Knots and braids

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Advanced School on Knot Theory and its Applications to Physics and Biology

ICTP, 11-22 May 2009

Carl Friedrich Gauss in 1830 drew this:

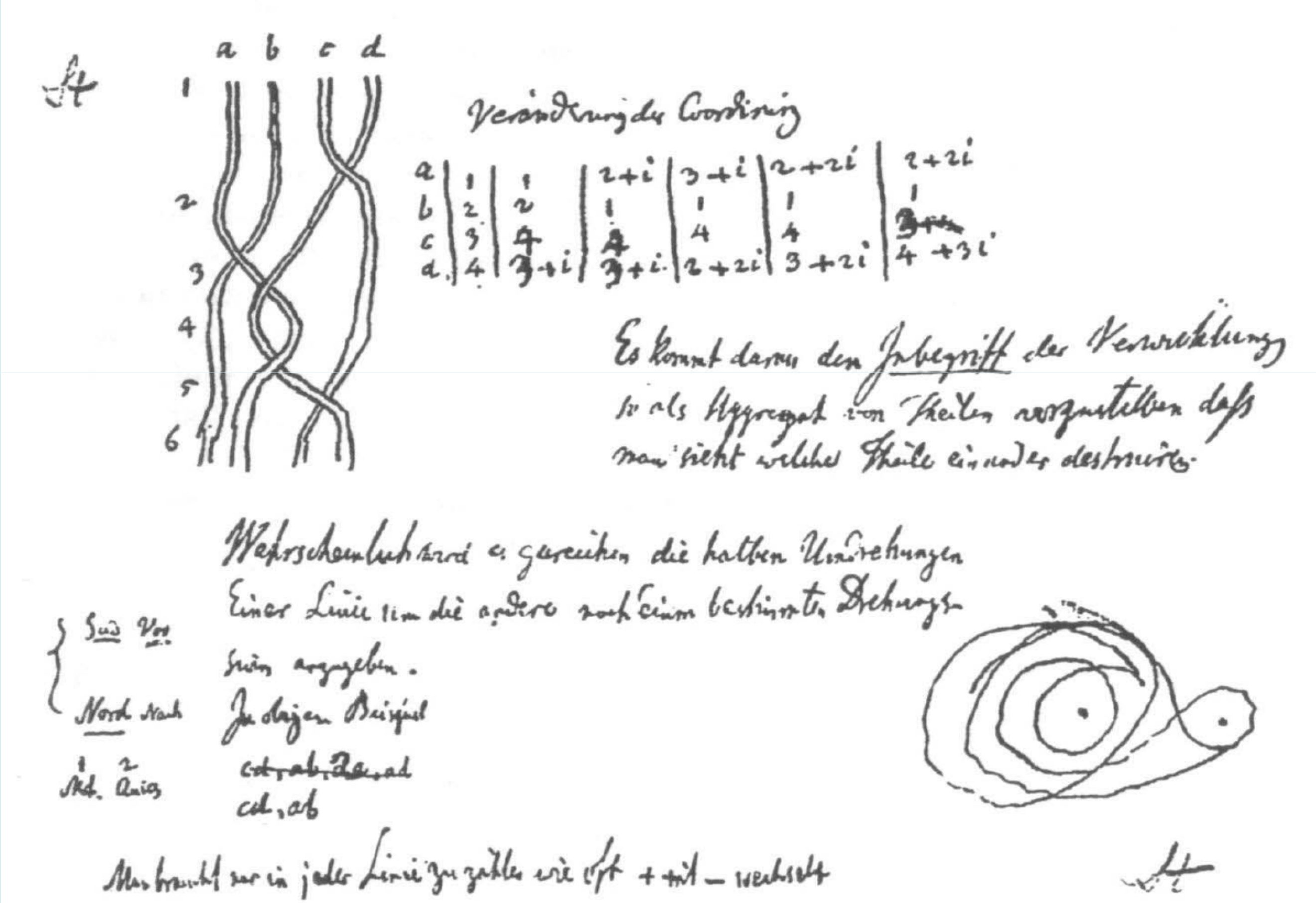
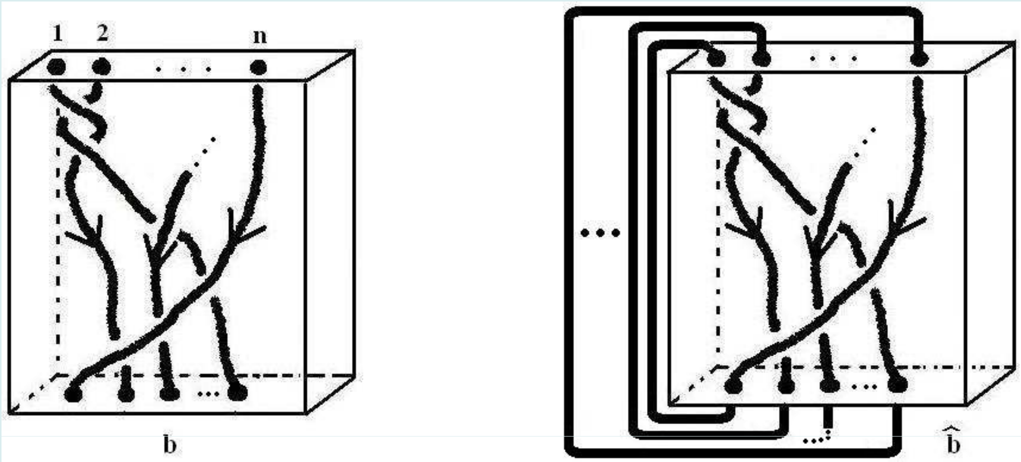


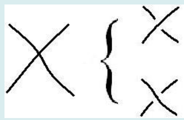
Figure 2. Page 283 of Gauss's Handbuch 7.

Braids – the braid group

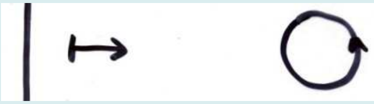


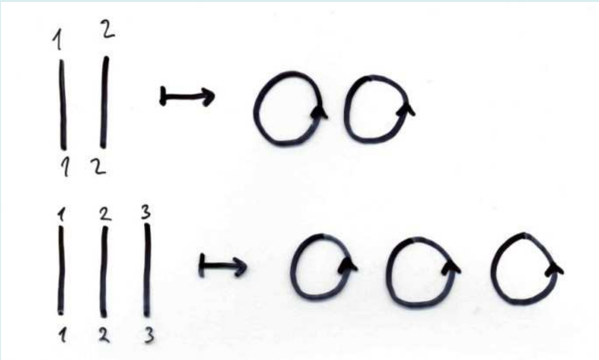
A **braid** on n strands is a homeomorphic image of n arcs in the interior of $[0,1]\times[0,\varepsilon]\times[0,1]$, $\varepsilon > 0$, such that the boundary of the image consists in n numbered points in $[0,1]\times[0,\varepsilon]\times\{1\}$ and n corresponding numbered points in $[0,1]\times[0,\varepsilon]\times\{0\}$, and it is monotonous, that is, no local maxima or minima.

