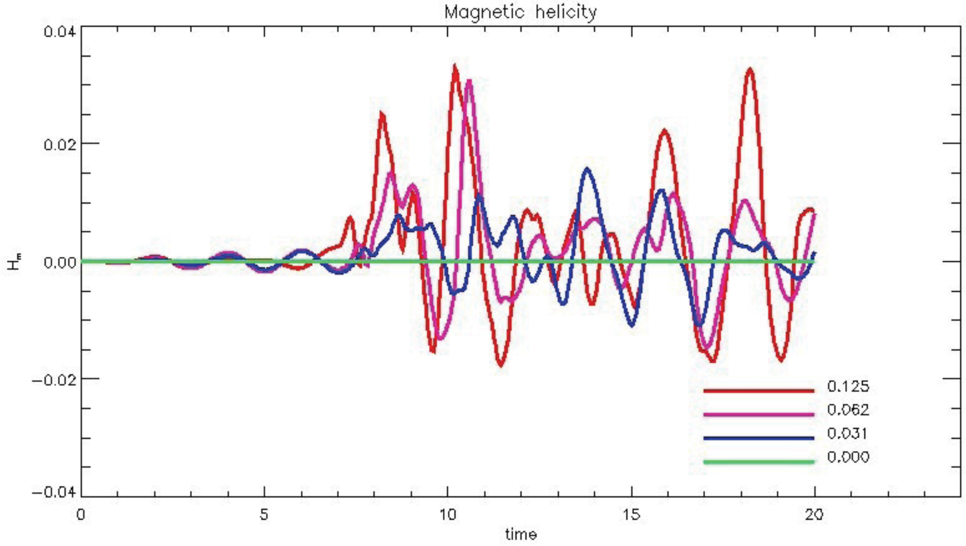
- Hybrid helicity can be split into

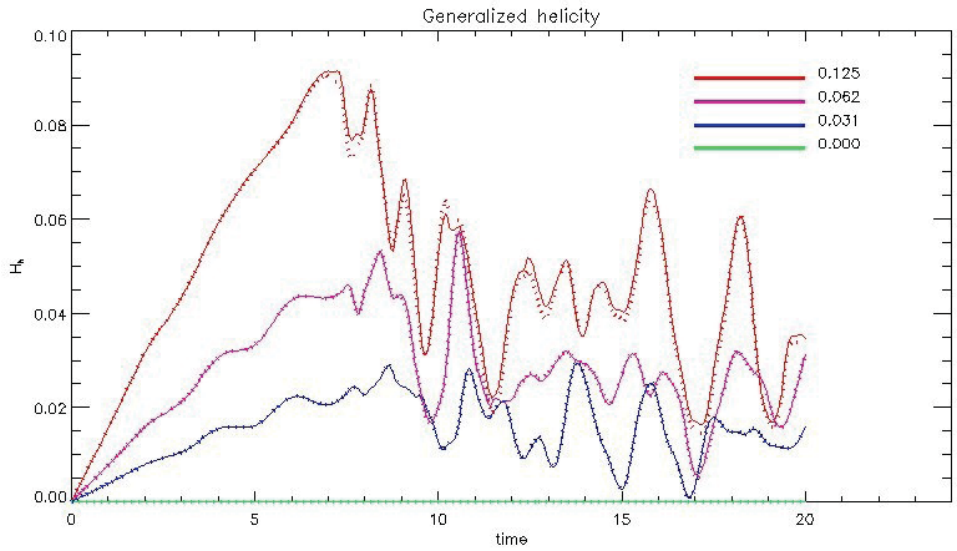
$$H_h = \frac{1}{2}\int d^3r\,(\vec{A}+\varepsilon\vec{U})\cdot(\vec{B}+\varepsilon\vec{\Omega}) = H_m + 2\varepsilon H_c + \varepsilon^2 H_k$$

