The shear dynamo problem

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1. Statement of the problem

Shearing sheet: Local approximation to a differentially rotating disc

Spacetime coordinates: $\boldsymbol{X} \ = \ (X_1, X_2, X_3)\,;$ time $= \ \tau$

Velocity field: $\boldsymbol{v}' \ = \ -2AX_1\boldsymbol{e}_2 \ + \ \boldsymbol{v}(\boldsymbol{X}, \tau)$

Fluctuations incompressible with zero mean: $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0\,; \quad \langle\boldsymbol{v}\rangle = \boldsymbol{0}$

Induction equation for the total magnetic field $\boldsymbol{B}'(\boldsymbol{X}, \tau)$:

$$\left(\frac{\partial}{\partial\tau} \ - \ 2AX_1\frac{\partial}{\partial X_2}\right)\boldsymbol{B}' \ + \ 2AB_1'\boldsymbol{e}_2 \ = \ \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B}') \ + \ \eta\boldsymbol{\nabla}^2\boldsymbol{B}'$$

2. Approach to the problem

- Mean field theory
- Non perturbative in the shear parameter A
- Shearing coordinate transformation
- Galilean invariance of the shearing sheet

3. The simplest problem

Case $\boldsymbol{v} = \mathbf{0}\,; \quad \eta = 0$

$$\left(\frac{\partial}{\partial \tau} \;-\; 2AX_1\frac{\partial}{\partial X_2}\right)\boldsymbol{B}' \;=\; -2AB_1'\boldsymbol{e}_2$$

Shearing transformation to new spacetime coordinates $(\boldsymbol{x}, t)$:

$$x_1 = X_1\,, \qquad x_2 = X_2 + 2A\tau X_1\,, \qquad x_3 = X_3\,, \qquad t = \tau$$

New variable: $\boldsymbol{H}'(\boldsymbol{x}, t) = \boldsymbol{B}'(\boldsymbol{X}, \tau)$

$$\frac{\partial \boldsymbol{H}'}{\partial t} \;=\; -2AH_1'\boldsymbol{e}_2$$

4. Solution

$$H_1' = f_1(\boldsymbol{x})\,; \quad H_3' = f_3(\boldsymbol{x})\,; \quad H_2' = f_2(\boldsymbol{x}) \;-\; 2At\, f_1(\boldsymbol{x})$$

$$B_1' \;=\; f_1(X_1\, X_2 + 2A\tau X_1, X_3)$$

$$B_3' \;=\; f_3(X_1, X_2 + 2A\tau X_1, X_3)$$

$$B_2' \;=\; f_2(X_1, X_2 + 2A\tau X_1, X_3) \;-\; 2A\tau\, f_1(X_1, X_2 + 2A\tau X_1, X_3)$$

Axisymmetric solutions have constant B_1' and B_3', while B_2' is generated from B_1' by the background shear at a constant rate.

Non–axisymmetric patterns in B_1' and B_3' are advected with velocity $-2AX_1\boldsymbol{e}_2$. In addition, B_2' is generated from B_1' by the background shear at a constant rate. Fine structure in X_1.

5. The shear dynamo problem: quasilinear kinematic theory

Reynolds averaging $\quad \boldsymbol{B}' = \boldsymbol{B} + \boldsymbol{b}\,, \quad \langle \boldsymbol{B}' \rangle = \boldsymbol{B}\,, \quad \langle \boldsymbol{b} \rangle = \boldsymbol{0}$

Mean field equation

$$\left(\frac{\partial}{\partial \tau} - 2AX_1 \frac{\partial}{\partial X_2} \right) \boldsymbol{B} + 2AB_1 \boldsymbol{e}_2 = \boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}}$$

Mean EMF $\quad \boldsymbol{\mathcal{E}} = \langle \boldsymbol{v} \times \boldsymbol{b} \rangle$

Fluctuating field equation

$$\left(\frac{\partial}{\partial \tau} - 2AX_1 \frac{\partial}{\partial X_2} \right) \boldsymbol{b} + 2Ab_1 \boldsymbol{e}_2 = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) + \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{b} - \boldsymbol{\mathcal{E}})$$

Calculate $\boldsymbol{\mathcal{E}}$ in terms of the statistical properties of the fluctuating velocity field using first–order smoothing.