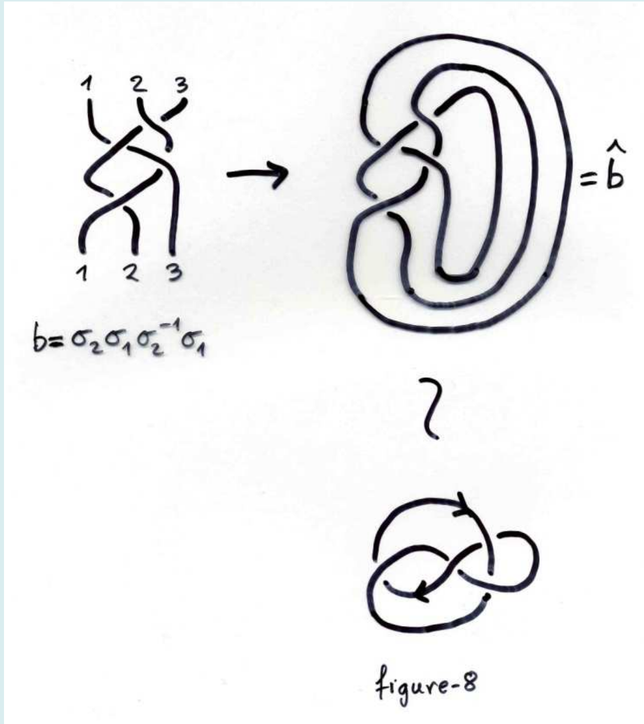
The **closure** of a braid consists in joining with simple arcs the corresponding endpoints.

A **braid diagram** is a regular projection on $[0,1]\times\{\varepsilon\}\times[0,1]$: 

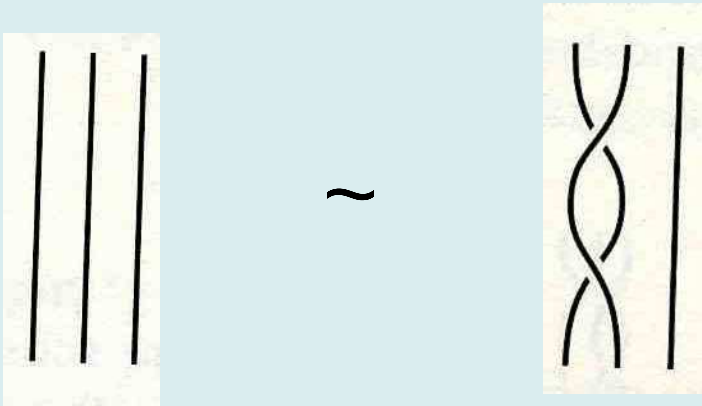
Closing a braid produces an oriented link







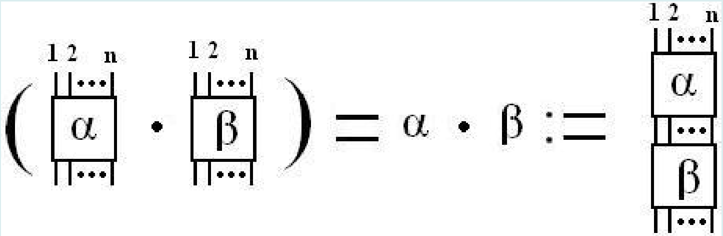
Two braids are **isotopic** through any isotopy in the inrerior of the braid box that fixes the endpoints and preserves the braid structure.



In the set of braids isotopic elements are considered the same.

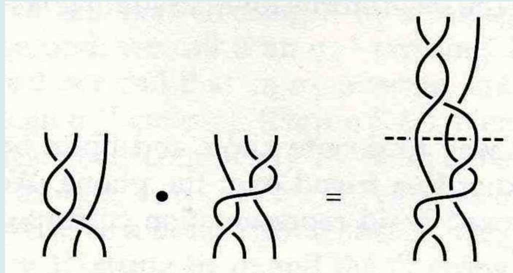
Let B_n denote the set of braids on n strands. Then:

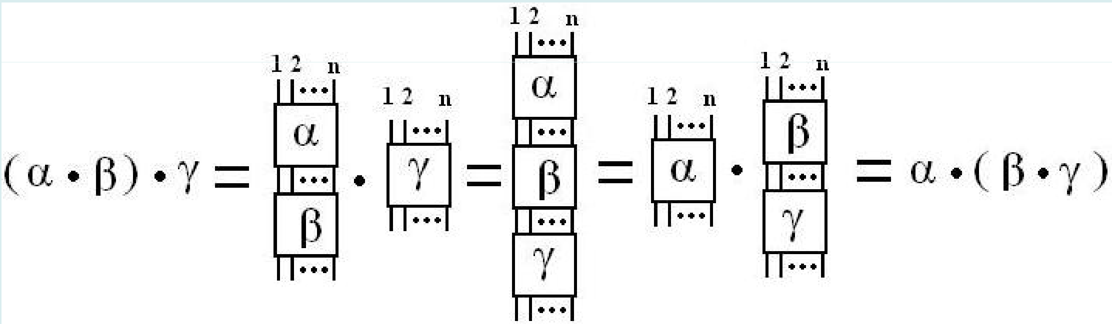
<u>Theorem</u> (Artin, 1926): B_n is a group.



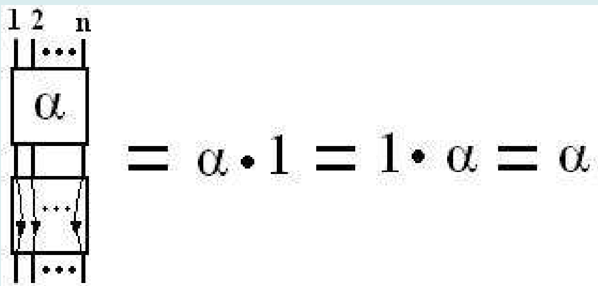
(the operation)

<u>Example</u>:

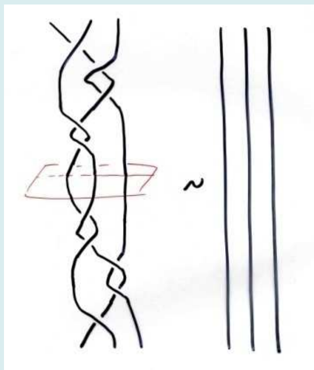




(associative)

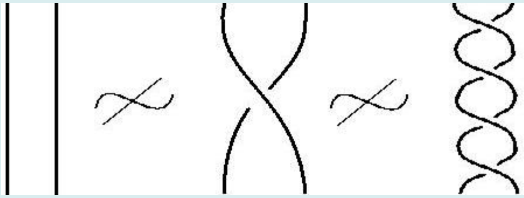


(identity
element)

