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Skein Module Motivation for Gram Determinants of Curves on Surfaces

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THE GRAM DETERMINANT OF THE TYPE B TEMPERLEY-LIEB ALGEBRA

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1. Introduction

In this paper, we solve a problem posed by the late Rodica Simion regarding type B Gram determinants, cf. [7]. We present this in a fashion influenced by the work of W.B.R.Lickorish on Witten-Reshetikhin-Turaev invariants of 3-manifolds. We will give a history of this problem in a sequel paper in which we also plan to address other related questions by Simion [8, 7] and connect the problem to Frenkel-Khovanov's work [2].

2. The type B Gram determinant

Let $\mathbf{A}_n$ be an annulus with $2n$ points, $a_1, \ldots, a_{2n}$, on the outer circle of the boundary, cf. Fig 1. Let $\mathbf{b}_n := \{b_1, b_2, \ldots, b_{\binom{2n}{n}}\}$ be the set of all possible diagrams,

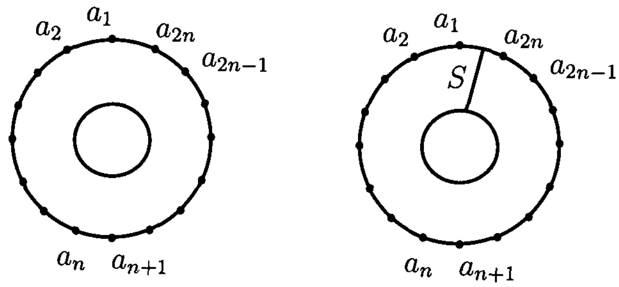

Figure 1. $\mathbf{A}_n$ and $\mathbf{A}_n$ with a segment S

up to deformation, in $\mathbf{A}_n$ with n non-crossing chords connecting these $2n$ points, Fig 2. We define a pairing $\langle\ ,\ \rangle$ on $\mathbf{b}_n$ as follows: Given $b_i, b_j \in \mathbf{b}_n$ we glue

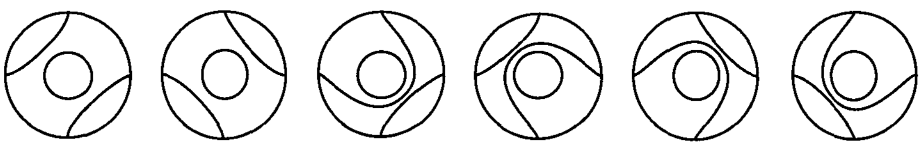

Figure 2. Connections in $\mathbf{b}_2 = \{b_1, b_2, b_3, b_4, b_5\}$ b_6

b_i with the inversion of b_j along the marked circle, respecting the labels of the marked points. The resulting picture is an annulus with two types of disjoint circles, homotopically non-trivial and trivial; compare Fig 3. The bilinear form $\langle\ ,\ \rangle$ is define by $\langle b_i, b_j\rangle = \alpha^m\delta^n$ where m and n are the number of homotopically non-trivial circles and homotopically trivial circles respectively.

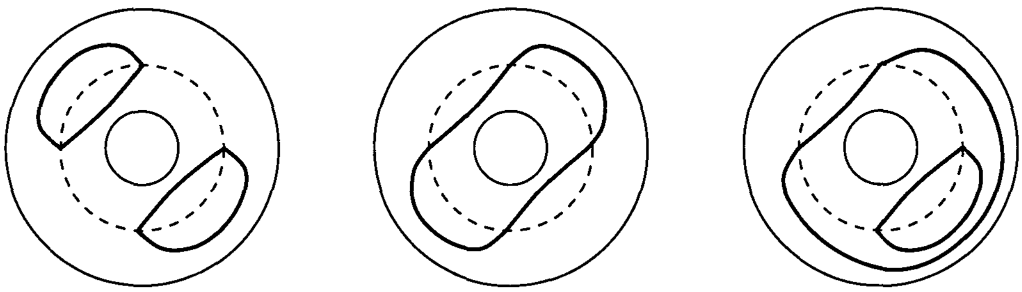

FIGURE 3. $\langle b_1, b_1\rangle = \delta^2$, $\langle b_1, b_2\rangle = \alpha$, $\langle b_1, b_3\rangle = \alpha\delta$; $b_i \in \mathbf{b}_2$

Let

$$G_n(\alpha,\delta) = \Big(\langle b_i, b_j\rangle\Big)_{1\leqslant i,j\leqslant \binom{2n}{n}}$$

be the matrix of the pairing on $\mathbf{b}_n$ called the Gram matrix of the type B Temperley-Lieb algebra. We denote its determinant by $D_n^B(\alpha,\delta)$. The roots of $D_n^B(\alpha,\delta)$ were predicted by Dąbkowski and Przytycki, and the complete factorization of $D_n^B(\alpha,\delta)$ was conjectured by G.Barad:

Conjecture 1 (G. Barad).

$$D_n^B(\alpha,\delta) = \prod_{i=1}^{n}\left(T_i(\delta)^2 - \alpha^2\right)^{\binom{2n}{n-i}}$$

where $T_i(\delta)$ is the Chebyshev (Tchebycheff) polynomial of the first kind:

$$T_0 = 2, \qquad T_1 = \delta, \qquad T_i = \delta\, T_{i-1} - T_{i-2}.$$

The rest of the paper is devoted to a proof of Conjecture 1. It follows directly from the following two lemmas, the first of which is proven in Section 3.

Lemma 1. *For $i \geqslant 1$, $\alpha = (-1)^{i-1}T_i(\delta)$ is a zero of $D_n^B(\alpha,\delta)$ of multiplicity at least $\binom{2n}{n-i}$.*

Lemma 2. *Let S be a line segment connecting the two boundary components of $\mathbf{A}_n$ such that S is disjoint from a_i, $1 \leqslant i \leqslant 2n$; see 1. Let $c(b_i)$ denote the number of chords in b_i that cut S, and let $P = (p_{ij})$ be a diagonal matrix defined by $p_{ii} = (-1)^{c(b_i)}$. Then $G_n(-\alpha,\delta) = PG_n(\alpha,\delta)P^{-1}$.*

Proof. The power of α in $\langle b_i, b_j\rangle$ is congruent to $c(b_i)+c(b_j)$ modulo 2, thus

$$\langle b_i, b_j\rangle|_{\alpha\mapsto-\alpha} = (-1)^{c(b_i)+c(b_j)}\langle b_i, b_j\rangle$$

and Lemma 2 follows. □

Proof of Conjecture 1. According to Lemma 2, $G_n(-\alpha,\delta)$ and $G_n(\alpha,\delta)$ are conjugate matrices. Hence $\alpha = (-1)^i T_i(\delta)$ is a zero of $D_n^B(\alpha,\delta)$ of the same multiplicity as $\alpha = (-1)^{i-1}T_i(\delta)$. Therefore, by this and Lemma 1 we have

$$D_n^B(\alpha,\delta) = p\prod_{i=1}^{n}\left(T_i(\delta)^2-\alpha^2\right)^{\binom{2n}{n-i}},$$

for some $p\in\mathbb{Z}[\alpha,\delta]$. The diagonal entries in $G_n(\alpha,\delta)$ are all equal to δ^n and they are of highest degree in each row thus $D_n^B(\alpha,\delta)$ is a monic polynomial in variable δ of degree $n\binom{2n}{n}$. Furthermore $T_i(\delta)$ is a monic polynomial of degree i. Couple these with a well known equality[†],

$$2\sum_{i=1}^{n} i\binom{2n}{n-i} = n\binom{2n}{n},$$

to conclude that $p=1$.

□

3. Proof of Lemma 1

It is enough to show that the nullity of $G_n((-1)^{i-1}T_i(\delta),\delta)$ is at least $\binom{2n}{n-i}$, which we prove by the theory of Kauffman Bracket Skein Module (KBSM); see [3, 6] for the definition and properties of KBSM. Denote the KBSM of a 3-manifold X by $\mathscr{S}(X)$. Let $\mathbf{A}$ be an annulus. For any two elements x, y in $\mathscr{S}(\mathbf{A}) = \mathbb{Z}[A^{\pm1},\alpha]$,[‡] let $H(x,y)$ be the element in $\mathscr{S}(S^3) = \mathbb{Z}[A,A^{-1}]$ obtained by decorating the two components of the Hopf link with x and y. Denote the k-th Jones-Wenzl idempotent by f_k, cf. [5]. Define a linear map

$$\phi_k : \mathscr{S}(\mathbf{A}) \to \mathbb{Z}[A,A^{-1}]$$

such that $\phi_k(x) = H(x,\hat{f}_k)$, where $\hat{f}_k\in\mathscr{S}(\mathbf{A})$ is the natural closure of f_k. For $b_i, b_j\in\mathbf{b}_n$, we will consider $\langle b_i,b_j\rangle$ as an element of $\mathscr{S}(\mathbf{A})$. If $\langle b_i,b_j\rangle = \alpha^m\delta^n$ then

$$\phi_k(\langle b_i,b_j\rangle) = (-A^{2(k+1)}-A^{-2(k+1)})^m(-A^2-A^{-2})^n\Delta_k, \tag{1}$$

[†]We use "telescoping" to get $2\sum_{i=1}^n i\binom{2n}{n-i} = 2\sum_{i=1}^n n(\binom{2n-1}{n-i}-\binom{2n-1}{n-i-1}) = 2n\binom{2n-1}{n-1} = n\binom{2n}{n}$.

[‡]For any surface F, we write $\mathscr{S}(F)$ for $\mathscr{S}(F\times[0,1])$. In $\mathscr{S}(\mathbf{A})$, α represents a nontrivial curve, 1 – the empty curve, and $\delta=-A^2-A^{-2}$ – a trivial curve. $\mathbf{b}_n$ is a basis of a relative KBSM, $\mathscr{S}(\mathbf{A}_n)$ as a module over $\mathbb{Z}[A^{\pm1},\alpha]$.

where $\Delta_k = (-1)^k(A^{2(k+1)} - A^{-2(k+1)})/(A^2 - A^{-2})$ is the Kauffman bracket of $\hat{f}_k$; see page 143 of [5]. To relate the Gram matrix $G_n(\alpha, \delta)$ to the map ϕ_k we substitute $\delta = -A^2 - A^{-2}$ to obtain $T_k(\delta) = (-1)^k(A^{2k} + A^{-2k})$. Let

$$F_{n,k} = \Big(\phi_{k-1}(\langle b_i, b_j\rangle)\Big)_{1\leqslant i,j\leqslant\binom{2n}{n}}.$$

Then

$$G_n((-1)^{k-1}T_k(-A^2 - A^{-2}), -A^2 - A^{-2}) = \frac{1}{\Delta_{k-1}}F_{n,k}.$$

Therefore, Lemma 1 follows from the next lemma.

Lemma 3. *The nullity of $F_{n,k}$ is at least $\binom{2n}{n-k}$.*

To prove Lemma 3 we need some linear maps defined on $\mathscr{S}(\mathbf{A}_n)$ and $\mathscr{S}(\mathbf{D}_{n,k})$, $k \geqslant 0$, where $\mathbf{D}_{n,k}$ is the disk with $2(n+k)$ points on its boundary. These points are labeled counter-clockwise by $a_1, \dots, a_{2n}$, $l_1, \dots, l_k$ and $u_k, \dots, u_1$.

Let

$$\psi_{n,k} : \mathscr{S}(\mathbf{A}_n) \to \mathscr{S}(\mathbf{D}_{n,k})$$

be a linear map defined as follows: Let $\psi'_{n,k}$ be an embedding of $\mathbf{A}_n$ into a neighborhood of the boundary of $\mathbf{D}_{n,k}$ such that the point a_i on $\mathbf{A}_n$ is mapped to a_i. Let $L \subset \mathbf{D}_{n,k}$ denote the lollipop consisting of the image of the inside boundary of $\mathbf{A}_n$ together with a line segment connecting it to a point between u_k and l_k. If $[x]$ is a diagram representing an element $x \in \mathscr{S}(\mathbf{A}_n)$ then $\psi_{n,k}(x)$ is represented by a diagram in $\mathbf{D}_{n,k}$ consisting of $\psi'_{n,k}[x]$ and k chords in $\mathbf{D}_{n,k}\backslash L$ parallel to L such that if a segment of $\psi'_{n,k}[x]$ intersects L then it intersects the k parallel chords in k over-crossings above the lollipop stick and k under-crossings beneath the stick. See Fig 4 for a value of $\psi_{3,2}$. We also need a linear map

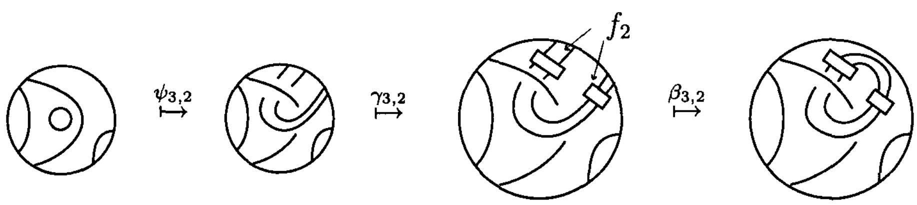

FIGURE 4.

$$\gamma_{n,k} : \mathscr{S}(\mathbf{D}_{n,k}) \to \mathscr{S}(\mathbf{D}_{n,k})$$

defined by inserting 2 copies of the Jones-Wenzl idempotents f_k close to $u_1, \dots u_k$ and $l_1, \dots, l_k$. See Fig 4 for a value of $\gamma_{3,2}$. The third map

$$\beta_{n,k} : \mathscr{S}(\mathbf{D}_{n,k}) \to \mathscr{S}(\mathbf{D}_{n,0})$$

is defined by connecting u_i to l_i outside $\mathbf{D}_{n,k}$ and then pushing these arcs into $\mathbf{D}_{n,k}$. See Fig 4 for a value of $\beta_{3,2}$. The fourth map

$$\zeta_n : \mathscr{S}(\mathbf{D}_{n,0}) \times \mathscr{S}(\mathbf{A}_n) \to \mathscr{S}(\mathbf{D}_{0,0})$$

is defined by gluing the two entries according to the marks. It is clear that for all $x, y \in \mathbf{A}_n$ we have

$$\phi_k(\langle x, y\rangle) = \zeta_n(\beta_{n,k} \circ \gamma_{n,k} \circ \psi_{n,k}(x), y). \tag{2}$$

Therefore, if one can show that $\gamma_{n,k-1} \circ \psi_{n,k-1}(b_{i_j})$, $1 \leqslant j \leqslant s$ for some integer s, are linearly dependent then so are the corresponding rows in $F_{n,k}$. This observation is used to prove Lemma 3.

