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**Introduction to Geometric Knot Theory 1:  
Knot invariants defined by minimizing curve invariants**

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# Introduction to Geometric Knot Theory 1: Knot invariants defined by minimizing curve invariants

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# Main question

Every field has a main question:

Open Question (Main Question of Topological Knot Theory)

*What are the isotopy classes of knots and links?*

Open Question (Main Question of Global Differential Geometry)

*What is the relationship between the topology and (sectional, scalar, Ricci) curvature of a manifold?*

Open Question (Main Question of Geometric Knot Theory)

*What is the relationship between the topology and geometry of knots?*

## Two Main Strands of Inquiry

### Strand 1 (Knot Invariants defined by minima)

*Given a geometric invariant of curves, define a topological invariant of knots by minimizing over all curves in a knot type. How are these invariants related?*

Examples: Crossing number, bridge number, total curvature, distortion, braid index, Möbius energy, ropelength.

### Strand 2 (Restricted Knot Theories)

*Restrict attention to curves obeying additional geometric hypotheses. Is every knot type realizable in this class? What are the isotopy types among curves in this class?*

Examples: Regular (nonvanishing curvature) isotopy, Legendrian and transverse knots, braid theory, polygonal knots, plumber's knots.

# An example: crossing number

## Definition

The crossing number of a knot or link is the minimum number of crossings in any projection of any configuration of the knot.

*... that's why I hate crossing number. (attributed to J.H. Conway)*

- 1 The unique knot of minimum crossing number is the unknot.
- 2 For any  $N$ , there only finitely many knot types with crossing number  $< N$ .

# An example: crossing number

## Definition

The crossing number of a knot or link is the minimum number of crossings in any projection of any configuration of the knot.

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Some useful properties of crossing number:

- 1 The unique knot of minimum crossing number is the unknot.
- 2 For any  $N$ , there only finitely many knot types with crossing number  $< N$ .

# Desirable Properties of Geometric Knot Invariants

## Definition

A *geometric knot invariant*  $MG([K])$  is a knot invariant defined by minimizing a geometric invariant  $G(K)$  of curves over all curves  $K$  in a knot type  $[K]$ .

## Definition

A geometric knot invariant  $MG([K])$  is

- *basic* if the minimum of  $MG([K])$  over all knot types is achieved uniquely for the unknot.
- *strong* if for each  $N$  there are only finitely many knot types with  $MG([K]) < N$ .

Adapted from Diao et. al. and covered in Dr. Rawdon's course.

# Desirable Properties of Geometric Knot Invariants

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## An old and hard open question: distortion

We now give another geometric curve invariant.

### Definition

The *distortion* of a curve  $\gamma$  is given by

$$\text{Dist}(\gamma) = \max_{p, q \in \gamma} \frac{\text{distance from } p \text{ to } q \text{ on } \gamma}{\text{distance from } p \text{ to } q \text{ in space}}.$$

The corresponding geometric knot invariant is denoted MDist.

### Open Question (Gromov, 1983)

*Is there is a universal upper bound on MDist for all knots?*

A popular question in the 80's, but pretty hard.

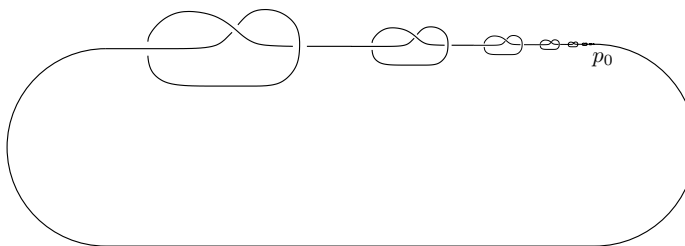
## What is known about distortion?

Theorem (Kusner and Sullivan 1999, Denne and Sullivan 2009)

*MDist is basic. The minimum distortion of an unknot is  $\pi/2$  (the round circle) while the distortion of a nontrivial tame knot is at least  $5\pi/3$ .*

Theorem (Gromov 1978, O'Hara 1992)

*MDist is not strong. (There are infinite families of prime knots with distortion bounded above.)*



## Conjecture: No upper bound on distortion for knots

To prove it:

- 1 Find a “sufficiently weak” knot invariant  $I$  that approaches infinity on  $(n, n - 1)$  torus knots. (Bridge number, crossing number, genus, etc are all ruled out already.)
- 2 Bound distortion below in terms of  $I$ .

### Definition

The *hull number* of a knot type  $[K]$  is the maximum  $N$  such that any curve  $K$  in  $[K]$  has some point  $p$  so that any plane through  $p$  cuts  $K$  at least  $2N$  times.

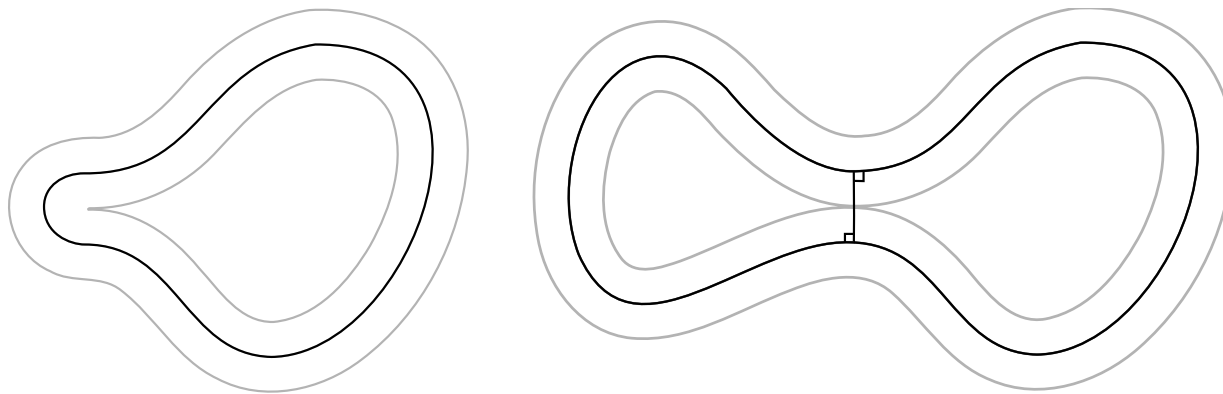
### Theorem (Izmestiev 2006)

*The hull number of a  $(p, q)$  torus knot is at least  $(1/4) \min(p, q)$ .*

# A geometric invariant of curves: Reach

## Definition (Federer 1959)

The *reach* of a space curve is the largest  $\epsilon$  so that any point in an  $\epsilon$ -neighborhood of the curve has a unique nearest neighbor on the curve.



## Idea

$\text{reach}(K)$  (also called *thickness*) is controlled by curvature maxima (kinks) and self-distance minima (struts).

# Ropelength

## Definition

The ropelength of  $K$  is given by  $\text{Rop}(K) = \text{Len}(K) / \text{reach}(K)$ .

