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Symmetric Extensions of Dihedral Quandles and Triple Points of Non-orientable Surfaces

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Abstract

Quandles with involutions that satisfy certain conditions, called good involutions, can be used to color non-orientable surface-knots. We use subgroups of signed permutation matrices to construct non-trivial good involutions on extensions of odd order dihedral quandles.

For the smallest example $\tilde{R}_3$ of order 6 that is an extension of the three-element dihedral quandle R_3, various symmetric quandle homology groups are computed, and applications to the minimal triple point number of surface-knots are given.

1 Introduction

In this paper, we construct an extension $\tilde{R}_m$ of the dihedral quandle R_m with a non-trivial good involution for each odd positive integer $m = 2n + 1$. The extensions $\tilde{R}_m$ we construct are not involutory, and in particular, not isomorphic to dihedral quandles. As an application, such an extension is used to study the minimal triple point numbers of non-orientable surface-knots in thickened 3-manifolds. Detailed definitions will be given in Section 2.

A quandle is a set with a binary operation that is self-distributive: $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$ and satisfies two other properties. The algebraic structure mimics the Reidemeister moves, and consequently quandles are a fundamental tool in knot theory. Quandle cohomology theories [2, 4] have been constructed, and applied to knots by using quandle elements as colors and cocycles as weights to define quandle cocycle invariants. The same construction was applied to surface-knots using triple points of the projections, and a variety of applications have been found.

The original definitions of quandle colorings and quandle cocycle invariants are dependent upon orientations of the diagrams, and in particular, the invariants were defined at first only for orientable surface-knots. A quandle is called *involutory* if $(x \triangleleft y) \triangleleft y = x$ holds for any elements x, y of the quandle. An involutory quandle is also called a *kei* [23], and has the property that colorings are defined for unoriented knots and surfaces. To generalize involutory quandles and quandle cocycle invariants for unoriented diagrams and non-orientable surfaces, quandles with good involutions were defined [15] and studied [16, 19]. Quandles with good involutions are called *symmetric quandles.*

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Constructions of symmetric quandles have depended mainly on computer calculations, or by hand for specific families of quandles, such as dihedral quandles. In particular, it was shown in [16] that any dihedral quandle of odd order only has the trivial (identity map) good involution. In this paper, we give a construction of symmetric quandles via extensions of dihedral quandles. Specifically, we prove that for any odd order dihedral quandle, there is an extension with a non-trivial good involution that is not involutory.

The smallest of such extensions is given by a two-to-one quandle homomorphism $\tilde{R}_3 \to R_3$, onto the three-element dihedral quandle. Various homology groups of this quandle $\tilde{R}_3$ are computed, and specific non-trivial cycles and cocycles are presented. Applications are given for the minimal triple point numbers of non-orientable surfaces.

The *minimal triple point number* $t(K)$ of a knotted or linked surface F is defined to be the smallest number of the triple point numbers among all the diagrams of the surface-link F, and denoted by $t(F)$. Quandle cocycle invariants [1] are used for studies of minimal triple point numbers of orientable surface-links; Satoh and Shima [22] determined the minimal triple point number of the 2-twist-spun trefoil, and Hatakenaka [11] gave a lower bound of 6 for the 2-twist-spun figure-eight knot. Kamada [15] proved that, for any positive integer N, there is an orientable 2-knot (an embedded sphere, a spherical surface-knot) K with $t(K) > N$. His argument (Alexander modules) does not immediately apply to higher genus surfaces or non-orientable surfaces.

Iwakiri [13] used quandle cocycle invariants to provide a surface-knots K with the triple point canceling number $\tau(K)$ as large as you please. The *triple point canceling number* is the minimal number of 1-handles needed to change a surface-knot into another with a projection that has no triple point. It was pointed out by Satoh that, since $\tau(K) \leq t(K)$, Iwakiri's result implies that for any positive integer N, there is an orientable surface-knot K with $t(K) > N$. Iwakiri's results can be applied to higher genus *orientable surfaces*, but not to non-orientable surfaces.

In [16, 19, 21], large minimal triple point numbers of (not necessarily orientable, and two-component) surface-links are realized. Their arguments, however, do not immediately apply to surface-knots. In this paper, we give surface-knots in thickened 3-manifolds with arbitrary large minimal triple point numbers.

The paper is organized as follows. A summary of definitions and known results necessary for this paper are given in Section 2. Explicit constructions are given in Section 3 to prove the existence of symmetric extensions of odd order dihedral quandles. Symmetric quandle homology groups are computed for the smallest such example in Section 4, and applications to the minimal triple points are presented.

