An Extension of the Kontsevich Integral to Knotted Trivalent Graphs

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Goal: $Z: \mathcal{K}(\Gamma) \to \mathcal{A}(\Gamma)$

knottings of trivalent graph Γ (oriented, framed)

$\mathcal{A}(\Gamma)$ = {chord diagrams on skeleton Γ} / 4T, VI

Z **homomorphic expansion**: should commute with operations delete, unzip, connected sum

i.e. $Z(d_e\Gamma) = d_e Z(\Gamma)$ etc.

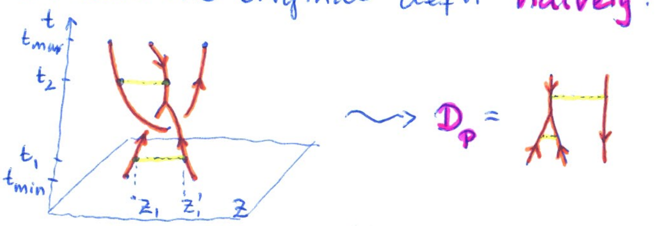
Bar-Natan (Thursday): "Z = choice of an associator!"

Murakami - Ohtsuki ('97): associator theory ⇝ construction of Z

Chmutov, 2 weeks ago: elementary construction of Z for knots

We do: elementary extension (easier than associators) mirroring the knot construction, but more complicated issues.

Extend the original def'n **naively**:



$$Z(\gamma) = \sum_{m=0}^{\infty} \int_{t_{min} < t_1 < \dots < t_m < t_{max}} \sum_{P=(z_i(t), z_i'(t))} \frac{(-1)^{P_\downarrow}}{(2\pi i)^m} D_P \bigwedge_{i=1}^{m} \frac{dz_i - dz_i'}{z_i - z_i'}$$

t_i non-crit non-vtx

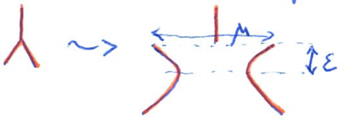
$Z(\gamma) \in \mathcal{A}(\Gamma)$ = {chord diag. on skeleton Γ} / 4T, VI

Problem I: divergent

short chords  ⇝ denominator → ∞

Solution: renormalize

fix $\mu \in \mathbb{R}$, "open up strands":



take limit $\varepsilon \to 0$

similarly for Y, ∩, ∪

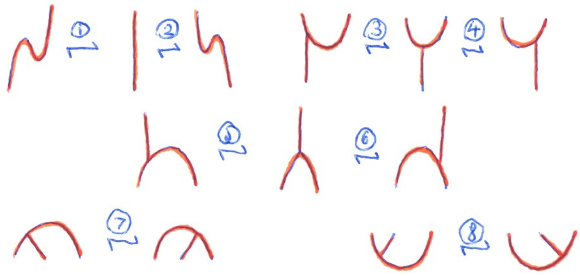
forbid V, Λ

⇒ **convergent**

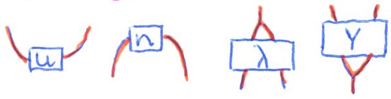
Problem II: not quite invariant.

Invariant under: horizontal defs,

Additional **moves** needed:



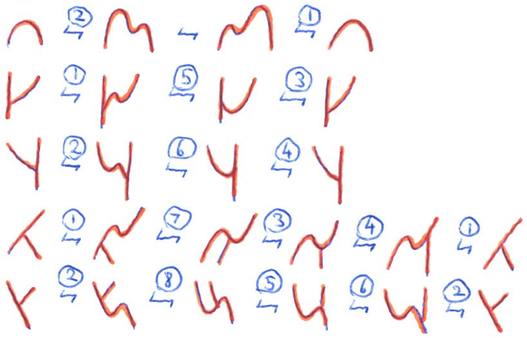
"**Correctors**" available:



$u, n \in \mathcal{A}(\uparrow)$
$\lambda, Y \in \mathcal{A}(\uparrow\uparrow)$

8 equations, 4 unknowns?