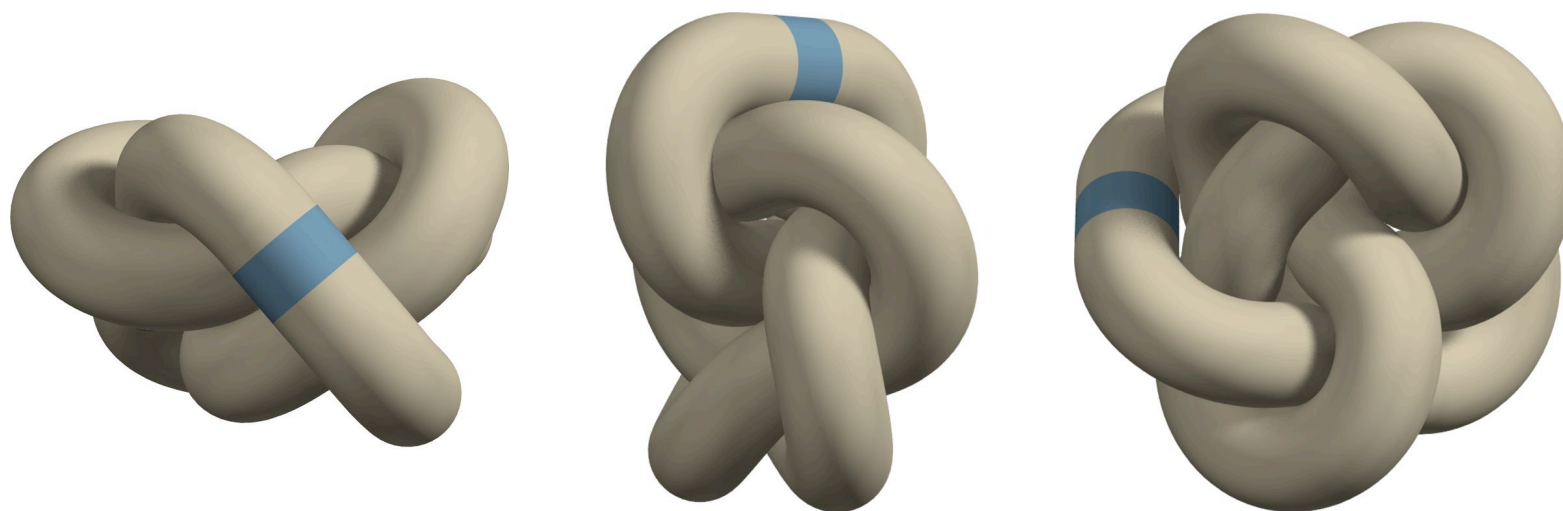
Theorem (with Kusner, Sullivan 2002, Gonzalez, De la Llave 2003, Gonzalez, Maddocks, Schuricht, Von der Mosel 2002)

*Ropelength minimizers (called tight knots) exist in each knot and link type and are  $C^{1,1}$ .*

## Open Question

*What is the smoothness of a tight knot? Current examples suggest that such a knot is piecewise smooth but not  $C^2$ .*

## Examples: Why only piecewise smooth?



Theorem (with Fu, Kusner, Sullivan, Wrinkle 2009, cf. Gonzalez, Maddocks 2000, Schuricht, Von der Mosel 2003)

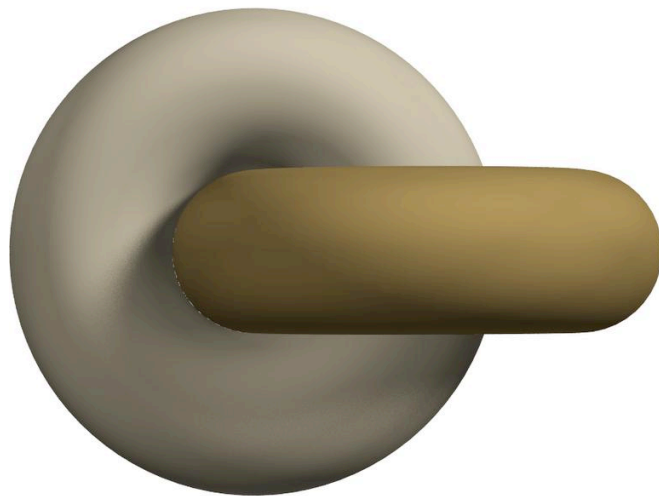
*Any open interval of a tight knot either: contains an endpoint of a strut, has curvature 1 almost everywhere, or is a straight line segment.*

# The Tight Hopf Link and Trefoil Knot

The Hopf Link  $2_1^2$

$$\text{Rop}([2_1^2]) = 8\pi$$

with Kusner, Sullivan 2002



The Trefoil Knot  $3_1$

$$31.32 \leq \text{Rop}([3_1]) \leq 32.743175$$

Denne, Diao and Sullivan 2006

Baranska, Przybyl, Pieranski 2008



# Lower Bounds on Ropelength

Theorem (Diao 2006)

$$\text{Rop}(K) \geq \frac{1}{2} \left( 17.334 + \sqrt{17.334^2 + 64\pi \text{Cr}(K)} \right).$$

Corollary

*Ropelength is basic and strong.*

Proof.

The ropelength of a tight unknot is  $4\pi = 12.566$ , less than any knot of higher crossing number. All knots with  $\text{Rop} < N$  have

$$\text{Cr} < \frac{0.000125}{\pi} N(500N - 8667).$$





## More consequences of this $\text{Cr} / \text{Rop}$ bound.

## Corollary

*Hopf link (Rop = 25.1327) is the tightest nontrivial link.*

## Proof.

Evaluating the formula in a few cases,

$\text{Cr}(K)$	3	4	5	...	10	11
$\text{Rop}(K) \geq$	23.698	25.286	26.735	...	32.704	33.73

So only  $\text{Rop}(3_1)$  could be lower than  $\text{Rop}(2_1)$ . But DDS show  $\text{Rop}(3_1) \geq 31.32 > 25.137$ .

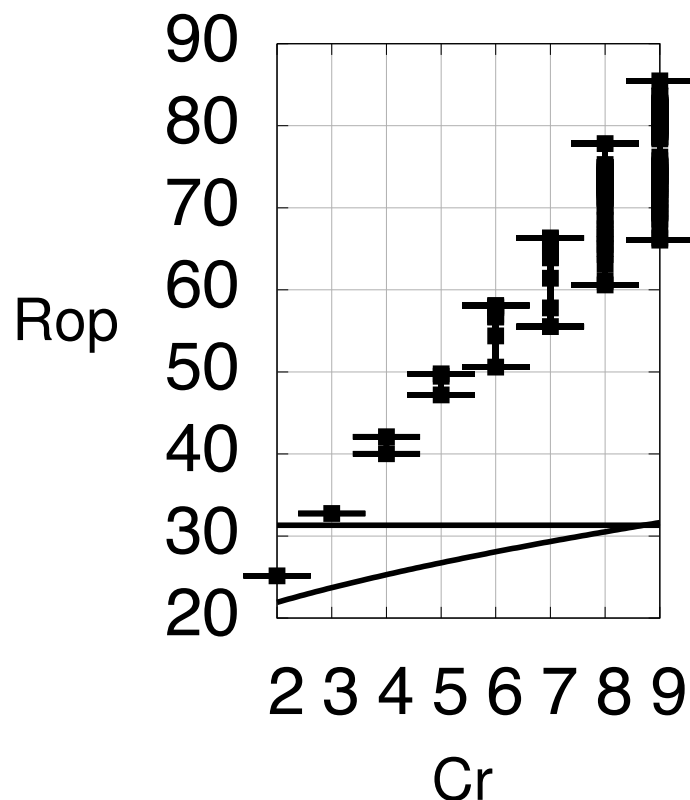
# Open Question

*Is the trefoil ( $R_{op} \simeq 32.74$ ) the tightest knot?*

# Ropelength and Crossing Number vs Data

## Open Question

*Find effective Rop bounds for simple ( $< 10$  crossing) knots.*



Cr	Rop	Links
3	32.74	$3_1$
4	[40.01, 42.09]	$4_1^2, 4_1$
5	[47.20, 49.77]	$5_1, 5_1^2$
6	[50.57, 58.1]	$6_3^3, 6_2^3$
7	[55.53, 66.33]	$7_7^2, 7_6^2$
8	[60.58, 77.83]	$8_7^3, 8_4^3$
9	[66.06, 85.47]	$9_{49}^2, 9_1^4$