2 Preliminaries

In this section we give a summary of necessary definitions and set up the notation.

2.1 Symmetric quandles

Definition 2.1 [1, 14, 18, 23] A *quandle*, X, is a set with a binary operation $(x, y) \mapsto x \triangleleft y$ such that
(I. IDEMPOTENCY) for any $x \in X$, $x \triangleleft x = x$,

(II. RIGHT-INVERTIBILITY) for any $x, y \in X$, there is a unique $z \in X$, denoted by $x \triangleleft \bar{y}$, such that $x = z \triangleleft y$, and
(III. SELF-DISTRIBUTIVITY) for any $x, y, z \in X$, we have $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$.

Typical examples are conjugations of groups $x \triangleleft y = y^{-1}xy$. Any subset of a group closed under conjugation, thus, is a quandle. In particular for the dihedral group D_{2m} for any positive integer m, the subset R_m of reflections forms a quandle by conjugation. In this paper, we will concentrate on the case $m = 2n + 1$ — an odd integer. It is known that D_{2m} has a presentation $\langle x, y : x^2 = y^m = (xy)^2 = 1\rangle$. The reflections and rotations of the regular m-gon are written as xy^j and y^j for $j = 0, \ldots, m-1$, respectively. Since $(xy^j)^{-1}(xy^i)(xy^j) = xy^{2j-i}$, the quandle R_m can be identified with $\mathbb{Z}_m$ with the operation $i \triangleleft j = 2j - i \pmod{m}$.

Let G be a group, H a subgroup, $s : G \to G$ an automorphism such that $s(h) = h$ for each $h \in H$. Define a binary operation on G by $a \triangleleft b = s(ab^{-1})b$. Then this defines a quandle structure on G. This passes to a well-defined quandle structure on the right cosets G/H that is given by $Ha \triangleleft Hb = Hs(ab^{-1})b$. In particular, if $\zeta \in Z(H) \cap H$ where $Z(H) = \{\zeta \in G : \zeta h = h\zeta \text{ for all } h \in H\}$, then $Ha \triangleleft Hb = Hab^{-1}\zeta b$ defines a quandle structure. Let us denote the resulting quandle by (G, H, ζ). This construction is found in [14, 18]. For $G = D_{2m}$ with $H = \langle x\rangle$ and $\zeta = x$, one computes $Hy^i \triangleleft Hy^j = Hy^i y^{-j} x y^j = Hxy^{2j-i} = Hy^{2j-i}$, so that we have $R_m = (D_{2m}, H, x)$.

Definition 2.2 [15] An involution $\rho : X \to X$ defined on a quandle is a *good involution* if $x \triangleleft \rho(y) = x \triangleleft \bar{y}$ and $\rho(x \triangleleft y) = \rho(x) \triangleleft y$. Such a pair (X, ρ) is called a quandle with a good involution or a *symmetric quandle*.

The associated group [8] of a quandle X is $G_X = \langle x \in X : x \triangleleft y = y^{-1}xy\rangle$. The associated group, $G_{(X,\rho)}$ of a symmetric quandle (X, ρ) is defined [16] by $G_{(X,\rho)} = \langle x \in X : x \triangleleft y = y^{-1}xy,\ \rho(x) = x^{-1}\rangle$. The natural map $\mu : X \to G_{(X,\rho)}$ is the composition of the inclusion map $X \to F(X)$ and the projection map $F(X) \to G_{(X,\rho)}$, where $F(X)$ is the free group on X. For a quandle X, an X-set [9] is a set Y equipped with a right action of the associated group G_X. For a symmetric quandle (X, ρ), an (X, ρ)-set is a set Y equipped with a right action of the associated group $G_{(X,\rho)}$. We denote by yg or by $y \cdot g$ the image of an element $y \in Y$ under the action of $g \in G_{(X,\rho)}$. The following three formulas hold: $y \cdot (x_1x_2) = (y \cdot x_1) \cdot x_2$, $y \cdot (x_1 \triangleleft x_2) = y \cdot (x_2^{-1}x_1x_2)$, and $y \cdot (\rho(x_1)) = y \cdot (x_1^{-1})$, for $x_1, x_2 \in X$ and $y \in Y$.

2.2 Homology theories for symmetric quandles

A cohomology theory of quandles was defined [4] as a modification of rack cohomology theory [9]. In this section we review homology groups for symmetric quandles defined in [15], see also [16].