6. Sheared coordinates and new variables

$$\boldsymbol{H}(\boldsymbol{x},t) \;=\; \boldsymbol{B}(\boldsymbol{X},\tau)\,, \qquad \boldsymbol{h}(\boldsymbol{x},t) \;=\; \boldsymbol{b}(\boldsymbol{X},\tau)\,, \qquad \boldsymbol{u}(\boldsymbol{x},t) \;=\; \boldsymbol{v}(\boldsymbol{X},\tau)$$

$$\frac{\partial \boldsymbol{h}}{\partial t} \;+\; 2Ah_1\boldsymbol{e}_2 \;=\; \left(\boldsymbol{H}\boldsymbol{\cdot}\frac{\partial}{\partial \boldsymbol{x}} + 2AtH_1\frac{\partial}{\partial x_2}\right)\boldsymbol{u} \;-\; \left(\boldsymbol{u}\boldsymbol{\cdot}\frac{\partial}{\partial \boldsymbol{x}} + 2Atu_1\frac{\partial}{\partial x_2}\right)\boldsymbol{H}$$

Solution in the kinematic limit:

$$h_m(\boldsymbol{x},t) \;=\; \int_0^t dt'\;[u'_{ml} \;-\; 2A(t-t')\delta_{m2}\,u'_{1l}]\,[H'_l \;+\; 2At'\delta_{l2}\,H'_1]$$

$$-\;\int_0^t dt'\;[u'_l \;+\; 2At'\delta_{l2}\,u'_1]\,[H'_{ml} \;-\; 2A(t-t')\delta_{m2}\,H'_{1l}]$$

where the primes denote evaluation at spacetime point $(\boldsymbol{x},t')$. We have also used notation $u_{ml} = (\partial u_m/\partial x_l)$ and $H_{ml} = (\partial H_m/\partial x_l)$.

7. Mean EMF

The expression for $\boldsymbol{h}$ should be substituted in $\boldsymbol{\mathcal{E}} = \langle \boldsymbol{v} \times \boldsymbol{b} \rangle = \langle \boldsymbol{u} \times \boldsymbol{h} \rangle$.

Following standard procedure, we allow $\langle \ \ \rangle$ to act only on the velocity variables but not the mean field; symbolically, it is assumed that $\langle \boldsymbol{uuH} \rangle = \langle \boldsymbol{uu} \rangle \, \boldsymbol{H}$.

$$
\begin{aligned}
\mathcal{E}_i(\boldsymbol{x}, t) \;&=\; \epsilon_{ijm} \, \langle u_j h_m \rangle \\
&=\; \int_0^t dt' \; \left[\widehat{\alpha}_{il}(\boldsymbol{x}, t, t') - 2A(t-t') \widehat{\beta}_{il}(\boldsymbol{x}, t, t') \right] \left[H'_l + 2At'\delta_{l2} \, H'_1 \right] \\
&-\; \int_0^t dt' \; \left[\widehat{\eta}_{iml}(\boldsymbol{x}, t, t') + 2At'\delta_{m2} \, \widehat{\eta}_{i1l}(\boldsymbol{x}, t, t') \right] \left[H'_{lm} - 2A(t-t')\delta_{l2} \, H'_{1m} \right]
\end{aligned}
$$

8. Transport coefficients

$$\begin{aligned}
\widehat{\alpha}_{il}(\boldsymbol{x},t,t') &= \epsilon_{ijm}\,\langle u_j(\boldsymbol{x},t)\,u_{ml}(\boldsymbol{x},t')\rangle \\
\widehat{\beta}_{il}(\boldsymbol{x},t,t') &= \epsilon_{ij2}\,\langle u_j(\boldsymbol{x},t)\,u_{1l}(\boldsymbol{x},t')\rangle \\
\widehat{\eta}_{iml}(\boldsymbol{x},t,t') &= \epsilon_{ijl}\,\langle u_j(\boldsymbol{x},t)\,u_{m}(\boldsymbol{x},t')\rangle
\end{aligned}$$

To obtain more specific expressions for the transport coefficients , we need to provide information on the $\boldsymbol{uu}$ velocity correlators. However, it is physically more transparent to consider velocity statistics in terms of $\boldsymbol{vv}$ velocity correlators, because this is referred to the lab frame, instead of the sheared coordinates.

By definition $\quad u_m(\boldsymbol{x},t) \;=\; v_m(\boldsymbol{X}(\boldsymbol{x},t),t)$

$$X_1 \;=\; x_1\,, \qquad X_2 \;=\; x_2 - 2Atx_1\,, \qquad X_3 \;=\; x_3\,, \qquad \tau \;=\; t$$

is the inverse of the shearing transformation.