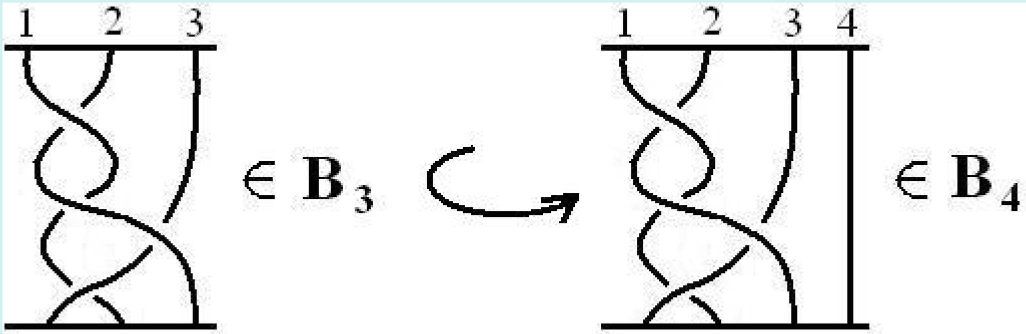


(inverse)

B_n is of infinite order. Indeed, B_2 is the free group on one generator:

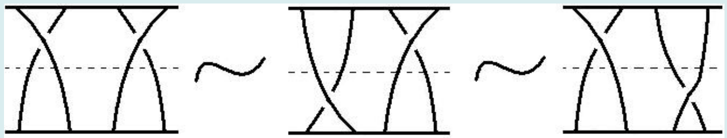


and B_n embeds in B_{n+1}:

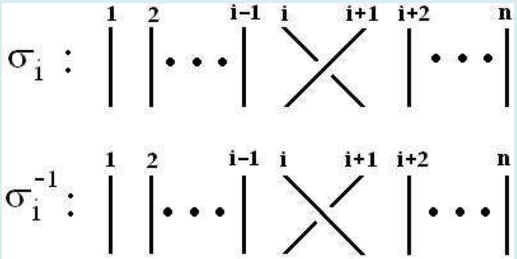


A presentation for B_n

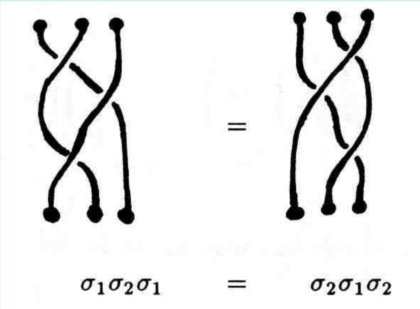
Every braid can be sliced into horizontal zones, such that in each zone there is only one crossing:



So, every braid is a product of the following elementary braids:



The basic braid relation:

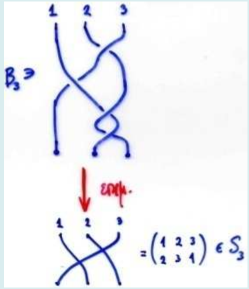


Theorem (Artin, 1926): B_n is presented by the generators $\sigma_1, \ldots, \sigma_{n-1}$, satisfying the relations:

$\sigma_i\sigma_j = \sigma_j\sigma_i$ for |i − j | > 1

$\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ for |i − j | = 1

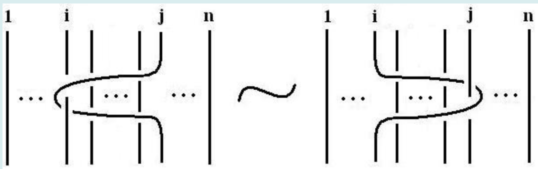
Proof (by Chow, 1948):



- There is a natural epimorphism $B_n \to S_n$
- The kernel is the pure braid group P_n so $P_n \lhd B_n$
- The following is a short exact sequence (Convince yourselves!)

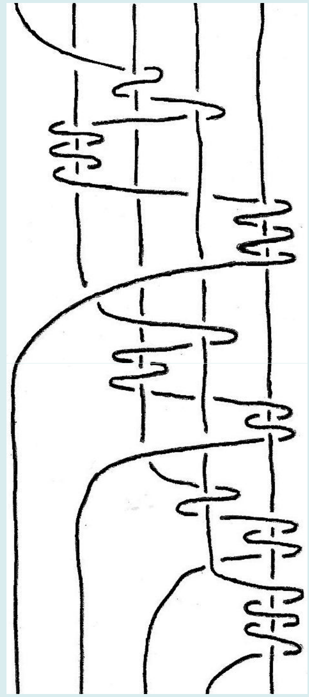
$$1 \to P_n \to B_n \to S_n \to 1$$

- P_n is generated by the loops



Think about relations in P_n

- S_n acts on P_n by permutation of the indices. (Show it is an action!)
- Apply the Schreier method.