Proof of Lemma 3. By the above argument it is enough to show that $\gamma_{n,k-1} \circ \psi_{n,k-1}(\mathbf{b}_n)$ is contained in a subspace of dimension $\binom{2n}{n} - \binom{2n}{n-k}$ in $\mathscr{S}(\mathbf{D}_{n,k-1})$. Therefore, it suffices to show that

$$\dim(\mathrm{Im}(\gamma_{n,k-1})) \leqslant \binom{2n}{n} - \binom{2n}{n-k}. \tag{3}$$

Let $\mathrm{NC}(\mathbf{D}_{n,k-1})$ be the set of non-crossing diagrams in $\mathbf{D}_{n,k-1}$ consisting of $n+k-1$ chords. Then $\mathrm{NC}(\mathbf{D}_{n,k-1})$ is a basis of $\mathscr{S}(\mathbf{D}_{n,k-1})$. If $x \in \mathrm{NC}(\mathbf{D}_{n,k-1})$ contains a chord connecting two u_i's or two l_i's then $\gamma_{n,k-1}(x) = 0$ by a well known property of the Jones-Wenzl idempotent, cf. Lemma 13.2 in [5]. Hence this lemma follows from the inequality

$$|\tilde{\mathrm{NC}}(\mathbf{D}_{n,k-1})| \leqslant \binom{2n}{n} - \binom{2n}{n-k}$$

where $\tilde{\mathrm{NC}}(\mathbf{D}_{n,k-1})$ is the set of diagrams in $\mathrm{NC}(\mathbf{D}_{n,k-1})$ with no chord connecting two u_i's or two l_i's. In fact, the equality holds: □

Lemma 4. *Assume the notation above. We have*

$$|\tilde{\mathrm{NC}}(\mathbf{D}_{n,k})| = \binom{2n}{n} - \binom{2n}{n-k-1}. \tag{4}$$

Proof. Lemma 4 is a standard combinatorial fact[§] but we give its proof for completeness. Recall that $\mathbf{A}_n$ denotes an annulus with $2n$ marks, labeled $a_1, \dots, a_{2n}$, on the outer boundary component. Fix a point x_0 between a_{2n} and a_1 on the marked boundary circle such that the arc containing x_0 has no other a_i's. Let S be a line segment connecting x_0 to the other boundary component of $\mathbf{A}_n$, see 1.. Let $\mathrm{NC}_{\leqslant k}(\mathbf{A}_n) \subset \mathbf{b}_n$ be the set of non-crossing diagrams in $\mathbf{A}_n$ consisting of n chords which intersect S at most k times. There is a 1-1 correspondence between $\mathrm{NC}_{\leqslant k}(\mathbf{A}_n)$ and $\tilde{\mathrm{NC}}(\mathbf{D}_{n,k})$. (Suppose $x \in \mathrm{NC}_{\leqslant k}(\mathbf{A}_n)$ intersects S at k' times. Draw $k - k'$ circles close and parallel to the unmarked boundary component of $\mathbf{A}_n$. Cut along S and we obtain a diagram in $\tilde{\mathrm{NC}}(\mathbf{D}_{n,k})$.)

Hence it is enough to show that the set $\mathrm{NC}_{\geqslant j}(\mathbf{A}_n) := \mathbf{b}_n \backslash \mathrm{NC}_{\leqslant j-1}(\mathbf{A}_n)$ has $\binom{2n}{n-j}$ elements. We construct a bijection between $\mathrm{NC}_{\geqslant j}(\mathbf{A}_n)$ and the choices of $n - j$ marks among the $2n$ marks on $\mathbf{A}_n$.

[§]It can be derived from the reflection principle by Desiré André, 1887; also compare [4, 9, 10, 1].

(i) Assume $n-j$ marks, $a_{i_1},...,a_{i_{n-j}}$, are chosen. We construct n chords as follows: If, for some s, the point a_{i_s+1} is not chosen, we draw an *oriented* chord from a_{i_s} to a_{i_s+1}, not cutting S. If a_{2n} is chosen but a_1 is not, then draw an oriented chord from a_{2n} to a_1 cutting S. At this stage at least one chord is drawn. Delete this chord together with its endpoint marks and repeat the process again until all chosen marks are used (they are the beginning marks of the constructed chords). We are left with $2j$ marks. Choose the mark with the largest index and draw a chord as before. All new j chords cut S so the constructed diagram cuts S in at least j points.

(ii) Conversely consider a diagram of n chords cutting S at least j times. Orient these chords counter-clockwise. Among the n chords there are $s \leqslant n-j$ of them not cutting S. Add to these s chords $n-j-s$ more chords which are as close to the outside circle of $\mathbf{A}_n$ as possible. The beginning marks of these $n-j$ chords are the marks corresponding to our diagram.

This ends the construction of the bijection. Hence we have
$|\mathrm{NC}_{\geqslant j}(\mathbf{A}_n)| = \binom{2n}{n-j}$. □

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THE GRAM MATRIX OF A TEMPERLEY-LIEB ALGEBRA IS SIMILAR TO THE MATRIX OF CHROMATIC JOINS

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Dedicated to the memory of Xiao-Song Lin (1957-2007)

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1. Introduction

Rodica Simion noticed experimentally that matrices of chromatic joins (introduced by W. Tutte in [Tu2]) and the Gram matrix of the Temperley-Lieb algebra, have the same determinant, up to renormalization. In the type A case, she was able to prove this by comparing the known formulas: by Tutte and R. Dahab [Tu2, Dah], in the case of chromatic joins, and by P. Di Francesco, and B. Westbury [DiF, We] (based on the work by K. H. Ko and L. Smolinsky [KS]) in the Temperley-Lieb case; see [CSS]. She then asked for a direct proof of this fact [CSS], [Sch], Problem 7.

The type B analogue was an open problem central to the work of Simion [Sch]. She demonstrated strong evidence that the type B Gram determinant of the Temperley-Lieb algebra is equal to the determinant of the matrix of type B chromatic joins, after a substitution similar to that in type A, cf. [Sch].

In this paper we show that the matrix J_n of chromatic joins and the Gram matrix G_n of the Temperley-Lieb algebra are similar (after rescaling), with the change of basis given by diagonal matrices. More precisely we prove the following two results:

Theorem A. *We have $J_n^A(\delta^2) = PG_n^A(\delta)P$, where $P = (p_{ij})$ is a diagonal matrix with $p_{ii}(\delta) = \delta^{\mathrm{bk}(\pi_i)-n/2}$; here $\mathrm{bk}(\pi_i)$ denotes the number of blocks in the type A non-crossing n-partition π_i; see 2.1 of Section 2 for precise definitions.*

Theorem B. *We have $J_n^B(\delta^2) = P^B G_n^B(1,\delta)P^B$ where $P^B = (p_{ij}^B)$ is a diagonal matrix with $p_{ii}^B(\delta) = \delta^{\mathrm{nzbk}(\pi_i)-n/2}$; here $\mathrm{nzbk}(\pi_i)$ denotes half of the number of non-zero blocks in the type B non-crossing n-partition π_i; see 2.2 of Section 2 for precise definitions.*

2. Definitions and notation

In the description of the matrix of chromatic joins (of type A and B) we follow V. Reiner, R. Simion and F. Schmidt [Rei, Sim, Sch].

2.1. **The type A case.** An *n-partition of type A* is a partition π of the n element set $\{1, 2, ..., n\}$ into blocks. The number of blocks is denoted by $\mathrm{bk}(\pi)$. To represent π pictorially, we place the numbers $1, 2, ..., n$ anti-clockwise around the boundary circle of the unit disk and draw a chord, called a *connection chord*, in the disk between two numbers $i < j$ if they are in the same block of π and there is no k in the same block with $i < k < j$. We say that π is *non-crossing* if all connection chords can be drawn without crossing each other. Notice that each block is represented by a tree. Denote the set of all non-crossing n-partitions of type A by Π_n^A. On the other hand if $n = 2m$ is even, we have *bipartitions* of $2m$ points of type A, those $2m$-partitions of type A with every block containing exactly 2 numbers. Denote the set of all non-crossing $2m$-bipartitions of type A by Γ_m^A. We have a bijection $\varphi_A : \Pi_n^A \to \Gamma_n^A$ realized by considering the boundary arcs of a regular neighborhood of the connection chords (see Fig. 1 and compare Fig. 2, Fig. 3).

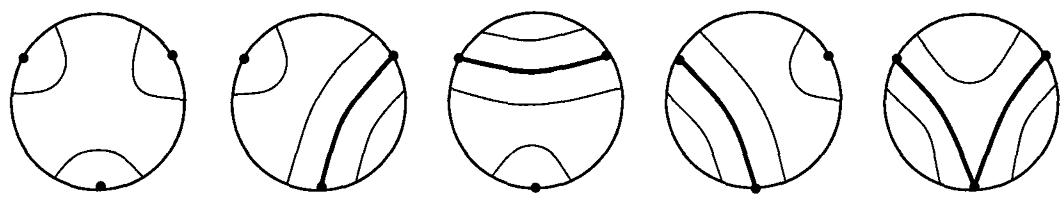


FIGURE 1. The bijection $\varphi_A : \Pi_3^A \to \Gamma_3^A$.

2.2. **The type B case.** An *n-partition of type B* is a partition π of the $2n$ element set $\{+1, +2, ..., +n, -1, -2, ..., -n\}$ into blocks with the property that for any block K of π, its opposite $-K$ is also a block of π, and that there is at most one invariant block (called the zero block[†]) for which $K = -K$. Since all non-zero blocks occur in pairs $\pm K$ one defines $\mathrm{nzbk}(\pi)$ as half of the number of all non-zero blocks. To represent π pictorially, we place the numbers $+1, +2, ..., +n, -1, -2, ..., -n$ anti-clockwise around the boundary circle of a disk and draw a connection chord in the disk between two numbers $i < j$[‡] if they are in the same block of π and there is no k in the same block with $i < k < j$. Then π is said to be *non-crossing* if all connection chords can be drawn without crossing each other.[§] Denote the set of all non-crossing n-partitions of type B by Π_n^B. The set Π_2^B is illustrated in Fig. 2.

[†]For topologists the term invariant block is more natural than zero block so we use this names interchangeably in the paper.

[‡]Here we use the order $+1 < +2 < \cdots < n < -1 < -2 < \cdots < -n$.

[§]The non-crossing condition forces a partition to have at most one zero block.

On the other hand if $n = 2m$ is even, we have bipartitions of $4m$ points of type

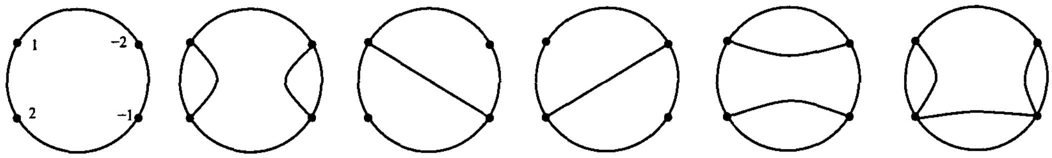


FIGURE 2. The pictorial representation of Π_2^B.

B, those $2m$-partitions of type B with every block containing exactly 2 numbers. Denote the set of all non-crossing $2m$-bipartitions of type B by Γ_m^B. Similar to the type A case we have a bijection $\varphi_B : \Pi_n^B \to \Gamma_n^B$ realized by considering the boundary arcs of a regular neighborhood of the connection chords (see Fig. 3).

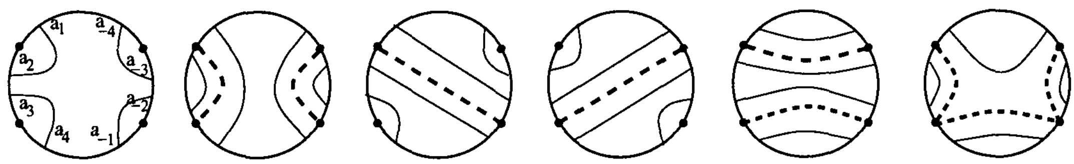


FIGURE 3. The bijection $\varphi_B : \Pi_2^B \to \Gamma_2^B$.

2.3. **The matrices.** For any n-partitions π and π' of type A (resp. B), denote by $\pi \vee \pi'$ the finest n-partition (not necessarily non-crossing) of type A (resp. B) that is coarser than both π and π'. The *matrix of chromatic joins* of type A and B are respectively:

$$(J_n^A(\delta))_{\pi,\pi'\in\Pi_n^A} = \delta^{\mathrm{bk}(\pi\vee\pi')} \quad \text{and} \quad (J_n^B(\delta))_{\pi,\pi'\in\Pi_n^B} = \delta^{\mathrm{nzbk}(\pi\vee\pi')}.$$

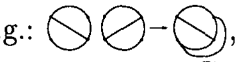
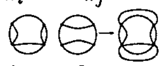
For any $2n$-bipartitions π and π' of type A (resp. B), one can glue them along the boundary circles respecting the labels. The result, denoted $\pi \vee \pi'$, is a collection of disjoint circles on a 2-sphere. The *Gram matrix of Temperley-Lieb algebra* of type A and B are respectively¶:

$$(G_n^A(\delta))_{\pi,\pi'\in\Gamma_n^A} = \delta^{c(\pi\vee\pi')} \quad \text{and} \quad (G_n^B(\alpha,\delta))_{\pi,\pi'\in\Gamma_n^B} = \alpha^{c_0(\pi\vee\pi')}\delta^{c_d(\pi\vee\pi')},$$

where $c(\pi \vee \pi')$ is the number of circles, $c_0(\pi \vee \pi')$ is the number of zero (i.e. invariant) circles C with $C = -C$, and $c_d(\pi \vee \pi')$ is the number of pairs of non-zero circles $C, -C$ with $C \neq -C$ in $\pi \vee \pi'$.

¶The matrix $G_n^A(\delta)$ was first used by H. Morton and P. Traczyk to find a basis of the Kauffman bracket skein module of a tangle [MT], and played an important role in Lickorish's approach to Witten-Reshetikhin-Turaev invariants of 3-manifolds [Li]. The matrix $G_n^B(1,\delta)$ was first considered by Rodica Simion in 1998; compare [Sch].