Let Y be an (X, ρ)-set which may be empty. Let $C_n(X)_Y$ be the free abelian group generated by $(y, x_1, \ldots, x_n)$, where $y \in Y$ and $x_1, \ldots, x_n \in X$. For a positive integer n, let $C_0(X)_Y = \mathbb{Z}(Y)$, the free abelian group generated by Y, and set $C_n(X)_Y = 0$ otherwise. (If Y is empty, then define $C_0(X) = 0$). Define the boundary homomorphism $\partial_n : C_n(X)_Y \longrightarrow C_{n-1}(X)_Y$ by

$$\begin{aligned}\partial_n(y, x_1, \ldots, x_n) = \sum_{i=1}^{n} (-1)^i [&(y, x_1, x_2, \ldots, x_{i-1}, \hat{x_i}, x_{i+1}, \ldots, x_n) \\ - \quad &(y \cdot x_i, x_1 \triangleleft x_i, x_2 \triangleleft x_i, \ldots, x_{i-1} \triangleleft x_i, \hat{x_i}, x_{i+1}, \ldots, x_n)]\end{aligned}$$

for $n \geq 1$ and $\partial_n = 0$ for $n \leq 1$. Then $C_*(X)_Y = \{C_n(X)_Y, \partial_n\}$ is a chain complex [9]. Let $D_n^Q(X)_Y$ be the subgroup of $C_n(X)_Y$ generated by $\cup_{i=1}^{n-1}\{(y, x_1, \ldots, x_n) \mid x_i = x_{i+1}\}$, and let $D_n^\rho(X)_Y$ be the subgroup of $C_n(X)_Y$ generated by n-tuples of the form

$$(y, x_1, \ldots, x_n) + (y \cdot x_i, x_1 \triangleleft x_i, \ldots, x_{i-1} \triangleleft x_i, \rho(x_i), x_{i+1}, \ldots, x_n)$$

where $y \in Y$, $x_1, \ldots, x_n \in X$, and $i \in \{1, \ldots, n-1\}$. Then $\{D_n^Q(X)_Y, \partial_n\}$ and $\{D_n^\rho(X)_Y, \partial_n\}$ are subcomplexes of C_n [16], and chain complexes $C_*^R(X)_Y$, $C_*^Q(X)_Y$, $C_*^{R,\rho}(X)_Y$, $C_*^{Q,\rho}(X)_Y$ are defined, respectively, from chain groups $C_n^R(X)_Y = C_n(X)_Y$, $C_n^Q(X)_Y = C_n(X)_Y / D_n^Q(X)_Y$, $C_n^{R,\rho}(X)_Y = C_n(X)_Y / D_n^\rho(X)_Y$, $C_n^{Q,\rho}(X)_Y = C_n(X)_Y / (D_n^Q(X)_Y + D_n^\rho(X)_Y)$. Their respective homology groups [16] are denoted by $H_*^R(X)_Y$, $H_*^Q(X)_Y$, $H_*^{R,\rho}(X)_Y$, and $H_*^{Q,\rho}(X)_Y$, respectively. When $Y = \emptyset$, this subscript is dropped. Corresponding cohomology groups are defined as usual, as well as (co)homology groups with other coefficient groups.

An *extension* of a quandle X is a surjective quandle homomorphism $f : E \to X$ such that for any element of X, the cardinality of the inverse image by f is constant. We also say that E is an extension of X. In [3], an interpretation of quandle 2-cocycles was given in terms of extensions of quandles, in a manner similar to group extensions by group 2-cocycles. It is, therefore, a natural question to ask for a relation between symmetric quandle 2-cocycles and extensions of symmetric quandles. Here we observe such an interpretation.

Let (X, ρ) be a symmetric quandle, and A be an abelian group, and $\phi : X^2 \to A$ be a symmetric quandle 2-cocycle. Specifically, ϕ satisfies

$$\phi(x_1, x_2) - \phi(x_1, x_3) - \phi(x_1 \triangleleft x_2, x_3) + \phi(x_1 \triangleleft x_3, x_2 \triangleleft x_3) = 0$$

for any $x_1, x_2, x_3 \in X$, and $\phi(x_1, x_2) + \phi(\rho(x_1), x_2) = 0$, $\phi(x_1, x_2) + \phi(x_1 \triangleleft x_2, \rho(x_2)) = 0$. An extension of a quandle X by a quandle 2-cocycle ϕ was defined in [3] by $(x, a) \triangleleft (y, b) = (x \triangleleft y, a + \phi(x, y))$. Define $\tilde{\rho} : X \times A \to X \times A$ by $\tilde{\rho}(x, a) = (\rho(x), -a)$.