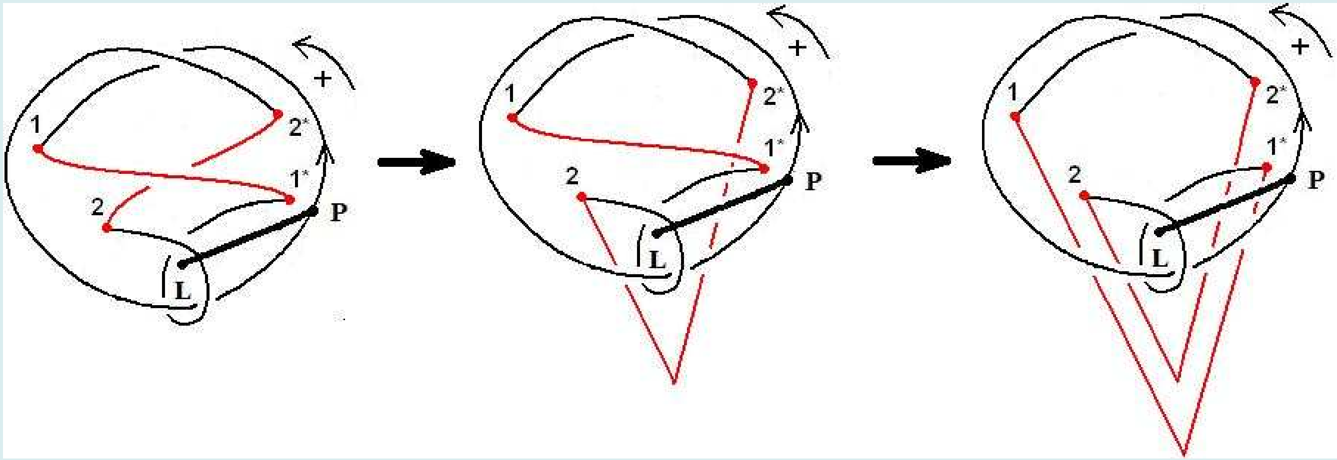


Artin's combing

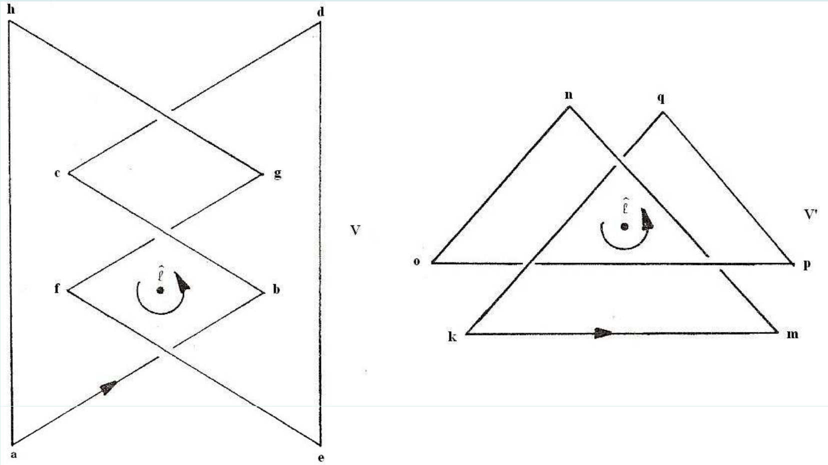
The Alexander theorem

Alexander theorem (1923): Every oriented knot or link may be isotoped to a closed braid.

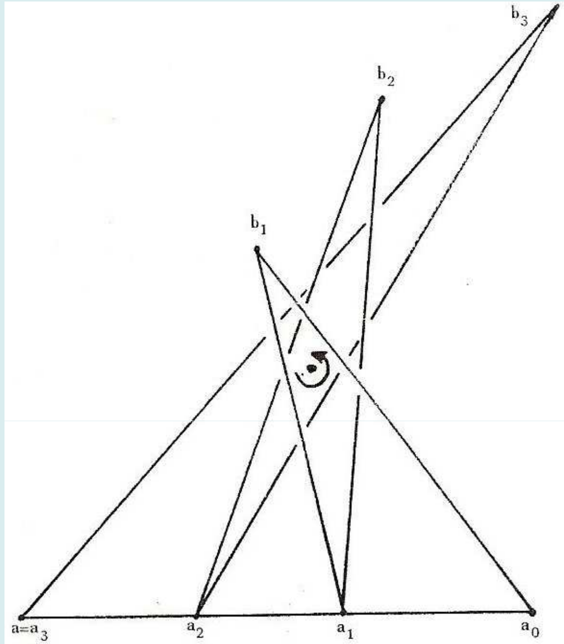
Proof 1 (Alexander, 1923): The braid axis is perpendicular to the blackboard. The result is a closed braid.



Proof 2 (Birman, 1976): based on Alexander's idea, but more rigorous.

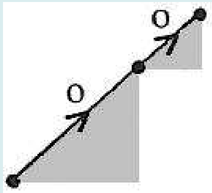


The height of a link is the number of `wrong' arcs.

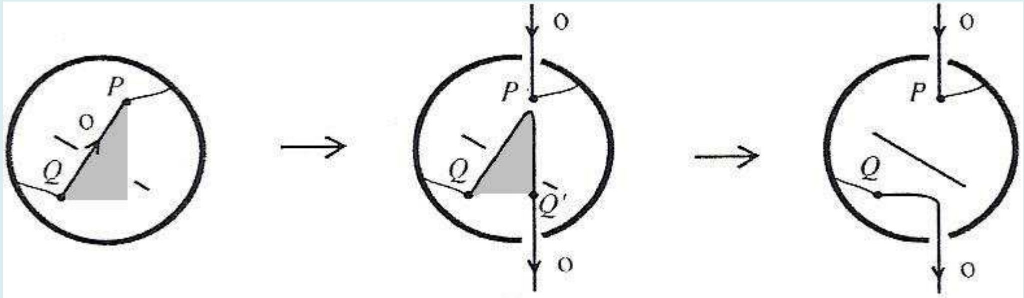


A sawtooth

Proof 3 (Lambropoulou, 1990): The braid axis is parallel and behind the blackboard. The result is an open braid.

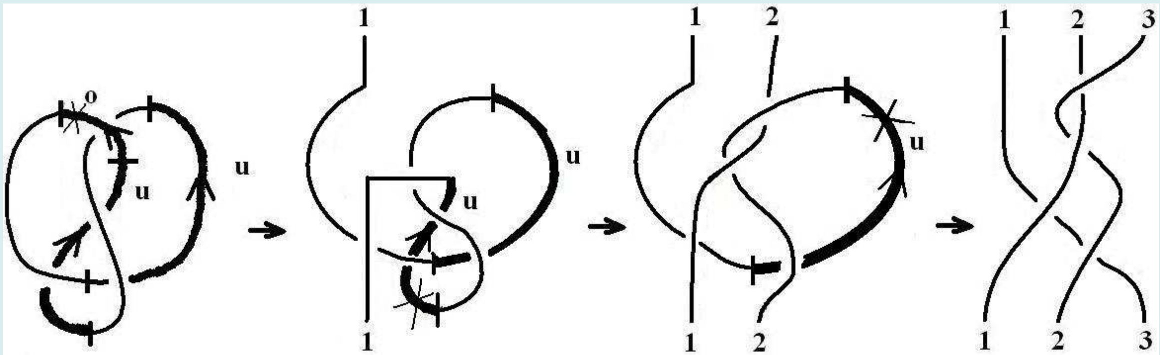


Subdivide up-arcs
(if necessary)

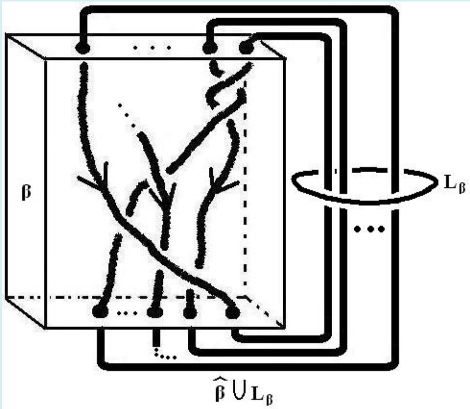


The elimination of an up-arc results a pair of corresponding braid strands

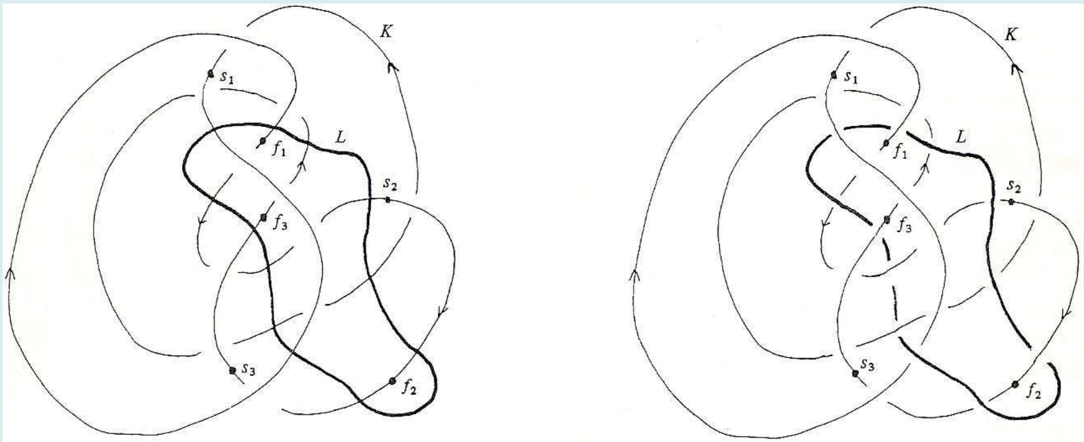
An example:



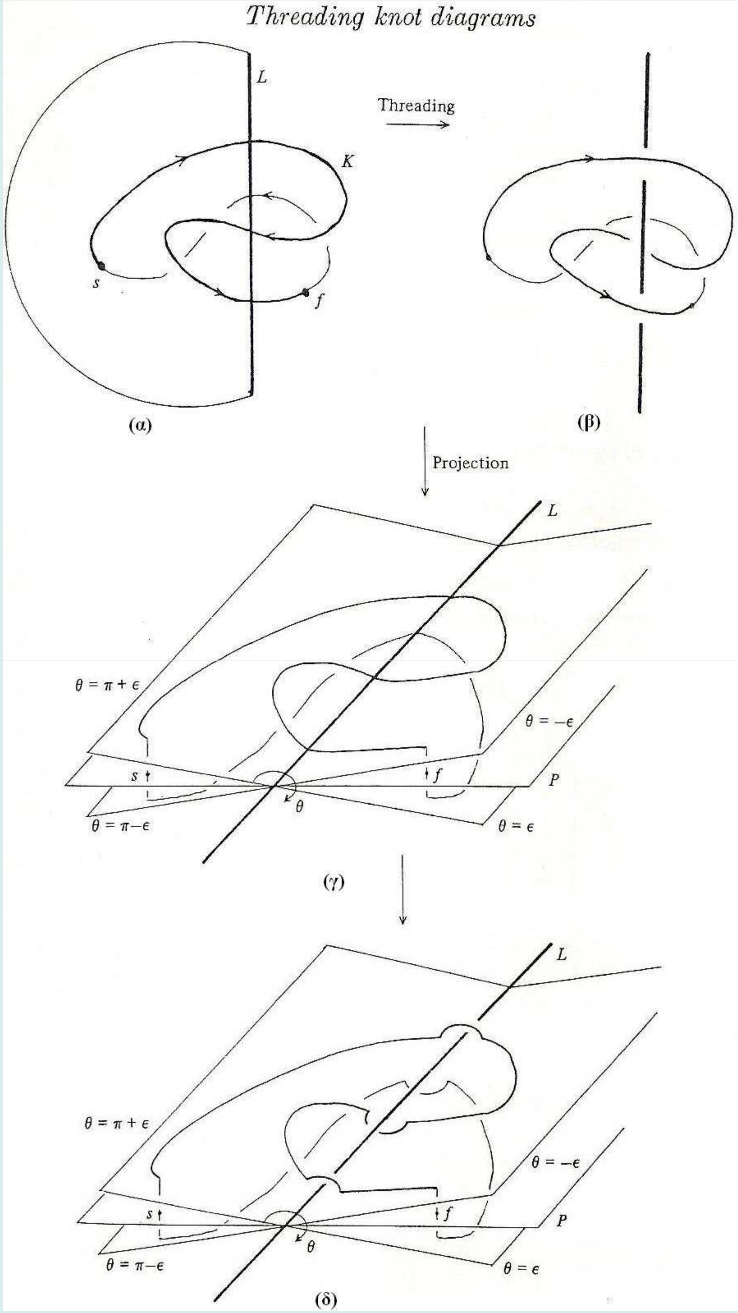
Proof 4 (Morton, 1986): The braid axis is a closed curve. The result is a closed braid.



Complete closure of a braid



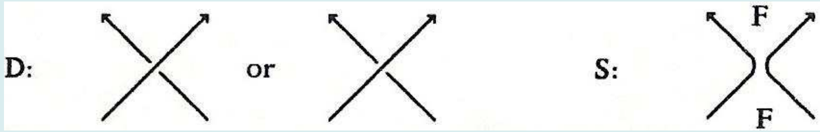
A threading results in a braided link (i.e. isotopic to the complete closure of a braid).

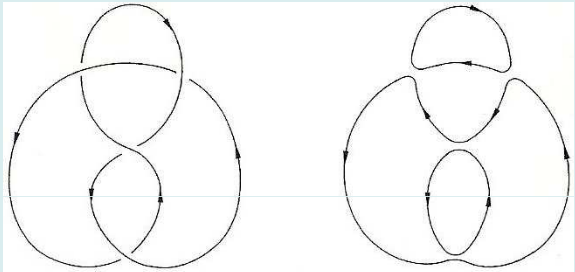


The polar coordinate is monotonically increasing, but is constant on the pages.

Proof 5 (Yamada,1987): The braid axis is perpendicular to the blackboard. The result is a closed braid. Uses Seifert circles.

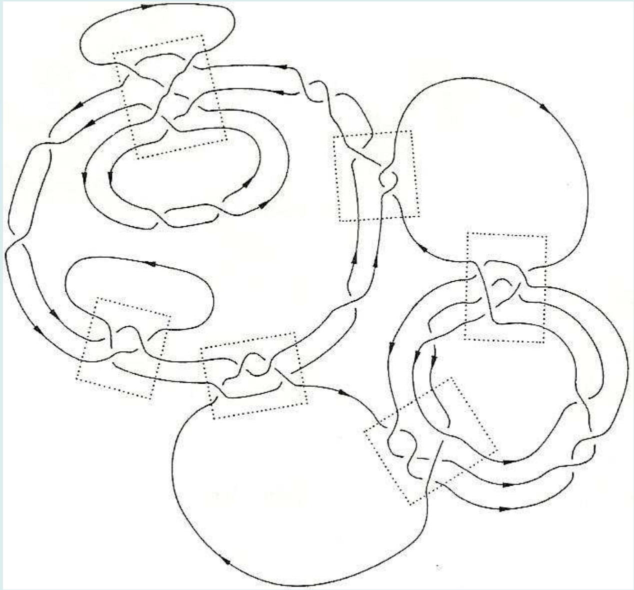
Smoothing each crossing in a diagram produces the Seifert circles