3. Proof of Theorems A and B

Proof of Theorem A. For $\pi_i \in \Pi_n^A$ let $b_i := \varphi_A(\pi_i) \in \Gamma_n^A$. Since $c(b_i \vee b_j)$ is also equal to the number of boundary components of the regular neighborhood of the pictorial representation of $\pi_i \vee \pi_j$ we have $2\,\mathrm{bk}(\pi_i \vee \pi_j) = c(b_i \vee b_j) + \mathrm{bk}(\pi_i) + \mathrm{bk}(\pi_j) - n = \mathrm{bk}(\pi_i) - \frac{n}{2} + c(b_i \vee b_j) + \mathrm{bk}(\pi_j) - \frac{n}{2}$. The formula can be obtained from the expression for the Euler characteristic of a plane graph: Let G_π be a graph corresponding to the non-crossing partition π. G_π is a forest of n vertices and $n - \mathrm{bk}(\pi)$ edges. Similarly, let $G_{\pi_i \vee \pi_j}$ be the graph corresponding to $\pi_i \vee \pi_j$. We should stress that $\pi_i \vee \pi_j$ does not have to be a noncrossing partition and that the graph $G_{\pi_i \vee \pi_j}$ is a plane graph obtained by putting G_{π_i} inside a disk and G_{π_j} outside the disk with $G_{\pi_i} \cap G_{\pi_j}$ composed of the n points on the unit circle
(e.g.: , or ).
By construction, $G_{\pi_i \vee \pi_j}$ is a plane graph of n vertices and $\mathrm{bk}(\pi_i \vee \pi_j)$ components. It has $E(G_{\pi_i \vee \pi_j}) = E(G_{\pi_i}) + E(G_{\pi_j}) = n - \mathrm{bk}(\pi_i) + n - \mathrm{bk}(\pi_j)$ edges. Furthermore, if we embed $G_{\pi_i \vee \pi_j}$ in a disjoint union of $\mathrm{bk}(\pi_i \vee \pi_j)$ 2-spheres (each component of $G_{\pi_i \vee \pi_j}$ in a different sphere) we can identify $c(b_i \vee b_j)$ with the number of regions of the embedded graph. The Euler characteristic is on the one hand equal to $2\mathrm{bk}(\pi_i \vee \pi_j)$ and on the other hand equal to $n - E(G_{\pi_i \vee \pi_j}) + c(b_i \vee b_j) = c(b_i \vee b_j) + \mathrm{bk}(\pi_i) + \mathrm{bk}(\pi_j) - n$, as needed.
Theorem A follows directly from the formula. □

Proof of Theorem B. For $\pi_i \in \Pi_n^B$ let $b_i := \varphi_B(\pi_i) \in \Gamma_n^B$ and $\mathrm{bk}_0(\pi_i)$ be the number of zero blocks of π_i. Recall that $c_0(b_i \vee b_j)$ is the number of zero (i.e. invariant) components of $b_i \vee b_j$. As in the type A case we have $2\,\mathrm{bk}(\pi_i \vee \pi_j) = c(b_i \vee b_j) + \mathrm{bk}(\pi_i) + \mathrm{bk}(\pi_j) - 2n$. Furthermore, we have (see Lemma 1):

$$2\,\mathrm{bk}_0(\pi_i \vee \pi_j) = c_0(b_i \vee b_j) + \mathrm{bk}_0(\pi_i) + \mathrm{bk}_0(\pi_j).$$

(Notice that $\mathrm{bk}_0(\pi_i \vee \pi_j)$ can be equal to 2, 1, or 0.) From these we conclude that: $2\,\mathrm{nzbk}(\pi_i \vee \pi_j) = c_d(b_i \vee b_j) + \mathrm{nzbk}(\pi_i) + \mathrm{nzbk}(\pi_j) - n$. Thus Theorem B follows. □

Lemma 1. *The zero blocks and zero components satisfy the following identity:*

$$2\,\mathrm{bk}_0(\pi_i \vee \pi_j) = c_0(b_i \vee b_j) + \mathrm{bk}_0(\pi_i) + \mathrm{bk}_0(\pi_j),$$

where $b_i = \varphi_B(\pi_i)$ and $b_j = \varphi_B(\pi_j)$.

Proof. The lemma reflects the basic properties of a 2-sphere with an involution fixing two points and its compact invariant submanifolds.

To demonstrate the formula we consider all cases of blocks of π_i, π_j and $\pi_i \vee \pi_j$ divided into four classes:
(1) If K is a non-zero (i.e. non-invariant) block of $\pi_i \vee \pi_j$ then all its constituent blocks in π_i and π_j are non-zero blocks and the boundary components of a regular neighborhood of the geometric realization of K (denoted ∂K), that is circles in $b_i \vee b_j$ corresponding to K, are non-invariant (non-zero) curves, i.e. not in $c_0(b_i \vee b_j)$.

(2) If K is a zero block of $\pi_i \vee \pi_j$ but all its constituent blocks in π_i and π_j are non-zero blocks, then exactly two components of ∂K are invariant curves.
(3) If K is a zero block of $\pi_i \vee \pi_j$ and exactly one constituent block is a zero block then exactly one component of ∂K is an invariant curve.
(4) If K is a zero block of $\pi_i \vee \pi_j$ and exactly two constituent blocks are zero-blocks (necessarily one in π_i and one in π_j) then no component of ∂K is an invariant curve.

These conditions taken together prove the formula in Lemma 1. □

4. Corollaries

Theorem B and the results of [MS, CP] allow us to answer Problems 1 and 2 of [Sch] about a formula for the determinant of the type-B matrix of chromatic joins:

Corollary 2.

$$\det(J_n^B(\delta^2)) = \prod_{i=1}^{n} \left(T_i(\delta)^2 - 1\right)^{\binom{2n}{n-i}}$$

where $T_i(\delta)$ is the Chebyshev polynomial of the first kind:

$$T_0 = 2, \qquad T_1 = \delta, \qquad T_i = \delta\, T_{i-1} - T_{i-2}.$$

The matrix $J_n^B(\delta)$ can be generalized to a matrix of two variables as follows:

$$(J_n^B(\alpha, \delta))_{\pi,\pi' \in \Pi_n^B} = \alpha^{\mathrm{bk}_0(\pi\vee\pi')}\delta^{\mathrm{nzbk}(\pi\vee\pi')}.$$

It follows from Lemma 1 that

$$J_n^B(\alpha^2, \delta^2) = P_n^B(\alpha, \delta) G_n^B(\alpha, \delta) P_n^B(\alpha, \delta),$$

where $P_n^B(\alpha, \delta) = (p_{ij})$ is a diagonal matrix with $p_{ii}(\alpha, \delta) = \alpha^{\mathrm{bk}_0(\pi_i)}\delta^{\mathrm{nzbk}(\pi_i)-n/2}$. Furthermore, $\det P_n^B(\alpha, \delta) = \alpha^{\frac{1}{2}\binom{2n}{n}}$** and thus we have:

Corollary 3.

$$\det J_n^B(\alpha^2, \delta^2) = \alpha^{\binom{2n}{n}} G_n^B(\alpha, \delta) = \alpha^{\binom{2n}{n}} \prod_{i=1}^{n} \left(T_i(\delta)^2 - \alpha^2\right)^{\binom{2n}{n-i}}.$$

Remark 4. Consider the Gram matrix of type B based on non-crossing connections in an annulus.†† This matrix is the same as the one considered before in Theorem B via the branched cover described in Fig. 4.

**This follows from Proposition 3 of [Rei], which asserts that there exists a fixed-point free involution γ on Π_n^B such that $\mathrm{bk}_0(\pi) + \mathrm{bk}_0(\gamma(\pi)) = 1$ and $\mathrm{nzbk}(\pi) + \mathrm{nzbk}(\gamma(\pi)) = n$.

††This interpretation of the Gram matrix of type B Temperley-Lieb algebra is mentioned in [Sch] as an annular skein matrix and utilized in [MS] and [CP].

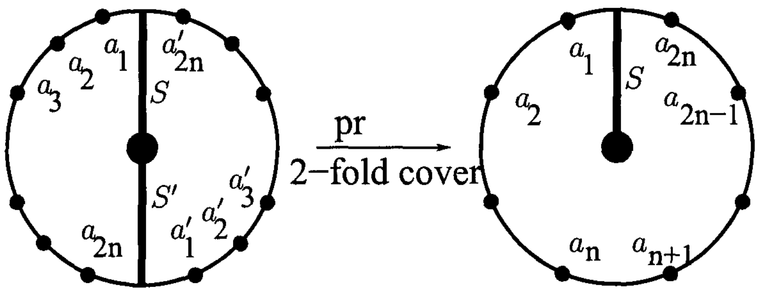


FIGURE 4. The double branch cover $pr : D^2 \to D^2$ with the "cutting" arc S

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GRAM DETERMINANT OF PLANAR CURVES

JÓZEF H. PRZYTYCKI AND XIAOQI ZHU

ABSTRACT. We investigate the Gram determinant of the bilinear form based on curves in a planar surface, with a focus on the disk with two holes. We prove that the determinant based on $n-1$ curves divides the determinant based on n curves. Motivated by the work on Gram determinants based on curves in a disk and curves in an annulus (Temperley-Lieb algebra of type A and B, respectively), we calculate several examples of the Gram determinant based on curves in a disk with two holes and advance conjectures on the complete factorization of Gram determinants.

1. INTRODUCTION

Let $F_{0,0}^n$ be a unit disk with $2n$ points on its boundary. Let $\mathbf{B}_{n,0}$ be the set of all possible diagrams, up to deformation, in $F_{0,0}^n$ with n non-crossing chords connecting these $2n$ points. It is well-known that $|\mathbf{B}_{n,0}|$ is equal to the n^{th} Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ [10]. Accordingly, we will call $\mathbf{B}_{n,0}$ the set of **Catalan states**.

We will now generalize this setup. Let $F_{0,k} \subset D^2$ be a plane surface with $k+1$ boundary components. $F_{0,0} = D^2$, and for $k \geq 1$, $F_{0,k}$ is equal to D^2 with k holes. Let $F_{0,k}^n$ be $F_{0,k}$ with $2n$ points, $a_0, \ldots, a_{2n-1}$, arranged counter-clockwise along the outer boundary, cf. Figure 1.

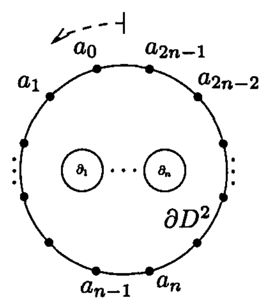


FIGURE 1. Throughout the paper, we number the points counter-clockwise beginning at the top of outer boundary. We label and differentiate between the holes.

Let $\mathbf{B}_{n,k}$ be the set of all possible diagrams, up to equivalence, in $F_{0,k}^n$ with n non-crossing chords connecting these $2n$ points. We define equivalence as follows: for each diagram $b \in \mathbf{B}_{n,k}$, there is a corresponding diagram $\gamma(b) \in \mathbf{B}_{n,0}$ obtained by filling the k holes in b. We call $\gamma(b)$ the underlying Catalan state of b (cf. Figure 2). In addition, a

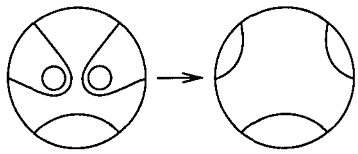


FIGURE 2. $b \mapsto \gamma(b)$

given diagram in $F^n_{0,k}$ partitions $F_{0,k}$ into $n+1$ regions. Two diagrams are equivalent if and only if they have the same underlying Catalan state and the labeled holes are distributed in the same manner across regions. Accordingly, $|\mathbf{B}_{n,k}| = (n+1)^{k-1}\binom{2n}{n}$. We remark that in the cases $k=0$ and $k=1$, two diagrams are equivalent if they are homotopic, but for $k>2$, this is not the case (for an example, see Figure 3).

In this paper, we define a pairing over $\mathbf{B}_{n,k}$ and investigate the Gram matrix of the pairing. This concept is a generalization of a problem posed by W. B. R. Lickorish for type A (based on a disk, i.e. $k=0$) Gram determinants, and Rodica Simion for type B (based on an annulus, i.e. $k=1$) Gram determinants, cf. [4, 5], [7, 8]. Significant research has been completed for the Gram determinants for type A and B. In particular, P. Di Francesco and B. W. Westbury gave a closed formula for the type A Gram determinant [3], [11]; a complete factorization of the type B Gram determinant was conjectured by Gefry Barad and a closed formula was proven by Q. Chen and J. H. Przytycki [1] (see also [6]). The type A Gram determinant was used by Lickorish to find an elementary construction of Reshetikhin-Turaev-Witten invariants of oriented closed 3-manifolds.

We specifically investigate the Gram determinant G_n of the bilinear form defined over $\mathbf{B}_{n,2}$ and prove that $\det G_{n-1}$ divides $\det G_n$ for $n>1$. Furthermore, we investigate the diagonal entries of G_n and give a method for computing terms of maximal degree in $\det G_n$. We conclude the paper by briefly discussing generalizations of the Gram determinant and presenting some open questions.

2. DEFINITIONS FOR $\mathbf{B}_{n,2}$

Consider $F^n_{0,2}$, a unit disk with two holes, along with $2n$ points along the outer boundary. Denote the holes in $F^n_{0,2}$ by ∂_{X_1} and ∂_{Y_1}, or more simply, just X_1 and Y_1. To differentiate between the two holes, we will always place X_1 to the left and Y_1 to the right if labels are not present.

Let $\mathbf{B}_n := \mathbf{B}_{n,2} := \{b^n_1, \ldots, b^n_{(n+1)\binom{2n}{n}}\}$, the set of all possible diagrams, up to equivalence in $F^n_{0,2}$ with n non-crossing chords connecting these $2n$ points. For simplicity, we will often use b_i instead of b^n_i, when the number of points along the outer boundary can be inferred from context.

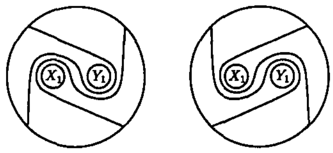


FIGURE 3. Two non-isotopic but equivalent diagrams in $F^2_{0,2}$. They correspond to the same state in $\mathbf{B}_2$. A complete specification of $\mathbf{B}_2$ can be found in the Appendix.

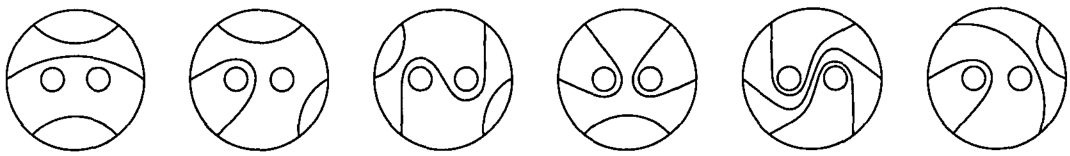


FIGURE 4. Pictorial representations of six states $\{b_1, b_2, b_3, b_4, b_5, b_6\} \subset \mathbf{B}_3$. We stress that this is not a natural ordering of states in $\mathbf{B}_3$.