# Ropelength and Crossing Number

Theorem (Buck, Simon 1999, Diao, Ernst, Yu 2003)

*There exist constants so  $c_1 \operatorname{Cr}^{3/4}(K) \leq \operatorname{Rop}(K) \leq c_2 \operatorname{Cr}^{3/2}(K)$ .*

Proof (sketch) of 3/4 power lower bound.

Scale the knot so  $\operatorname{reach}(K) = 1$ . Then  $\operatorname{Rop}(K) = \operatorname{Len}(K)$ .

$$\begin{aligned} \operatorname{Cr}(K) &\leq A\operatorname{Cr}(K) = \frac{1}{4\pi} \iint \frac{|K'(s) \times K'(t) \cdot (K(s) - K(t))|}{|K(s) - K(t)|^3} ds dt \\ &\leq \frac{1}{4\pi} \iint \frac{1}{|K(s) - K(t)|^2} ds dt. \end{aligned}$$

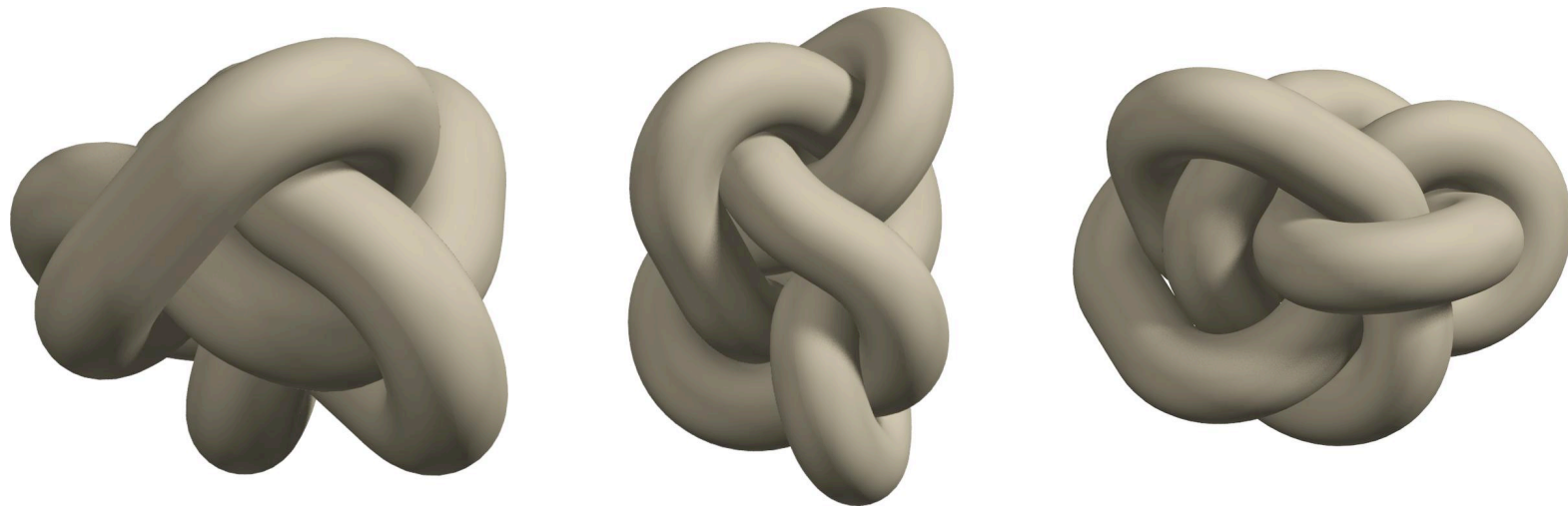
Now we estimate this integral above in terms of  $\operatorname{Len}(K)$ . □

# Proof (sketch) of $3/4$ power lower bound

We start by estimating

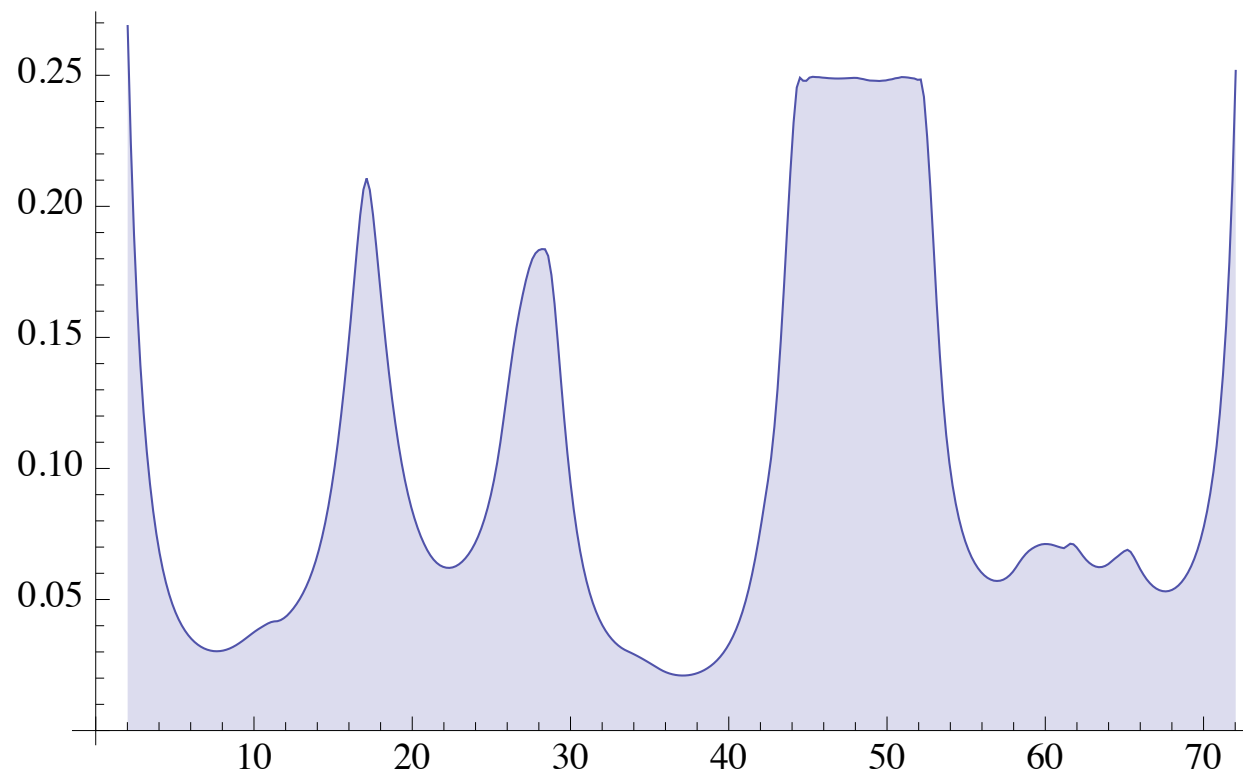
$$\int_{d(s,t)>2} \frac{1}{|K(s) - K(t)|^2} ds$$

where  $d(s, t)$  is the arclength distance along  $K$ . Our example plots will come from this  $9_{49}$  knot:



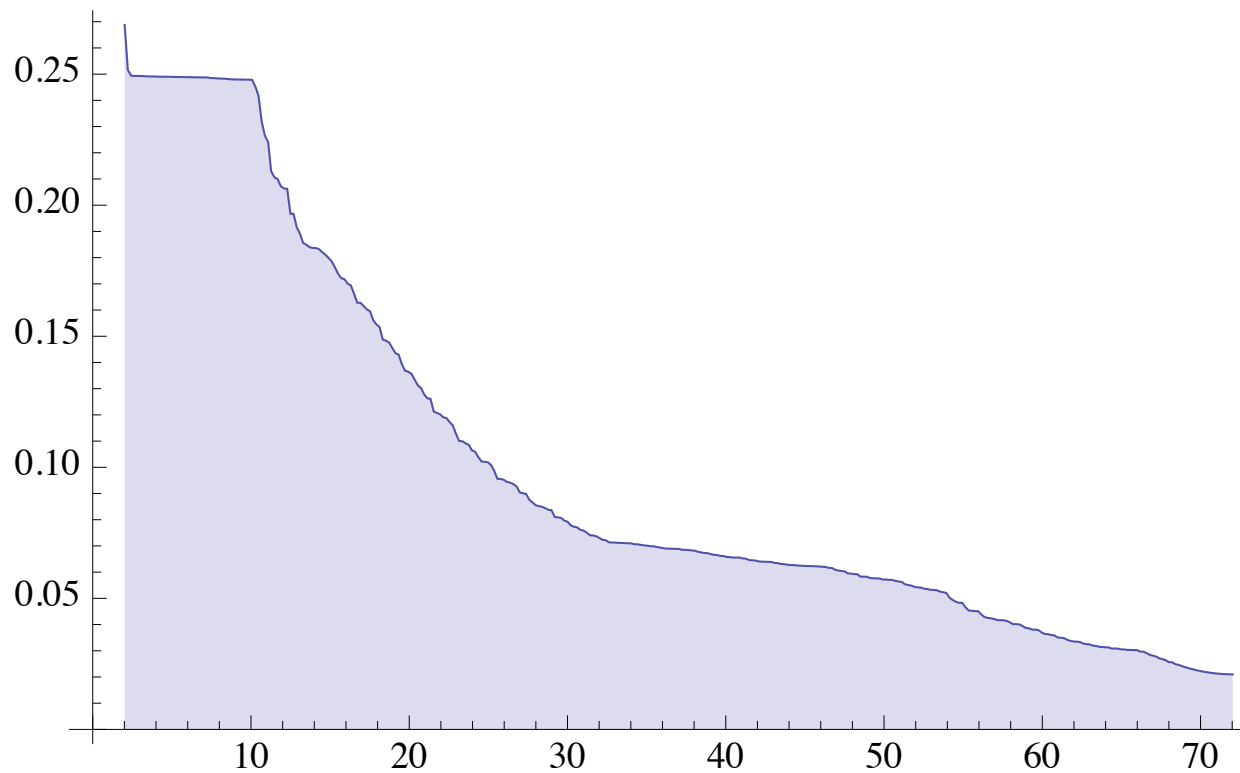
## Proof (sketch) of $3/4$ power lower bound

Here is a graph of the inverse square distance from  $K(0)$  to  $K(s)$  for the  $9_{49}$  knot above:



# Proof (sketch) of $3/4$ power lower bound

Without changing the integral, we can take a monotone rearrangement of the function:



## Proof (sketch) of $3/4$ power lower bound

A (possibly disconnected) section of tube of total arclength  $s$  has volume  $\pi s$ . If that section of tube is within distance  $r$  of the origin, then this tube is all packed in the sphere of radius  $r + 1$ , which has volume  $(4/3)\pi(r + 1)^3$ . Assuming  $r > 2$ ,

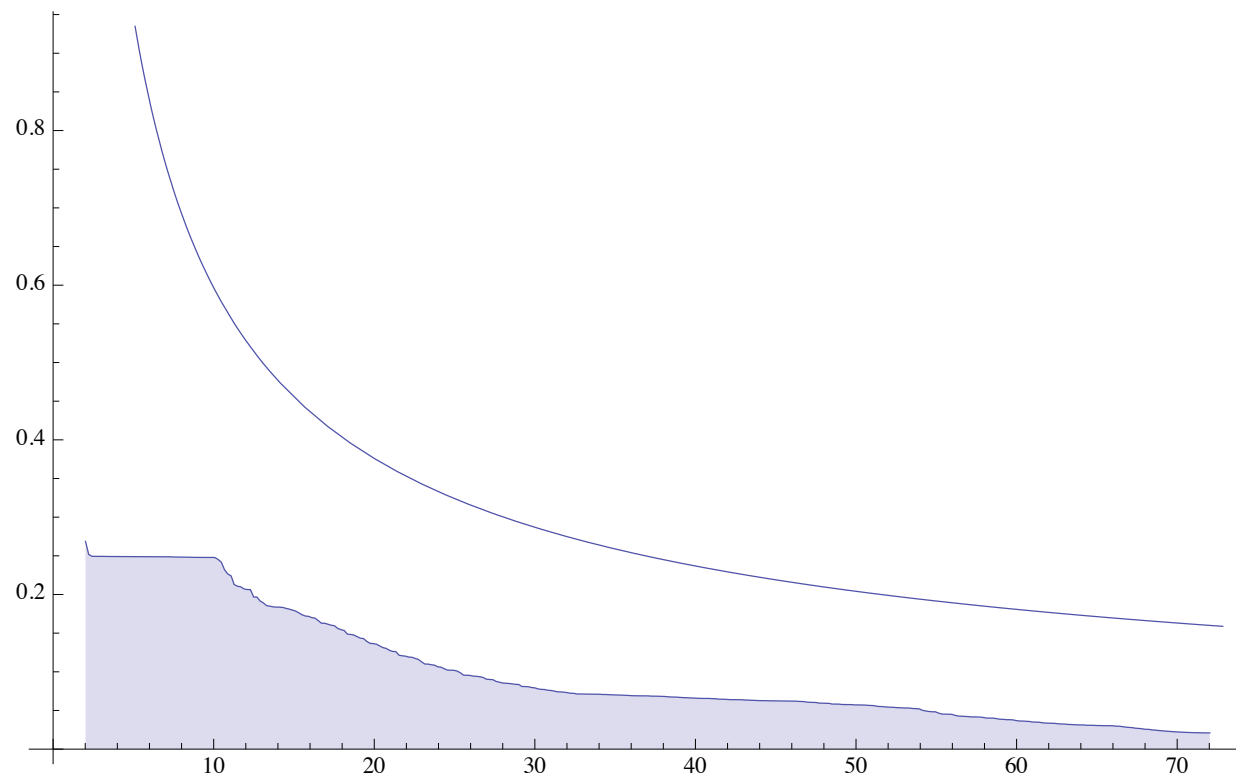
$$\pi s < \frac{4}{3}\pi (r + 1)^3 < 4.5r^3,$$

we can rearrange to get

$$2.73 s^{-\frac{2}{3}} > \frac{1}{r^2}.$$

# Proof (sketch) of $3/4$ power lower bound

This estimate shows that our rearranged distance function is less than  $2.73 s^{-2/3}$  (when  $1/r^2 < 0.25$ ):