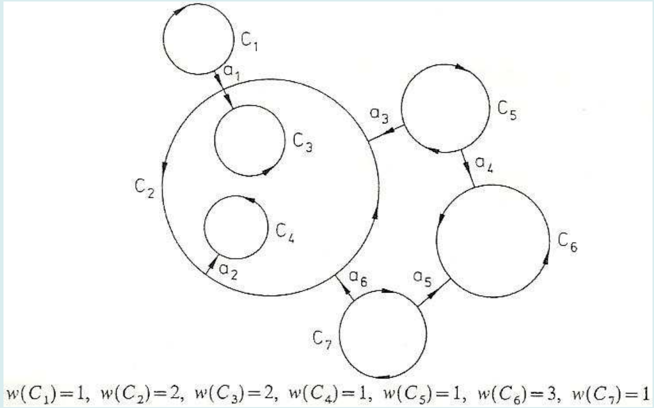




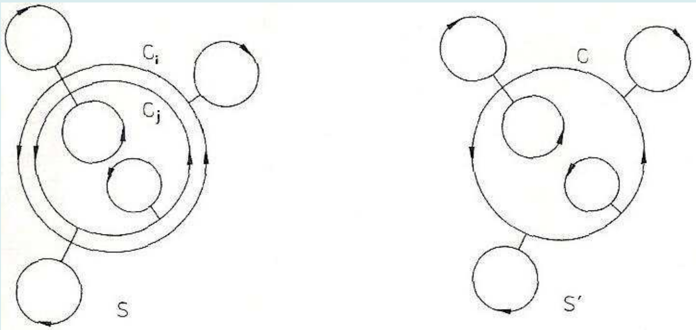
The *Seifert number* s(L) is the least s(D) for any diagram D of L.
The *braid index* b(L) is the least number of strands among all braid representations of L. Yamada's braiding algorithm implies the following:
<u>Theorem</u> (Yamada,1987): b(L) = s(L).

Which is the easy way round: $b(L) \leq s(L)$ or $s(L) \leq b(L)$?

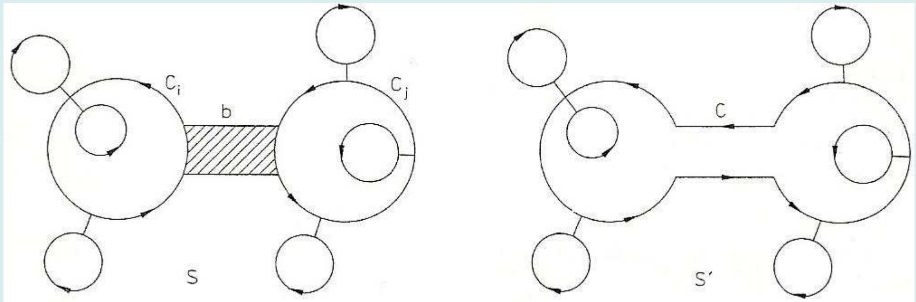




A weighted system of circles



1st grouping operation



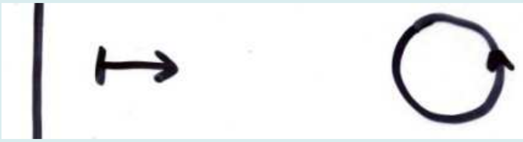
2nd grouping operation

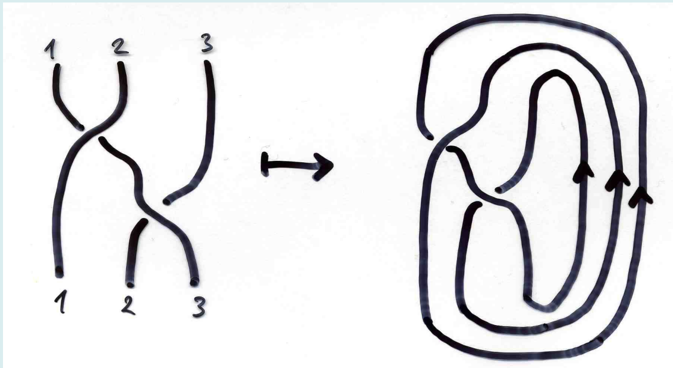
- A grouping operation can be always applied on a non-trivial system.
- A closed braid gives rise to a trivial system.

Proof 6 (Vogel, 1990): Based on Yamada's algorithm. Measures complexity by trees.

The Markov theorem

Different braids close to isotopic links.
For example conjugate braids or the braids below.





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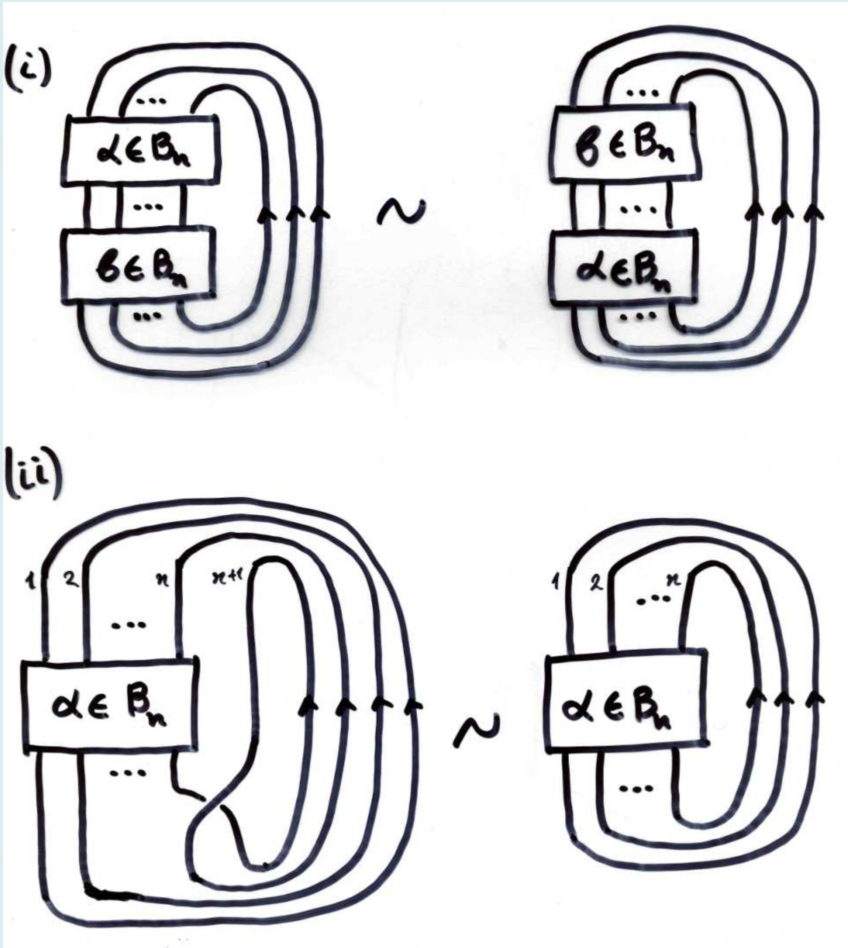
How are such braids related?