Proposition 2.3 *$(X \times A, \tilde{\rho})$ is a symmetric quandle.*

This is proved by direct calculations.

2.3 Colorings of surface-knots by symmetric quandles

A knot diagram for a classical knot ($n = 1$) or for a surface-knot ($n = 2$) is the image of a general position map from a closed n-manifold (collection of circles or surfaces) into $\mathbb{R}^{n+1}$ with crossing information indicated by breaking the under-arc or under-sheet (see [7] for details). Let a surface diagram D of a surface-knot F be given. We cut the diagram further into *semi-sheets* by considering the upper sheets also to be broken along the double point arcs. Observe that in the local picture of a branch point there are two semi-sheets, at a double point there are 4 semi-sheets, and at a triple point, there are 12 semi-sheets.

Let (X, ρ) denote a symmetric quandle, and let Y denote an (X, ρ)-set. The surface diagram D has elements of X assigned to the sheets and elements of Y assigned to regions separated by the projection such that the following conditions are satisfied.

- (Quandle coloring rule) Suppose that two adjacent under-sheets e_1 and e_2 are separated along a double curve and are labeled by x_1 and x_2. Suppose that one of the two semi-sheets coming from the over sheet of D, say e_3, is labeled by x_3. We assume that a local normal orientation of e_3 points from e_1 to e_2. If the normal orientations of e_1 and e_2 are coherent, then $x_1 \triangleleft x_3 = x_2$, otherwise $x_1 \triangleleft x_3 = \rho(x_2)$.
- (Region colors) Suppose that two adjacent regions r_1 and r_2 which are separated by a semi-sheet, say e, are labeled by y_1 and y_2, where $y_1, y_2 \in Y$. Suppose that the semi-sheet e is labeled by x. If the normal orientation of e points from r_1 to r_2, then $y_1 \cdot x = y_2$.
- An equivalence relation (of a local normal orientation assigned to each semi-sheet and a quandle element associated to this local orientation) is generated by the following rule (*basic inversion*): Suppose that two adjacent semi-sheets coming from an over-sheet of D about a double curve are labeled by x_1 and x_2. If the normal orientations are coherent, then $x_1 = x_2$, otherwise $x_1 = \rho(x_2)$.

A *symmetric quandle coloring*, or an *$(X, \rho)_Y$-coloring,* of a surface-knot diagram is such an equivalence class of symmetric quandle colorings. See Fig. 1.

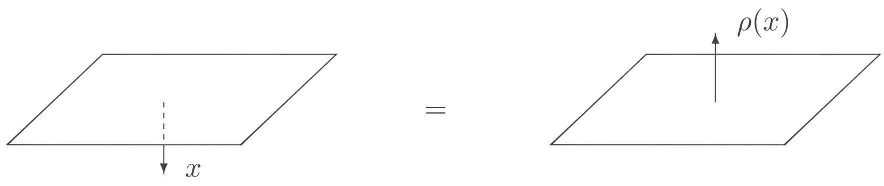
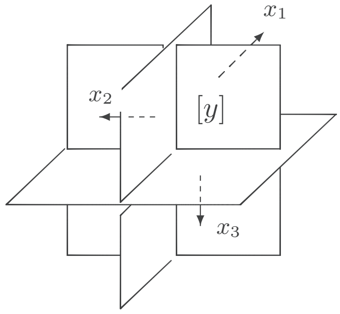


Figure 1: A basic inversion

We call a diagram D with an $(X, \rho)_Y$-coloring, C_D, an *$(X, \rho)_Y$-colored diagram* and denote it by (D, C_D). Let (D, C_D) and $(D', C_{D'})$ be $(X, \rho)_Y$-colored diagrams.

We say that (D, C_D) and $(D', C_{D'})$ are *Roseman move equivalent* if they are related by a finite sequence of Roseman moves [20] (see also [7]) such that the colors are kept constant outside of each local move.