where $H_c = \frac{1}{2}\int d^3r\,\vec{U}\cdot\vec{B}$ cross-helicity

$H_k = \frac{1}{2}\int d^3r\,\vec{U}\cdot\vec{\Omega}$ kinetic helicity

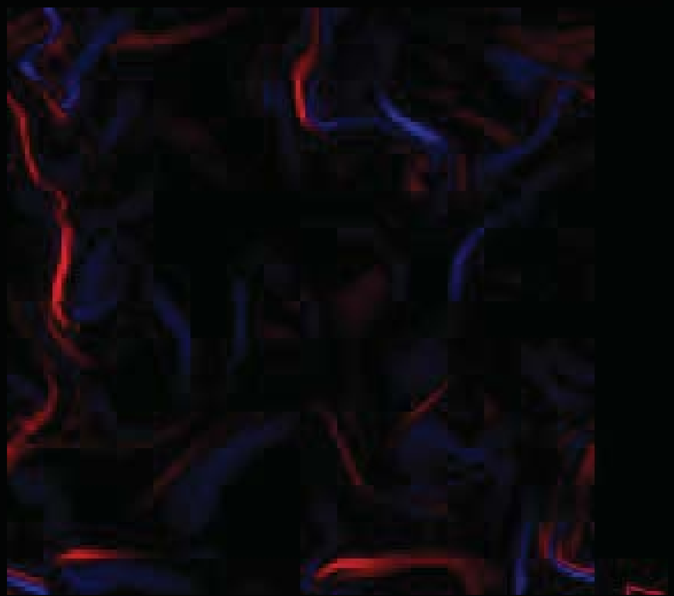
- Note that what is being mostly injected is H_c , which is not an exact invariant in Hall MHD.

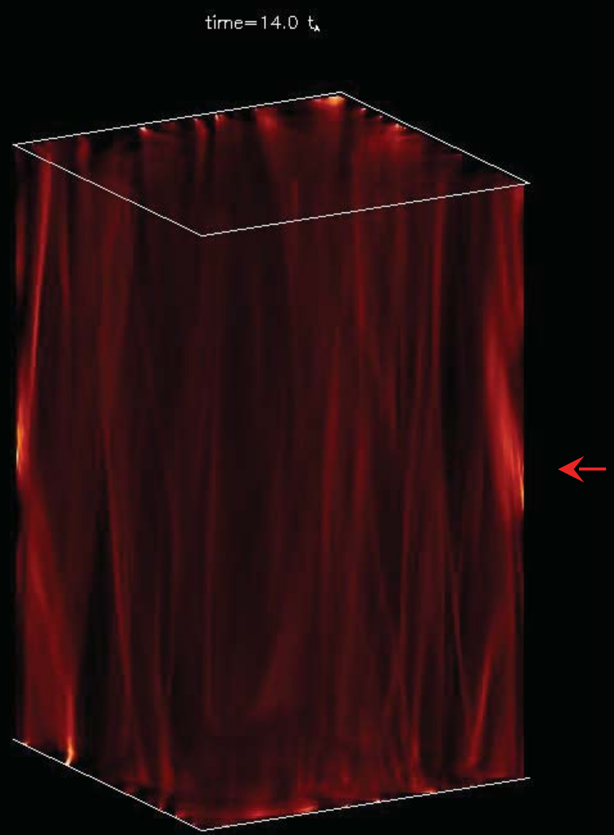




Dissipative structures: current sheets in 2D

- Most of the energy dissipation takes place in current sheets. We display the current density (upflows & downflows) along the loop in a transverse cut.