# Proof (sketch) of $3/4$ power lower bound

Integrating over  $[0, \text{Len}(K)]$ , we get

$$\begin{aligned} \int_{d(s,t)>2} \frac{1}{|K(s) - K(t)|^2} ds &< \int_0^{\text{Rop}(K)} 2.77 s^{-2/3} ds \\ &< 8.177 \text{Rop}(K)^{1/3}. \end{aligned}$$

and so

$$\text{Cr}(K) \leq \frac{1}{4\pi} \iint \frac{1}{|K(s) - K(t)|^2} ds dt < 0.651 \text{Rop}(K)^{4/3}.$$

and taking care of the pairs  $d(s, t) < 2$  with another argument,

$$1.38 \text{Cr}^{3/4} + (\text{lower order terms}) \leq \text{Rop}(K).$$

## Open Question: What's the best bound of this type?

The actual bound of Buck and Simon is

Theorem (Buck and Simon 1999)

$$\text{Rop}(K) \geq 2.205 \text{Cr}(K)^{3/4}$$

One can easily improve the argument above to

$$\text{Rop}(K) \geq 2.5357 \text{Cr}(K)^{3/4} + (\text{lower order terms})$$

but the lower order terms are significant.

Open Question

*What is the largest  $c_1$  so that  $c_1 \text{Cr}(K)^{3/4} \leq \text{Rop}(K)$  for all  $K$ ?*

## A remark

### Theorem (Diao 2006)

$$\text{Rop}(K) \geq \frac{1}{2} \left( 17.334 + \sqrt{17.334^2 + 64\pi \text{Cr}(K)} \right).$$

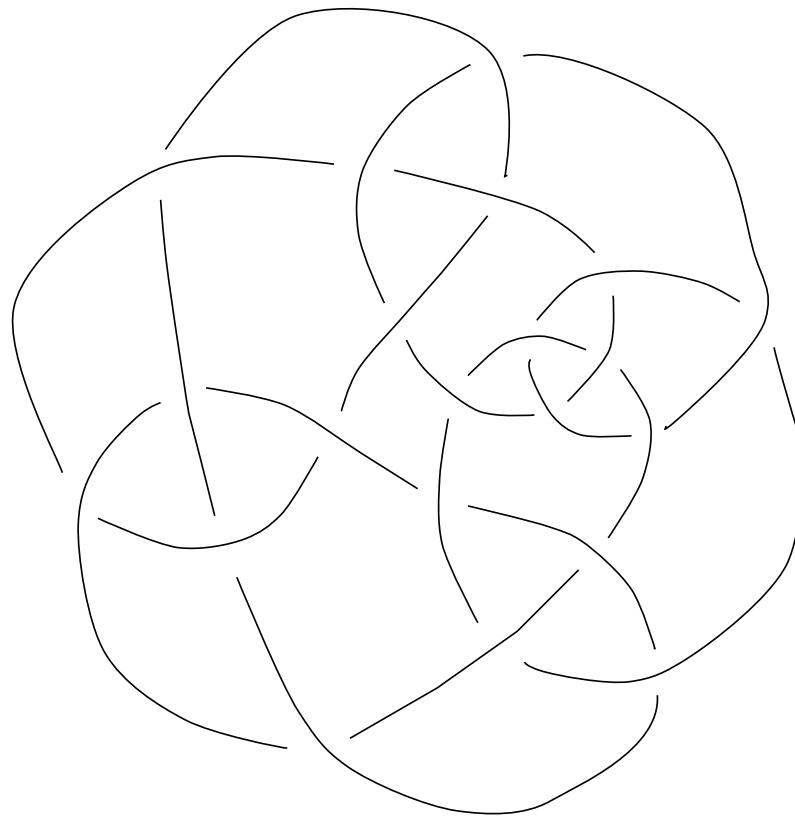
is also proved by bounding

$$\begin{aligned} \text{Cr}(K) \leq \text{ACr}(K) &= \frac{1}{4\pi} \iint \frac{|K'(s) \times K'(t) \cdot (K(s) - K(t))|}{|K(s) - K(t)|^3} ds dt \\ &\leq \frac{1}{4\pi} \iint \frac{1}{|K(s) - K(t)|^2} ds dt. \end{aligned}$$

but in Diao's proof the lower order terms dominate the bound.

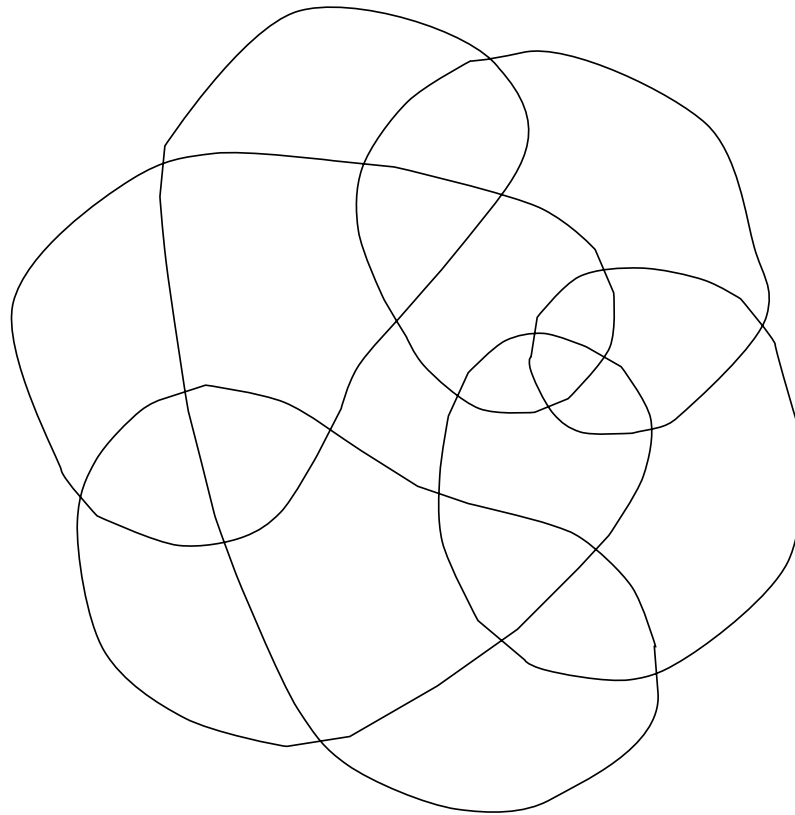
## Proof (sketch) of $3/2$ power upper bound

We must find an algorithm for constructing “short” embeddings of knots, such as this one:  $21_{4,385,281}$