Syzygies: relations between the equations

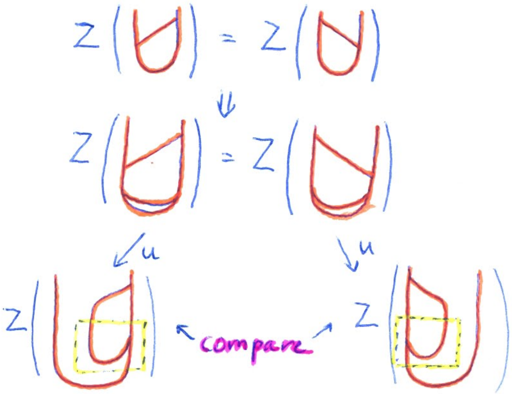


⇒ **reduced to ①, ③ ④**

①: Solve as in knot case

Problem:  not independent, but ③ ⇏ ④

Idea in solving ③ and ④ simultaneously:



⇝ 1-parameter family of solutions u, n, λ, Y

Theorem We have constructed a **1-param. family of invariants**, all of which are

- **universal** finite type invts of knotted trivalent graphs
- **well-behaved** under the KTG operations (homomorphic expansion)

In particular, we have constructed an associator Z().

Thank you!

ALGEBRAIC KNOT THEORY & THE KONTSEVICH INTEGRAL

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Joint work with Dror Bar-Natan

A bit of history

Vassiliev 1990: notion of finite type invariants

Fundamental Thm (Kontsevich 1993)

finite type invariants $\overset{1-1}{\longleftrightarrow}$ "weight systems" = linear functions on "chord diagrams"

Big tool used to prove this: Kontsevich Integral "universal finite type invariant"

Question (open): Does the Kontsevich Integral separate knots?

Murakami - Ohtsuki 1997: extended K.I. to knotted trivalent graphs using associators

Idea (Bar-Natan): Use extra structure on trivalent graphs and come up w/ structure preserving invariants to learn more about knots "Algebraic Knot Theory"

The Kontsevich Integral

Z: {knots} $\to \mathcal{A}$ "algebra of chord diagrams"

chord diagram:  circle w/ chords, i.e. a pairing of pts on the circle (only combinatorial info). # of chords called **order**

$\mathcal{A}$ = v. space gen. by all chord diagrams / 1T, 4T relations

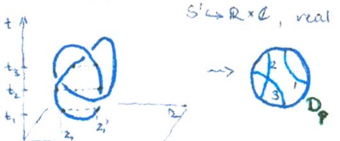
multiplication: connected sum # (well-def by 4T!)

Values of Z: formal infinite series in $\mathcal{A}$

Definition of Z

$\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$, K oriented Morse knot

$S^1 \hookrightarrow \mathbb{R} \times \mathbb{C}$, real coordinate Morse fctn



$$Z(K) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int_{\substack{t_{min}<t_1<\dots<t_m<t_{max} \\ t_i \text{ non-crit}}} \sum_{P=\{(z_i,z_i')\}} (-1)^{P_\downarrow} D_P \bigwedge_{i=1}^{m} \frac{dz_i - dz_i'}{z_i - z_i'}$$

t_{min}, t_{max}: min & max of t on K

Int. domain: m-dim simplex divided by crit. values into conn. cpnents. # of summands const. on each cpnent

P: In plane $\{t=t_i\}$ choose unordered pair of pts on K (z_i,t_i) & (z_i',t_i) s.t. $z_i(t_i)$, $z_i'(t_i)$ cont. on each component

$P_\downarrow$: # of pts (z_i,t_i) or (z_i',t_i) where t decreases along K

D_P: chord diagram associated to P

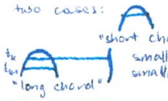
Convergence & Invariance

Claim: The integral always convergent.

Proof sketch: $\frac{1}{z_i - z_i'}$ bounded unless near $\cap$ or $\cup$

two cases: "short chord" $\leadsto D_P = 0$ (1T)

"long chord": smallness of domain for kth chord cancels out smallness of the denominator for k+1th chord.