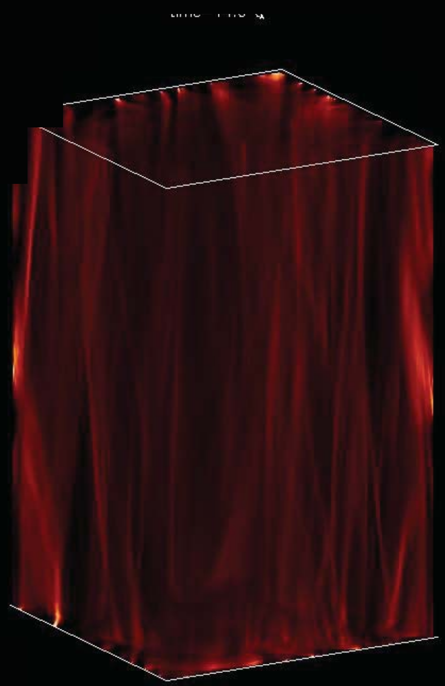


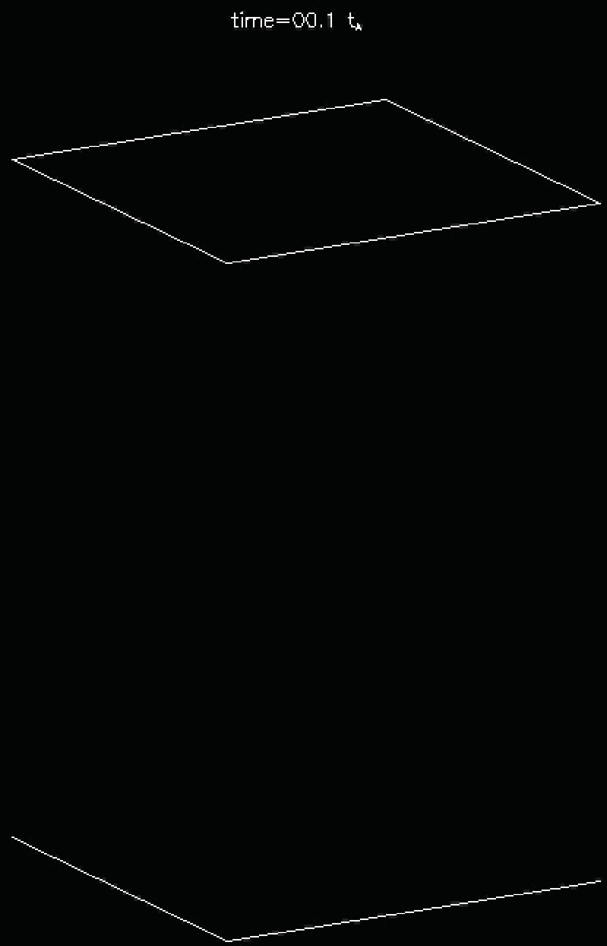
- Versus height.

- Versus time.

Dissipative structures: current sheets in 3D

- 3D distribution of the energy dissipation rate.
- We display the dissipation rate during 20 Alfven times with a cadence of 0.1 t_A.