Let (D, C_D) be an $(X, \rho)_Y$-colored diagram of an $(X, \rho)_Y$-colored surface-link (F, C). For a triple point of D, define the *weight* as follows: Choose one of eight 3-dimensional regions around the triple point and call the region a *specified region.* There exist 12 semi-sheets around the triple points. Let S_T, S_M and S_B be the three of them that face the specified region, where S_T, S_M and S_B are in the top sheet, the middle sheet and the bottom sheet at the triple point, respectively. Let n_T, n_M and n_B be the normal orientations of S_T, S_M and S_B which point away from the specified region. Consider a representative of C_D such that the normal orientations of S_T, S_M and S_B are n_T, n_M and n_B and let x_1, x_2 and x_3 be the labels assigned the semi-sheets S_B, S_M and S_T, with the normal orientations n_B, n_M and n_T, respectively. Let y be the label assigned to the specified region. The weight is $\epsilon(y, x_1, x_2, x_3)$, where ϵ is $+1$ (or -1) if the triple of the normal orientations (n_T, n_M, n_B) does (or does not, respectively) match the orientation of $\mathbb{R}^3$. See Figure 2. The sum $\sum_\tau \epsilon(y, x_1, x_2, x_3)$ of the weight over all triple points of a colored diagram (D, C_D) represents a 3-cycle $[c_D] \in C_3^{Q,\rho}(X)_Y$ [16]. Colored diagrams and cycles represented by colored diagrams are similarly defined when region colors are absent, or equivalently, when $Y = \emptyset$.



$+(y, x_1, x_2, x_3)$

Figure 2: A weight of a triple point

Lemma 2.4 [16, 19] *If two colored diagrams (D, C_D) and $(D', C_{D'})$ are Roseman move equivalent, then they represent homologous 3-cycles, $[C_D] = [C_{D'}] \in H_3^{Q,\rho}(X)_Y$.*

For surface-knots or link in $\mathbb{R}^4$, it is known [20] that two diagrams of equivalent (ambiently isotopic) surface-knot or a link are Roseman move equivalent.

2.4 Triple point numbers

Let F be a surface-link and D a diagram of F. The minimal triple point number of F is evaluated by quandle invariants with symmetric quandle cocycles as follows:

Lemma 2.5 [16, 19] *Let (X, ρ) be a symmetric quandle. Let $\theta : \mathbb{Z}(X^3) \to \mathbb{Z}$ be a symmetric quandle 3-cocycle $\theta \in C^3_{Q,\rho}(X)$ of (X, ρ) such that $\theta(a, b, c) \in \{0, -1, 1\}$ for any $(a, b, c) \in X^3$. For a symmetric quandle coloring C_D of the diagram D, if $\theta([C_D]) = \alpha$ for $\alpha \in \mathbb{Z}$, then $t(F) \geq |\alpha|$.*

For surface-knots and links in a thickened 3-manifold $M \times [0, 1]$, where M is a closed 3-manifold, we take the natural projection $p : M \times [0, 1] \to M$ in the direction of the unit interval to define the diagrams. Then the minimal triple point number is defined in the same manner as above. By [12], we may assume that diagrams in M of equivalent (ambiently isotopic) surface-knot or link in $M \times [0, 1]$ are Roseman move equivalent.

3 Extensions of dihedral quandles with good involutions

In this section we prove:

Theorem 3.1 *For each positive integer n, there is an extension $\tilde{R}_{2n+1}$ of R_{2n+1} with a non-trivial good involution ρ that is not involutory.*

The proof follows a construction of a group extension of the dihedral group and the definition of a quandle structure on group cosets that were described in Section 2.

Definition 3.2 Let e_j denote the column vector in $\mathbb{R}^m$ whose jth entry is 1 and the remaining entries are each 0. A *signed permutation matrix* is a square matrix of size m matrix whose columns are of the form $(\pm e_{\sigma(1)}, \pm e_{\sigma(2)}, \ldots, \pm e_{\sigma(m)})$ where $\sigma \in \Sigma_m$ is a permutation. The set of signed permutation matrices form a group H_m of order $2^m m!$ that is called the *hyper-octahedral group*. Define the group SH_m to be the signed permutation matrices of determinant 1.