Let X_2 and Y_2 be the inversions[1] of X_1 and Y_1, respectively, with respect to the unit disk, and let $\mathcal{S} = \{X_1, X_2, Y_1, Y_2\}$. Given $b_i \in \mathbf{B}_n$, let $b_i{}^*$ denote the inversion of b_i. Given $b_i, b_j \in \mathbf{B}_n$, we glue b_i with $b_j{}^*$ along the outer boundary, respecting the labels of the marked points. b_i and b_j each contains n non-crossing chords, so $b_i \circ b_j{}^*$ can have at most n closed curves. The resulting diagram, denoted by $b_i \circ b_j{}^*$, is then a set of up to n closed curves in the 2-dimensional sphere $(D^2 \cup (D^2)^*)$ with four holes: X_1, X_2, Y_1, Y_2 (we disregard the outer boundary, ∂D^2). Each closed curve partitions the set $\mathcal{S}$ into two sets. Two closed curves are of the same type if they partition $\mathcal{S}$ the same way. For each $b_i \circ b_j{}^*$, there are then up to eight types of disjoint closed curves, whose multiplicities we index by the following variables:

$$
\begin{aligned}
n_d &= \text{the number of curves with } \{X_1, X_2, Y_1, Y_2\} \text{ on the same side} \\
n_{x_1} &= \text{the number of curves that separate } \{X_1\} \text{ from } \{X_2, Y_1, Y_2\} \\
n_{x_2} &= \text{the number of curves that separate } \{X_2\} \text{ from } \{X_1, Y_1, Y_2\} \\
n_{y_1} &= \text{the number of curves that separate } \{Y_1\} \text{ from } \{X_1, X_2, Y_2\} \\
n_{y_2} &= \text{the number of curves that separate } \{Y_2\} \text{ from } \{X_1, X_2, Y_1\} \\
n_{z_1} &= \text{the number of curves that separate } \{X_1, X_2\} \text{ from } \{Y_1, Y_2\} \\
n_{z_2} &= \text{the number of curves that separate } \{X_1, Y_1\} \text{ from } \{X_2, Y_2\} \\
n_{z_3} &= \text{the number of curves that separate } \{X_1, Y_2\} \text{ from } \{X_2, Y_1\}
\end{aligned}
$$

[1]Inversion is an involution defined on the sphere $\mathbb{C} \cup \infty$ by $z \leftrightarrow \frac{z}{|z|^2}$.

Let $R := \mathbb{Z}[d, x_1, x_2, y_1, y_2, z_1, z_2, z_3]$, and $R\mathbf{B}_n$ be the free module over the ring R with basis $\mathbf{B}_n$. We define a bilinear form $\langle , \rangle : R\mathbf{B}_n \times R\mathbf{B}_n \to R$ by:

$$\langle b_i, b_j \rangle = d^{n_d} {x_1}^{n_{x_1}} {x_2}^{n_{x_2}} {y_1}^{n_{y_1}} {y_2}^{n_{y_2}} {z_1}^{n_{z_1}} {z_2}^{n_{z_2}} {z_3}^{n_{z_3}}$$

$\langle b_i, b_j \rangle$ is a monomial of degree at most n. Some examples of paired diagrams and their corresponding monomials, using examples from Figure 4, are given in Figure 5.

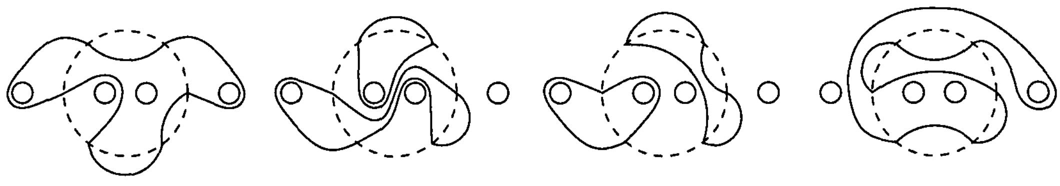


FIGURE 5. From left to right:
$\langle b_2, b_4 \rangle = x_1 \quad \langle b_5, b_2 \rangle = x_1 x_2 \quad \langle b_6, b_2 \rangle = d z_1 \quad \langle b_1, b_3 \rangle = x_2$

Let

$$G_n = (g_{ij}) = \left(\langle b_i, b_j \rangle \right)_{1 \le i,j \le (n+1)\binom{2n}{n}}$$

be the Gram matrix of the pairing on $\mathbf{B}_n$. For example,

$$G_1 = \begin{bmatrix} d & y_2 & x_2 & z_2 \\ y_1 & z_1 & z_3 & x_1 \\ x_1 & z_3 & z_1 & y_1 \\ z_2 & x_2 & y_2 & d \end{bmatrix} \text{ up to ordering of } \mathbf{B}_1 \text{ and}$$

$$\begin{aligned} \det G_1 &= ((d + z_2)(z_1 + z_3) - (x_1 + y_1)(x_2 + y_2)) \\ &\quad ((d - z_2)(z_1 - z_3) - (x_1 - y_1)(x_2 - y_2)). \end{aligned}$$

We remark that for $b_i, b_j \in \mathbf{B}_n$, $\langle b_j, b_i \rangle$ can be obtained by taking $b_i \circ {b_j}^*$ and interchanging the roles of X_1 and Y_1 with X_2 and Y_2, respectively. Let h_t be an involution on the entries of G_n which interchanges the variables x_1 with x_2 and y_1 with y_2. It follows that $\langle b_i, b_j \rangle = h_t(\langle b_j, b_i \rangle)$. The transpose matrix is then given by:

$${}^tG_n = \left(h_t(\langle b_i, b_j \rangle) \right)$$

We note that the variables d, z_1, z_2, z_3 are preserved by h_t (cf. Theorem 3.2(4)).

We can define more generally: given $A = \{b_{n_1}, b_{n_2}, \dots, b_{n_p}\} \subseteq \mathbf{B}_n$ and $B = \{b_{m_1}, b_{m_2}, \dots, b_{m_q}\} \subseteq \mathbf{B}_n$, let $\langle A, B \rangle$ be an $p \times q$ submatrix of G_n given by:

$$\langle A, B \rangle = \left(\langle b_{n_i}, b_{m_j} \rangle \right)_{1 \le i \le p, 1 \le j \le q}$$

For example, we can express the matrix G_n as $\langle \mathbf{B}_n, \mathbf{B}_n \rangle$. The i^{th} row of G_n can be written as $\langle b_i, \mathbf{B}_n \rangle$.

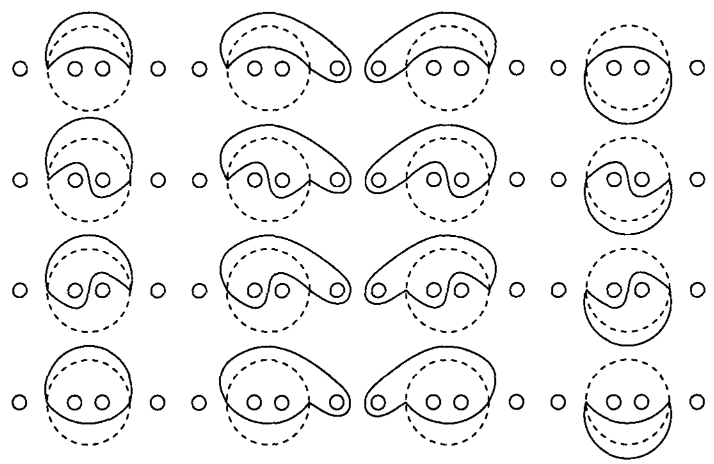


FIGURE 6. A pictorial representation of curves used to define G_1

This paper is mostly devoted to exploring possible factorizations of $\det G_n$, and is the first step toward computing $\det G_n$ in full generality, which we conjecture to have a nice decomposition.

Let $i_0 : \mathbf{B}_n \to \mathbf{B}_{n+1}$ be the **embedding** map defined as follows: for $b_i \in \mathbf{B}_n$, $i_0(b_i) \in \mathbf{B}_{n+1}$ is given by adjoining to b_i a non-crossing chord close to the outer boundary that intersects the outer circle at two points between a_0 and a_{2n-1}, cf. upper part of Figure 8.

We will also use a generalization of i_0, for which we need first the following definition. For any real number α, consider the homeomorphism $r_\alpha : \mathbb{C} \to \mathbb{C}$ on the annulus $R' \leq |z| \leq 1$, which we call the **α-Dehn Twist**, defined by:

$$r_\alpha(z) = ze^{i\alpha(1-(1-|z|)/(1-R'))}$$

Note that $r_\alpha(z) = z$ as $|z| = R'$. Therefore, we can extend the domain of r_α to D^2 by defining $r_\alpha(z) = z$ for $0 \leq |z| \leq R'$. Fix R' such that a circle of radius R' encloses X_1 and Y_1. Then r_α acts on $b_i \in \mathbf{B}_n$ as a clockwise rotation of a diagram close to the outer boundary.

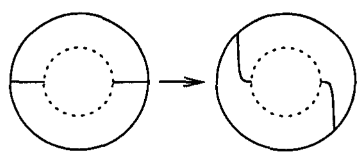


FIGURE 7. A $\pi/4$-Dehn Twist. Note that $r_{2\pi}(b_i) = b_i$ (cf. Figure 3).

Consider the k-**conjugated embedding** $i_k : \mathbf{B}_n \to \mathbf{B}_{n+1}$ defined by:

$$i_k(b_i) = r_{\pi/n+1}{}^k i_0 r_{\pi/n}{}^{-k}(b_i)$$

Intuitively, if for $b_i \in \mathbf{B}_{n+1}$ there exists $b_j \in \mathbf{B}_n$ such that $i_k(b_j) = b_i$, then b_i is composed of b_j and a non-crossing chord close to the outer boundary connecting a_k and a_{k-1}[2], Figure 8.

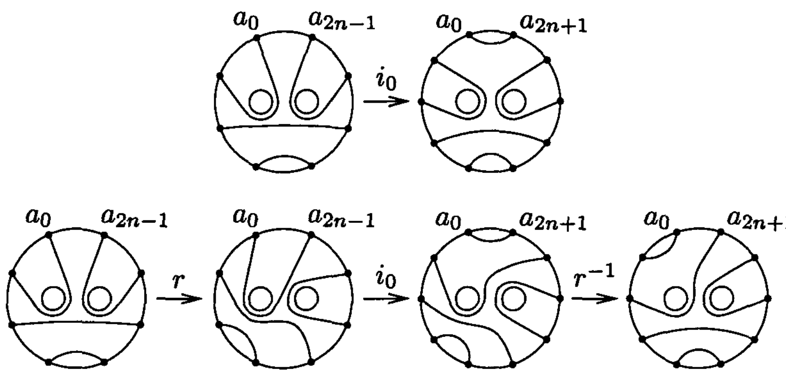


FIGURE 8. An embedding $b_i \mapsto i_0(b_i)$, top; a 1-conjugated embedding $b_i \mapsto i_1(b_i)$, bottom; $b_i \in \mathbf{B}_4$.

For every $b_i \in \mathbf{B}_n$, let $p_k(b_i)$ be the diagram obtained by gluing to b_i a non-crossing chord connecting a_k and a_{k-1} outside the circle, and pushing the chord inside the circle. The properties of p_k will be explored in greater detail in Section 4. We conclude this section with a basic identity linking i_0 and p_0:

Proposition 2.1. *For any $b_i \in \boldsymbol{B}_n$, $b_j \in \boldsymbol{B}_{n-1}$, $b_i \circ i_0(b_j)^* = p_0(b_i) \circ b_j{}^*$.*

3. Basic Properties of Gram Determinant

In this section, we prove basic properties of $\det G_n$. In particular, we show that the determinant of G_n is nonzero.

Lemma 3.1. *$\langle b_i, b_j\rangle$ is a monomial of maximal degree if and only if $\gamma(b_i) = \gamma(b_j)$.*

Proof. $b_i \circ b_j{}^*$ has n closed curves if and only if each closed curve is formed by exactly two arcs, one in b_i and one in $b_j{}^*$. Hence, any two points connected by a chord in b_i must also be connected by a chord in b_j, so $\gamma(b_i) = \gamma(b_j)$. □

Theorem 3.1. $\det G_n \neq 0$ *for all integers $n \geq 1$.*

[2]Throughout this paper, we use a_k and a_{k-1} to denote two adjacent points along the outer boundary, where k is taken modulo $2n$.

Proof. Assume $\langle b_i, b_j\rangle$ is a monomial of maximal degree consisting only of the variables d and z_1. Because $\gamma(b_i) = \gamma(b_j)$ by Lemma 3.1, it follows that any two points connected in b_i are also connected in b_j. Each connection in b_i can be drawn in four different ways with respect to X and Y, since there are two ways to position the chord relative to each hole. Because $\langle b_i, b_j\rangle$ is assumed to consist only of the variables d and z_1, it follows that each pair of arcs that form a closed curve in $b_i \circ b_j{}^*$ either separates $\{X_1, X_2\}$ from $\{Y_1, Y_2\}$ or has $\{X_1, X_2, Y_1, Y_2\}$ on the same side of the curve. One can check each of the four cases to see that this condition implies that any two arcs that form a closed curve in $b_i \circ b_j{}^*$ must be equal, so $b_i = b_j$. Using Laplacian expansion, this implies that the product of the diagonal of G_n is the unique summand of degree $n(n+1)\binom{2n}{n}$ in $\det G_n$ consisting only of the variables d and z_1. ☐

We need the following notation for the next theorem: let $f : \alpha_1 \leftrightarrow \alpha_2$ denote a function f which acts on the entries of G_n by interchanging variables α_1 with α_2. We can extend the domain of f to G_n. Let $f(G_n)$ denote the matrix formed by applying f to all the individual entries of G_n.

Let h_1, h_2, h_3 be involutions acting on the entries of G_n with the following definitions:

(1) $h_1 : x_1 \leftrightarrow y_1 \quad z_1 \leftrightarrow z_3$
(2) $h_2 : x_2 \leftrightarrow y_2 \quad z_1 \leftrightarrow z_3$
(3) $h_3 = h_1h_2 : x_1 \leftrightarrow y_1 \quad x_2 \leftrightarrow y_2$
(4) $h_t : x_1 \leftrightarrow x_2 \quad y_1 \leftrightarrow y_2$

Theorem 3.2.