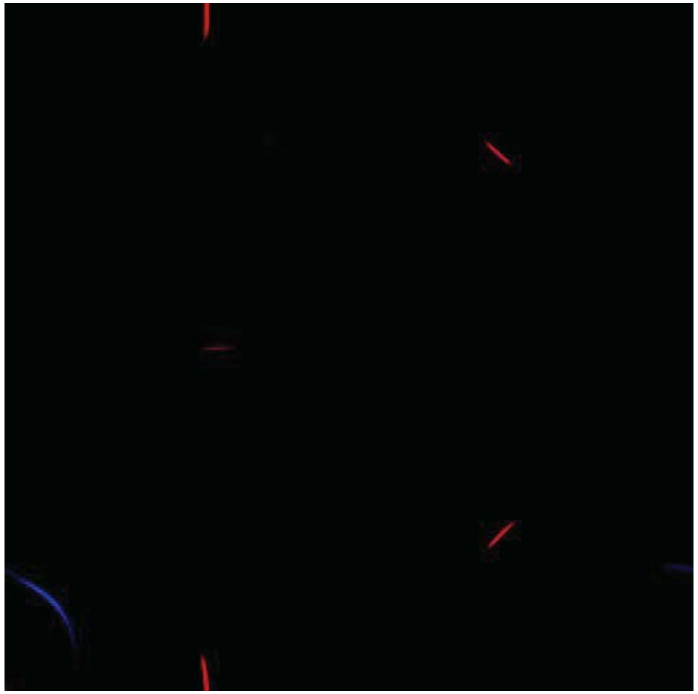


Current sheets in RHMHD

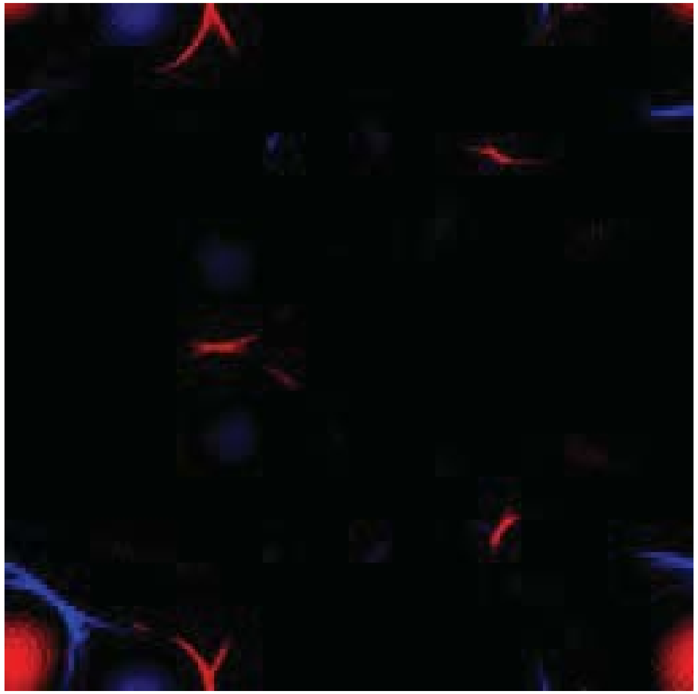
- Energy dissipation concentrates on very small structures known as current sheets, in which current density flows almost parallel to z.

  The picture shows **positive** and **negative** current density in a transverse cut at $z = 1/2$, for pure RMHD (i.e. $\varepsilon = 0$).

- When the Hall effect is considered, current sheets display the typical Petschek-like structure.