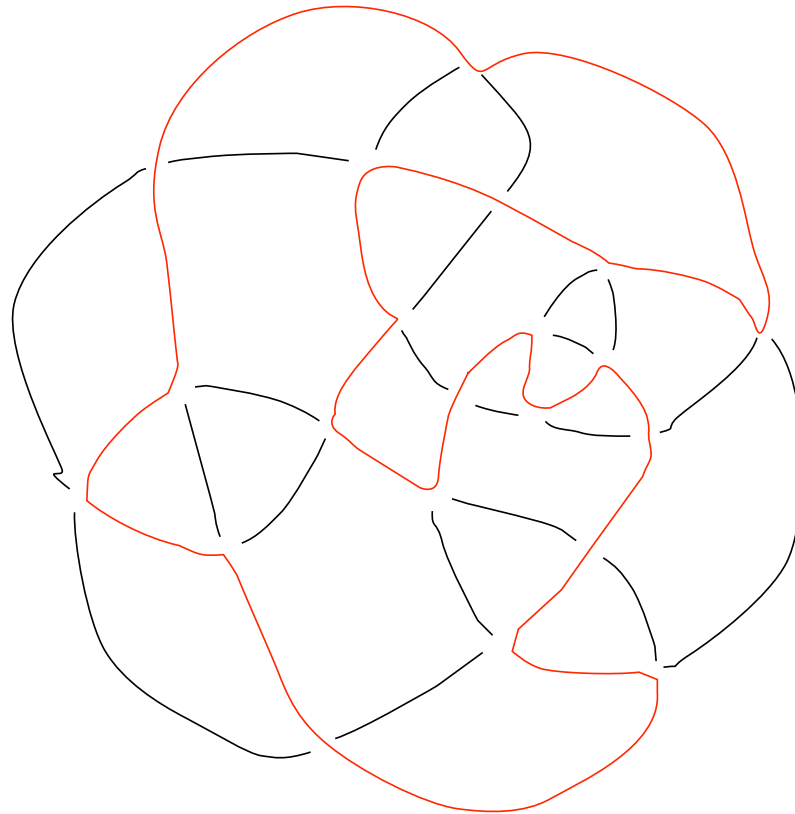
# Proof (sketch) of $3/2$ power upper bound

Start by converting it to a planar 4-regular graph:



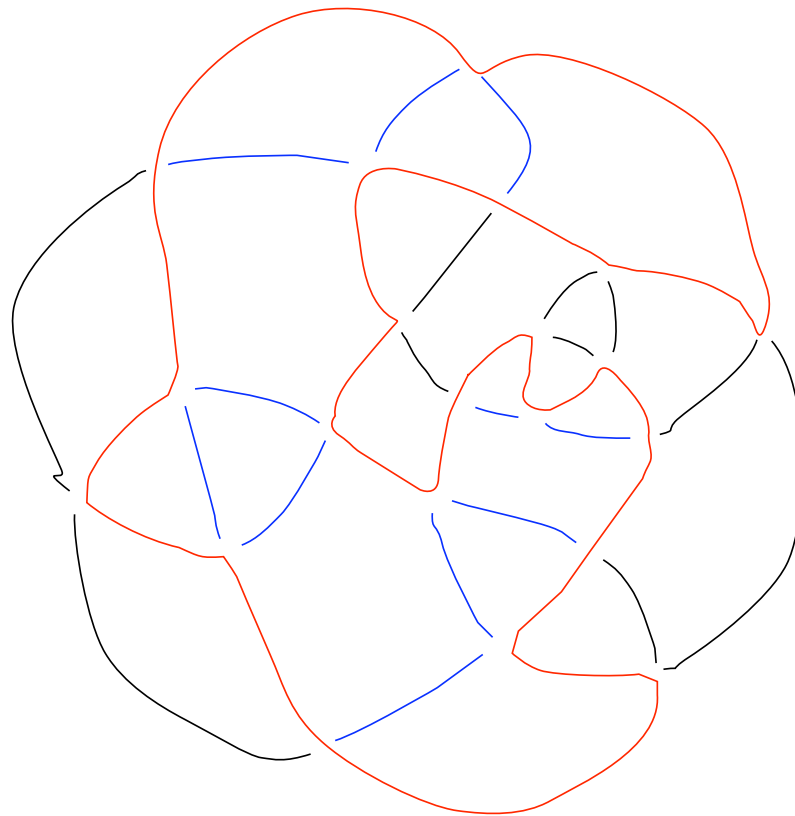
## Proof (sketch) of $3/2$ power upper bound

We can arrange for such graphs to always be Hamiltonian (by adding extra verts if needed):



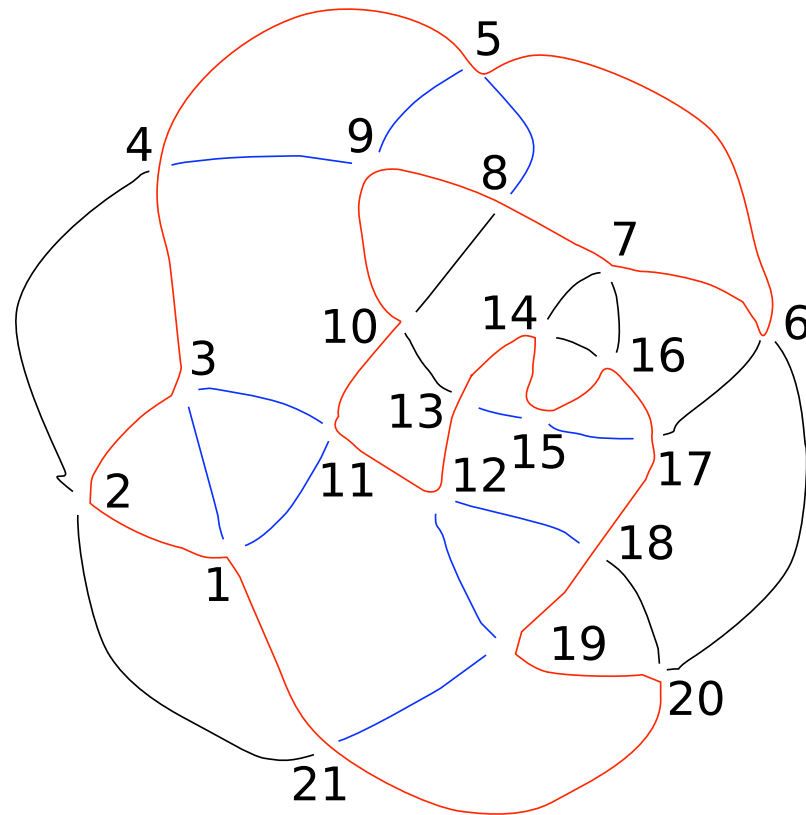
## Proof (sketch) of $3/2$ power upper bound

The edges not on the Hamiltonian circuit are either “inner” or “outer” edges.