(1) $\det h_1(G_1) = -\det G_1$, *and for* $n > 1$, $\det h_1(G_n) = \det G_n$.
(2) $\det h_2(G_1) = -\det G_1$, *and for* $n > 1$, $\det h_2(G_n) = \det G_n$.
(3) $\det h_3(G_n) = \det G_n$.
(4) $\det h_t(G_n) = \det G_n$.

Proof. For (1), note that $h_1(G_n)$ corresponds to exchanging the positions of the holes X_1 and Y_1 for all $b_i \in \mathbf{B}_n$. $b_j{}^*$ is unchanged, so h_1 can be realized by a permutation of rows. For states where X_1 and Y_1 lie in the same region, their corresponding rows are unchanged by h_1. The number of such states is given by $\frac{1}{n+1}|\mathbf{B}_n|$. Thus, the total number of row transpositions is equal to

$$\frac{1}{2}\left(|\mathbf{B}_n| - \left(\frac{1}{n+1}\right)|\mathbf{B}_n|\right) = \frac{n}{2}\binom{2n}{n} = \left(\frac{n(n+1)}{2}\right)C_n$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$. It is a known combinatorial fact that C_n is odd if and only if $n = 2^m - 1$ for some m, [2]. Hence, C_n is odd implies that

$$\frac{n(n+1)}{2} = \frac{2^m(2^m-1)}{2} = 2^{m-1}(2^m-1)$$

which is even for all $m > 1$. Thus, $h_1(G_n)$ can be obtained from G_n by an even permutation of rows for $n > 1$, so $\det h_1(G_n) = \det G_n$. $h_1(G_1)$ is given by an odd number of row transpositions on G_1, so $\det h_1(G_1) = -\det G_1$.

(2) can be shown using the same method of proof as before. $h_2(G_n)$ corresponds to exchanging the positions of the holes X_2 and Y_2 for all $b_i \in \mathbf{B}_n$. h_2 can thus be realized by a permutation of columns, and the rest of the proof follows in a similar fashion as the previous one. Since $h_2(G_n)$ can be obtained from G_n by an even permutation of columns for $n > 1$, $\det h_2(G_n) = \det G_n$. $h_2(G_2)$ is given by an odd number of column transpositions on G_1, so $\det h_2(G_1) = -\det G_1$, which proves (2).

Since $h_3 = h_1 h_2$, it follows immediately that $\det h_3(G_n) = \det G_n$ for $n > 1$. The sum of two odd permutations is even, so the equality also holds for $n = 1$, which proves (3). (4) follows because $\det h_t(G_n) = \det {}^tG_n = \det G_n$. □

Theorem 3.3. $\det G_n$ *is preserved under the following involutions on variables:*

(1) $g_1 : x_1 \leftrightarrow -x_1, x_2 \leftrightarrow -x_2, z_2 \leftrightarrow -z_2, z_3 \leftrightarrow -z_3$
(2) $g_2 : y_1 \leftrightarrow -y_1, y_2 \leftrightarrow -y_2, z_2 \leftrightarrow -z_2, z_3 \leftrightarrow -z_3$
(3) $g_3 : x_1 \leftrightarrow -x_1, y_2 \leftrightarrow -y_2, z_1 \leftrightarrow -z_1, z_2 \leftrightarrow -z_2$
(4) $g_1g_2 : x_1 \leftrightarrow -x_1, x_2 \leftrightarrow -x_2, y_1 \leftrightarrow -y_1, y_2 \leftrightarrow -y_2$
(5) $g_1g_3 : x_2 \leftrightarrow -x_2, y_2 \leftrightarrow -y_2, z_1 \leftrightarrow -z_1, z_3 \leftrightarrow -z_3$
(6) $g_2g_3 : x_1 \leftrightarrow -x_1, y_1 \leftrightarrow -y_1, z_1 \leftrightarrow -z_1, z_3 \leftrightarrow -z_3$
(7) $g_1g_2g_3 : x_2 \leftrightarrow -x_2, y_1 \leftrightarrow -y_1, z_1 \leftrightarrow -z_1, z_2 \leftrightarrow -z_2$

Proof. To prove (1), we show that g_1 can be realized by conjugating the matrix G_n by a diagonal matrix P_n of all diagonal entries equal to ± 1. The diagonal entries of P_n are defined as

$$p_{ii} = (-1)^{q(b_i, F_x)}$$

where $q(b_i, F_x)$ is the number of times b_i intersects F_x modulo 2, cf. Figure 9. The theorem follows because curves corresponding to the variables x_1, x_2, z_2 and z_3 intersect $F_x \cup {F_x}^*$ in an odd number of points, whereas curves corresponding to the variables d, z_2, y_1 and y_2 cut it an even number of times.

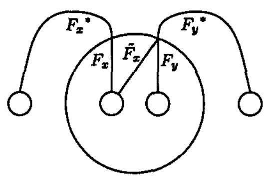


FIGURE 9.

More precisely, for

$$g_{ij} = \langle b_i, b_j \rangle = d^{n_d} {x_1}^{n_{x_1}} {x_2}^{n_{x_2}} {y_1}^{n_{y_1}} {y_2}^{n_{y_2}} {z_1}^{n_{z_1}} {z_2}^{n_{z_2}} {z_3}^{n_{z_3}},$$

the entry g'_{ij} of $P_n G_n {P_n}^{-1}$ satisfies:

$$\begin{aligned} g'_{ij} &= p_{ii} g_{ij} p_{jj} \\ &= p_{ii} p_{jj} g_{ij} \\ &= (-1)^{q(b_i,F_x)+q(b_j,F_x)} g_{ij} \\ &= (-1)^{n_{x_1}+n_{x_2}+n_{z_2}+n_{z_3}} g_{ij} \\ &= d^{n_d} (-x_1)^{n_{x_1}} (-x_2)^{n_{x_2}} {y_1}^{n_{y_1}} {y_2}^{n_{y_2}} {z_1}^{n_{z_1}} (-z_2)^{n_{z_2}} (-z_3)^{n_{z_3}} \end{aligned}$$

For (2) and (3), we use the same method of proof as for (1). In (2), we use F_y and $F_y \cup {F_y}^*$. In (3), we use $\tilde{F}_x$ and $\tilde{F}_x \cup {F_y}^*$. (4) through (7) follow directly from (1), (2) and (3). □

4. Terms of Maximal Degree in $\det G_n$

Theorem 3.1 proves that the product of the diagonal entries of G_n is the unique term of maximal degree, $n(n+1)\binom{2n}{n}$, in $\det G_n$ consisting only of the variables d and z_1. More precisely, the product of the diagonal of G_n is given by

$$\delta(n) = \prod_{b_i \in \mathbf{B}_n} \langle b_i, b_i \rangle = d^{\alpha(n)} z_1^{\beta(n)}$$

with $\alpha(n) + \beta(n) = n(n+1)\binom{2n}{n}$. $\delta(n)$ for the first few n are given here:

$$\delta(1) = d^2 z_1^2 \qquad \delta(2) = d^{20} z_1^{16} \qquad \delta(3) = d^{144} z_1^{96} \qquad \delta(4) = d^{888} z_1^{512}$$

Computing the general formula for $\delta(n)$ can be reduced to a purely combinatorial problem. We conjectured that $\beta(n) = (2n)4^{n-1}$ and this was in fact proven by Louis Shapiro using an involved generating function argument [9]. The result is stated formally below.

Theorem 4.1.

$$\delta(n) = d^{n(n+1)\binom{2n}{n} - (2n)4^{n-1}} z_1^{(2n)4^{n-1}}$$

Let $h(\det G_n)$ denote the truncation of $\det G_n$ to terms of maximal degree, that is, of degree $n(n+1)\binom{2n}{n}$. Each term is a product of $(n+1)\binom{2n}{n}$ entries in G_n, each of which is a monomial of degree n. By Lemma 3.1, $\langle b_i, b_j \rangle$ has degree n if and only if b_i and b_j have the same underlying Catalan state. There are $C_n = \frac{1}{n+1}\binom{2n}{n}$ elements in $\mathbf{B}_{n,0}$. Divide $\mathbf{B}_n$ into subsets corresponding to underlying Catalan states, that is, into subsets $A_1, \ldots, A_{C_n}$, such that for all $b_i, b_j \in A_k$, $\gamma(b_i) = \gamma(b_j)$. Then from Lemma 3.1 we have

Proposition 4.1.

$$h(\det G_n) = \prod_{k=1}^{C_n} \det\langle A_k, A_k \rangle$$

Note that $\langle A_k, A_k \rangle$ are simply blocks in G_n whose determinants can be multiplied together to give the highest terms in $\det G_n$. Finding the terms of maximal degree in $\det G_n$ can give insight into decomposition of $\det G_n$ for large n.

Example 1. *$\boldsymbol{B}_1$ corresponds to the single Catalan state in $\boldsymbol{B}_{1,0}$. Thus,* $\det G_1 = h(\det G_1)$, *a homogeneous polynomial of degree 4.*

Example 2. *$\boldsymbol{B}_2$ can be divided into two subsets, corresponding to the two Catalan states in $\boldsymbol{B}_{2,0}$. We can thus find $h(\det G_2)$ by computing two 9×9 block determinants. The two Catalan states in $\boldsymbol{B}_{2,0}$ are equivalent up to rotation, so the two block determinants are equal. Specifically, we have:*

$$\begin{aligned} h(\det G_2) &= d^6(x_1x_2 + x_2y_1 + x_1y_2 + y_1y_2 - dz_1 - z_1z_2 - dz_3 - z_2z_3)^4 \\ &\quad (-x_1x_2 + x_2y_1 + x_1y_2 - y_1y_2 + dz_1 - z_1z_2 - dz_3 + z_2z_3)^4 \\ &\quad (-x_1x_2z_1 - y_1y_2z_1 + dz_1{}^2 + x_2y_1z_3 + x_1y_2z_3 - dz_3{}^2)^2 \\ &\quad (-2x_1x_2y_1y_2 + dx_1x_2z_1 + dy_1y_2z_1 - d^2z_1{}^2 + dx_2y_1z_3 + dx_1y_2z_3 - d^2z_3{}^2)^2 \\ &= d^6 \det G_1{}^4(-x_1x_2z_1 - y_1y_2z_1 + dz_1{}^2 + x_2y_1z_3 + x_1y_2z_3 - dz_3{}^2)^2 \\ &\quad (-2x_1x_2y_1y_2 + dx_1x_2z_1 + dy_1y_2z_1 - d^2z_1{}^2 + dx_2y_1z_3 + dx_1y_2z_3 - d^2z_3{}^2)^2 \end{aligned}$$

Example 3. *$\boldsymbol{B}_3$ can be divided into five subsets, corresponding to the five Catalan states in $\boldsymbol{B}_{3,0}$. We can thus find $h(\det G_3)$ by computing the determinants of five blocks in $\boldsymbol{B}_3$. The determinant of each block gives a homogeneous polynomial of degree $240/5 = 48$. $\boldsymbol{B}_{3,0}$ forms two equivalence classes up to rotation, so there are only two unique block determinants. For precise terms, we refer the reader to the Appendix.*

5. $\det G_{n-1}$ Divides $\det G_n$

In this section, we prove that the Gram determinant for $n-1$ chords divides the Gram determinant for n chords. We need several lemmas:

Lemma 5.1. *For any $b_i \in \boldsymbol{B}_n$, $p_0(b_i) \in \boldsymbol{B}_{n-1}$ if and only if b_i contains no chord connecting a_0 and a_{2n-1}.*

Proof. Suppose a_0 and a_{2n-1} are not connected by a chord in b_i, say, a_0 is connected to a_j and a_{2n-1} is connected to a_k. Then $p_0(b_i)$ connects a_0 and a_{2n-1} by a chord outside the outer boundary, and this chord does not form a closed curve. Because a_j is connected to a_0 and a_k is connected to a_{2n-1}, $p_0(b_i)$ contains single path from a_k to a_j, which we can deform through isotopy so that it fits inside the outer circle. Thus, $p_0(b_i) \in \mathbf{B}_{n-1}$, cf. Figure 10.

If b_i contains an arc connecting a_0 and a_{2n-1}, then $p_0(b_i)$ contains a closed curve enclosing some subset of $\{X_1, Y_1\}$, and cannot be in $\mathbf{B}_n$. □

Lemma 5.2. *For any $b_i \in \boldsymbol{B}_n$, if $p_0(b_i) \notin \boldsymbol{B}_{n-1}$, there exists $b_{\alpha(i)} \in \boldsymbol{B}_{n-1}$ such that, for all $b_j \in \boldsymbol{B}_{n-1}$, one of the following is true:*

1. $\langle p_0(b_i), b_j\rangle = d\langle b_{\alpha(i)}, b_j\rangle$
2. $\langle p_0(b_i), b_j\rangle = x_1\langle b_{\alpha(i)}, b_j\rangle$
3. $\langle p_0(b_i), b_j\rangle = y_1\langle b_{\alpha(i)}, b_j\rangle$
4. $\langle p_0(b_i), b_j\rangle = z_2\langle b_{\alpha(i)}, b_j\rangle$.

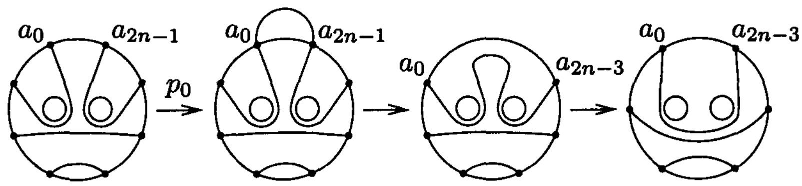


FIGURE 10. From b_i, we obtain $p_0(b_i)$ by adjoining a chord outside the outer boundary between a_0 and a_{2n-1}, and pushing the chord inside the boundary. If b_i does not contain a chord connecting a_0 and a_{2n-1}, then $p_0(b_i) \in \mathbf{B}_{n-1}$.

Proof. By Lemma 5.1, b_i contains a chord connecting points a_0 and a_{2n-1}, so $p_0(b_i)$ must consist of some diagram in $\mathbf{B}_{n-1}$ and a closed curve enclosing some subset of $\{X_1, Y_1\}$. The former is given by $\langle b_{\alpha(i)}, b_j \rangle$ for some $b_{\alpha(i)} \in \mathbf{B}_{n-1}$, and the latter curve is given by one of the following variables: d, x_1, y_1, z_2. □

The previous two lemmas, combined with Proposition 2.1, leads to the following corollary.