- 

  $\varepsilon = 0.0$

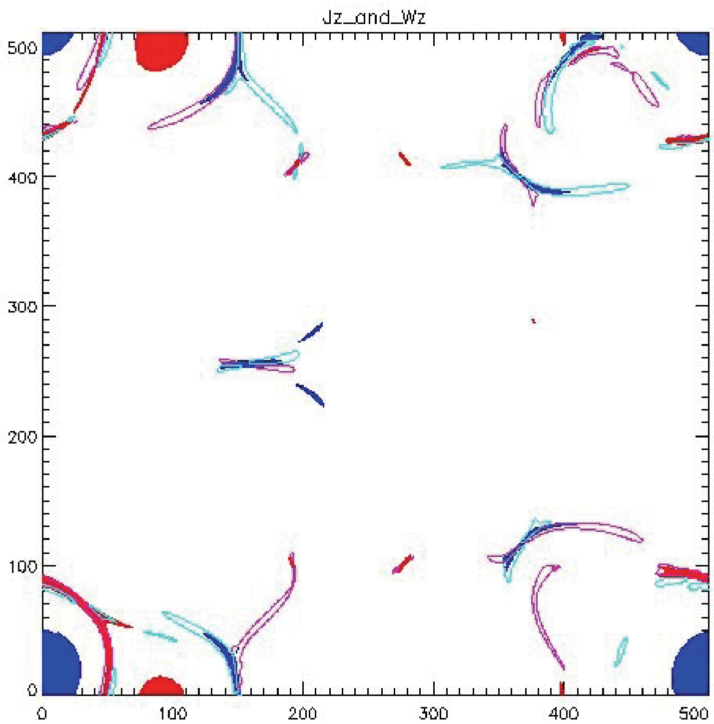
  

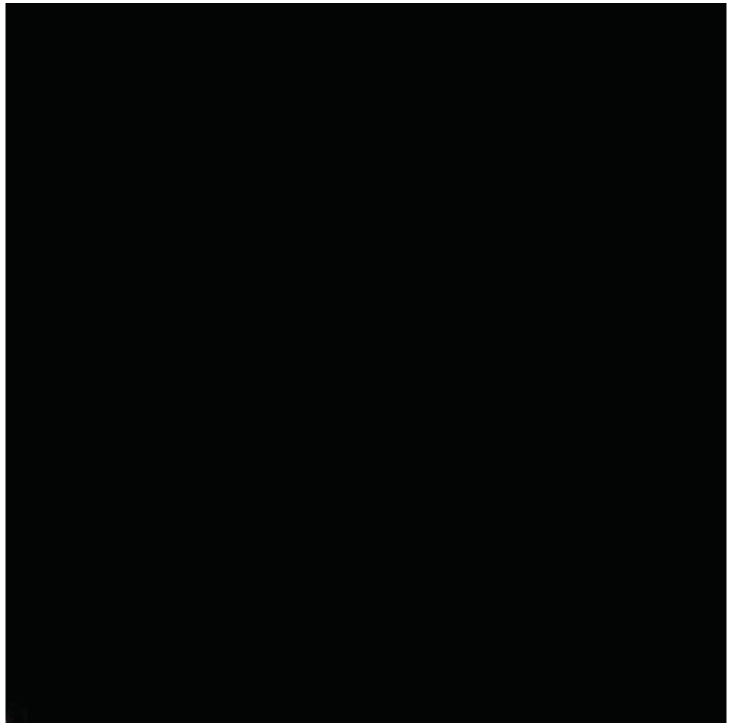
  $\varepsilon = 0.1$

Current sheets and vorticity in RMHD

- We can also display an animation of how current sheets evolve in time. In this case for $\varepsilon = 0$.

- The flow in the surroundings of each of these current sheets displays the behavior sketched by Parker's model, which is consistent with a quadrupolar structure for the vorticity field.



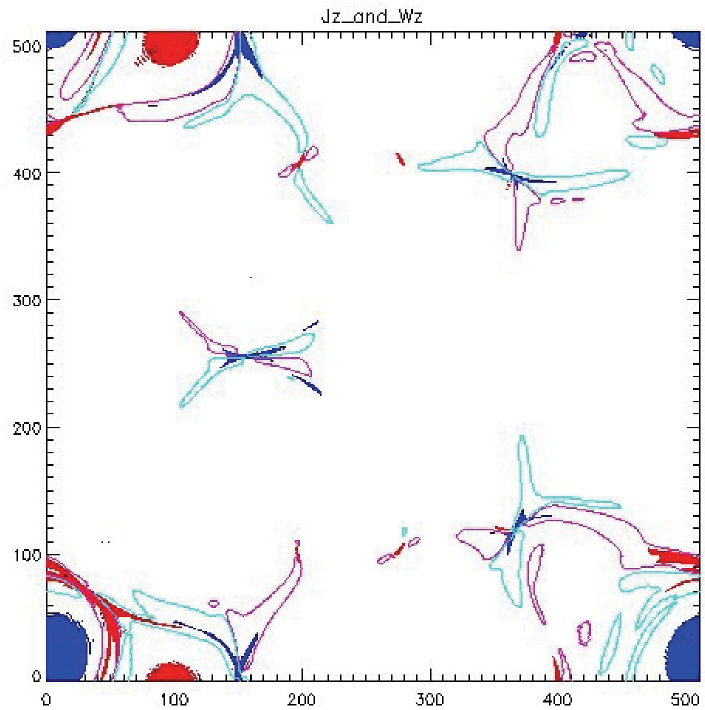


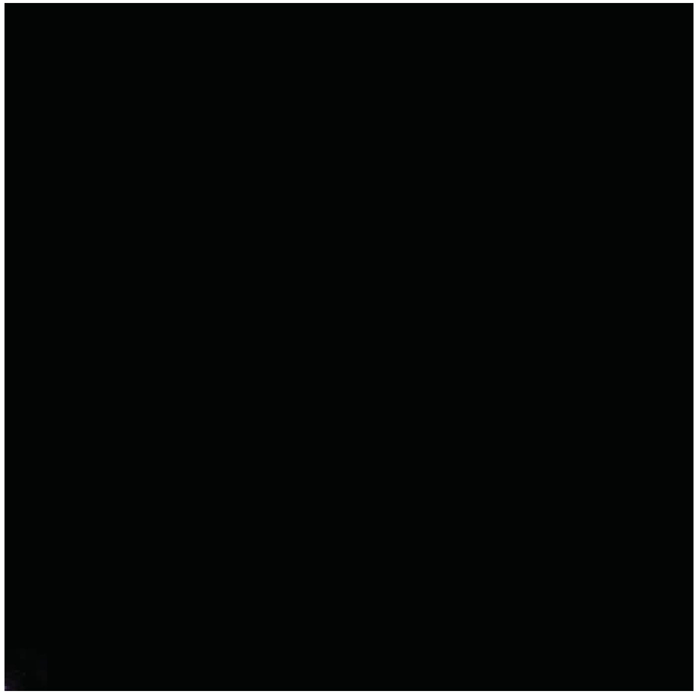
- Current density is displayed in **red** and **blue**, while contours coloured in **light blue** and **pink** correspond to vorticity.