<u>Markov theorem</u> (1936): Two oriented knots or links are isotopic iff any two corresponding braids differ by finitely many of the moves:



Conjugation in each braid group:

$$B_n \ni \quad \sigma_i^{-1} \alpha \sigma_i \sim \alpha \quad \in B_n$$

The Markov move (or stabilization move):

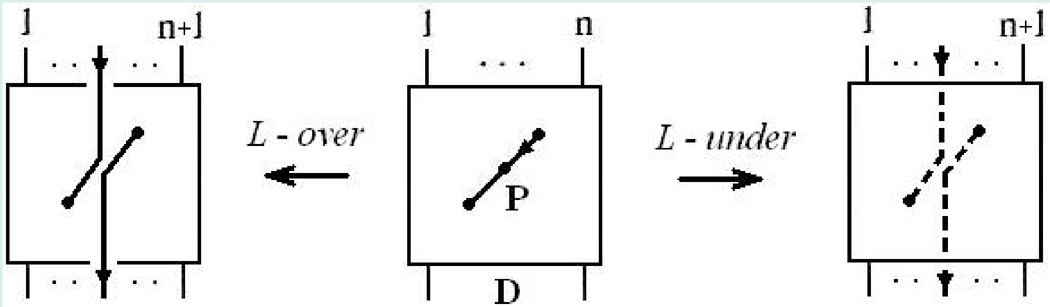
$$B_n \ni \quad \alpha \sim \alpha \sigma_n^{\pm 1} \quad \in B_{n+1}$$

Proofs of Markov's theorem by:

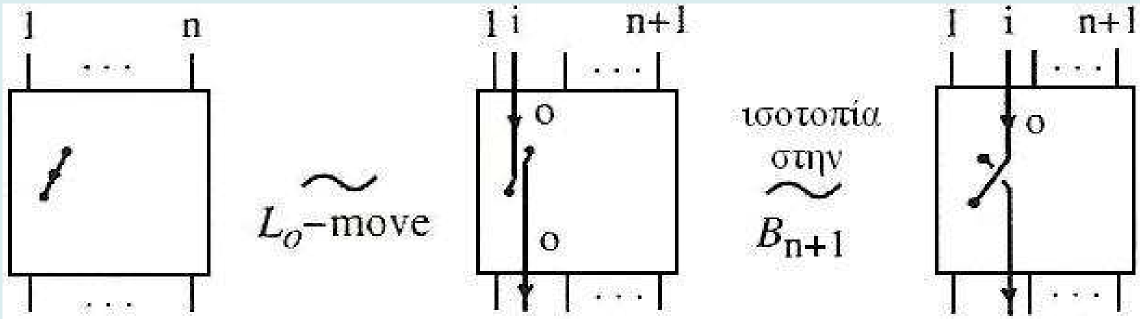
- A.A. Markov, 1936 – **3 moves** (using Alexander's algorithm)
- N. Weinberg, 1936 – reduced to the **2 known moves**
- Joan Birman, 1976 (rigorous, filled in all details)
- Bennequin, 1982 (using 3-dimensional contact topology)
- Morton, 1986 (using the threading algorithm)
- Lambropoulou & Rourke, 1997 – **1 move** (using Lambropoulou's algorithm)
- Traczyk, 1998 (using Vogel's algorithm)
- Birman & Menasco, 2002 (using Bennequin's ideas)

<u>1-move Markov theorem</u> (Lambropoulou & Rourke, 1997): Two oriented knots or links are isotopic iff any two corresponding braids differ by the L-moves.

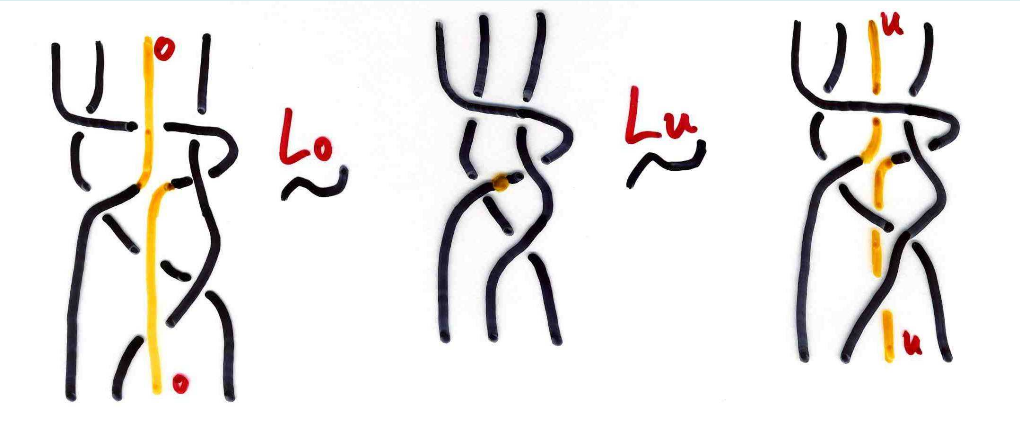
The L-moves



L-equivalent braids have isotopic closures. (Can you see this?)

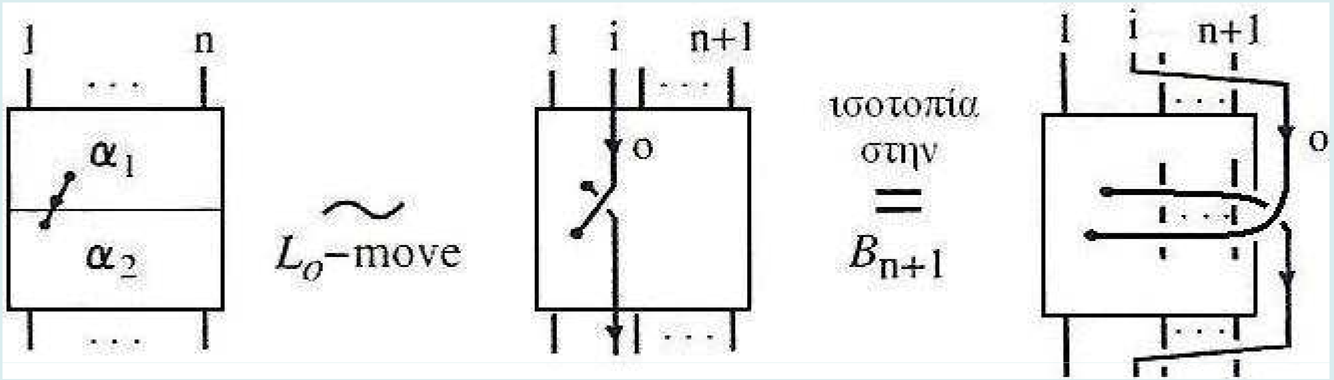


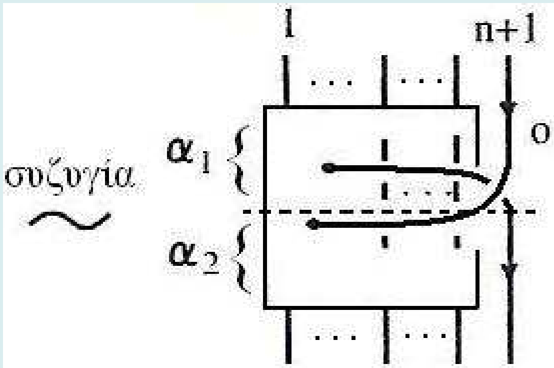
An L-move is the same as introducing a crossing inside the braid box.



