Galactic Dynamos

Question 1. One-dimensional compression of magnetised gas

(a) **Compression perpendicular to the magnetic field.** Use the conservation of mass and magnetic flux to derive the dependence of magnetic field strength on gas density under one-dimensional compression perpendicular to the magnetic field in a slab shown in the left-hand panel of Fig. 1. Assume that magnetic field is frozen into the flow.

(b) Compression parallel to the magnetic field. Apply similar arguments to one-dimensional compression parallel to the magnetic field, as shown in the right-hand panel of Fig. 1, to find out how compression affects magnetic field in this case.

Solution

(a) The system is governed by the following conservation laws:

$$\begin{aligned} \text{mass conservation:} \quad M = & \quad \rho h L d = \text{const}, \quad L, h = \text{const} , \\ \text{magnetic flux conservation:} \quad \Phi = & \quad B h d = \text{const} , \end{aligned}$$

where the notation is clear from the left panel of Fig. 1. With the initial values of the variables denoted by zero subscript, we obtain

$$M = \text{const} \quad \Rightarrow \quad \frac{\rho}{\rho_0} = \frac{d_0}{d} . \qquad \Phi = \text{const} \quad \Rightarrow \quad B = B_0 \frac{d_0}{d} = B_0 \frac{\rho}{\rho_0} ,$$

i.e., magnetic field strength is proportional to gas density.

(b) In this case the conservation laws are as follows:

$$\Phi = Bhd = \text{const}, \quad M = \rho h L d = \text{const}, \quad h,\ d = \text{const} .$$

Therefore, $\rho = \rho_0 L_0 / L$, but $B = \text{const}$, i.e., B is independent of ρ. In other words, the gas slides along the magnetic field lines without affecting the field.

Question 2. Refraction of magnetic field in a spiral arm

Consider a model of the effect of a spiral arm on the large-scale galactic magnetic field, illustrated in Fig. 2. Assume that gas densities within the arm and in the interarm region are ρ_a and ρ_i, respectively; neglect the curvature of the arms. The magnetic field in the interarm space has a strength $3\,\mu$G, makes an angle $p_\mathrm{i} = 10^\circ$ with the arm axis as shown in Fig. 2, and is carried into the arm by gas flow. Assume that magnetic field is frozen into the gas and that the gas in the arm is four times as dense as in the interarm space, $\rho_\mathrm{a} = 4\rho_\mathrm{i}$. Using the results of Question 1, calculate the strength of magnetic field within the arm and its angle p_a with the arms axis by considering separately magnetic field components parallel and perpendicular to the arm.

Solution

As follows from the results of Question 1, the normal and tangential components of magnetic field are differently affected by the one-dimensional compression. Denote these components by superscripts '(n)' and '(t)', respectively; subscript 'i' will refer to the interarm region, and 'a' refers to the arm, as shown in Fig. 2.

The component of the interarm magnetic field normal to the arm, $B_\mathrm{i}^{(n)} = B_\mathrm{i} \sin p_\mathrm{i}$, is unaffected by the gas compression, whereas the tangential component, $B_\mathrm{i}^{(t)} = B_\mathrm{i} \cos p_\mathrm{i}$, increases in proportion to the density:

$$B_\mathrm{a}^{(t)} = B_\mathrm{i}^{(t)} \frac{\rho_\mathrm{a}}{\rho_\mathrm{i}} .$$

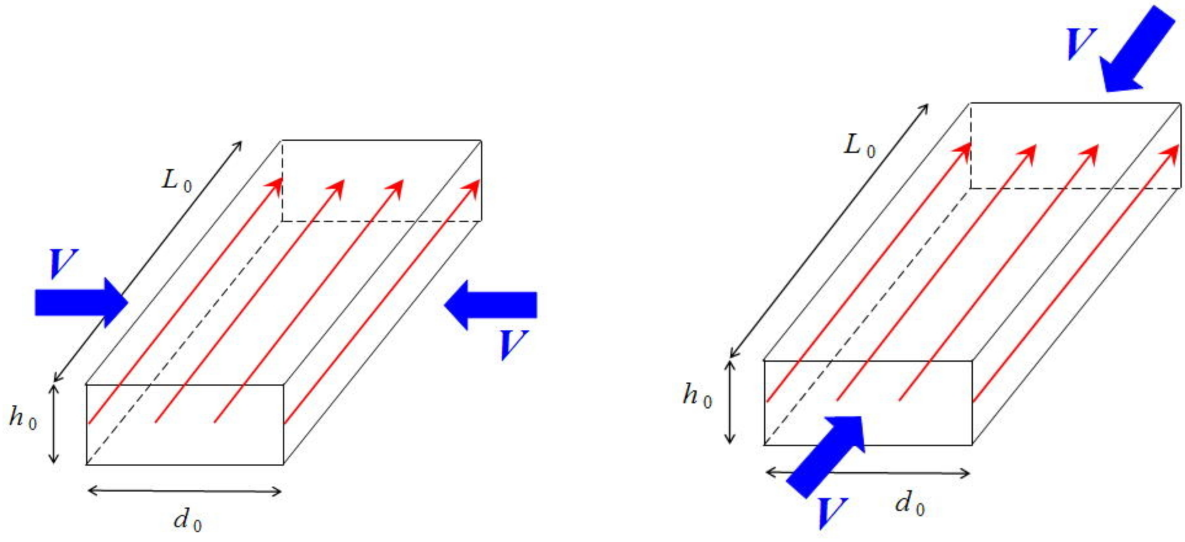

Figure 1: One-dimensional compression of a slab of magnetised gas in a direction perpendicular to the magnetic field (left panel) and parallel to the magnetic field (right panel).

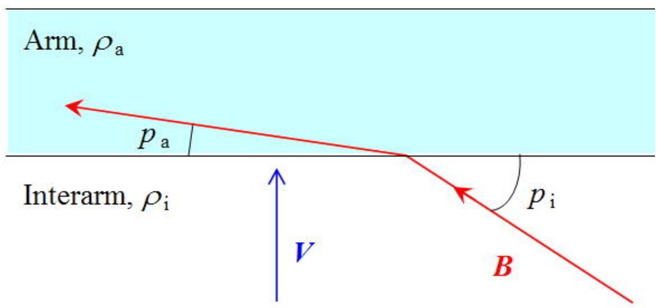

Figure 2: Refraction of magnetic lines in a spiral arm due to one-dimensional compression.

Then the normal and tangential components of magnetic field within the arm follow as

$$B_{\rm a}^{(n)} = B_{\rm i} \sin p_{\rm i} \ , \quad B_{\rm a}^{(t)} = B_{\rm i} \frac{\rho_{\rm a}}{\rho_{\rm i}} \cos p_{\rm i} \ .$$

This yields the field strength within the arm

$$B_{\rm a} = \sqrt{{B_{\rm a}^{(n)}}^2 + {B_{\rm a}^{(t)}}^2} = B_{\rm i} \sqrt{\sin^2 p_{\rm i} + \frac{\rho_{\rm a}^2}{\rho_{\rm i}^2} \cos^2 p_{\rm i}} \approx 3.9 B_{\rm i} \approx 12\,\mu{\rm G} \ .$$

The angle to the arm axis is obtained as

$$\tan p_{\rm a} = \frac{B_{\rm a}^n}{B_{\rm a}^t} = \frac{\rho_{\rm i}}{\rho_{\rm a}} \tan p_{\rm i} \approx 0.044 \quad \Rightarrow \quad p_{\rm a} \approx 2.5^\circ \ .$$

Thus, the refraction of magnetic lines due to gas compression in spiral arms leads to a closer alignment of the magnetic field with the arm axis. The tight alignment of the large-scale magnetic field with spiral arms is typical of spiral galaxies.