Corollary 5.1. *Let $\mathcal{A} = \{1, d, x_1, y_1, z_2\}$. For any $b_i \in \boldsymbol{B}_n$, there exists $b_{\alpha(i)} \in \boldsymbol{B}_{n-1}$ and $c \in \mathcal{A}$ such that $\langle b_i, i_0(\boldsymbol{B}_{n-1}) \rangle = c \langle b_{\alpha(i)}, i_0(\boldsymbol{B}_{n-1}) \rangle$.*

That is, the rows of $\langle \mathbf{B}_n, i_0(\mathbf{B}_{n-1}) \rangle$ are each either equal to some row of G_{n-1}, or to some row of G_{n-1} multiplied by one of the following variables: d, x_1, y_1, z_2. We now have all the lemmas needed for our main result of this section.

Theorem 5.1. *For $n > 1$, $\det G_{n-1} | \det G_n$.*

Proof. We begin by proving that for every row of the matrix G_{n-1}, there exists an equivalent row in the submatrix $\langle \mathbf{B}_n, i_0(\mathbf{B}_{n-1}) \rangle$ of G_n. Fix $b_i \in \mathbf{B}_{n-1}$ and take the row of G_{n-1} given by $\langle b_i, \mathbf{B}_{n-1} \rangle$. We claim that the row in $\langle \mathbf{B}_n, i_0(\mathbf{B}_{n-1}) \rangle$ given by $\langle i_1(b_i), i_0(\mathbf{B}_{n-1}) \rangle$ is equal to $\langle b_i, \mathbf{B}_{n-1} \rangle$. In other words, $\langle i_1(b_i), i_0(\mathbf{B}_{n-1}) \rangle$ is equal to the i^{th} row of G_{n-1}, a fact which we leave to the reader for the moment, but will demonstrate explicitly in the next section, cf. Theorem 6.1.

Reorder the elements of $\mathbf{B}_n$ so that $\langle i_0(\mathbf{B}_{n-1}), i_0(\mathbf{B}_{n-1}) \rangle$ forms an upper-leftmost block of G_n and $\langle i_1(\mathbf{B}_{n-1}), i_0(\mathbf{B}_{n-1}) \rangle$ forms a block directly underneath $\langle i_0(\mathbf{B}_{n-1}), i_0(\mathbf{B}_{n-1}) \rangle$.

This is illustrated below:

$$G_n = \begin{pmatrix} \langle i_0(\mathbf{B}_{n-1}), i_0(\mathbf{B}_{n-1})\rangle & * & * & * & * & * \\ \langle i_1(\mathbf{B}_{n-1}), i_0(\mathbf{B}_{n-1})\rangle & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix}$$

$$= \begin{pmatrix} \langle i_0(\mathbf{B}_{n-1}), i_0(\mathbf{B}_{n-1})\rangle & * & * & * & * & * \\ G_{n-1} & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix}$$

Corollary 5.1 implies that every row of $\langle \mathbf{B}_n, i_0(\mathbf{B}_{n-1})\rangle$ is a multiple of some row in G_{n-1}. Let $j_1, \dots, j_k$ denote the indices of all rows of $\langle \mathbf{B}_n, i_0(\mathbf{B}_{n-1})\rangle$ other than those in $\langle i_1(\mathbf{B}_{n-1}), i_0(\mathbf{B}_{n-1})\rangle$. Let $G_n{}'$ be the matrix obtained by properly subtracting multiples of rows in $\langle i_1(\mathbf{B}_{n-1}), i_0(\mathbf{B}_{n-1})\rangle$ from rows $j_1, \dots, j_k$ of G_n so that the submatrix obtained by restricting $G_n{}'$ to rows $j_1, \dots, j_k$ and columns corresponding to states in $i_0(\mathbf{B}_{n-1})$ is equal to 0:

$$G_n{}' = \begin{pmatrix} 0 & * & * & * & * & * \\ G_{n-1} & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \end{pmatrix}$$

Thus, $G_n{}'$ restricted to the columns corresponding to states in $i_0(\mathbf{B}_{n-1})$ contains precisely $n\binom{2n-2}{n-1}$ nonzero rows, each equal to some unique row of G_{n-1}. The determinant of this submatrix is equal to $\det G_{n-1}$. Since $\det G_{n-1} | \det G_n{}'$ and $\det G_n{}' = \det G_n$, this completes the proof. □

6. Further Relation Between $\det G_{n-1}$ and $\det G_n$

As was first noted in the proof of Theorem 5.1, there exists a submatrix of G_n equal to G_{n-1}. This section will be focused on identifying multiple nonoverlapping submatrices in G_n equal to multiples of G_{n-1}. This will prove useful for simplifying the computation of $\det G_n$. We start with the main lemma for this section and for Theorem 5.1.

Lemma 6.1. *For any $b_i, b_j \in B_{n-1}$, $\langle i_0(b_i), i_1(b_j)\rangle = \langle i_1(b_i), i_0(b_j)\rangle = \langle b_i, b_j\rangle$.*

Proof. We begin with the equality $\langle i_1(b_i), i_0(b_j)\rangle = \langle b_i, b_j\rangle$. By Proposition 2.1, $i_1(b_i) \circ i_0(b_j)^* = p_0 i_1(b_i) \circ b_j{}^*$, so it suffices to prove that $p_0 i_1(b_i) = p_0 r_{\pi/n} i_0 r_{\pi/n-1}{}^{-1}(b_i) = b_i$. This is demonstrated pictorially:

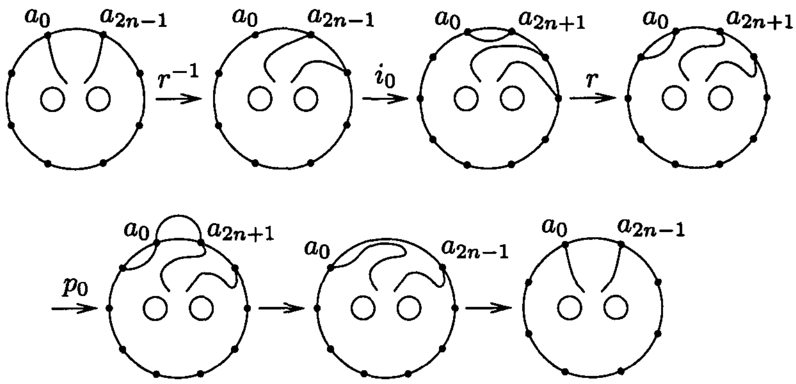


FIGURE 11.

Thus, $\langle i_1(b_i), i_0(b_j)\rangle = \langle b_i, b_j\rangle$. Recall that $\langle b_i, b_j\rangle = h_t(\langle b_j, b_i\rangle)$. From this and the previous equality, it follows that

$$\langle i_0(b_i), i_1(b_j)\rangle = h_t(\langle i_1(b_j), i_0(b_i)\rangle) = h_t(\langle b_j, b_i\rangle) = {h_t}^2(\langle b_i, b_j\rangle) = \langle b_i, b_j\rangle.$$

□

Corollary 6.1. $\langle i_0(B_{n-1}), i_1(B_{n-1})\rangle = \langle i_1(B_{n-1}), i_0(B_{n-1})\rangle = G_{n-1}$.

Lemma 6.2. *For any $b_i, b_j \in B_{n-1}$, $\langle i_0(b_i), i_0(b_j)\rangle = \langle i_1(b_i), i_1(b_j)\rangle = d\langle b_i, b_j\rangle$.*

Proof. $i_0(b_i) \circ i_0(b_j)^*$ is composed of $b_i \circ b_j{}^*$ in addition to a chord close to the boundary glued with its inverse. The latter pairing gives a trivial circle. Thus, $\langle i_0(b_i), i_0(b_j)\rangle = d\langle b_i, b_j\rangle$ for all $b_i, b_j \in \mathbf{B}_{n-1}$.

By symmetry, $\langle i_1(\mathbf{B}_{n-1}), i_1(\mathbf{B}_{n-1})\rangle = dG_{n-1}$. □

Corollary 6.2. $\langle i_0(B_{n-1}), i_0(B_{n-1})\rangle = \langle i_1(B_{n-1}), i_1(B_{n-1})\rangle = dG_{n-1}$.

Using these two facts, we can construct from G_n a $(|B_n| - 2|B_{n-1}|) \times (|B_n| - 2|B_{n-1}|)$ matrix whose determinant is equal to $\det G_n/(1-d^2)^{n\binom{2n-2}{n-1}} \det {G_{n-1}}^2$. This allows us to compute $\det G_n$ with greater ease, assuming we know $\det G_{n-1}$. This process is shown in the next theorem.

Theorem 6.1. *There exists an integer $k \geq 0$[3] such that, for all integers $n > 1$,*

$$\det {G_{n-1}}^2 \,|\, \det G_n (1-d^2)^k.$$

---

[3]Clearly k is bounded above by $(n+1)\binom{2n}{n}$, or even better, by $|\mathbf{B}_n| - 2|\mathbf{B}_{n-1}|$. There are obviously better approximations possible, but we do not address them in this paper.

Proof. Order the elements of $\mathbf{B}_n$, (or equivalently, the rows and columns of G_n) as shown in Theorem 5.1. We apply the procedure from Theorem 5.1 to construct $G_n{}'$, whose form is given roughly below:

$$G_n{}' = \begin{pmatrix} 0 & (1-d^2)G_{n-1} & * & * & * & * \\ G_{n-1} & dG_{n-1} & * & * & * & * \\ 0 & \boxed{*} & * & * & * & * \\ 0 & \boxed{*} & * & * & * & * \\ 0 & \boxed{*} & * & * & * & * \\ 0 & \boxed{*} & * & * & * & * \end{pmatrix}$$

Consider the block in $G_n{}'$ whose columns correspond to states in $i_1(\mathbf{B}_{n-1})$ and whose rows correspond to states in neither $i_0(\mathbf{B}_{n-1})$ nor $i_1(\mathbf{B}_{n-1})$ (boxed above). Every row in this submatrix is a linear combination of two rows from G_{n-1}. More precisely, each row is of the form $a_1l_1 - a_2dl_2$, where l_1 and l_2 are two rows, not necessarily distinct, in G_{n-1}, and $a_1, a_2 \in \mathcal{A} = \{1, d, x_1, y_1, z_2\}$. If we assume $(1-d^2)$ is invertible in our ring, then each row is a linear combination of two rows from $(1-d^2)G_{n-1}$. We then simplify $G_n{}'$ as follows.

Let $G_n{}''$ be the matrix obtained by properly subtracting linear combinations of the first $n\binom{2n-2}{n-1}$ rows of $G_n{}'$ from the rows which correspond to states in neither $i_0(\mathbf{B}_{n-1})$ nor $i_1(\mathbf{B}_{n-1})$ so that the submatrix obtained by restricting $G_n{}''$ to columns corresponding to states in $i_1(\mathbf{B}_{n-1})$ and rows corresponding to states in neither $i_0(\mathbf{B}_{n-1})$ nor $i_1(\mathbf{B}_{n-1})$ is equal to 0:

$$G_n{}'' = \begin{pmatrix} 0 & (1-d^2)G_{n-1} & * & * & * & * \\ G_{n-1} & dG_{n-1} & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \end{pmatrix}$$

The block decomposition at this point proves that $\det G_n{}''$ is equal to $(1-d^2)^{n\binom{2n-2}{n-1}}(\det G_{n-1})^2$ times the determinant of the boxed block, which we denote by $\bar{G_n}$. The latter contains a power of $(1-d^2)^{-1}$, whose degree is unspecified. Thus, $\det G_{n-1}{}^2 | \det G_n{}''(1-d^2)^k$ for some integer $k \geq 0$. We remind the reader that $G_n{}''$ is obtained from $G_n{}'$ via determinant preserving operations, and hence $\det G_n{}' = \det G_n$. □

Note that if $\det \bar{G_n}$ has fewer than $n\binom{2n-2}{n-1}$ powers of $(1-d^2)^{-1}$, then $\det G_{n-1}{}^2 | \det G_n$. It remains an open problem as to whether this is true. For an example of this decomposition, we refer the reader to the Appendix.

7. Future Directions

In this section, we discuss briefly generalizations of the Gram determinant and present a number of open questions and conjectures.

7.1. **The case of a disk with k holes.** We can generalize our setup by considering $F^n_{0,k}$, a unit disk with k holes, in addition to $2n$ points, $a_0, \dots, a_{2n-1}$, arranged in a similar way to points in $F^n_{0,2}$. For $b_i, b_j \in \mathbf{B}_{n,k}$, let $b_i \circ b_j{}^*$ be defined in the same way as before. Each paired diagram $b_i \circ b_j{}^*$ consists of up to n closed curves on the 2-sphere $(D^2 \cup (D^2)^*)$ with $2k$ holes. Let $\mathcal{S}$ denote the set of all $2k$ holes. We differentiate between the closed curves based on how they partition $\mathcal{S}$. We define a bilinear form by counting the multiplicities of each type of closed curve in the paired diagram. In the case $k = 2$, we assigned to each paired diagram a corresponding element in a polynomial ring of eight variables, each variable representing a type of closed curve. In the general case, the number of types of closed curves is equal to

$$\frac{2^{|\mathcal{S}|}}{2} = \frac{2^{2k}}{2} = 2^{2k-1}$$

so we can define the Gram matrix of the bilinear form for a disk with k holes and $2n$ points with $(n+1)^{k-1}\binom{2n}{n} \times (n+1)^{k-1}\binom{2n}{n}$ entries, each belonging to a polynomial ring of 2^{2k-1} variables. We denote this Gram matrix by $G_n^{F_{0,k}}$. For $n = 1$ and $k = 3$, we can easily write this 8×8 Gram matrix. For purposes of notation, let us denote the holes in $F^n_{0,3}$ by ∂_1, ∂_2 and ∂_3, and their inversions by ∂_{-1}, ∂_{-2} and ∂_{-3}, respectively. Hence, each closed curve in the surface encloses some subset of $\mathcal{S} = \{\partial_1, \partial_{-1}, \partial_2, \partial_{-2}, \partial_3, \partial_{-3}\}$. Let x_{a_1,a_2,a_3} denote a curve separating the set of holes $\{\partial_{a_1}, \partial_{a_2}, \partial_{a_3}\}$ from $\mathcal{S} - \{\partial_{a_1}, \partial_{a_2}, \partial_{a_3}\}$. We can similarly define x_{a_1,a_2} and x_{a_1}. The Gram matrix is then:

$$G_1^{F_{0,3}} = \begin{pmatrix}
d & x_{-3} & x_{-2} & x_{-2,-3} & x_{-1} & x_{-1,-3} & x_{-1,-2} & x_{1,2,3} \\
x_3 & x_{3,-3} & x_{-2,3} & x_{1,-1,2} & x_{-1,3} & x_{1,2,-2} & x_{1,2,-3} & x_{1,2} \\
x_2 & x_{2,-3} & x_{2,-2} & x_{1,-1,3} & x_{-1,2} & x_{1,-2,3} & x_{1,3,-3} & x_{1,3} \\
x_{2,3} & x_{1,-1,-2} & x_{1,-1,-3} & x_{1,-1} & x_{1,-2,-3} & x_{1,-2} & x_{1,-3} & x_1 \\
x_1 & x_{1,-3} & x_{1,-2} & x_{1,-2,-3} & x_{1,-1} & x_{1,-1,-3} & x_{1,-1,-2} & x_{2,3} \\
x_{1,3} & x_{1,3,-3} & x_{1,-2,3} & x_{-1,2} & x_{1,-1,3} & x_{2,-2} & x_{2,-3} & x_2 \\
x_{1,2} & x_{1,2,-3} & x_{1,2,-2} & x_{1,-3} & x_{1,-1,2} & x_{-2,3} & x_{3,-3} & x_3 \\
x_{1,2,3} & x_{-1,-2} & x_{-1,-3} & x_{-1} & x_{-2,-3} & x_{-2} & x_{-3} & d
\end{pmatrix}$$

It would be tempting to conjecture that the determinant of the above matrix has a straightforward decomposition of the form $(u+v)(u-v)$. We found that it is the case for the substitution $x_{a_1} = x_{a_1,a_2} = 0$ with $a_1, a_2 \in \{-3,-2,-1,1,2,3\}$ (see Appendix). However in general, the preliminary calculation suggests that $\det G_n^{F_{0,3}}$ may be an irreducible polynomial.