## Proof (sketch) of $3/2$ power upper bound

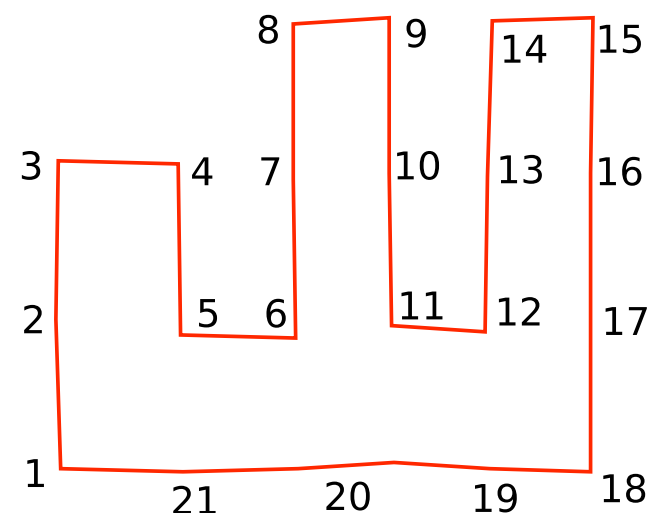
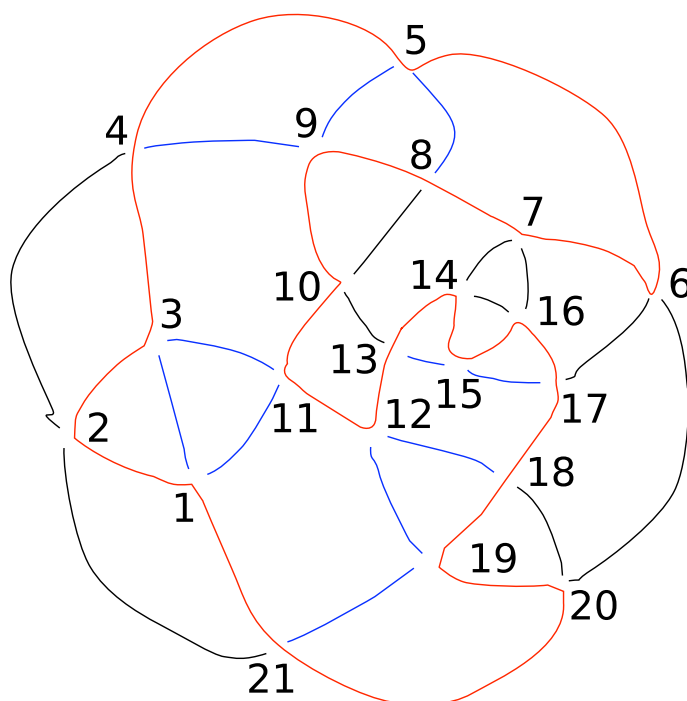
Now we can start the embedding. First, number the vertices to help us keep track:





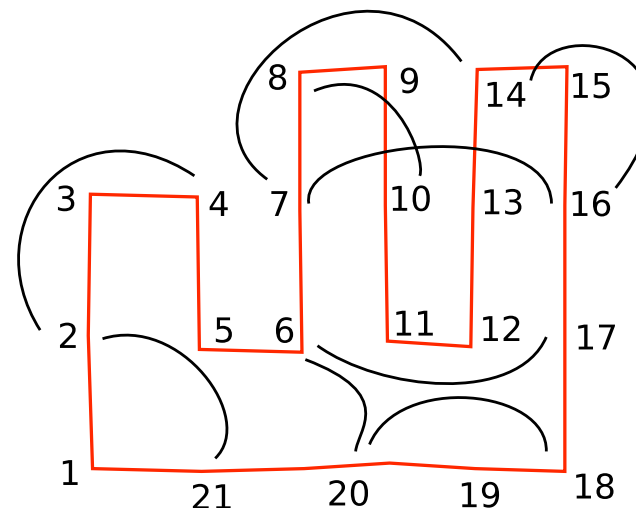
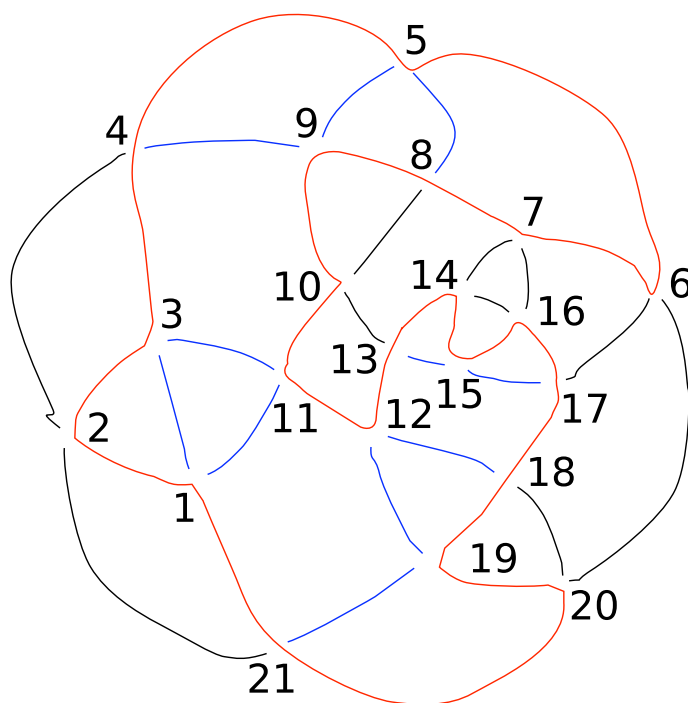
# Proof (sketch) of $3/2$ power upper bound

We can lay out the Hamiltonian circuit in a compact manner on a  $\sqrt{Cr} \times \sqrt{Cr}$  grid in the  $xy$  plane with length  $\sim Cr$ .



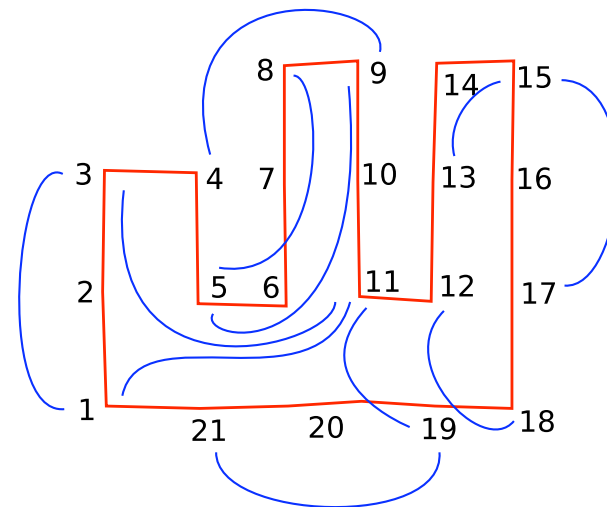
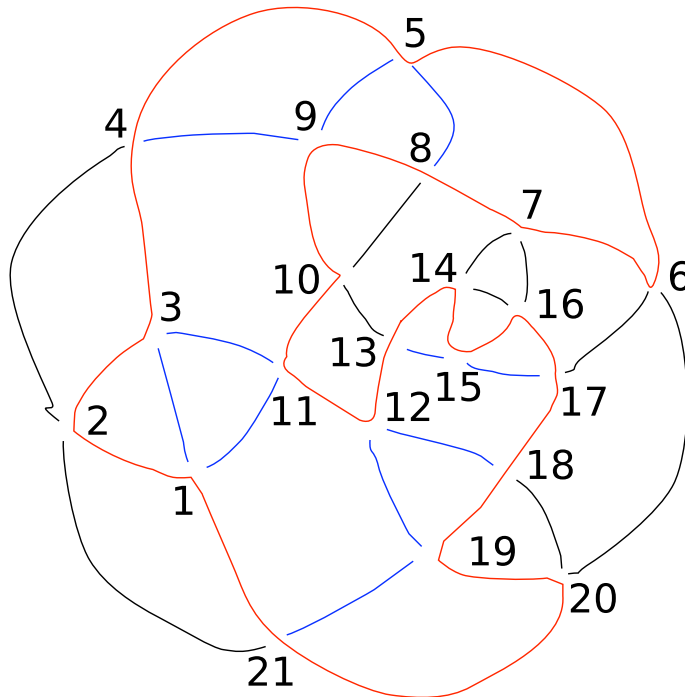
# Proof (sketch) of $3/2$ power upper bound

The “outer edges” are embedded above the  $xy$  plane. They have length  $\sim \sqrt{Cr}$ , and there are at most  $Cr$  of them.