Question 3. Non-isotropic collapse through a sequence of equilibrium states

The collapse of a flattened, magnetised cloud, initially along an external magnetic field, is governed by the following conservation laws (Mestel & Paris, *Astron. Astrophys,* **136**, 98, 1984):

$$\begin{aligned}
\text{mass conservation:} \quad M &= 2\pi\rho h R^2 = \text{const} \,, \\
\text{magnetic flux conservation:} \quad \Phi &= \pi B R^2 = \text{const} \\
\text{gravity–gas pressure balance:} \quad 2\pi G\rho h^2 &= c_\mathrm{s}^2 = \text{const} \,,
\end{aligned}$$

where h is the cloud half-thickness (see Fig. 3), c_s is the isothermal sound speed, and G is the gravitational constant, and the field is assumed to be frozen into the gas. The difference from the spherically symmetric collapse discussed in the *Basic MHD Problem Set Q. 2(a)* is that the cloud no longer collapses freely. Instead, its thickness h is controlled by the equilibrium between the gravity force and internal pressure.

Derive a relation between magnetic field strength and density within such a cloud.

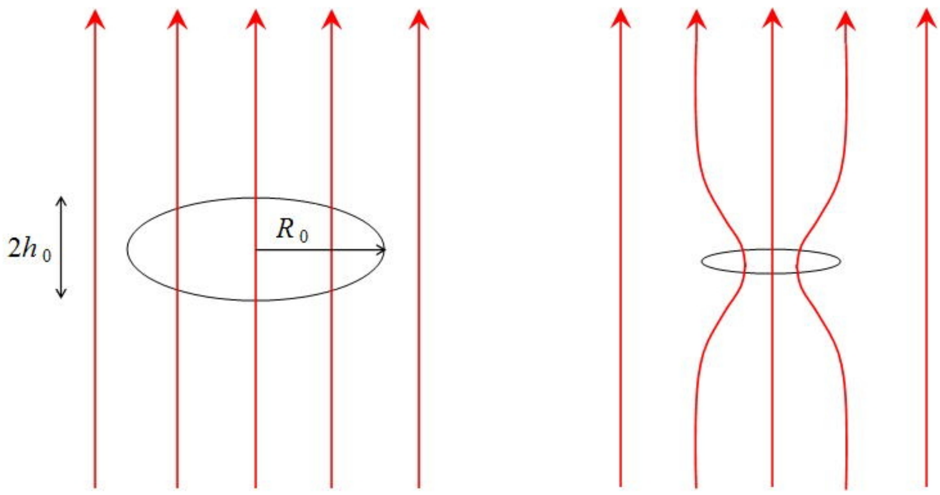

Figure 3: A non-isotropic collapse of a magnetised cloud.

Solution

Now, the conservation laws take the form (with subscript zero denoting initial values)

$$\begin{aligned}
M &= \text{const} \quad \Rightarrow \quad \left(\frac{R_0}{R}\right)^2 = \frac{\rho h}{\rho_0 h_0} \,, \\
\rho h^2 &= \text{const} \quad \Rightarrow \quad \frac{h}{h_0} = \left(\frac{\rho_0}{\rho}\right)^{1/2} \,, \\
\Phi &= \text{const} \quad \Rightarrow \quad B = B_0\left(\frac{R_0}{R}\right)^2 = B_0\frac{\rho h}{\rho_0 h_0} = B_0\left(\frac{\rho}{\rho_0}\right)^{1/2} \,.
\end{aligned}$$

Thus, magnetic field frozen into a collapsing flattened cloud grows with gas density as

$$B = B_0(\rho/\rho_0)^{1/2} \,. \tag{1}$$

Question 4. Magnetic field trapped by a young star

Consider a young star produced by the collapse of an interstellar gas cloud whose magnetic field and density are $B_0 = 10\,\mu\mathrm{G}$ and $\rho_0 = 6\times10^{-22}\,\mathrm{g\,cm^{-3}}$, respectively. As argued by Mestel & Paris (*Astron. Astrophys,* **136**, 98, 1984), the scaling of magnetic field strength with gas density derived in

Question 3 is applicable to such a cloud. The size and mass of the star can be assumed to be similar to those of the Sun, $R = 7 \times 10^{10}$ cm and $M = 2 \times 10^{33}$ g.

Estimate magnetic field in the star assuming that magnetic field is frozen into the collapsing gas and compare the result with the typical strength of a stellar magnetic field of $B = 10^3$ G. Derive estimates for both spherically symmetric collapse and a non-isotropic collapse discussed in Question 3.

By comparing the results with the typical stellar magnetic field given above, decide whether magnetic flux is conserved during the collapse.

Solution

For spherically symmetric collapse, $B = B_0(\rho/\rho_0)^{2/3}$. The mean gas density within the star is given by

$$\rho = \frac{M}{4\pi R^3/3} \approx 1.4\,\mathrm{g\,cm^{-3}}\ ,$$

so $\rho/\rho_0 \approx 2\times10^{21}$, hence $B = 10\,\mu\mathrm{G}\times(2\times10^{21})^{2/3} \approx 1.6\times10^9$ G in the case of spherically symmetric collapse. This value is a factor 10^6 larger than the typical stellar magnetic field strength.

The field enhancement is weaker if the collapse in non-isotropic, with $B = B_0(\rho/\rho_0)^{1/2} \approx 4.5 \times 10^5$ G , which is significantly smaller than the field strength obtained under spherically symmetric collapse, but still too large to match a stellar field.

We conclude that magnetic flux is not conserved during the collapse, and most of it leaks from the newly formed star. This conclusion has far reaching consequences for theory of star formation which, unfortunately, are beyond the scope of our short course.

Question 5. Could the galactic magnetic field have originated in the galactic centre?

Chakrabarti et al. (*Nature,* **368**, 434, 1994) proposed that the large-scale galactic magnetic field can be a result of an outflow of magnetised gas from an active galactic nucleus (powered by a massive black hole). They argue that the azimuthal magnetic field within $r = 3 \times 10^{11}$ cm of the galactic centre can be $B_0 = 3 \times 10^5$ G strong, and it can be carried out to the outer parts of the galaxy by galactic wind (outflow of interstellar gas). These authors suggest that the strength of azimuthal magnetic field at a radius $R = 10\,\mathrm{kpc} = 3\times10^{22}$ cm (similar to the distance of the Sun to the Galactic centre) can be estimated from magnetic flux conservation as $B = B_0 r/R \approx 3\times10^{-6}$ G, which is close to the observed field strength.

Do you think this estimate is correct? If not, derive the correct estimate of magnetic field strength that might result from this process.

Hint: the half-thickness of the magnetised galactic disc can be adopted as $h = 1\,\mathrm{kpc} = 3\times10^{21}$ cm.

Solution

The estimate appears to be wrong. The magnetic flux of the azimuthal magnetic field through a meridional plane at the centre is given by $\Phi \approx r^2 B_0$ (here and below we omit factors of order unity, such as π or 2). When this surface has been stretched throughout the galaxy by the outflow, so that its radial and vertical sizes become R and h, its area becomes $S \approx hR$, and the flux of the azimuthal magnetic field through it is estimated as $\Phi \approx hRB$, Given that the magnetic flux is conserved, we obtain the strength of magnetic field at a radius R as

$$B \approx B_0 \frac{r^2}{hR} \approx 3 \times 10^{-16}\,\mathrm{G}\ ,$$

a value 10 orders of magnitude smaller than that suggested by Chakrabarti et al. When making the estimate, they have apparently forgotten that the thickness of the magnetised galactic disc is much larger than the size of the magnetised region near the black hole at the centre, and hence omitted the factor $r/h \approx 10^{-10}$.