9. Transport coefficients, continued

Work out

$$u_{ml} = \left(\frac{\partial}{\partial X_l} - 2A\tau\, \delta_{l1} \frac{\partial}{\partial X_2} \right) v_m = v_{ml} - 2A\tau\, \delta_{l1}\, v_{m2}$$

where $v_{ml} = (\partial v_m / \partial X_l)$.

$$\begin{aligned}
\widehat{\alpha}_{il}(\boldsymbol{x}, t, t') &= \epsilon_{ijm} \left[\langle v_j(\boldsymbol{X}, t)\, v_{ml}(\boldsymbol{X}', t') \rangle - 2At'\, \delta_{l1} \, \langle v_j(\boldsymbol{X}, t)\, v_{m2}(\boldsymbol{X}', t') \rangle \right] \\
\widehat{\beta}_{il}(\boldsymbol{x}, t, t') &= \epsilon_{ij2} \left[\langle v_j(\boldsymbol{X}, t)\, v_{1l}(\boldsymbol{X}', t') \rangle - 2At'\, \delta_{l1} \, \langle v_j(\boldsymbol{X}, t)\, v_{12}(\boldsymbol{X}', t') \rangle \right] \\
\widehat{\eta}_{iml}(\boldsymbol{x}, t, t') &= \epsilon_{ijl} \, \langle v_j(\boldsymbol{X}, t)\, v_m(\boldsymbol{X}', t') \rangle
\end{aligned}$$

where $\boldsymbol{X}$ and $\boldsymbol{X}'$ are shorthand for

$$\boldsymbol{X} = (x_1 \,, x_2 - 2Atx_1 \,, x_3) \,, \qquad \boldsymbol{X}' = (x_1 \,, x_2 - 2At'x_1 \,, x_3)$$

10. Galilean invariance (GI)

The origin of a comoving observer translates with uniform velocity:

$$\boldsymbol{X}_c(\tau) \;=\; (\xi_1\,,\xi_2 - 2A\tau\xi_1\,,\xi_3)$$

Galilean transformation (relation between comoving and lab coordinates):

$$\tilde{\boldsymbol{X}} \;=\; \boldsymbol{X} \;-\; \boldsymbol{X}_c(\tau)\,, \qquad \tilde{\tau} \;=\; \tau - \tau_0$$

Transformation of fields:

$$\left[\tilde{\boldsymbol{B}}(\tilde{\boldsymbol{X}},\tilde{\tau})\,,\tilde{\boldsymbol{b}}(\tilde{\boldsymbol{X}},\tilde{\tau})\,,\tilde{\boldsymbol{v}}(\tilde{\boldsymbol{X}},\tilde{\tau})\right] \;=\; [\boldsymbol{B}(\boldsymbol{X},\tau)\,,\boldsymbol{b}(\boldsymbol{X},\tau)\,,\boldsymbol{v}(\boldsymbol{X},\tau)]$$

Induction equation is invariant under the simultaneous transformation of coordinates and fields

11. GI velocity correlators

$$
\begin{aligned}
\langle v_i(\boldsymbol{R}, \tau)\, v_j(\boldsymbol{R}', \tau')\rangle &= \langle v_i(\boldsymbol{R} + \boldsymbol{X}_c(\tau), \tau)\, v_j(\boldsymbol{R}' + \boldsymbol{X}_c(\tau'), \tau')\rangle \\
\langle v_i(\boldsymbol{R}, \tau)\, v_{jl}(\boldsymbol{R}', \tau')\rangle &= \langle v_i(\boldsymbol{R} + \boldsymbol{X}_c(\tau), \tau)\, v_{jl}(\boldsymbol{R}' + \boldsymbol{X}_c(\tau'), \tau')\rangle
\end{aligned}
$$

for all $(\boldsymbol{R}, \boldsymbol{R}', \tau, \tau', \boldsymbol{\xi})$.

Choose $\boldsymbol{R} = \boldsymbol{R}' = \boldsymbol{0}\,, \tau = t\,, \tau' = t'$ and $\boldsymbol{\xi} = \boldsymbol{x}$. Then

$$
\begin{aligned}
\boldsymbol{X}_c(\tau) &= (x_1\,, x_2 - 2Atx_1\,, x_3) = \boldsymbol{X} \\
\boldsymbol{X}_c(\tau') &= (x_1\,, x_2 - 2At'x_1\,, x_3) = \boldsymbol{X}'
\end{aligned}
$$

12. GI velocity correlators and transport coefficients

$$
\begin{aligned}
\langle v_i(\boldsymbol{X},t)\, v_j(\boldsymbol{X}',t')\rangle &= \langle v_i(\mathbf{0},t)\, v_j(\mathbf{0},t')\rangle = R_{ij}(t,t') \\
\langle v_i(\boldsymbol{X},t)\, v_{jl}(\boldsymbol{X}',t')\rangle &= \langle v_i(\mathbf{0},t)\, v_{jl}(\mathbf{0},t')\rangle = S_{ijl}(t,t')
\end{aligned}
$$

are independent of space.