Finally, we observe that many results we have proven for $\det G_n^{F_{0,2}}$ holds for general $\det G_n^{F_{0,k}}$. For example, $\det G_n^{F_{0,k}} \neq 0$ and $\det G_n^{F_{0,k-1}} | \det G_n^{F_{0,k}}$. In the specific case of $\det G_n^{F_{0,3}}$ we conjecture that the diagonal term is of the form $\delta(n) = d^{\alpha(n)}(x_{1,-1}x_{2,-2}x_{3,-3})^{\beta(n)}$, where $\alpha(n) + 3\beta(n) = n(n+1)^2\binom{2n}{n}$ and $\beta(n) = n(n+1)4^{n-1}$.

7.2. **Speculation on factorization of $\det G_n$.** Section 5 establishes that $\det G_{n-1} | \det G_n$, but we conjecture that there are many more powers of $\det G_{n-1}$ in $\det G_n$. Indeed, even in the base case, $\det {G_1}^k | \det G_2$ for k up to 4. Finding the maximal power of $\det G_{n-1}$ in $\det G_n$ in the general case is an open problem and can be helpful toward computing the full decomposition of $\det G_n$.

Examining the terms of highest degree in $\det G_n$, that is, $h(\det G_n)$ may also yield helpful hints toward the full decomposition. In particular, we note that:

$$\det {G_1}^4 | h(\det G_2) \quad \text{and} \quad \left(\frac{h(\det G_2)^6}{\det {G_1}^9}\right) | h(\det G_3)$$

We can conjecture that

$$\left(\frac{\det {G_2}^6}{\det {G_1}^9}\right) | \det G_3$$

so it follows that $\det {G_1}^{15} | \det G_3$. We therefore offer the following conjecture:

Conjecture 1. $\det {G_1}^{\binom{2n}{n-1}} | \det G_n$ *for* $n \geq 1$.

In addition, we also offer the following conjecture, motivated by observations of $\det G_1$ and $\det G_2$:

Conjecture 2. *Let* H_n *denote the factors of* $\det G_n$ *not in* $\det G_{n-1}$, *that is,* $H_n | \det G_n$ *and* $\gcd(H_n, \det G_{n-1}) = 0$. *Then* $(H_{n-1})^{2n} | \det G_n$.

Conjecture 3. *Let, as before,* $R = \mathbb{Z}[d, x_1, x_2, y_1, y_2, z_1, z_2, z_3]$ *and* R_1 *be a subgroup of* R *of elements invariant under* h_1, h_2, h_t, *and* g_1, g_2, g_3. *Similarly, let* R_2 *be a subgroup of* R *composed of elements* $w \in R$ *such that* $h_1(w) = h_2(w) = -w$ *and* $h_t(w) = g_1(w) = g_2(w) = g_3(w)$. *Then*

(1) $\det G_n = u^2 - v^2$, *where* $u \in R_1$ *and* $v \in R_2$.
(2) $\det G_n = \prod_\alpha (u_\alpha^2 - v_\alpha^2)$, *where* $u_\alpha \in R_1$ *and* $v_\alpha \in R_2$, *and* $u_\alpha - v_\alpha$ *and* $u_\alpha + v_\alpha$ *are irreducible polynomials.*
(3) $\det G_n = \prod_{i=1}^n ({u_i}^2 - {v_i}^2)^{\binom{2n}{n-i}}$, *where* $u_i \in R_1$ *and* $v_i \in R_2$.

Notice that if $w_1 = u_1^2 - v_1^2$ *and* $w_2 = u_2^2 - v_2^2$, *then* $w_1 w_2 = (u_1u_2 + v_1v_2)^2 - (u_1v_2 + u_2v_1)^2$.

We have little confidence in Conjecture 3(3). It is closely, maybe too closely, influenced by the Gram determinant of type B ($\det G_n^B = \det G_n^{F_{0,1}}$). That is

Theorem 7.1. *([1, 6])*

$$\det G_n^B = \prod_{i=1}^n \left(T_i(d)^2 - a^2\right)^{\binom{2n}{n-i}}$$

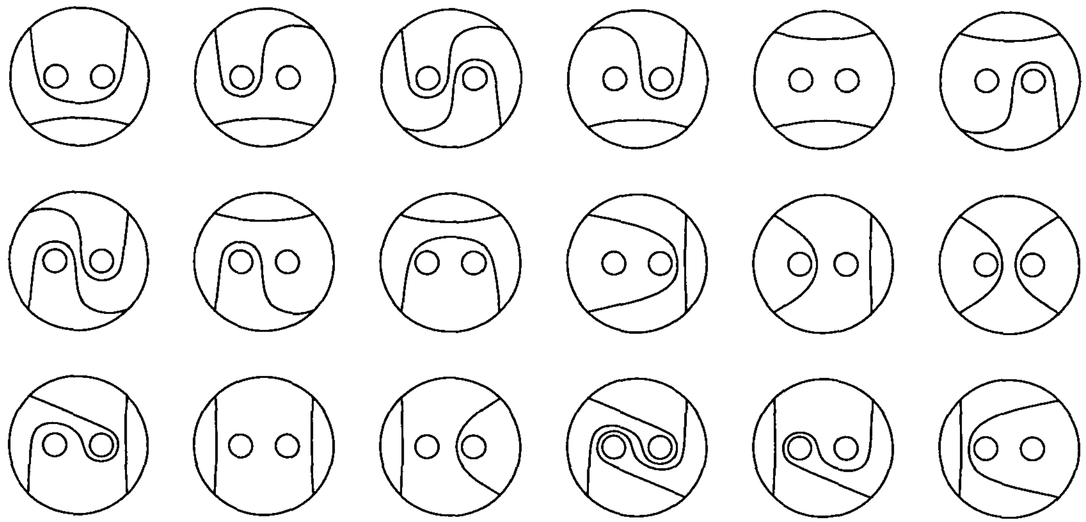
where $T_i(d)$ *is the Chebyshev polynomial of the first kind:*

$$T_0 = 2, \qquad T_1 = d, \qquad T_i = d\,T_{i-1} - T_{i-2};$$

d and a in the formula, correspond to the trivial and the nontrivial curves in the annulus $F_{0,1}$, *respectively.*

8. Appendix

8.1. $\mathbf{B}_2$



8.2. G_2

$$\begin{pmatrix}
d^2 & dy_2 & dx_2 & dz_2 & z_2 & x_2 & y_2 & d & y_2z_2 & x_2z_2 & {z_2}^2 & z_2 & x_2 & y_2 & {y_2}^2 & {x_2}^2 & z_2 & z_2 \\
dy_1 & dz_1 & dz_3 & dx_1 & x_1 & z_3 & z_1 & y_1 & x_1y_2 & x_1x_2 & x_1z_2 & x_1 & z_3 & z_1 & y_2z_1 & x_2z_3 & x_1 & x_1 \\
dx_1 & dz_3 & dz_1 & dy_1 & y_1 & z_1 & z_3 & x_1 & y_1y_2 & x_2y_1 & y_1z_2 & y_1 & z_1 & z_3 & y_2z_3 & x_2z_1 & y_1 & y_1 \\
dz_2 & dx_2 & dy_2 & d^2 & d & y_2 & x_2 & z_2 & dy_2 & dx_2 & dz_2 & d & y_2 & x_2 & x_2y_2 & x_2y_2 & d & d \\
z_2 & x_2 & y_2 & d & d^2 & dy_2 & dx_2 & dz_2 & y_2 & x_2 & z_2 & {z_2}^2 & x_2z_2 & y_2z_2 & z_2 & z_2 & {x_2}^2 & {y_2}^2 \\
x_1 & z_3 & z_1 & y_1 & dy_1 & dz_1 & dz_3 & dx_1 & z_1 & z_3 & x_1 & x_1z_2 & x_1x_2 & x_1y_2 & x_1 & x_1 & x_2z_3 & y_2z_1 \\
y_1 & z_1 & z_3 & x_1 & dx_1 & dz_3 & dz_1 & dy_1 & z_3 & z_1 & y_1 & y_1z_2 & x_2y_1 & y_1y_2 & y_1 & y_1 & x_2z_1 & y_2z_3 \\
d & y_2 & x_2 & z_2 & dz_2 & dx_2 & dy_2 & d^2 & x_2 & y_2 & d & dz_2 & dx_2 & dy_2 & d & d & x_2y_2 & x_2y_2 \\
y_1z_2 & x_2y_1 & y_1y_2 & dy_1 & y_1 & z_1 & z_3 & x_1 & dz_1 & dz_3 & dx_1 & y_1 & z_1 & z_3 & x_2z_1 & y_2z_3 & y_1 & y_1 \\
x_1z_2 & x_1x_2 & x_1y_2 & dx_1 & x_1 & z_3 & z_1 & y_1 & dz_3 & dz_1 & dy_1 & x_1 & z_3 & z_1 & x_2z_3 & y_2z_1 & x_1 & x_1 \\
{z_2}^2 & x_2z_2 & y_2z_2 & dz_2 & z_2 & x_2 & y_2 & d & dx_2 & dy_2 & d^2 & z_2 & x_2 & y_2 & {x_2}^2 & {y_2}^2 & z_2 & z_2 \\
z_2 & x_2 & y_2 & d & {z_2}^2 & x_2z_2 & y_2z_2 & dz_2 & y_2 & x_2 & z_2 & d^2 & dy_2 & dx_2 & z_2 & z_2 & {y_2}^2 & {x_2}^2 \\
x_1 & z_3 & z_1 & y_1 & x_1z_2 & x_1x_2 & x_1y_2 & dx_1 & z_1 & z_3 & x_1 & dy_1 & dz_1 & dz_3 & x_1 & x_1 & y_2z_1 & x_2z_3 \\
y_1 & z_1 & z_3 & x_1 & y_1z_2 & x_2y_1 & y_1y_2 & dy_1 & z_3 & z_1 & y_1 & dx_1 & dz_3 & dz_1 & y_1 & y_1 & y_2z_3 & x_2z_1 \\
{y_1}^2 & y_1z_1 & y_1z_3 & x_1y_1 & z_2 & x_2 & y_2 & d & x_1z_1 & x_1z_3 & {x_1}^2 & z_2 & x_2 & y_2 & {z_1}^2 & {z_3}^2 & z_2 & z_2 \\
{x_1}^2 & x_1z_3 & x_1z_1 & x_1y_1 & z_2 & x_2 & y_2 & d & y_1z_3 & y_1z_1 & {y_1}^2 & z_2 & x_2 & y_2 & {z_3}^2 & {z_1}^2 & z_2 & z_2 \\
z_2 & x_2 & y_2 & d & {x_1}^2 & x_1z_3 & x_1z_1 & x_1y_1 & y_2 & x_2 & z_2 & {y_1}^2 & y_1z_1 & y_1z_3 & z_2 & z_2 & {z_1}^2 & {z_3}^2 \\
z_2 & x_2 & y_2 & d & {y_1}^2 & y_1z_1 & y_1z_3 & x_1y_1 & y_2 & x_2 & z_2 & {x_1}^2 & x_1z_3 & x_1z_1 & z_2 & z_2 & {z_3}^2 & {z_1}^2
\end{pmatrix}$$

