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# School on Astrophysical Turbulence and Dynamos

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## Tutorial 1: Basic Fluid Dynamics

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# Random Processes

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- A random process is one whose outcome does not seem to follow a deterministic pattern. Of course, it could well be that the underlying laws are deterministic, and even simple to state. However, our knowledge of the parameters of the system is often limited, and if the system is sufficiently sensitive to those parameters, it virtually becomes impossible to predict the outcome of the process in a deterministic manner.

- There is a class of random processes that follow statistical determinism. What it means is that although we are not be able to predict the outcome of a given experiment, we are still able to say something about the relative frequencies of various possible outcomes in a probabilistic manner.

- Let us imagine a process that has discrete outcomes that belong to a set $U$, that is $X_i \in U \equiv \{X_1, X_2, X_3, \ldots\}$. By repeatedly drawing outcomes we can construct frequencies $p_i$ through

$$p_i = \lim_{N\to\infty} \frac{N_{X_i}}{N} \tag{1}$$

  If the limit exists and is well defined then $p_i$ is called the probability of obtaining $X_i$ and $P = \{p_i\}$ is called the probability distribution function (PDF)

  Examples: Throwing a dice, drawing cards randomly from a deck, tossing a coin, etc.

- The expectation value of a function $F$ of $X_i$ is defined as

$$\langle F \rangle = \sum_i p_i F(X_i) \tag{2}$$

- From a logical point of view it is useful to define the concept of an *ensemble* of identically prepared systems. The limit over $N$ could then be thought of as limit over the ensemble. This makes it possible to talk about the time evolution of the PDF.

- If the PDF for a system is invariant with respect to time then the process is called stationary. For stationary processes limit over ensemble is identical to the limit taken in time. As an example of a non-stationary process consider a dice that accumulates dirt and grime unevenly over a period of time. In this case its PDF might evolve with time.
- An ensemble is a useful theoretical device, however, often it is impossible to attain in practice. The assumption of stationarity then makes it possible to estimate the PDFs for such systems. In other cases there might be theoretical reasons to believe that the distribution has a certain form. In such cases the PDF is a theoretical model for the random process and has to be verified by experiments.

Continuous Processes

- A continuous random process $X$ is one whose outcome is not discrete. In this case the PDF $P(X)$ is a probability density in the following manner. The probability of obtaining an outcome in the range $X$ and $X + dX$ is given by
$$dp = P(X)dX \tag{3}$$
- By an obvious generalization, the expectation value of a function $F(X)$ is
$$\langle F \rangle = \int P(X)F(X)dX \tag{4}$$
In particular the expectation value of $F(X) = X$, called the mean, is
$$\mu = \langle X \rangle = \int XP(X)dX \tag{5}$$
A random process that has zero mean is called centered. The variance of random variable $X$ is defined as
$$\sigma^2 = \langle (X - \mu)^2 \rangle = \int (X - \mu)^2 P(X)dX \tag{6}$$
and $\sigma$ is known as the standard deviation.

- The mean and variance are the most commonly used statistical descriptors of random processes. In general we define the $m$th moment of the random variable $X$ as

$$\langle X^m \rangle = \int X^m P(X) dX \tag{7}$$

It is easy to show that $\sigma^2 = \langle X^2 \rangle - \langle X \rangle^2$

- The characteristic function of $P(X)$ is defined as the expectation value of $\exp(iZX)$

$$K(Z) = \langle \exp(iZX) \rangle = \int \exp(iZX) P(X) dX \tag{8}$$

The characteristic function is the Fourier transform of the PDF. Since the PDF is normalized, we see that the characteristic function always exist. However, the moments of the random process (derived below) may not exist.

- It is easy to see that

$$\langle X^m \rangle = \left(\frac{1}{i}\right)^m \frac{d}{dZ} K(Z)\Big|_{Z=0} \tag{9}$$

- A centered Gaussian random process is one for which

$$K(Z) = \exp\left(-\frac{1}{2}\sigma^2 Z^2\right) \tag{10}$$

The PDF for a centered Gaussian process is easily seen to be

$$P(X) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{X^2}{2\sigma^2}\right) \tag{11}$$

Multivariate random processes

- It the outcome of a random process is not a scalar but a vector $\boldsymbol{X} = \{X_1, X_2, X_3, \ldots\}$, where $X_i$s are continuous variables, then it is called a multivariate random process.

- A multivariate random process is Gaussian if for any arbitrary vector $\mathbf{Z}$, $\mathbf{Z}\cdot\mathbf{X}$ is a Gaussian random variable.