GI transport coefficients:

$$
\begin{aligned}
\widehat{\alpha}_{il}(t,t') &= \epsilon_{ijm}\,[S_{jml}(t,t') - 2At'\,\delta_{l1}\,S_{jm2}(t,t')] \\
\widehat{\beta}_{il}(t,t') &= \epsilon_{ij2}\,[S_{j1l}(t,t') - 2At'\,\delta_{l1}\,S_{j12}(t,t')] \\
\widehat{\eta}_{iml}(t,t') &= \epsilon_{ijl}\,R_{jm}(t,t')
\end{aligned}
$$

are also independent of space.

13. GI mean EMF

Define new combinations of velocity correlators:

$$
\begin{aligned}
C_{jml}(t,t') &= S_{jml}(t,t') \;-\; 2A(t-t')\delta_{m2}\, S_{j1l}(t,t') \\
D_{jm}(t,t') &= R_{jm}(t,t') \;+\; 2At'\delta_{m2}\, R_{j1}(t,t')
\end{aligned}
$$

Mean EMF

$$
\begin{aligned}
\mathcal{E}_i(\boldsymbol{x},t) \;=\; & \epsilon_{ijm}\int_0^t dt'\; C_{jml}(t,t')\, H'_l \\
& - \int_0^t dt'\; \left[\epsilon_{ijl} - 2A(t-t')\delta_{l1}\epsilon_{ij2}\right]\; D_{jm}(t,t')\, H'_{lm}
\end{aligned}
$$

14. Mean–field induction equation

$$\frac{\partial H_i}{\partial t} + 2A\delta_{i2}H_1 = (\boldsymbol{\nabla}\times\boldsymbol{\mathcal{E}})_i + \eta\boldsymbol{\nabla}^2 H_i$$

$$\begin{aligned}
(\boldsymbol{\nabla}\times\boldsymbol{\mathcal{E}})_i \;&=\; \int_0^t dt'\,[C_{iml} - C_{mil}]\,[H'_{lm} + 2At\delta_{m1}H'_{l2}] \;+ \\
&+\; \int_0^t dt'\,D_{jm}\left[H'_{ijm} + 2At\delta_{j1}H'_{i2m}\right] \\
&-\; 2A\delta_{i2}\int_0^t dt'\,(t-t')D_{jm}\left[H'_{1jm} + 2At\delta_{j1}H'_{12m}\right]
\end{aligned}$$

15. Galilean invariance and the shearing coordinate transformation

$$\left\langle \tilde{v}_i(\boldsymbol{K}, \tau)\, \tilde{v}_j^*(\boldsymbol{K}', \tau') \right\rangle \;=\; (2\pi)^6\, \delta(\boldsymbol{k} - \boldsymbol{k}')\, \Pi_{ij}(\boldsymbol{k}, \tau, \tau')$$

$$\begin{aligned} k_1 &= K_1 - 2A\tau K_2\,, \qquad & k_2 &= K_2\,, \qquad & k_3 &= K_3\,, \qquad & t &= \tau \\ k_1' &= K_1' - 2A\tau' K_2'\,, \qquad & k_2' &= K_2'\,, \qquad & k_3' &= K_3'\,, \qquad & t' &= \tau' \end{aligned}$$

$$\begin{aligned} \Pi_{ij}(\boldsymbol{k}, \tau, \tau') &= \Pi_{ij}^*(-\boldsymbol{k}, \tau, \tau') \;=\; \Pi_{ji}(-\boldsymbol{k}, \tau', \tau) \\ K_i\, \Pi_{ij}(\boldsymbol{k}, \tau, \tau') &= (k_i + 2A\tau\, \delta_{i1} k_2)\, \Pi_{ij}(\boldsymbol{k}, \tau, \tau') \;=\; 0 \\ K_j'\, \Pi_{ij}(\boldsymbol{k}, \tau, \tau') &= (k_j + 2A\tau'\, \delta_{j1} k_2)\, \Pi_{ij}(\boldsymbol{k}, \tau, \tau') \;=\; 0 \end{aligned}$$

16. Some comments

- Theory for non zero η

- Dynamical theory for velocity correlators (momentum forcing)

- Compressibility, Rotation, Convection, Lorentz force, Turbulence

- Supernovae explosions in a disc (energy forcing)