An example of L-equivalent braids

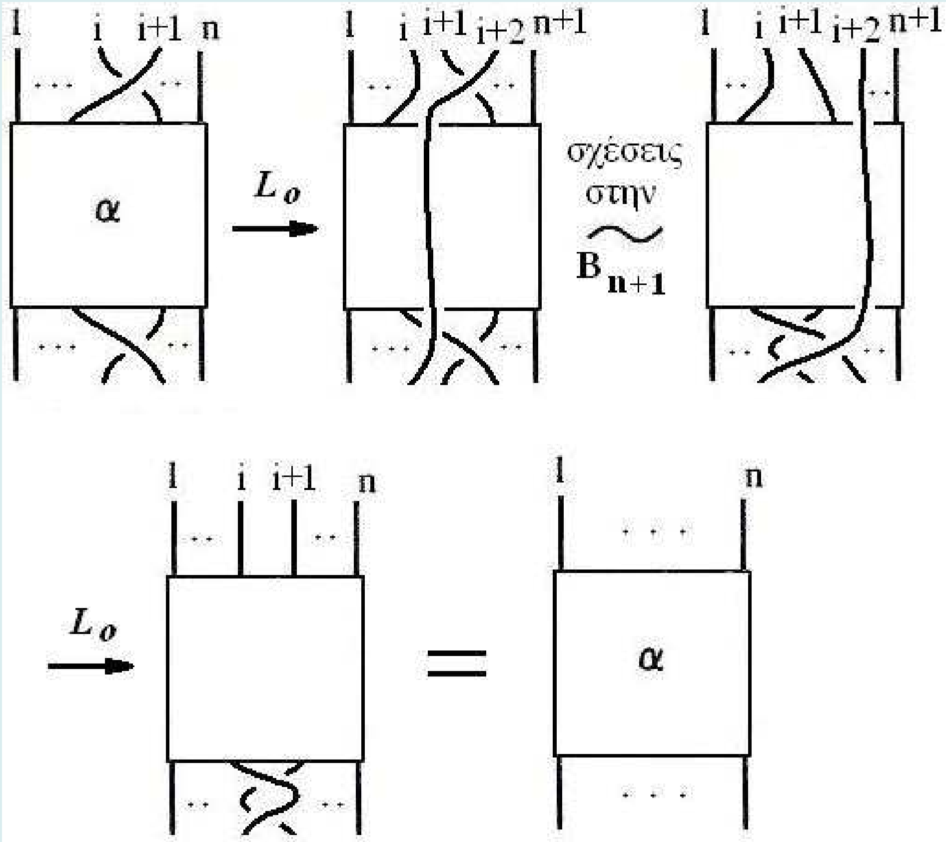
An L-move is built from the classical moves:



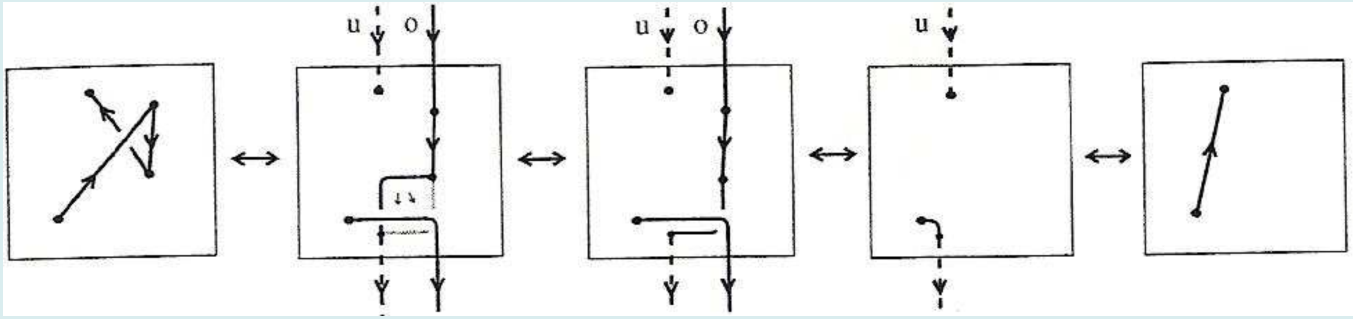


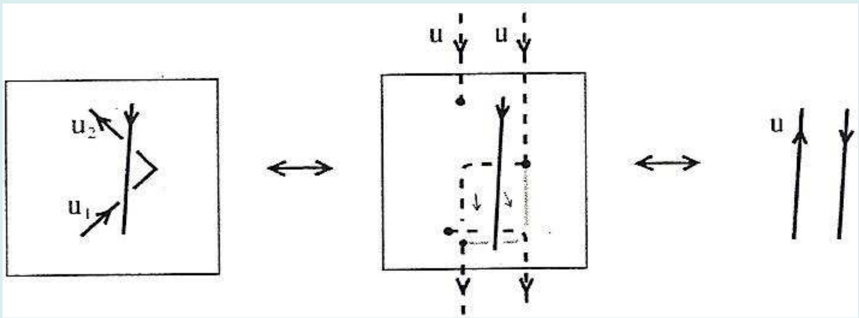
The classical moves are built from L-moves:

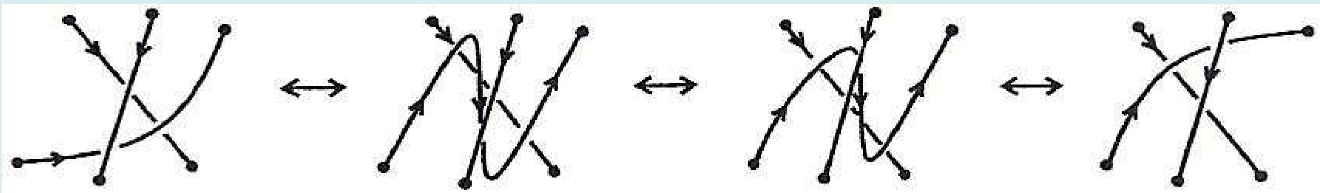
Indeed, a Markov move is simply a special case of an L-move (in the crossing form). Conjugation is illustrated below.



Reidemeister moves on the diagram correspond to L-moves on the braid level, thus the 1-move Markov theorem follows.







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