Question 6. Induction equation with the Hall term

Ohm's law can be written as

$$\vec{\mathbf{J}} = \sigma\left(\vec{\mathbf{E}} + \frac{1}{c}\vec{\mathbf{V}} \times \vec{\mathbf{B}}\right) - \sigma_{\mathrm{H}}\vec{\mathbf{J}} \times \vec{\mathbf{B}}\,,$$

where the last term on the right-hand side arises from the effect of magnetic field on plasma motion, known as the Hall effect. (The factor σ_{H} is inversely proportional to the magnetic field strength B, so that the Hall effect in fact depends on the direction of magnetic field $\hat{\vec{\mathbf{B}}} = \vec{\mathbf{B}}/B$ rather than on its strength.)

Derive the induction equation with allowance for the Hall effect.

Solution

As usual, we use Ohm's law, now in the given form, to write out an expression for electric field:

$$\vec{\mathbf{E}} = \frac{1}{\sigma}\vec{\mathbf{J}} - \frac{1}{c}\vec{\mathbf{V}} \times \vec{\mathbf{B}} + \frac{\sigma_{\mathrm{H}}}{\sigma}\vec{\mathbf{J}} \times \vec{\mathbf{B}}\,,$$

where we use Ampere's law $\nabla \times \vec{\mathbf{B}} = \dfrac{4\pi}{c}\vec{\mathbf{J}}$ to eliminate $\vec{\mathbf{J}}$:

$$\vec{\mathbf{E}} = \frac{c}{4\pi\sigma}\nabla \times \vec{\mathbf{B}} - \frac{1}{c}\vec{\mathbf{V}} \times \vec{\mathbf{B}} + \frac{c}{4\pi\sigma}\sigma_{\mathrm{H}}(\nabla \times \vec{\mathbf{B}}) \times \vec{\mathbf{B}}\,,$$

and then Faraday's law $\dfrac{1}{c}\dfrac{\partial\vec{\mathbf{B}}}{\partial t} = -\nabla \times \vec{\mathbf{E}}$ yields the induction equation:

$$\frac{\partial\vec{\mathbf{B}}}{\partial t} = \nabla \times (\vec{\mathbf{V}} \times \vec{\mathbf{B}}) - \nabla \times (\eta\nabla \times \vec{\mathbf{B}}) - \nabla \times \eta_{\mathrm{H}}[(\nabla \times \vec{\mathbf{B}}) \times \vec{\mathbf{B}}]\,,$$

where the last term arises from the Hall effect, and $\eta = \dfrac{c^2}{4\pi\sigma}$, $\eta_{\mathrm{H}} = \eta\sigma_{\mathrm{H}}$.

Question 7. Helical motion

Find the helicity $H = \vec{\mathbf{v}} \cdot (\nabla \times \vec{\mathbf{v}})$, where $\vec{\mathbf{v}}$ is the velocity of motion, for a helical motion whose trajectory is represented in parametric form as $x = \cos\omega t$, $y = \sin\omega t$, $z = ut$, where ω and u are given constants, the angular velocity of rotation in the xy-plane and linear motion along the z-axis, respectively.

Solution

The velocity field is obtained from $x(t), y(t)$ and $z(t)$ by differentiation with respect to t, time: $\vec{\mathbf{v}} = (-\omega\sin\omega t, \omega\cos\omega t, u) = (-\omega y, \omega x, u)$, and then $\nabla \times \vec{\mathbf{v}} = (0, 0, 2\omega)$, so that $H = 2\omega u$.

Question 8. Magnetic field in a shear flow

Consider a constant initial magnetic field, $\vec{\mathbf{B}} = B_0\,\hat{\mathbf{y}}$ at $t = 0$, embedded in a fluid of zero resistivity, so that $\eta = 0$. Solve the induction equation with the velocity field $\vec{\mathbf{V}} = V_0\exp(-y^2/2)\,\hat{\mathbf{x}}$ with the above initial condition to determine how the magnetic field evolves with time.

Integrate equation for magnetic lines, $\dfrac{dx}{B_x} = \dfrac{dy}{B_y} = \dfrac{dz}{B_z}$ to obtain the equation of magnetic lines and sketch the magnetic lines at $t > 0$.

Solution

With the velocity field directed along the x-axis and dependent on y alone, the x- and y-components of the induction equation (with $\eta = 0$) reduce to:

$$\begin{aligned}\frac{\partial B_x}{\partial t} + V_x \frac{\partial B_x}{\partial x} &= SB_y \;, \qquad S = \frac{\partial V_x}{\partial y} \;, &(2)\\ \frac{\partial B_y}{\partial t} + V_x \frac{\partial B_x}{\partial x} &= 0 \;, &(3)\end{aligned}$$

and the z-component implies that $B_z = 0$. With the form of the velocity field given, $V_x = V_0 \exp(-y^2/2)$, we have the shear rate in the form

$$S = -V_0 y \exp(-y^2/2) \; .$$

The characteristic curves of Eqs (2) and (3) in the (t, x)-plane are given by

$$\frac{dx}{dt} = V_x \quad \Rightarrow \quad x = V_x t + \text{const} \;,$$

and we introduce new variables (τ, ξ), where $\tau = t$ and ξ is constant on the characteristics:

$$\tau = t; \qquad \xi = x - V_x t \; .$$

In terms of the new variables, we obtain, using the standard arguments (see any textbook on first-order partial differential equations):

$$\begin{aligned}\frac{\partial B_x}{\partial t} &= \frac{\partial B_x}{\partial \tau}\frac{\partial \tau}{\partial t} + \frac{\partial B_x}{\partial \xi}\frac{\partial \xi}{\partial t} = \frac{\partial B_x}{\partial \tau} - V_x \frac{\partial B_x}{\partial \xi} \;,\\ \frac{\partial B_x}{\partial x} &= \frac{\partial B_x}{\partial \tau}\frac{\partial \tau}{\partial x} + \frac{\partial B_x}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{\partial B_x}{\partial \xi} \; .\end{aligned}$$

Thus, Eqs (2) and (3) reduce to

$$\frac{\partial B_x}{\partial \tau} - V_x \frac{\partial B_x}{\partial \xi} + V_x \frac{\partial B_x}{\partial \xi} = SB_y \quad \Rightarrow \quad \frac{\partial B_x}{\partial \tau} = SB_y \;, \qquad \text{and, likewise,} \qquad \frac{\partial B_y}{\partial \tau} = 0 \; .$$

These equations imply that B_y does not changes along the characteristics, whereas B_x grows along the characteristics at the rate SB_y. The variables (τ, ξ) can be interpreted as coordinates in a reference frame moving together with the fluid; such a frame is called the *Lagrangian frame*, and τ and ξ are called the *Lagrangian coordinates.*