- All contours correspond to 20% of the maximum value.

Current sheets and vorticity in RHMHD

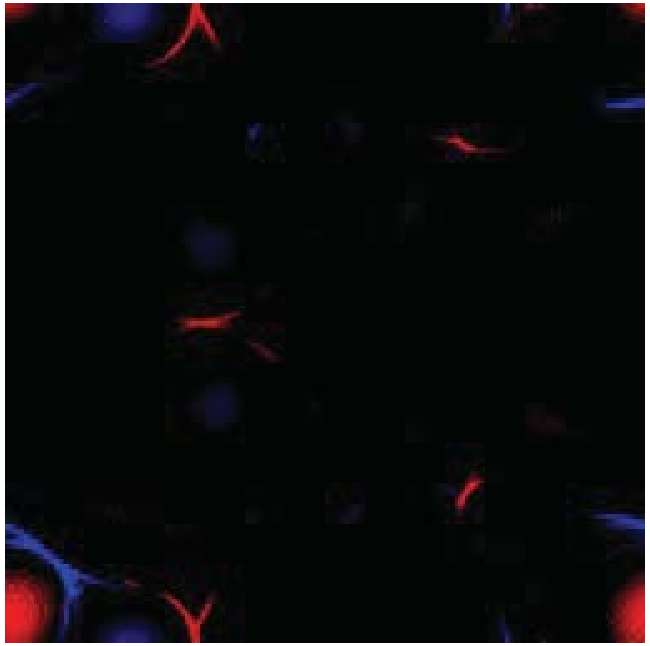
- This is an animation of current density vs time for $\varepsilon = 0.1$
- The quadrupolar structure for vorticity persists, but it gets strongly distorted with respect to the pure RMHD case.

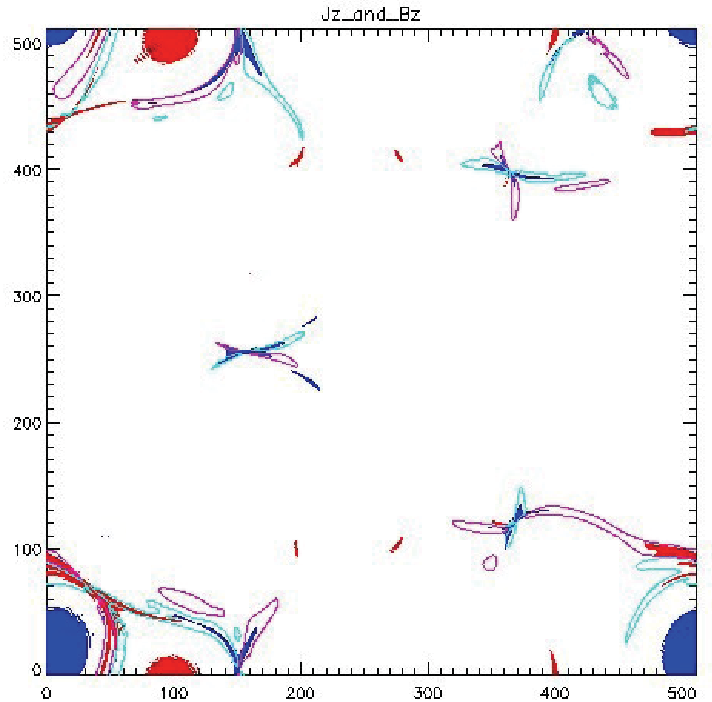




- Current density is displayed in **red** and **blue**, while contours coloured in **light blue** and **pink** correspond to vorticity.