Theorem: Z invariant under deformations that do not change the # of critical points.

invariance under horizontal deformations: use Z for tangles, chop up knot, use Stokes thm for braid components.

Then prove we can move crits by shrinking / extending sharp needles.

Problem: Z not invariant under 

Solution: $Z(K) = Z(K) \cdot Z(\infty)$ 

Note: for any K, the degree 0 part of Z(K) is by convention $\bigcirc \cdot 1$, the unit of $\mathcal{A}$.

$\Rightarrow$ Z(K) has inverse by $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$

$\Rightarrow$ To fix problem, replace Z by $Z(K) \cdot Z(\infty)^{c/2}$, where c = # of crits in K.

New Z is a knot invariant!

Nice properties:
1) $Z(K_1 \# K_2) = Z(K_1) Z(K_2)$ (Fubini)
2) for K framed, c cabling, and appropriate definition of c on $\mathcal{A}$: $Z(c(K)) = c(Z(K))$.

Problem #, c not enough structure on knots ...

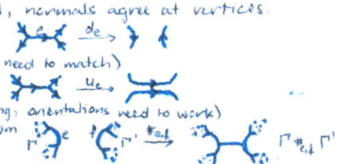
Possible solution (Bar-Natan): extend to knotted trivalent graphs

Γ trivalent graph:
- 3 edges meeting at each vtx
- allow multiple edges, circles ⊖ ○
- each edge oriented

knotting: $\Gamma \hookrightarrow \mathbb{R}^3$ embedding, up to isotopy

framing: edges framed, normals agree at vertices.

Operations:
- edge delete $\xrightarrow{d_e}$ (orientations need to match)
- edge unzip $\xrightarrow{u_e}$ (needs framing, orientations need to work)
- connected sum $\Gamma \#_{e,f} \Gamma'$



Interesting knot properties (e.g. genus, unknotting #, ribbon property) defineable through the graphs and these operations.

Algebraic knot theory

We want: structure preserving invariants
- "algebraic" spaces $\mathcal{A}(\Gamma)$ for each graph Γ
- homomorphisms $d_e: \mathcal{A}(\Gamma) \to \mathcal{A}(d_e\Gamma)$, $u_e: \mathcal{A}(\Gamma) \to \mathcal{A}(u_e\Gamma)$, $\#_{e,f}: \mathcal{A}(\Gamma) \otimes \mathcal{A}(\Gamma') \to \mathcal{A}(\Gamma \#_{e,f} \Gamma')$ (cheating here)
- knotted graph invariant $Z: K(\Gamma) \to \mathcal{A}(\Gamma)$ (where K(Γ) = {knottings of Γ}) s.t. Z commutes w/ operations

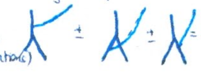
$$\begin{array}{ccc} K(\Gamma) & \xrightarrow{d_e} & K(d_e\Gamma) \\ \downarrow Z & & \downarrow Z \\ \mathcal{A}(\Gamma) & \xrightarrow{d_e} & \mathcal{A}(d_e\Gamma) \end{array}$$

commutative, same for d_e, u_e, $\#_{e,f}$

$\leadsto$ Could obtain algebraic conditions for knot properties.

Extend Kontsevich Integral to obtain such a Z

$\mathcal{A}(\Gamma)$ = ⟨chord diagrams on skeleton Γ⟩ / relations

relations:
- drop 1T (don't want framing independence)
- keep 4T
- add **vertex invariance**  (signs depend on edge orientations)

Operations: $d_e: \mathcal{A}(\Gamma) \to \mathcal{A}(d_e\Gamma)$

$d_e(D) = \begin{cases} 0 & \text{if any chord ends on e} \\ D \text{ w/ e ommitted} & \text{if no chords end on e} \end{cases}$