The general solution of the equation for B_y is

$$B_y = g_y(\xi) = f(x - V_x t) \;,$$

where f is an arbitrary function. The meaning of this solution is that B_y is just advected (carried along) by the flow. With the initial condition $B_y|_{t=0} = B_0$, we obtain $f = B_0$ and, thus, the desired solution for B_y follows as

$$B_y = B_0 \; . \tag{4}$$

Now consider equation for B_x whose general solution easily follows as

$$B_x = SB_y\tau + g_x(\xi) = SB_0 t + g(x - V_x t) \;,$$

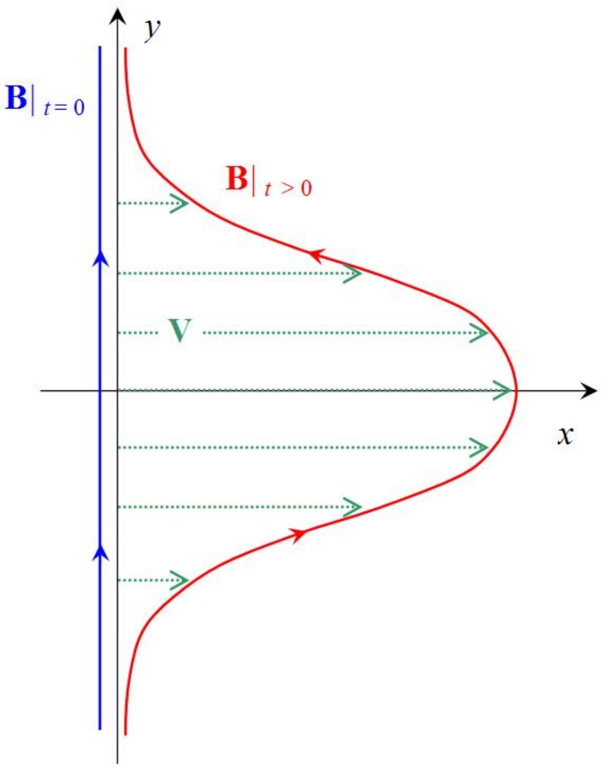

Figure 4: The magnetic lines in Question 2, at $t = 0$ (the vertical line), and that at $t > 0$, a Gaussian curve. The shape of the magnetic lines at $t > 0$ is the same as the profile of $V_x(y)$ indicated with dotted lines: magnetic field is frozen into the flow and follows the elementary volumes of the fluid.

where g is an arbitrary function which is determined from the initial condition $B_x|_{t=0} = 0$ as $g = 0$, so that

$$B_x = SB_0 t = -tV_0 B_0 y \exp(-y^2/2) \,. \tag{5}$$

The magnetic field strength grows with t as

$$B = \sqrt{B_x^2 + B_y^2} = B_0\sqrt{1 + V_0^2 y^2 e^{-y^2} t^2}.$$

We can now determine the form of magnetic lines by solving

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z} \,,$$

where the last ratio with $B_z = 0$ implies that the magnetic lines do not deviate from the (x, y)-plane, i.e., $dz = 0$. From the first two ratios, using Eqs (5) and (4), we obtain

$$\frac{dx}{dy} = \frac{B_x}{B_y} = -V_0 t y e^{-y^2/2} \,,$$

which can easily be integrated for $t = \text{const}$:

$$x = V_0 t e^{-y^2/2} \,,$$

i.e., magnetic lines acquire the shape of a Gaussian curve in the (x, y)-plane similar to the profile of the shear flow, as shown in Fig. 4.

Question 9. Perturbation solutions for $\alpha\omega$-dynamos

The $\alpha\omega$-dynamo in a thin disc is governed by the following equations (written in cylindrical polar coordinates):

$$\begin{aligned}\gamma B_r &= -\frac{d}{dz}(\alpha B_\phi) + \frac{d^2 B_r}{dz^2}\ , &(6)\\ \gamma B_\phi &= D B_r + \frac{d^2 B_\phi}{dz^2}\ , &(7)\end{aligned}$$

with vacuum boundary conditions at the disc surface $|z| = 1$,

$$B_r|_{z=1} = B_\phi|_{z=1} = 0\ ,$$

Derive perturbation solutions of these equations for

$$\alpha = z$$

assuming that $|D| \ll 1$. Specifically, derive an approximate dependence of γ and $\vec{\mathbf{B}}$ on the dynamo number D. Also obtain an approximate expression for the magnetic pitch angle $p = \arctan(B_r/B_\phi)$.

Consider separately solutions of dipolar parity, where the boundary (symmetry) conditions at the disc midplane $z = 0$ are given by

$$B_r|_{z=0} = B_\phi|_{z=0} = 0\ , \tag{8}$$

and those of quadrupolar parity, where

$$\left.\frac{\partial B_r}{\partial z}\right|_{z=0} = \left.\frac{\partial B_\phi}{\partial z}\right|_{z=0} = 0\ . \tag{9}$$

Compare your results for the growth rates γ of the dipolar and quadrupolar solutions to decide which of the two can be generated in galactic discs, given that typically $D \approx -10$ in spiral galaxies.

Solution

In order to apply the perturbation techniques to the system of equations (6) and (7), we have to isolate the unperturbed operator $\widehat{W}$ and the perturbation operator $\widehat{V}$, so that the equations can be written in the form

$$\gamma\vec{\mathbf{B}} = (\widehat{W} + \epsilon\widehat{V})\vec{\mathbf{B}}\ ,$$

where $|\epsilon| \ll 1$ is a small parameter. For this purpose we introduce new variable B'_ϕ such that $B_\phi = B'_\phi |D|^{1/2}$. As discussed in the *Tutorial*, this reduces the governing equations to the form (where we have dropped the dash at B'_ϕ to simplify the notation):

$$\begin{aligned}\gamma B_r &= -|D|^{1/2}\frac{d}{dz}(\alpha B_\phi) + \frac{d^2 B_r}{dz^2}\ ,\\ \gamma B_\phi &= |D|^{1/2}\,\mathrm{sign}\,(D)\, B_r + \frac{d^2 B_\phi}{dz^2}\ ,\end{aligned}$$

which can be rewritten as

$$\gamma\vec{\mathbf{B}} = \left(\widehat{W} + |D|^{1/2}\widehat{V}\right)\vec{\mathbf{B}}\ , \tag{10}$$

$$\widehat{W} = \begin{pmatrix} \dfrac{d^2}{dz^2} & 0 \\ 0 & \dfrac{d^2}{dz^2}\end{pmatrix}, \qquad \widehat{V} = \begin{pmatrix} 0 & -\dfrac{d}{dz}(\alpha\cdot\ldots) \\ \mathrm{sign}\, D & 0\end{pmatrix},$$

so that $\epsilon = |D|^{1/2}$. These equations can be solved using the perturbation techniques based on the idea that equation with $\epsilon = 0$ (known as the unperturbed equation) can be solved more or less easily, and then an approximate solution of the system with $|\epsilon| \ll 1$ can be found in the form of a series expansion over the unperturbed solutions. The number of terms to be retained in the series depends on the required accuracy of the solution.