# Proof (sketch) of $3/2$ power upper bound

The “inner edges” are embedded below the  $xy$  plane. They also have length  $\sim \sqrt{Cr}$ , and there are at most  $Cr$  of them.

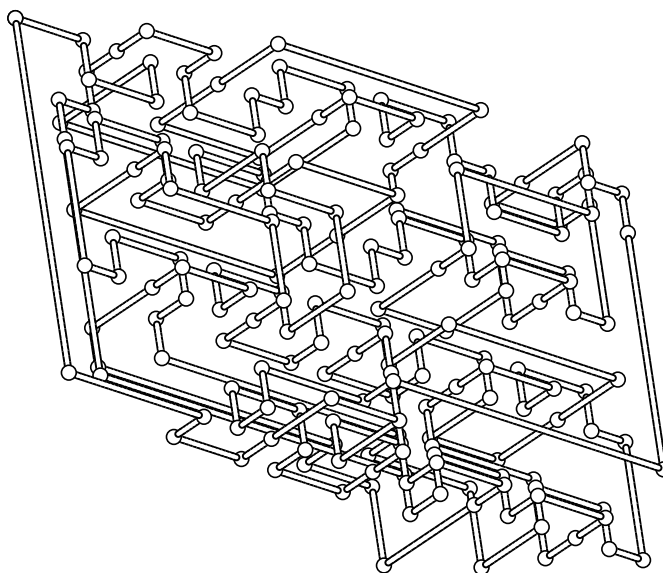


## Proof (sketch) of $3/2$ power upper bound

This gives a total ropelength  $\sim Cr^{3/2}$  (modulo plenty of details).  
The actual upper bound is

$$\text{Rop}(K) \leq 272 \text{Cr}(K)^{3/2} + 168 \text{Cr}(K) + 44\sqrt{\text{Cr}(K)} + 22.$$

Diao et. al. have implemented this algorithm, producing:



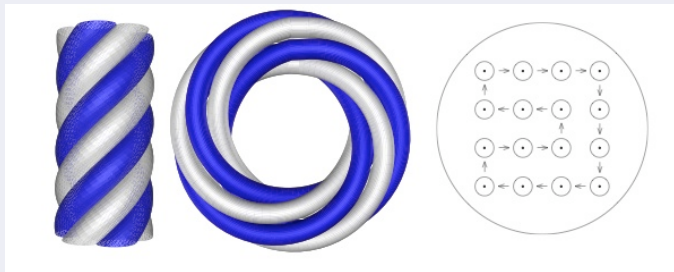
# The asymptotic relationship between Rop and Cr

Theorem (with Kusner, Sullivan 1998, Diao, Ernst 1998)

*There are examples of infinite families of knots with  $\text{Rop}(K_n) \sim \text{Cr}^p(K_n)$  for every  $p$  between  $3/4$  and  $1$ .*

Example for  $p = 3/4$ .

In a solid torus of radii  $r$  and  $R$  “cabled” with unit tubes forming an  $(n, n - 1)$  torus knot,  $r \sim \sqrt{n}$  and  $R \sim r$ .



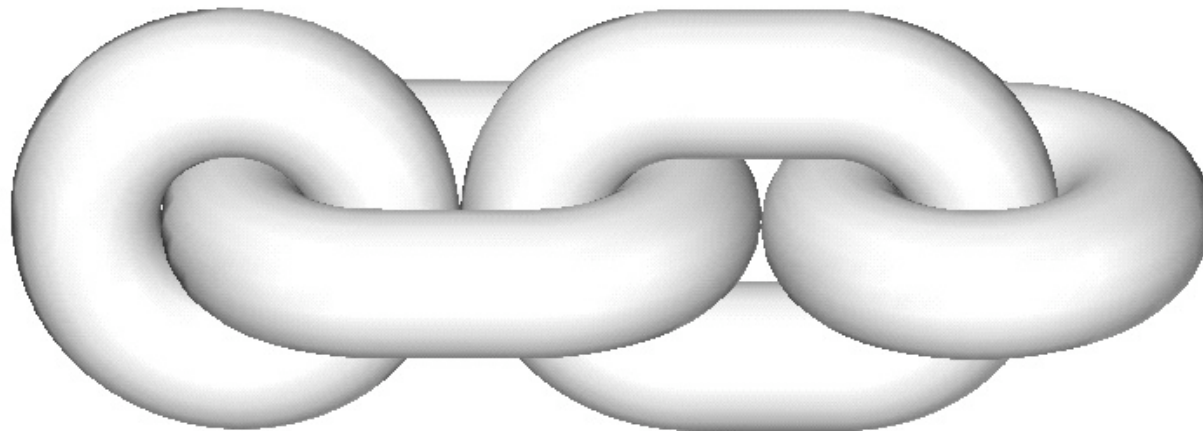
$$\text{Rop} \sim nR \sim n^{3/2}, \quad \text{Cr} = n(n - 2) \sim n^2, \quad \text{Rop} \sim \text{Cr}^{3/4}.$$

# An example family of links with $\text{Rop}(L) = a\text{Cr}(L) + b$

Theorem (with Kusner, Sullivan 2002)

*The minimum ropelength of a chain  $L_n$  of  $n$  links is*

$$\text{Rop}(L_n) = (4\pi + 4)n - 8.$$



The inner “stadium curves” have length  $4\pi + 4$  while the end rings have length  $4\pi$ . Proof discussed later.

## Open Questions: Ropelength and Crossing Number

- 1 Is there any family of links with  $\text{Rop}(L_n) \sim \text{Cr}(L_n)^p$  for  $p > 1$ ?
- 2 Can you find an upper bound so that  $\text{Rop}(K) \leq c_2 \text{Cr}(K)^p$  for  $p < 3/2$ ? The graph embedding literature suggests that a better bound should be possible.
- 3 The example family with  $\text{Rop}(L_n) \sim \text{Cr}(L_n)^{3/4}$  was very nonalternating. Is it true that  $\text{Rop}(L_n) \geq c \text{Cr}(L_n)$  for alternating knots?
- 4 The simple chain of 3 links has ropelength  $12\pi + 4$ . It is a connect sum of two Hopf links, each with ropelength  $8\pi$ . Is it always true that

$$\text{Rop}(K_1 \# K_2) \leq \text{Rop}(K_1) + \text{Rop}(K_2) - (4\pi - 4)?$$

# Thank you for coming!

Slides on the web at:

`http://www.jasoncantarella.com/`

under “Courses” and “Geometric Knot Theory”.

Topics for Lecture 2:

- ① Ropelength bounds in terms of other knot invariants.
- ② Computation of approximate ropelength minimizers.
- ③ Gordian unknots and local minima for ropelength.
- ④ Geometry of tight knots.