$u_e: \mathcal{A}(\Gamma) \to \mathcal{A}(u_e\Gamma)$

if no chords end on e → just unzip

if chord ends on e → replace by

$\#_{e,f}$: take connected sum of the graphs, do nothing to the chords. (well def. by 4T, VI)

$Z: K(\Gamma) \to \mathcal{A}(\Gamma)$ extend naively

$$Z(\gamma) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int_{\substack{t_{min}<t_1<\dots<t_m<t_{max} \\ t_j \text{ non-crit non-vtx}}} \sum_{P=\{(z_j,z_j')\}} (-1)^{P_\downarrow} D_P \bigwedge_{j=1}^{m} \frac{dz_j - dz_j'}{z_j - z_j'}$$

for $\gamma \in K(\Gamma)$

Good properties
- $Z(d_e\gamma) = d_e Z(\gamma)$ ... obvious
- $Z(u_e\gamma) = u_e Z(\gamma)$ ... easy
- $Z(\gamma \#_{e,f} \gamma') = Z(\gamma) \#_{e,f} Z(\gamma')$ ... Fubini

But: Divergent.

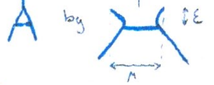
A: 1T doesn't help anymore

⅄: same problem

Renormalization

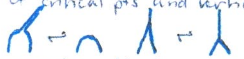
For this need to restrict to vertices of ⅄ or Y shape (will later cause problems w/ invariance)

When integrating, replace (for fixed scale μ)



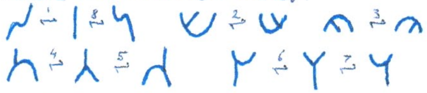
and take $\varepsilon \to 0$ ... do same for Y, ∪, ∩

Proposition: The renormalized Z is
- convergent
- invariant under horizontal defs that leave crits & vertices fixed, and under rigid motions of critical pts and vertices:



- commutes with d_e, u_e, $\#_{e,f}$
- well-behaved under changing the scale μ.

Problem: Not invariant under:



... Fix by corrections!

Looking for corrections:

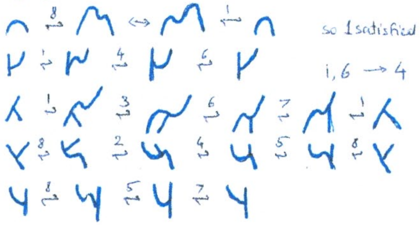
need to satisfy 8 equations,

e.g. $nu Z(\mathcal{N}) \overset{①}{=} Z(/)$

(for this equ $n = u = Z(\infty)^{-1/2}$ works)

8 eqns, 4 degrees of freedom - is it doable?

Syzygies:



so 1 satisfied ⇒ so is 8

1, 6 → 4

1, 6, 7 → 3

8, 4, 5 → 2

8, 7 → 5

So 1, 6, 7 → 8, 4, 5, 2, 3

Enough to satisfy 1, 6, 7!

After corrections: invariant with all the good properties of algebraic knot theory!

Further directions

Problem: $\mathcal{A}(\Gamma)$ poorly understood → computations very difficult

Question: How to find quotients of $\mathcal{A}(\Gamma)$ where computations are accessible, but enough information remains?

References

M. Kontsevich, Vassiliev's Knot Invariants, Adv. in Sov. Math. 16(2) (1993) 137-150

D. Bar-Natan, On the Vassiliev Knot Invariants, Topology 34 (1995) 423-472

S.V. Chmutov, D. Duzhin, The Kontsevich Integral, Acta Applicandae Math 66 (2) 155-190 April 2001

J. Murakami, T. Ohtsuki, Topological Quantum Field Theory for The Universal Quantum Invariant, Comm. in Math Physics, 188-3 (1997) 501-520

D. Bar-Natan, Algebraic Knot Theory: a Call for Action, paperlet available at www.math.toronto.edu/drorbn (also more info on related topics and the Kontsevich Integral)

This handout will be available at: www.math.toronto.edu/~zsuzsi

Thank you!