(1) Dipolar solutions

(a) The unperturbed solution (the free-decay modes of dipolar parity)

We start with solving the *unperturbed* system $\lambda\vec{\mathbf{b}} = \widehat{W}\vec{\mathbf{b}}$ with the same boundary conditions as the desired solution, i.e., for the dipolar modes,

$$\lambda b_r = \frac{d^2 b_r}{dz^2} \,, \qquad b_r(0) = b_r(1) = 0 \,,$$
$$\lambda b_\phi = \frac{d^2 b_\phi}{dz^2} \,, \qquad b_\phi(0) = b_\phi(1) = 0 \,.$$

The equations for b_r and b_ϕ are decoupled and identical to each other, and their general solutions are, e.g. for b_r

$$b_r = A\cos(z\sqrt{-\lambda}) + C\sin(z\sqrt{-\lambda}) \,,$$

where A and C are constants. The boundary conditions now yield

$$b_r(0) = 0 \quad \Rightarrow \quad A = 0 \,, \qquad b_r(1) = 0 \quad \Rightarrow \quad \sqrt{-\lambda} = \pi n, \;\; n = 1, 2, \ldots \,,$$

where we exclude $n = 0$ to have a nontrivial solution for b_r. Thus, we have found the eigenvalues of the unperturbed system as

$$\lambda_n = -\pi^2 n^2, \quad n = 1, 2, 3 \ldots \,. \tag{11}$$

The corresponding solution for b_r is $b_r = C_n \sin(\pi n z)$. Since the equation for b_ϕ (as well as for b_r) does admit the trivial solution, one admissible solution of the unperturbed system is given by

$$\mathbf{b}_n = C_n \begin{pmatrix} \sin(\pi n z) \\ 0 \end{pmatrix} .$$

In a similar manner we obtain the other, linearly independent, unperturbed eigenfunction corresponding to the same eigenvalue:

$$\mathbf{b}'_n = C'_n \begin{pmatrix} 0 \\ \sin(\pi n z) \end{pmatrix} .$$

It is convenient to normalize the eigenfunctions, i.e., to fix the constants C_n and C'_n so that

$$\int_0^1 |\vec{\mathbf{b}}_n|^2 \, dz = \int_0^1 |\vec{\mathbf{b}}'_n|^2 \, dz = 1 \,.$$

For example,

$$\int_0^1 |\vec{\mathbf{b}}_n|^2 \, dz = C_n^2 \int_0^1 \sin^2(\pi n z) \, dz = C_n^2 \tfrac{1}{2} \int_0^1 [1 - \cos(2\pi n z)] \, dz = \tfrac{1}{2} C_n^2 \,,$$

and likewise for $\vec{\mathbf{b}}'_n$. Thus, the normalization condition yields $C_n = C'_n = \sqrt{2}$ and the eigenfunctions normalized to unity are given by

$$\mathbf{b}_n = \begin{pmatrix} \sqrt{2}\sin(\pi n z) \\ 0 \end{pmatrix}, \qquad \mathbf{b}'_n = \begin{pmatrix} 0 \\ \sqrt{2}\sin(\pi n z) \end{pmatrix} . \tag{12}$$

It can be verified by direct calculation that

$$\int_0^1 \vec{\mathbf{b}}_n \cdot \vec{\mathbf{b}}_m \, dz = 0 \quad \text{for } n \neq m \;, \qquad \int_0^1 \vec{\mathbf{b}}'_n \cdot \vec{\mathbf{b}}'_m \, dz = 0 \quad \text{for } n \neq m \;, \qquad \vec{\mathbf{b}}_n \cdot \vec{\mathbf{b}}_m = 0 \quad \text{for any } n, m \;,$$

i.e., each family of eigenfunctions represents an orthonormal set of functions, and the two families are orthogonal to each other.

(b) The perturbed solution

Solution of Eqs (6) and (7) can be represented in the form

$$\vec{\mathbf{B}} = \sum_{n=1}^{\infty} (C_n \vec{\mathbf{b}}_n + C'_n \vec{\mathbf{b}}'_n)$$

with certain constants C_n and C'_n, where two terms are included for each n since the unperturbed solution is doubly degenerate (i.e., two eigenfunctions correspond to each eigenvalue). Here we restrict ourselves to the lowest approximation retaining the minimum number of modes in the expansion,

$$\mathbf{B} \approx C\mathbf{b}_1 + C'\mathbf{b}'_1 \;,$$

where

$$\mathbf{b}_1 = \begin{pmatrix} \sqrt{2}\sin\pi z \\ 0 \end{pmatrix}, \qquad \mathbf{b}'_1 = \begin{pmatrix} 0 \\ \sqrt{2}\sin\pi z \end{pmatrix}.$$

Substitute the expansion into the governing equation (10) and use $\widehat{W}\mathbf{b}_1 = \lambda_1 \mathbf{b}_1$ and $\widehat{W}\mathbf{b}'_1 = \lambda_1 \mathbf{b}'_1$ (we replace the sign of approximate equality by $=$):

$$C(\gamma - \lambda_1)\mathbf{b}_1 + C'(\gamma - \lambda_1)\mathbf{b}'_1 = |D|^{1/2}[C\widehat{V}\mathbf{b}_1 + C'\widehat{V}\mathbf{b}'_1] \;. \tag{13}$$

Scalar multiplication with $\mathbf{b}_0$ and integration over z then yields, upon using the orthogonality and normalization conditions:

$$C(\gamma - \lambda_1) = D|^{1/2}(CV_{11} + C'V_{11'}) \;,$$

where the matrix elements of the perturbation operator are given by

$$V_{11} = \int_0^1 \mathbf{b}_1 \cdot \widehat{V}\mathbf{b}_1 \, dz \;, \qquad V_{11'} = \int_0^1 \mathbf{b}_1 \cdot \widehat{V}\mathbf{b}'_1 \, dz \;.$$

Further,

$$\begin{aligned} \widehat{V}\mathbf{b}_1 &= \begin{pmatrix} 0 & -\dfrac{d}{dz}(\alpha \cdot \ldots) \\ \operatorname{sign} D & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2}\sin\pi z \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \sqrt{2}\,\operatorname{sign}(D)\sin(\pi z) \end{pmatrix}, \end{aligned}$$

and hence $\mathbf{b}_1 \cdot \widehat{V}\mathbf{b}_1 = 0 \quad \Rightarrow \quad V_{11} = 0.$

The other matrix element is calculated as follows:

$$\begin{aligned} \widehat{V}\mathbf{b}'_1 &= \begin{pmatrix} 0 & -\dfrac{d}{dz}(\alpha \cdot \ldots) \\ \operatorname{sign}(D) & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{2}\sin\pi z \end{pmatrix} \\ &= \begin{pmatrix} -\sqrt{2}\dfrac{d}{dz}\alpha\sin(\pi z) \\ 0 \end{pmatrix}, \end{aligned}$$

and hence

$$\begin{aligned}
\mathbf{b}_1 \cdot \widehat{V}\mathbf{b}_1' &= -2\sin(\pi z)\frac{d}{dz}[\alpha\sin(\pi z)] \qquad (14)\\
&= -2\sin(\pi z)\frac{d}{dz}[z\sin(\pi z)]\\
&= -2\sin(\pi z)[\sin(\pi z)+\pi z\cos(\pi z)]\\
&= -2\sin^2(\pi z) - 2\pi z\sin(\pi z)\cos(\pi z)\\
&= -[1-\cos(2\pi z)] - \pi z\sin(2\pi z)\\
&= -1+\cos 2\pi z - \pi z\sin\pi z\ .
\end{aligned}$$

Now we are ready to integrate:

$$\begin{aligned}
V_{11'} = \int_0^1 \mathbf{b}_1\cdot\widehat{V}\mathbf{b}_1'\,dz &= \int_0^1[-1+\cos(2\pi z)-\pi z\sin(\pi z)]\,dz\\
&= -1+\frac{1}{2\pi}\sin(2\pi z)\Big|_0^1 - \pi\int_0^1 z\sin(2\pi z)\,dz\\
&= -1+\pi\frac{1}{2\pi}\int_0^1 z\,d(\cos 2\pi z) \quad \text{(integrating by parts)}\\
&= -1+\tfrac{1}{2}\left[z\cos(2\pi z)|_0^1 - \int_0^1\cos(2\pi z)\,dz\right]\\
&= -1+\tfrac{1}{2}\left[1-\frac{1}{2\pi}\sin(2\pi z)\Big|_0^1\right]\\
&= -\tfrac{1}{2}\ .
\end{aligned}$$

This yields

$$C(\gamma-\lambda_1) = -\tfrac{1}{2}|D|^{1/2}C' \quad\Rightarrow\quad C(\gamma-\lambda_1)+C'\tfrac{1}{2}|D|^{1/2}=0\ .$$