To avoid extra subscripts, we write $(\pm e_{\sigma(1)}, \pm e_{\sigma(2)}, \ldots, \pm e_{\sigma(m)})$ as $(\pm\sigma(1), \pm\sigma(2), \ldots, \pm\sigma(m))$. Then the matrix multiplication, in this notation, is written by

$$(\epsilon_1 \cdot \sigma(1),\ \ldots,\ \epsilon_m \cdot \sigma(m)) \cdot (\delta_1 \cdot \tau(1),\ \ldots,\ \delta_m \cdot \tau(m)) = (\epsilon_{\tau(1)}\delta_1 \cdot \sigma(\tau(1)),\ \ldots,\ \epsilon_{\tau(m)}\delta_m \cdot \sigma(\tau(m))),$$

where $\epsilon_i = \pm 1$ and $\delta_j = \pm 1$ for $i, j = 1, \ldots, m$. The product is obtained by looking at the entry in the $\tau(1)$ position of $(\epsilon_1 \cdot \sigma(1),\ \epsilon_2 \cdot \sigma(2),\ \ldots,\ \epsilon_m \cdot \sigma(m))$ and write that entry first after having been multiplied by δ_1, then look at the entry in the $\tau(2)$ position and write that second after having been multiplied by δ_2, and the process continues to the mth position. For example, $(1, 5, 4, -3, -2)\cdot(5, 1, 2, 3, 4) = (-2, 1, 5, 4, -3)$ while $(5, 1, 2, 3, 4)\cdot(1, 5, 4, -3, -2) = (5, 4, 3, -2, -1)$.

We identify a subgroup of SH_m that maps onto the dihedral group. Let $m = 2n+1$. Consider the subgroup G_{2n+1} of SH_{2n+1} that is generated by the pair of elements $a = (1, 2n+1, 2n, \ldots, n+2, -(n+1), \ldots, -3, -2)$ and $b = (2n+1, 1, 2, \ldots, 2n)$.

The dihedral group, $D_{2(2n+1)}$, will be identified with the image of its faithful representation in permutation matrices. Specifically, we identify the reflection x with $(1, 2n+1, 2n, \ldots, 2)$, the rotation y with $(2n+1, 1, 2, \ldots, 2n)$, and $D_{2(2n+1)}$ with the subgroup of permutation matrices generated by these two elements. Then the map that takes each matrix (a_{ij}) to $(|a_{ij}|)$ defines a group homomorphism onto the dihedral group: $f : G_{2n+1} \to D_{2(2n+1)}$, such that $f(a) = x$ and $f(b) = y$.

Lemma 3.3 *The order of G_{2n+1} is $(2n+1) \cdot 2^{2n+1}$. The centralizer of a, $C(a) = \{c \in G_{2n+1} : ac = ca\}$, is generated by the elements*

$$(1,\ \epsilon_2\ (2n+1),\ \epsilon_3\ (2n),\ \ldots,\ \epsilon_{n+1}\ (n+2),\ -\epsilon_{n+1}\ (n+1),\ \ldots,\ -\epsilon_2\ (2))$$

where $\epsilon_j = \pm 1$ for $j = 2, \ldots, n+1$. In particular, $|C(a)| = 2^{n+1}$.

Proof. Let $I_{\vec{\epsilon}} = (\epsilon_1\,(1),\ \epsilon_2\,(2),\ \ldots,\ \epsilon_{2n+1}\,(2n+1))$ where $\epsilon_j = \pm 1$ for $j = 1, \ldots, 2n+1$, such that $\Pi_{j=1}^{2n+1} \epsilon_j = 1$ (an even number of entries are negative). There are

$$\binom{2n+1}{0} + \binom{2n+1}{2} + \cdots + \binom{2n+1}{2n} = 2^{2n}$$

such elements. We show that these elements are in G_{2n+1}. Since $a^2 = (1, -2, -3, \ldots, -(2n+1))$, for $i = 1, \ldots, 2n$, $a^2(b^{-i}a^2b^i)$ has negative signs at the first and the $(i+1)$th entries, and positive signs otherwise. Hence $b^{-j}(a^2b^{-i}a^2b^i)b^j$ has negative signs at the $(j+1)$th and $(i+j+1)$th entries, and positive signs elsewhere. By multiplying elements of these forms, any $I_{\vec{\epsilon}}$ with an even number of negative signs can be obtained. Since $b^i a = ab^{-i} I_{\vec{\epsilon}}$ for such an $I_{\vec{\epsilon}}$, any element of G_{2n+1} is written uniquely as $b^j I_{\vec{\epsilon}}$ or $ab^j I_{\vec{\epsilon}}$. This is analogous to elements of D_{2n+1} having the form y^j and xy^j. In total, we have $|G_{2n+1}| = (2n+1) \cdot 2^{2n+1}$.