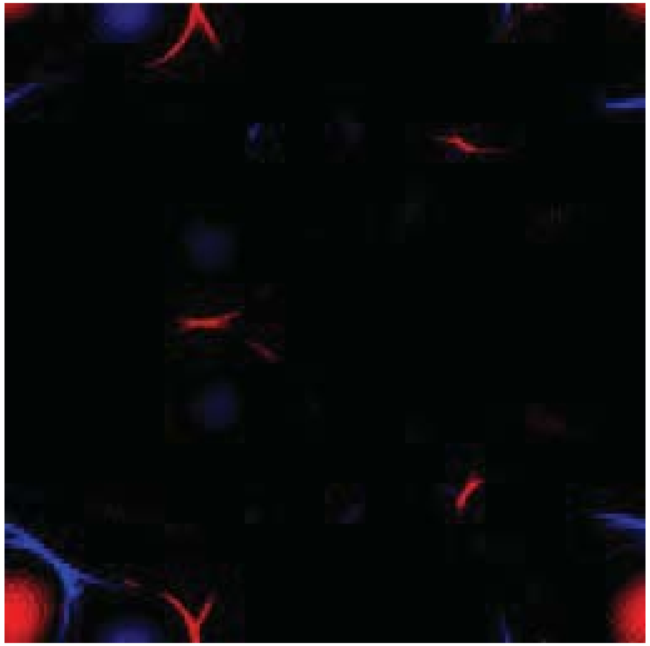
Parallel magnetic field

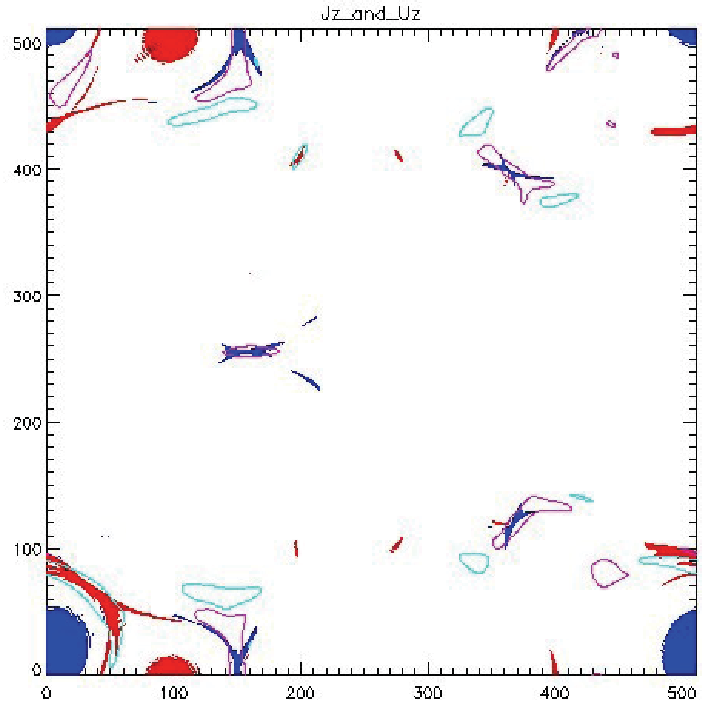




- The variable part of magnetic field along z (i.e. b), also display a quadrupolar structure around current sheets.
- Current density is displayed in **red** and **blue**, while contours coloured in **light blue** and **pink** correspond to b.