Now, scalar multiplication of Eq. (13) with $\mathbf{b}_1'$ and integration over z yield the other algebraic equation for C and C':

$$C'(\gamma-\lambda_1) = |D|^{1/2}(CV_{1'1}+C'V_{1'1'})\ ,$$

where

$$V_{1'1} = \int_0^1 \mathbf{b}_1'\cdot\widehat{V}\mathbf{b}_1\,dz\ ,\qquad V_{1'1'} = \int_0^1 \mathbf{b}_1'\cdot\widehat{V}\mathbf{b}_1'\,dz\ .$$

As above, it can easily be shown that

$$V_{1'1'}\equiv 0$$

and

$$\begin{aligned}
\widehat{V}\mathbf{b}_1 &= \begin{pmatrix} 0 & -\dfrac{d}{dz}(\alpha\cdot\ldots)\\ \operatorname{sign}(D) & 0\end{pmatrix}\begin{pmatrix}\sqrt{2}\sin\pi z\\ 0\end{pmatrix}\\
&= \begin{pmatrix}0\\ \sqrt{2}\operatorname{sign}(D)\sin(\pi z)\end{pmatrix},
\end{aligned}$$

and therefore

$$\begin{aligned}
\mathbf{b}_1'\cdot\widehat{V}\mathbf{b}_1 &= 2\operatorname{sign}(D)\sin^2(\pi z)\\
&= \operatorname{sign}(D)[1-\cos(2\pi z)]\ ,
\end{aligned}$$

and

$$V_{1'1} = \operatorname{sign}(D)\int_0^1[1-\cos(2\pi z)]\,dz = \operatorname{sign}(D)\ .$$

This yields

$$C'(\gamma-\lambda_1)=|D|^{1/2}\mathrm{sign}\,(D)C \quad \Rightarrow \quad -C|D|^{1/2}\mathrm{sign}\,(D)+C'(\gamma-\lambda_1)=0\ .$$

The resulting system of equations for C and C' is given by

$$\begin{aligned} C(\gamma-\lambda_1)+C'\tfrac{1}{2}|D|^{1/2} &= 0\ , \\ -C|D|^{1/2}\mathrm{sign}\,(D)+C'(\gamma-\lambda_1) &= 0\ , \end{aligned}$$

and it has nontrivial solutions if its determinant vanishes:

$$\begin{vmatrix} \gamma-\lambda_1 & \tfrac{1}{2}|D|^{1/2} \\ -|D|^{1/2}\mathrm{sign}\,(D) & \gamma-\lambda_1 \end{vmatrix}=0 \quad \Rightarrow \quad (\gamma-\lambda_1)^2+\tfrac{1}{2}D=0\ ,$$

since $|D|\mathrm{sign}\,(D)\equiv D$. Thus,

$$\gamma=\lambda_1\pm\sqrt{-\tfrac{1}{2}D}\ .$$

Since $\lambda_1=-\pi^2<0$, growing solutions, i.e., those with $\gamma>0$, only occur if we choose the upper sign (and we require that $-D>0$, which is generally true in spiral galaxies and accretion discs):

$$\gamma\approx-\pi^2+\sqrt{-\tfrac{1}{2}D}\ ,$$

where we have restored the approximate equality sign as appropriate.

Growing solutions are possible if $|D|$ is large enough, i.e., if the dynamo action is strong enough:

$$\gamma>0 \quad \Rightarrow \quad D<-2\pi^4\approx-195\ . \tag{15}$$

The spatial form of the solution is clarified as follows. The first of the algebraic equations for C and C' gives

$$C(\gamma-\lambda_1)+C'\tfrac{1}{2}|D|^{1/2}=0 \quad \Rightarrow \quad C'=-C\frac{\gamma-\lambda_1}{\tfrac{1}{2}|D|^{1/2}}=-C\frac{\sqrt{-D/2}}{\tfrac{1}{2}\sqrt{-D}}=-\sqrt{2}C\ ,$$

so that

$$\mathbf{B}\approx C\vec{\mathbf{b}}_1+C'\vec{\mathbf{b}}_1'=C\sqrt{2}\begin{pmatrix}\sin\pi z\\ -\sqrt{2}\sin\pi z\end{pmatrix}=C\sqrt{2}\begin{pmatrix}1\\ -\sqrt{2}\end{pmatrix}\sin(\pi z)\ .$$

Restoring the original field components $B_r\to R_\alpha B_r$ and $B_\phi\to|D|^{1/2}B_\phi$, we obtain

$$\vec{\mathbf{B}}\approx C\sqrt{2}\begin{pmatrix}R_\alpha\\ -\sqrt{2}|D|^{1/2}\end{pmatrix}\sin(\pi z)\ .$$

Thus (note that $D=R_\alpha R_\omega$)

$$\frac{B_r}{B_\phi}\approx-\frac{1}{\sqrt{2}}\frac{R_\alpha}{|D|^{1/2}}=-\frac{1}{\sqrt{2}}\sqrt{\frac{R_\alpha}{R_\omega}}\ ,$$

so that the magnetic pitch angle follows, for $R_\alpha=0.6$ and $R_\omega=-15$, as

$$p=\arctan\frac{B_r}{B_\phi}\approx-\frac{1}{\sqrt{2}}\sqrt{\frac{R_\alpha}{|R_\omega|}}\approx-8^\circ\ .$$

(2) Quadrupolar solutions

The idea and many details of the solution for quadrupolar modes are exactly as for the dipolar modes. The main difference is that the boundary conditions at $z = 0$ differ from those above, which affects the unperturbed solution.

(a) The unperturbed solution (the free-decay modes of quadrupolar parity)

Equations for b_r and b_ϕ remain the same, but the boundary conditions at $z = 0$ are given by (9):

$$\lambda b_r = \frac{d^2 b_r}{dz^2}\ , \qquad \left.\frac{db_r}{dz}\right|_{z=0} = b_r(1) = 0\ ,$$
$$\lambda b_\phi = \frac{d^2 b_\phi}{dz^2}\ , \qquad \left.\frac{db_\phi}{dz}\right|_{z=0} = b_\phi(1) = 0\ .$$

Proceed as for the dipolar modes:

$$b_r = C\cos(z\sqrt{-\lambda}) + A\sin(z\sqrt{-\lambda})\ , \qquad \frac{db_r}{dz} = -C\sqrt{-\lambda}\sin(z\sqrt{-\lambda}) + A\sqrt{-\lambda}\cos(z\sqrt{-\lambda})\ ,$$

$$\left.\frac{db_r}{dz}\right|_{z=0} = 0 \quad\Rightarrow\quad A = 0\ , \qquad b_r(1) = 0 \quad\Rightarrow\quad \sqrt{-\lambda} = \tfrac{1}{2} + \pi n,\ \ n = 0, 1, 2, \dots\ ,$$

where $n = 0$ is allowed. Thus,

$$\lambda_n = -\pi^2(n + \tfrac{1}{2})^2, \quad n = 0, 1, 2, 3\dots\ . \tag{16}$$

The eigenfunctions are again doubly degenerate; being normalized to unity, they are given by:

$$\mathbf{b}_n = \begin{pmatrix} \sqrt{2}\cos[\pi(n + \frac{1}{2})z] \\ 0 \end{pmatrix}\ , \qquad \mathbf{b}'_n = \begin{pmatrix} 0 \\ \sqrt{2}\cos[\pi(n + \frac{1}{2})z] \end{pmatrix}. \tag{17}$$

As for the dipolar modes, each family of eigenfunctions represents an orthonormal set of functions, and the two families are orthogonal to each other.