If $ac = ca$, then c must be of the form $I_{\vec{\epsilon}}$ or $aI_{\vec{\epsilon}}$. From the equation $a \cdot aI_{\vec{\epsilon}} = aI_{\vec{\epsilon}} \cdot a$, we have $\epsilon_j = \epsilon_{2n+3-j}$. Since the determinants of the matrices are all $+1$, the initial sign, ϵ_1 must be positive; or else, an odd number of the remaining ϵ_j are negative, but these signs agree in pairs. Thus

$$aI_{\vec{\epsilon}} = (1,\ \epsilon_2\ (2n+1),\ \ldots,\ \epsilon_{n+1}\ (n+1),\ -\epsilon_{n+1}(n),\ \ldots,\ -\epsilon_2\ (2)).$$

A similar computation gives that $I_{\vec{\epsilon}} = (1,\ \epsilon_2\ (2),\ \ldots,\ \epsilon_{n+1}\ (n+1),\ \epsilon_{n+1}\ (n+2),\ \ldots,\ \epsilon_2\ (2n+1))$ are the only diagonal signed permutation matrices that commutes with a. These are products of the $aI_{\vec{\epsilon}}$ that commute with a. This completes the proof. □

Let $H = C(a)$ denote the centralizer of a. Consider the quandle structure $\tilde{R}_{2n+1} = (G_{2n+1}, H, a)$ given by $Hu \triangleleft Hv = Huv^{-1}av$. From the preceding lemma, we have $|\tilde{R}_{2n+1}| = (2n+1)2^n$.

Lemma 3.4 *There is a surjective quandle homomorphism* $f : \tilde{R}_{2n+1} \to R_{2n+1}$.

Proof. The group homomorphism $f : G_{2n+1} \to D_{2(n+1)}$ defined earlier, by $f(a) = x$ and $f(b) = y$, satisfies $f(C(a)) = \langle x \rangle$. Hence f induces a quandle homomorphism (written by the same letter) $f : \tilde{R}_{2n+1} = (G_{2n+1}, C(a), a) \to R_{2n+1} = (D_{2(2n+1)}, \langle x \rangle, x)$. □

Lemma 3.5 *The quandle* $\tilde{R}_{2n+1}$ *is not involutory for any positive integer* n.

Proof. It is sufficient to show that the equality $(Ha \triangleleft Hb) \triangleleft Hb = Ha$ does not hold in $\tilde{R}_{2n+1}$. Since $(Ha \triangleleft Hb) \triangleleft Hb = Hab^{-1}abb^{-1}ab = Hb^{-1}a^2b$, we show that $b^{-1}a^2b \notin H$. One computes

$$\begin{aligned} b^{-1}a^2b &= (2, \ldots, 2n+1, 1)(1, -2, -3, \ldots, -(2n+1))(2n+1, 1, \ldots, 2n) \\ &= (2, \ldots, 2n+1, 1)(-(2n+1), 1, -2, -3, \ldots, -2n) \\ &= (-1, 2, -3, \ldots, -(2n+1)). \end{aligned}$$

By Lemma 3.3, H is generated by some square matrices of size $2n+1$ whose $(1,1)$-entries are 1, and $(1,i)$-entries and $(j,1)$-entries are 0 for any $i, j \neq 1$. Thus the matrix $(-1, 2, -3, \ldots, -(2n+1))$ is not an element of H. Therefore, $\tilde{R}_{2n+1}$ is not involutory. □

Lemma 3.6 *For any positive integer* n*, the quandle* $\tilde{R}_{2n+1} = (G_{2n+1}, H, a)$ *has a good involution.*

Proof. Define a map $\rho : \tilde{R}_{2n+1} \to \tilde{R}_{2n+1}$ by

$$\rho(Hu) = \begin{cases} H(-1, -2, \ldots, -(n+1), n+2, \ldots, 2n+1)u & \text{if } n \text{ is an odd number,} \\ H(1, -2, \ldots, -(n+1), n+2, \ldots, 2n+1)u & \text{if } n \text{ is an even number.} \end{cases}$$