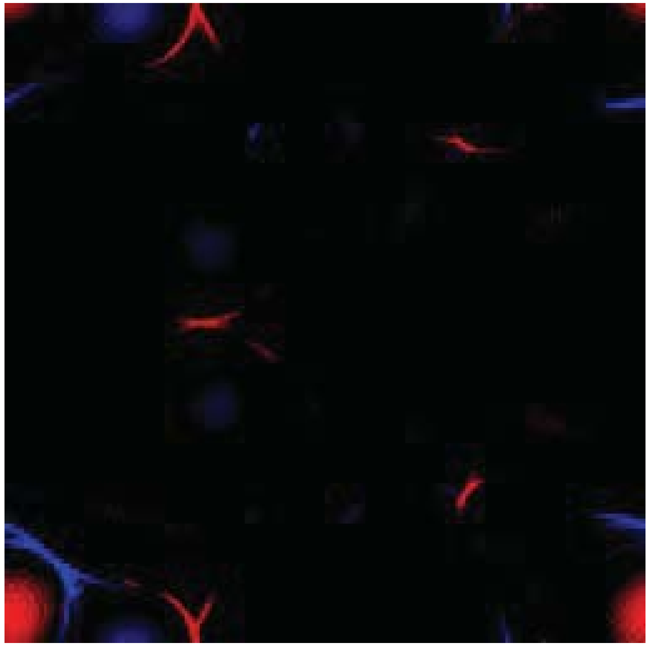
Parallel velocity field

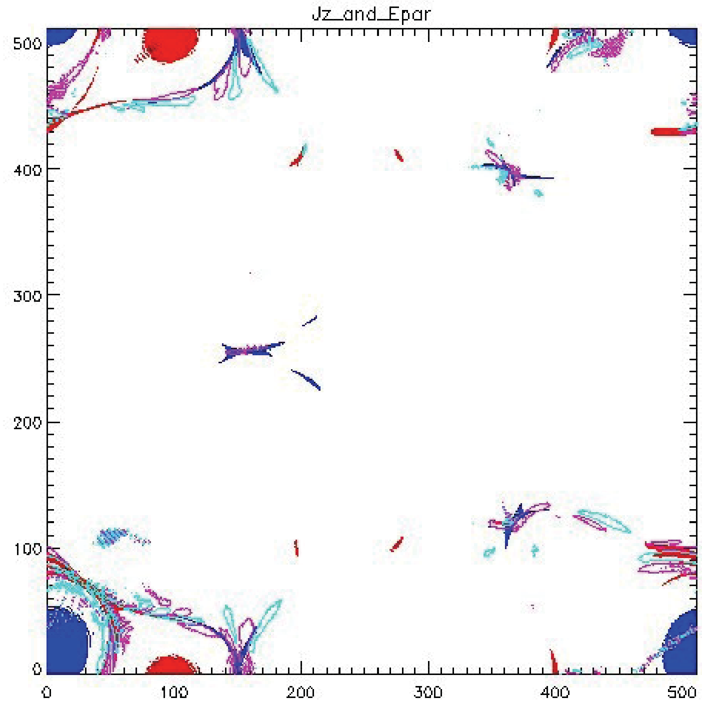




- The velocity component along z (i.e. u), instead, tends to behave as a net flow along the current sheet.
- Current densiity is displayed in **red** and **blue**, while contours coloured in **light blue** and **pink** correspond to u.

Parallel electric field





- One of the important new features of the Hall effect, is the presence of a parallel electric field, i.e. $E_{//} = \frac{E \cdot B}{|B|}$
- To order α^2 it can be computed as $E_{//} = \varepsilon(\partial_z b - [a,b])$

  and of course can potentially accelerate particles along magnetic field lines.
- Current densiity is displayed in **red** and **blue**, while contours coloured in **light blue** and **pink** correspond to the parallel electric field.

Conclusions

- We have derived a new set of equations, which we call reduced Hall-MHD (RHMHD), following the asymptotic procedure used in deriving the conventional RMHD.
- The RHMHD equations describe the slow dynamics of Hall plasmas embedded in a strong magnetic field (Gomez, Mahajan & Dmitruk 2008, PoP 15 102303).
- They contain additional physics (when compared to pure RMHD) such as parallel electric fields and normal modes like whistlers and ion-cyclotron waves.
- Since ion-cyclotron waves have a phase speed which decreases with wavenumber, its associated electric field can be potentially relevant in accelerating and heating particles.
- We report the results of numerical simulations with different values of the Hall parameter, to study changes in the plasma dynamics as the Hall effect becomes progressively more important.
- We believe that this model will be potentially quite useful for a number of applications such as fusion devices and astrophysical plasmas.