(b) The perturbed solution

Retaining the minimum admissible number of modes in the eigenfunction expansion,

$$\mathbf{B} \approx C\mathbf{b}_0 + C'\mathbf{b}'_0\ ,$$

proceed exactly as above to show that

$$C(\gamma - \lambda_0) = D|^{1/2}(CV_{00} + C'V_{00'})\ ,$$

where the matrix elements of the perturbation operator are given by

$$V_{00} = \int_0^1 \mathbf{b}_0 \cdot \widehat{V}\mathbf{b}_0\, dz\ , \qquad V_{00'} = \int_0^1 \mathbf{b}_0 \cdot \widehat{V}\mathbf{b}'_0\, dz\ .$$

As shown in the *Tutorial,* $V_{00} = 0$.

Further,

$$\begin{aligned}\widehat{V}\mathbf{b}'_0 &= \begin{pmatrix} 0 & -\dfrac{d}{dz}(\alpha\cdot\ldots) \\ \operatorname{sign} D & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \sqrt{2}\cos(\pi z/2) \end{pmatrix} \\ &= \begin{pmatrix} -\sqrt{2}\dfrac{d}{dz}[\alpha\cos(\pi z/2)] \\ 0 \end{pmatrix},\end{aligned}$$

and hence

$$\begin{aligned}
\mathbf{b}_0 \cdot \widehat{V}\mathbf{b}_0' &= -2\cos(\pi z/2)\frac{d}{dz}[\alpha\cos(\pi z/2)] \qquad (18) \\
&= -2\cos(\pi z/2)\frac{d}{dz}[z\cos(\pi z/2)] \\
&= -2\cos(\pi z/2)[\cos(\pi z/2) - \tfrac{1}{2}\pi z\sin(\pi z/2)] \\
&= -2\cos^2(\pi z/2) + \pi z\sin(\pi z/2)\cos(\pi z/2) \\
&= -[1+\cos(\pi z)] + \tfrac{1}{2}\pi z\sin(\pi z)\ ,
\end{aligned}$$

so the matrix element follows as

$$\begin{aligned}
V_{00'} = \int_0^1 \mathbf{b}_0 \cdot \widehat{V}\mathbf{b}_0'\,dz &= \int_0^1 [-1-\cos(\pi z) + \tfrac{1}{2}\pi z\sin(\pi z)]\,dz \\
&= -1 - \tfrac{1}{2}\int_0^1 z\,d(\cos\pi z) \quad \text{(integrating by parts)} \\
&= -1 - \tfrac{1}{2}\left[z\cos(\pi z)|_0^1 - \int_0^1 \cos(\pi z)\,dz\right] \\
&= -1 - \tfrac{1}{2}\left[-1 - \frac{1}{\pi}\sin(\pi z)\Big|_0^1\right] \qquad (19) \\
&= -\tfrac{1}{2}\ . \qquad (20)
\end{aligned}$$

This yields

$$C(\gamma-\lambda_0) = -\tfrac{1}{2}|D|^{1/2}C' \quad \Rightarrow \quad C(\gamma-\lambda_0) + C'\tfrac{1}{2}|D|^{1/2} = 0\ .$$

As before, the other algebraic equation for C and C' follows as

$$C'(\gamma-\lambda_0) = |D|^{1/2}(CV_{0'0} + C'V_{0'0'})\ ,$$

where

$$V_{0'0} = \int_0^1 \mathbf{b}_0' \cdot \widehat{V}\mathbf{b}_0\,dz\ , \qquad V_{0'0'} = \int_0^1 \mathbf{b}_0' \cdot \widehat{V}\mathbf{b}_0'\,dz\ .$$

As above,

$$V_{0'0'} \equiv 0$$

and

$$\begin{aligned}
\widehat{V}\mathbf{b}_0 &= \begin{pmatrix} 0 & -\frac{d}{dz}(\alpha\cdot\ldots) \\ \mathrm{sign}\,(D) & 0 \end{pmatrix}\begin{pmatrix} \sqrt{2}\cos(\pi z/2) \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \sqrt{2}\,\mathrm{sign}\,(D)\cos(\pi z/2) \end{pmatrix},
\end{aligned}$$

and thus

$$\mathbf{b}_0' \cdot \widehat{V}\mathbf{b}_0 = 2\,\mathrm{sign}\,(D)\cos^2(\pi z/2) \quad \Rightarrow \quad V_{0'0} = \mathrm{sign}\,(D)\ .$$

This yields

$$-C|D|^{1/2}\mathrm{sign}\,(D) + C'(\gamma-\lambda_0) = 0\ .$$

The resulting system of equations for C and C' has the same form as for the dipolar solutions, but now with a different λ:

$$\begin{aligned}
C(\gamma-\lambda_0) + C'\tfrac{1}{2}|D|^{1/2} &= 0\ , \\
-C|D|^{1/2}\mathrm{sign}\,(D) + C'(\gamma-\lambda_0) &= 0\ ,
\end{aligned}$$

and it has nontrivial solutions if its determinant vanishes:

$$(\gamma - \lambda_0)^2 + \tfrac{1}{2}D = 0 \ ,$$

so a growing solution has

$$\gamma \approx \lambda_0 + \sqrt{-\tfrac{1}{2}D} = -\tfrac{1}{4}\pi^2 + \sqrt{-\tfrac{1}{2}D} \ ,$$

and

$$\gamma > 0 \quad \Rightarrow \quad D < -\tfrac{1}{8}\pi^4 \approx -12 \ . \tag{21}$$

This is useful to compare with the condition for the dipolar modes to grow, $D < -195$, Eq. (15). It is clear that dipolar modes require much stronger dynamo action. Since dynamo numbers typical of spiral galaxies are about -10, we can conclude that quadrupolar modes should dominate in spiral galaxies. This conclusion is confirmed by detailed observational analysis on the large-scale magnetic field in the Milky Way.

The spatial form of the solution:

$$C' = -C\frac{\gamma - \lambda_0}{\frac{1}{2}|D|^{1/2}} = -\sqrt{2}C$$

as for the dipolar modes. This results in the same magnetic pitch angle as for the dipolar modes:

$$p = \arctan\frac{B_r}{B_\phi} \approx -\frac{1}{\sqrt{2}}\sqrt{\frac{R_\alpha}{|R_\omega|}} \approx -8^\circ \ .$$

Question 10. Perturbation solution for α^2-dynamos

The α^2-dynamo in a thin disc is governed by the following equations (written in cylindrical polar coordinates):

$$\gamma B_r = -R_\alpha\frac{d}{dz}(\alpha B_\phi) + \frac{d^2 B_r}{dz^2} \ , \tag{22}$$

$$\gamma B_\phi = R_\alpha\frac{d}{dz}(\alpha B_r) + \frac{d^2 B_\phi}{dz^2} \ , \tag{23}$$

with vacuum boundary conditions at the disc surface $|z| = 1$,

$$B_r|_{z=1} = B_\phi|_{z=1} = 0 \ ,$$

1. Derive perturbation solutions of these equations for

   $$\alpha = z$$

   assuming that $|R_\alpha| \ll 1$. Specifically, derive an approximate dependence of the growth rate γ and $\vec{\mathbf{B}}$ on R_α.

2. For the approximate solution obtained, $\mathcal{B} = e^{\gamma t}\,\mathbf{B}$, derive the physical components of magnetic field by isolating the real part of $\mathcal{B}$ and interpret the result in terms of the temporal variation of magnetic field.