8.3. $\bar{G_2}$ (defined in Theorem 6.1), after simplification

$$\left(\begin{array}{ccccc}
x_2y_2 - dz_2 & -(-1+d)d(1+d) & -d(-1+y_2)(1+y_2) & -d(-1+z_2)(1+z_2) & -(-1+d)(1+d)y_2 \\
0 & -(d-y_1)(d+y_1) & -(y_2-z_1)(y_2+z_1) & (x_1-z_2)(x_1+z_2) & -dy_2+y_1z_1 \\
0 & -(d-z_2)(d+z_2) & (x_2-y_2)(x_2+y_2) & (d-z_2)(d+z_2) & -dy_2+x_2z_2 \\
-dx_1+x_2z_1-y_1z_2+y_2z_3 & -2(-1+d)(1+d)y_1 & 2y_1-y_1{y_2}^2-dy_2z_1 & 2y_1-dx_1z_2-y_1{z_2}^2 & -dy_1y_2+2z_1-d^2z_1 \\
0 & -dy_1+x_1z_2 & -y_2z_1+x_2z_3 & dy_1-x_1z_2 & x_1x_2-dz_1 \\
-{x_1}^2-{y_1}^2+{z_1}^2+{z_3}^2 & -2(dx_1y_1-z_2) & -x_1y_2z_1+2z_2-y_1y_2z_3 & -(-2+{x_1}^2+{y_1}^2)z_2 & 2x_2-dx_1z_1-dy_1z_3 \\
-dy_1+y_2z_1-x_1z_2+x_2z_3 & -2(-1+d)(1+d)x_1 & 2x_1-x_1{y_2}^2-dy_2z_3 & 2x_1-dy_1z_2-x_1{z_2}^2 & -dx_1y_2+2z_3-d^2z_3 \\
0 & -dx_1+y_1z_2 & x_2z_1-y_2z_3 & dx_1-y_1z_2 & x_2y_1-dz_3 \\
-d^2+{x_2}^2+{y_2}^2-{z_2}^2 & -2(-1+d)(1+d)z_2 & -dx_2y_2+2z_2-{y_2}^2z_2 & -z_2(-2+d^2+{z_2}^2) & 2x_2-d^2x_2-dy_2z_2 \\
0 & -(d-x_1)(d+x_1) & -(y_2-z_3)(y_2+z_3) & (y_1-z_2)(y_1+z_2) & -dy_2+x_1z_3
\end{array}\right.$$

$$\left.\begin{array}{ccccc}
y_2-dx_2z_2 & -(-1+d)(1+d)x_2 & x_2-dy_2z_2 & -(-1+d)(1+d)z_2 & -d(-1+x_2)(1+x_2) \\
-x_2z_2+x_1z_3 & -dx_2+y_1z_3 & x_1z_1-y_2z_2 & x_1y_1-dz_2 & -(x_2-z_3)(x_2+z_3) \\
dy_2-x_2z_2 & -dx_2+y_2z_2 & dx_2-y_2z_2 & 0 & -(x_2-y_2)(x_2+y_2) \\
-dx_1x_2+2z_1-x_2y_1z_2 & -dx_2y_1+2z_3-d^2z_3 & -dx_1y_2-y_1y_2z_2+2z_3 & 2x_1-d^2x_1-dy_1z_2 & 2y_1-{x_2}^2y_1-dx_2z_3 \\
-x_1x_2+dz_1 & x_1y_2-dz_3 & -x_1y_2+dz_3 & 0 & y_2z_1-x_2z_3 \\
-x_2(-2+{x_1}^2+{y_1}^2) & 2y_2-dy_1z_1-dx_1z_3 & -(-2+{x_1}^2+{y_1}^2)y_2 & -d(-2+{x_1}^2+{y_1}^2) & -x_2y_1z_1+2z_2-x_1x_2z_3 \\
-dx_2y_1-x_1x_2z_2+2z_3 & -dx_1x_2+2z_1-d^2z_1 & -dy_1y_2+2z_1-x_1y_2z_2 & 2y_1-d^2y_1-dx_1z_2 & 2x_1-x_1{x_2}^2-dx_2z_1 \\
-x_2y_1+dz_3 & y_1y_2-dz_1 & -y_1y_2+dz_1 & 0 & -x_2z_1+y_2z_3 \\
-x_2(-2+d^2+{z_2}^2) & 2y_2-d^2y_2-dx_2z_2 & -y_2(-2+d^2+{z_2}^2) & -d(-2+d^2+{z_2}^2) & -dx_2y_2+2z_2-{x_2}^2z_2 \\
y_1z_1-x_2z_2 & -dx_2+x_1z_1 & -y_2z_2+y_1z_3 & x_1y_1-dz_2 & -(x_2-z_1)(x_2+z_1)
\end{array}\right)$$

$$\det \bar{G_2} = \frac{\det G_2}{(1-d^2)^4 \det {G_1}^2}$$

8.4. $\det G_2$

$$\begin{aligned}
\det G_2 \quad = \quad & -d^2(-x_1x_2+x_2y_1+x_1y_2-y_1y_2+dz_1-z_1z_2-dz_3+z_2z_3)^4(-x_1x_2-x_2y_1-x_1y_2-y_1y_2+dz_1+z_1z_2+dz_3+z_2z_3)^4 \\
& (-x_1x_2z_1-y_1y_2z_1+d{z_1}^2+x_2y_1z_3+x_1y_2z_3-d{z_3}^2)^2 \\
& (8d^2-2d^4-8{x_1}^2+2d^2{x_1}^2-8{x_2}^2+2d^2{x_2}^2+2{x_1}^2{x_2}^2+8dx_1y_1-2d^3x_1y_1-2dx_1{x_2}^2y_1-8{y_1}^2+2d^2{y_1}^2+2{x_2}^2{y_1}^2+ \\
& 8dx_2y_2-2d^3x_2y_2-2d{x_1}^2x_2y_2+2d^2x_1x_2y_1y_2-2dx_2{y_1}^2y_2-8{y_2}^2+2d^2{y_2}^2+2{x_1}^2{y_2}^2-2dx_1y_1{y_2}^2+2{y_1}^2{y_2}^2+ \\
& 2dx_1x_2z_1-d^3x_1x_2z_1-4x_2y_1z_1+2d^2x_2y_1z_1-4x_1y_2z_1+2d^2x_1y_2z_1+2dy_1y_2z_1-d^3y_1y_2z_1+8{z_1}^2-6d^2{z_1}^2+d^4{z_1}^2- \\
& 8d^2z_2+2d^4z_2+dx_1x_2z_1z_2-2x_2y_1z_1z_2-2x_1y_2z_1z_2+dy_1y_2z_1z_2+4{z_1}^2z_2-d^2{z_1}^2z_2+8{z_2}^2-2d^2{z_2}^2-4x_1x_2z_3+ \\
& 2d^2x_1x_2z_3+2dx_2y_1z_3-d^3x_2y_1z_3+2dx_1y_2z_3-d^3x_1y_2z_3-4y_1y_2z_3+2d^2y_1y_2z_3-2x_1x_2z_2z_3+dx_2y_1z_2z_3+dx_1y_2z_2z_3- \\
& 2y_1y_2z_2z_3+8{z_3}^2-6d^2{z_3}^2+d^4{z_3}^2+4z_2{z_3}^2-d^2z_2{z_3}^2) \\
& (8d^2-2d^4-8{x_1}^2+2d^2{x_1}^2-8{x_2}^2+2d^2{x_2}^2+2{x_1}^2{x_2}^2-8dx_1y_1+2d^3x_1y_1+2dx_1{x_2}^2y_1-8{y_1}^2+2d^2{y_1}^2+2{x_2}^2{y_1}^2- \\
& 8dx_2y_2+2d^3x_2y_2+2d{x_1}^2x_2y_2+2d^2x_1x_2y_1y_2+2dx_2{y_1}^2y_2-8{y_2}^2+2d^2{y_2}^2+2{x_1}^2{y_2}^2+2dx_1y_1{y_2}^2+2{y_1}^2{y_2}^2+ \\
& 2dx_1x_2z_1-d^3x_1x_2z_1+4x_2y_1z_1-2d^2x_2y_1z_1+4x_1y_2z_1-2d^2x_1y_2z_1+2dy_1y_2z_1-d^3y_1y_2z_1+8{z_1}^2-6d^2{z_1}^2+d^4{z_1}^2+ \\
& 8d^2z_2-2d^4z_2-dx_1x_2z_1z_2-2x_2y_1z_1z_2-2x_1y_2z_1z_2-dy_1y_2z_1z_2-4{z_1}^2z_2+d^2{z_1}^2z_2+8{z_2}^2-2d^2{z_2}^2+4x_1x_2z_3- \\
& 2d^2x_1x_2z_3+2dx_2y_1z_3-d^3x_2y_1z_3+2dx_1y_2z_3-d^3x_1y_2z_3+4y_1y_2z_3-2d^2y_1y_2z_3-2x_1x_2z_2z_3-dx_2y_1z_2z_3-dx_1y_2z_2z_3- \\
& 2y_1y_2z_2z_3+8{z_3}^2-6d^2{z_3}^2+d^4{z_3}^2-4z_2{z_3}^2+d^2z_2{z_3}^2)
\end{aligned}$$

8.5. Terms of maximal degree in $\det G_3$

$$\begin{aligned}
h(\det G_3) &= h(\det G_2)^6 \det {G_1}^{-9} d^{30} w^3 \bar{w}^3 \\
&= d^{66}(-x_1x_2 + x_2y_1 + x_1y_2 - y_1y_2 + dz_1 - z_1z_2 - dz_3 + z_2z_3)^{15} \\
&\quad (-x_1x_2 - x_2y_1 - x_1y_2 - y_1y_2 + dz_1 + z_1z_2 + dz_3 + z_2z_3)^{15} \\
&\quad (-x_1x_2z_1 - y_1y_2z_1 + dz_1^2 + x_2y_1z_3 + x_1y_2z_3 - d{z_3}^2)^{12} \\
&\quad (2x_1x_2y_1y_2 - dx_1x_2z_1 - dy_1y_2z_1 + d^2{z_1}^2 - dx_2y_1z_3 - dx_1y_2z_3 + d^2{z_3}^2)^{12} \\
&\quad (x_1x_2y_1y_2z_1 - dx_1x_2{z_1}^2 - dy_1y_2{z_1}^2 + d^2{z_1}^3 - x_1x_2y_1y_2z_3 + dx_2y_1{z_3}^2 + dx_1y_2{z_3}^2 - d^2{z_3}^3)^3 \\
&\quad (x_1x_2y_1y_2z_1 - dx_1x_2z_1^2 - dy_1y_2z_1^2 + d^2{z_1}^3 + x_1x_2y_1y_2z_3 - dx_2y_1{z_3}^2 - dx_1y_2{z_3}^2 + d^2{z_3}^3)^3
\end{aligned}$$

8.6. $\det G_3$ with substitution $x_1 = x_2 = y_1 = y_2 = z_2 = 0$

$$\begin{aligned}
\det G_3|_{x_1=x_2=y_1=y_2=z_2=0} &= (-2+d)^{16}(-1+d)^4 d^{60}(1+d)^4(2+d)^{16}(-3+d^2)^6(z_1-z_3)^{30}(z_1+z_3)^{30} \\
&\quad ({z_1}^2 - z_1z_3 + {z_3}^2)({z_1}^2 + z_1z_3 + {z_3}^2)(-2d^2 - 2{z_1}^2 + d^2{z_1}^2 - 2{z_3}^2 + d^2{z_3}^2)^{12} \\
&\quad (-3d^2 - {z_1}^2 + d^2{z_1}^2 + z_1z_3 - d^2z_1z_3 - {z_3}^2 + d^2{z_3}^2)^2(-3d^2 - {z_1}^2 + d^2{z_1}^2 - z_1z_3 + d^2z_1z_3 - {z_3}^2 + d^2{z_3}^2)^2
\end{aligned}$$

8.7. $\det G_1^{F_{0,3}}$ with substitution $x_{a_1} = x_{a_1,a_2} = 0$ for all variables of the form x_{a_1} and x_{a_1,a_2}

$$\begin{aligned}
\det G_1^{F_{0,3}}|_{x_{a_1}=x_{a_1,a_2}=0} &= -(d - x_{1,2,3})(d + x_{1,2,3}) \\
&\quad (x_{1,2,-2}x_{1,-1,3}x_{1,-1,-2} + x_{1,3,-3}x_{1,-1,2}x_{1,-1,-3} - x_{1,2,-3}x_{1,-1,3}x_{1,-1,-3} - \\
&\quad x_{1,-1,-2}x_{1,-1,-2}x_{1,-2,3} - x_{1,2,-2}x_{1,3,-3}x_{1,-2,-3} + x_{1,2,-3}x_{1,-2,3}x_{1,-2,-3})^2
\end{aligned}$$

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THE GRAM DETERMINANT OF CURVES ON THE MOBIUS BAND

1. Jozef: Feb 1, 2009

Let b_1, b_2, b_3 be the base for $n = 1$. b_1 and b_2 the same as for annulus and b_3 cutting the core of Mobius band once. In this basis (see Fig. 1) and basis on the Klein bottle given by d - trivial curve, a as old a, that is combination $b_1 \circ b_2$ and of $b_2 \circ b_1$, and further x corresponding to $b_3 \circ b_1 = b_3 \circ b_2$; y corresponding to $b_1 \circ b_3 = b_2 \circ b_3$ and z to $b_3 \circ b_3$.

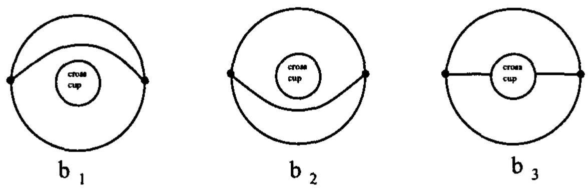

Figure 1

The first Gram matrix and determiant are:

$$\det \begin{bmatrix} d & a & y \\ a & d & y \\ x & x & z \end{bmatrix} = (d-a)(z(d+a) - 2xy)$$

For $x = y = 0$ we get $z \det G_1^B$.

Of course it is a general property so for $x = y = 0$ we know our matrix from the B-type. Similarly we can argue that $\det G_n^{Mb}$ divides $\det G_{n+1}^{Mb}$.

So this are obvious observations, but probably we do not have a nice formula for the dimensions of the matrix (size of basis) in general. Maybe generated function of this is not that difficult?

2. Qi: April 3, 2009

Denote the set of configurations, up to isotopies fixing the boundary, of n non-crossing intervals on the Mobius band by $\mathscr{M}_n$. Let $\mathscr{M}_{n,i}$ be the subset consists of

i intervals through the cross-cup. Then obviously we have

$$\mathscr{M}_n = \bigsqcup_{i=0}^{n} \mathscr{M}_{n,i}.$$

So when $i = 0$ we get the type B. It is easy to see that $|\mathscr{M}_{n,i}| = \binom{2n}{n-i}$. Hence the dimension of our problem is

$$|\mathscr{M}_n| = \sum_{i=0}^{n} \binom{2n}{n-i} = 2^{2n-1} + \frac{1}{2}\binom{2n}{n}.$$

Denote

$$\mathbf{D}_{n,0} = \det G_n^B = \prod_{k=1}^{n} \left(T_k(d)^2 - a^2\right)^{\binom{2n}{n-k}},$$

and when $i > 0$

$$\mathbf{D}_{n,i} = \prod_{k=i+1}^{n} \left(T_{2k}(d) - 2\right)^{\binom{2n}{n-k}}.$$

Also denote

$$\mathbf{O}_{n,2i} = T_{2i}(z) - \frac{2(2-a)}{T_{2i}(d) - a}, \qquad \mathbf{O}_{n,2i+1} = T_{2i+1}(z) - \frac{2xy}{T_{2i+1}(d) + a}.$$

Then the Gram determinant on $\mathscr{M}_n$ is

$$\mathbf{D}_n = \prod_{i=0}^{n} \mathbf{D}_{n,i} \prod_{j=1}^{n} \mathbf{O}_{n,j}^{\binom{2n}{n-j}}.$$

This formula agrees with Maple calculations for $n = 1, 2, 3, 4$.