- The characteristic function is given by

$$K(\mathbf{Z}) = \langle \exp\left(i\,\mathbf{Z}\cdot\mathbf{X}\right)\rangle = \exp\left(-\frac{1}{2}\langle\mathbf{Z}\cdot\mathbf{X}\rangle^2\right) = \exp\left(-\frac{1}{2}Z_iZ_j\langle X_iX_j\rangle\right) \tag{12}$$

- Defining $\sigma_{ij} = \langle X_iX_j\rangle$ as the covariance tensor, we find that the characteristic function of a multivariate random process is completely specified by the covariance tensor. In fact, all the higher moments of a Gaussian random field can be derived from the second moment.

Random Fields

- Till now we have talked of only scalar and vector random processes where the outcome of an experiment is either a scalar or a vector. In general random processes could be functions of space as well as time.

- Let us first consider a random scalar field $\Phi(\mathbf{x}, t)$. What it means is that at a given space point, $\Phi$ is a random variable that has different values are different times. And at a fixed time it has different values at different space points.

- The PDF for such a process is a more complicated object called a probability functional that assigns probability for the process returning a value between $\Phi(\mathbf{x}, t)$ and $\Phi(\mathbf{x}, t) + d\Phi(\mathbf{x}, t)$. As we have learnt, for Gaussian random processes all the information is encoded in the two point correlation function that we had earlier called the covariance tensor; which in this case is defined as

$$\xi(\mathbf{x}, \mathbf{x}', t, t') = \langle\Phi(\mathbf{x}, t)\Phi(\mathbf{x}', t')\rangle \tag{13}$$

- If the correlation function depends only on the difference $\mathbf{x} - \mathbf{x}'$ then the process is called homogenous, similarly, if the correlation function depends only on the difference in time $t - t'$ then the process is called stationary. For a moment let us imagine a random process that is independent of time. Then for a homogenous process

$$\xi(\mathbf{x} - \mathbf{x}') = \langle \Phi(\mathbf{x})\Phi(\mathbf{x}') \rangle \tag{14}$$

- Furthermore, if the correlation function depends only on the length $r = |\mathbf{x} - \mathbf{x}'|$ then the process is called isotropic. Thus, for an isotropic process

$$\xi(|\mathbf{x} - \mathbf{x}'|) = \langle \Phi(\mathbf{x})\Phi(\mathbf{x}') \rangle \tag{15}$$

- Power spectrum: Consider the Fourier transform of $\Phi(\mathbf{x})$

$$\Phi_{\mathbf{k}} = \int_{R^3} \Phi(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} \, d^3r \tag{16}$$

and the inverse transform as

$$\Phi(\mathbf{x}) = \frac{1}{(2\pi)^3} \int_{R^3} \Phi_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} \, d^3k \tag{17}$$

We should note here that for a homogenous random field, which extends over infinite space, the Fourier transform does not exist. We must assume a finite support for the process. However, taking the volume to be large our formal results, derived below through the Fourier transform, still hold. For a finite volume Fourier integral is replaced with the Fourier series leading to similar results.

- Let us consider the quantity

$$\begin{aligned} \langle \Phi_{\mathbf{k}} \Phi^{\star}_{\mathbf{k}'} \rangle &= \int_{R^3}\int_{R^3} \langle \Phi(\mathbf{x})\Phi(\mathbf{x}') \rangle e^{i(\mathbf{k}\cdot\mathbf{x} - \mathbf{k}'\cdot\mathbf{x}')} d^3r d^3r' \\ &= \int_{R^3}\int_{R^3} \xi(\mathbf{x} - \mathbf{x}') e^{i(\mathbf{k}\cdot\mathbf{x} - \mathbf{k}'\cdot\mathbf{x}')} d^3r d^3r' \end{aligned}$$

By changing coordinates to $\mathbf{u} = \mathbf{x} - \mathbf{x}'$ and $\mathbf{x}' = \mathbf{x}'$, this can be cast in the form

$$\langle \Phi_{\mathbf{k}} \Phi^{\star}_{\mathbf{k}'} \rangle = \int_{R^3}\int_{R^3} \xi(\mathbf{u}) e^{i[(\mathbf{k} - \mathbf{k}')\cdot\mathbf{x}' + \mathbf{k}\cdot\mathbf{u}]} d^3u d^3r' \tag{18}$$

This change of coordinates has to be considered carefully. We have to ensure that the Jacobian is properly taken into account and the tiling of the coordinate space is done properly. For details look for Faltung theorem in any standard book.

- Performing the integration over $\mathbf{x}'$ we obtain

$$\langle \Phi_{\mathbf{k}} \Phi^{\star}_{\mathbf{k}'} \rangle = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}') \int_{R^3} \xi(\mathbf{u}) e^{i\mathbf{k}\cdot\mathbf{u}} d^3u = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}')\, P(\mathbf{k}) \tag{19}$$

  where $P(\mathbf{k})$ is the power spectrum. For an isotropic process $P(\mathbf{k}) = P(k)$, and is given by the Fourier transform of the two point correlation function $\xi(r)$. This remarkable result is known as Wiener-Khinchin theorem.