Consider solutions of quadrupolar parity, where the boundary (symmetry) conditions at the disc midplane $z = 0$ are given by

$$\left.\frac{\partial B_r}{\partial z}\right|_{z=0} = \left.\frac{\partial B_\phi}{\partial z}\right|_{z=0} = 0 \; . \tag{24}$$

Solution

First, we write the governing equations in the matrix operator form to isolate the unperturbed operator and the perturbation:

$$\gamma\vec{\mathbf{B}} = (\widehat{W} + R_\alpha\widehat{V})\vec{\mathbf{B}} \; , \tag{25}$$

with

$$\widehat{W} = \begin{pmatrix} \dfrac{d^2}{dz^2} & 0 \\ 0 & \dfrac{d^2}{dz^2} \end{pmatrix}, \qquad \widehat{V} = \begin{pmatrix} 0 & -\dfrac{d}{dz}(\alpha\cdot\ldots) \\ \dfrac{d}{dz}(\alpha\cdot\ldots) & 0 \end{pmatrix}. \tag{26}$$

The unperturbed eigenvalues and eigenfunctions are exactly as in Question 9, as given in Eqs (11) and (12) for the dipolar modes, and Eqs (16) and (17) for the quadrupolar modes. Thus, we can proceed to the perturbed solution:

$$\mathbf{B} = C\mathbf{b}_0 + C'\mathbf{b}_0' \; ,$$

where

$$\mathbf{b}_0 = \begin{pmatrix} \sqrt{2}\cos\pi z/2 \\ 0 \end{pmatrix}, \qquad \mathbf{b}_0' = \begin{pmatrix} 0 \\ \sqrt{2}\cos\pi z/2 \end{pmatrix}.$$

Substitute the expansion into the governing equations and use $\widehat{W}\mathbf{b}_0 = \lambda_0\mathbf{b}_0$ and $\widehat{W}\mathbf{b}_0' = \lambda_0\mathbf{b}_0'$:

$$C(\gamma-\lambda_0)\mathbf{b}_0 + C'(\gamma-\lambda_0)\mathbf{b}_0' = R_\alpha C\widehat{V}\mathbf{b}_0 + R_\alpha C'\widehat{V}\mathbf{b}_0' \; .$$

Scalar multiplication with $\mathbf{b}_0$ and integration over z then yields, upon noting that the eigenfunctions are normalized to unity and $\mathbf{b}_n\cdot\mathbf{b}_n' = 0$:

$$C(\gamma-\lambda_0) = R_\alpha(CV_{00} + C'V_{00'}) \; ,$$

where the matrix elements of the perturbation operator are given by

$$V_{00} = \int_0^1 \mathbf{b}_0\cdot\widehat{V}\mathbf{b}_0\,dz \; , \qquad V_{00'} = \int_0^1 \mathbf{b}_0\cdot\widehat{V}\mathbf{b}_0'\,dz \; .$$

Further,

$$\begin{aligned}\widehat{V}\mathbf{b}_0 &= \begin{pmatrix} 0 & -\dfrac{d}{dz}(\alpha\cdot\ldots) \\ \dfrac{d}{dz}(\alpha\cdot\ldots) & 0 \end{pmatrix}\begin{pmatrix} \sqrt{2}\cos\pi z/2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \sqrt{2}\dfrac{d}{dz}[\alpha\cos(\pi z/2)] \end{pmatrix},\end{aligned}$$

and hence $V_{00} = \mathbf{b}_0\cdot\widehat{V}\mathbf{b}_0 = 0$.

The other matrix element:

$$\begin{aligned}\widehat{V}\mathbf{b}_0' &= \begin{pmatrix} 0 & -\dfrac{d}{dz}(\alpha\cdot\ldots) \\ \dfrac{d}{dz}(\alpha\cdot\ldots) & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \sqrt{2}\cos\pi z/2 \end{pmatrix} \\ &= \begin{pmatrix} -\sqrt{2}\dfrac{d}{dz}[\alpha\cos(\pi z/2)] \\ 0 \end{pmatrix},\end{aligned}$$

and hence
$$\mathbf{b}_0 \cdot \widehat{V}\mathbf{b}_0' = -2\cos(\pi z/2)\frac{d}{dz}[z\cos(\pi z/2)] \ .$$
This is the same expression as in Eq. (18), and so we obtain the same result as in Eq. (20):
$$V_{00'} = -\tfrac{1}{2} \ .$$

This yields
$$C(\gamma - \lambda_0) + \tfrac{1}{2}R_\alpha C' = 0 \ .$$
Now, the other algebraic equation for C and C' follows as
$$C'(\gamma - \lambda_0) = R_\alpha(CV_{0'0} + C'V_{0'0'}) \ ,$$
As above,
$$V_{0'0'} \equiv 0$$
and
$$V_{0'0} = \int_0^1 2\cos(\pi z/2)\frac{d}{dz}[z\cos(\pi z/2)]\,dz = -V_{00'} = \tfrac{1}{2} \ .$$
The resulting system of equations for C and C' is given by
$$C(\gamma - \lambda_0) + C'\tfrac{1}{2}R_\alpha = 0 \ , \qquad -C\tfrac{1}{2}R_\alpha + C'(\gamma - \lambda_0) = 0 \ .$$
The solvability condition is the vanishing of the determinant of the system, which yields
$$(\gamma - \lambda_0)^2 + \left(\tfrac{1}{2}R_\alpha\right)^2 = 0 \ \Rightarrow \ \gamma - \lambda_0 = \pm i\tfrac{1}{2}R \ \Rightarrow \ \gamma = -\tfrac{1}{4} \pm i\tfrac{1}{2}R_\alpha \ . \tag{27}$$
The first of the algebraic equations then gives
$$\pm Ci\tfrac{1}{2}R_\alpha + \tfrac{1}{2}R_\alpha C' = 0 \quad \Rightarrow \quad C' = \mp iC \ ,$$
and hence
$$\mathbf{B} \approx C\begin{pmatrix} 1 \\ \mp i \end{pmatrix}\cos\left(\frac{\pi}{2}z\right) \ . \tag{28}$$
To complete the solution, we isolate the real part of the magnetic field, including both the temporal and spatial dependencies. Since
$$e^{\pm iR_\alpha t/2} = \cos\left(R_\alpha t/2\right) \pm i\sin\left(R_\alpha t/2\right) \ ,$$
we obtain from Eqs (28) and (27)
$$\mathcal{B} = e^{\gamma t}\,\mathbf{B} = C\cos\left(\frac{\pi}{2}z\right)e^{-t\pi^2/4}\left[\cos\left(\tfrac{1}{2}R_\alpha t\right) \pm i\sin\left(\tfrac{1}{2}R_\alpha t\right)\right]\begin{pmatrix} 1 \\ \mp i \end{pmatrix},$$
so that
$$\mathrm{Re}\,\mathcal{B}_r = C\cos\left(\frac{\pi}{2}z\right)e^{-t\pi^2/4}\cos\left(\tfrac{1}{2}R_\alpha t\right) \ ,$$
$$\mathrm{Re}\,\mathcal{B}_\phi = C\cos\left(\frac{\pi}{2}z\right)e^{-t\pi^2/4}\sin\left(\tfrac{1}{2}R_\alpha t\right) \ .$$
The magnetic field exhibits decaying oscillations and represents a standing helical wave. For $t = 0$, $\mathrm{Re}\,\mathcal{B}_r = \max$ for a given z, whereas $\mathrm{Re}\,\mathcal{B}_\phi = 0$, so the oscillations in the two components have a phase shift. This means that the magnetic field vector rotates with time (a standing helical wave).

AS **28/4/2009**