Note that the elements inserted $(-1, -2, \ldots, -(n+1), n+2, \ldots, 2n+1)$ and $(1, -2, \ldots, -(n+1), n+2, \ldots, 2n+1)$, respectively, are indeed elements of G_{2n+1}, as they have even numbers of negative signs. We prove that this map is a good involution of $\tilde{R}_{2n+1}$.
(i) It is an involution by

$$(\rho \circ \rho)(Hu) = H(\varepsilon 1, -2, \ldots, -(n+1), n+2, \ldots, 2n+1)^2 u = Hu,$$

where ε is the sign $\pm$.
(ii) For any element Hu and Hv in $\tilde{R}_{2n+1}$,

$$\begin{aligned} \rho(Hu) \triangleleft Hv &= H(\varepsilon 1, -2, \ldots, -(n+1), n+2, \ldots, 2n+1)u \triangleleft Hv \\ &= H(\varepsilon 1, -2, \ldots, -(n+1), n+2, \ldots, 2n+1)uv^{-1}av. \end{aligned}$$

On the other hand,

$$\rho(Hu \triangleleft Hv) = \rho(Huv^{-1}av) = H(\varepsilon 1, -2, \ldots, -(n+1), n+2, \ldots, 2n+1)uv^{-1}av.$$

Hence, $\rho(Hu) \triangleleft Hv = \rho(Hu \triangleleft Hv)$ is satisfied.
(iii) For any element Hu and Hv in $\tilde{R}_{2n+1}$,

$$\begin{aligned} Hu \triangleleft \rho(Hv) &= Hu \triangleleft H(\varepsilon 1, -2, \dots, -(n+1), n+2, \dots, 2n+1)v \\ &= Huv^{-1}(\varepsilon 1, -2, \dots, -(n+1), n+2, \dots, 2n+1) \\ &\quad\ a\ (\varepsilon 1, -2, \dots, -(n+1), n+2, \dots, 2n+1)v \\ &= Huv^{-1}a^{-1}v. \end{aligned}$$

The last equality is satisfied by

$$(\varepsilon 1, -2, \dots, -(n+1), n+2, \cdots, 2n+1)\ a\ (\varepsilon 1, -2, \dots, -(n+1), n+2, \cdots, 2n+1) = a^{-1}.$$

The equality

$$(Hu \triangleleft \rho(Hv)) \triangleleft Hv = Huv^{-1}a^{-1}v \triangleleft Hv = Huv^{-1}a^{-1}vv^{-1}av = Hu$$

implies $Hu \triangleleft \rho(Hv) = Hu \triangleleft \overline{Hv}$. □

Theorem 3.1 follows from Lemmas 3.4, 3.5 and 3.6.

Example 3.7 The extension $\tilde{R}_3 = (G_3, C(a), a)$, where $H = C(a)$, $a = (1, -3, 2)$ and $b = (3, 1, 2)$, consists of 6 elements. Let $I_1 = (-1, -2, 3)$. The six elements are represented by 0 through 5 as $\{0 = H, 1 = Hb, 2 = Hb^2, 3 = HI_1, 4 = HbI_1, 5 = Hb^2I_1\}$ with the quandle operation given by the following table.

$R \triangleleft C$	0	1	2	3	4	5
0	0	5	1	0	2	4
1	2	1	3	5	1	0
2	4	0	2	4	3	2
3	3	2	4	3	5	1
4	5	4	0	2	4	3
5	1	3	5	4	0	5

The map $f : \tilde{R}_3 \to R_3$ is given by $f(i) \equiv i \pmod 3$. The good involution is the involution $\rho = (0\ 3)(1\ 4)(2\ 5)$.

4 Homology groups of $\tilde{R}_3$ and triple point numbers

For $\tilde{R}_3$, computer calculations give the results below on symmetric quandle homology groups. Let $\chi_{(x,y,z)} \in C^3_{Q,\rho}(\tilde{R}_{2n+1}, \mathbb{Z})$ be the characteristic function. Define a 3-cochain

$$\begin{aligned} A(x,y,z) &= \chi_{(x,y,z)} - \chi_{(\rho(x),y,z)} - \chi_{(x \triangleleft y, \rho(y), z)} - \chi_{(x \triangleleft z, y \triangleleft z, \rho(z))} \\ &\quad + \chi_{(\rho(x) \triangleleft y, \rho(y), z)} + \chi_{(\rho(x) \triangleleft y, y \triangleleft z, \rho(z))} \\ &\quad + \chi_{((x \triangleleft y) \triangleleft z, \rho(y) \triangleleft z, \rho(z))} - \chi_{((\rho(x) \triangleleft y) \triangleleft z, \rho(y) \triangleleft z, \rho(z))}. \end{aligned}$$

Then *Mathematica* calculations show:

Lemma 4.1 *Let* $\tilde{R}_3$ *be as above.*

(i) $H_2^{Q,\rho}(\tilde{R}_3, \mathbb{Z}) = 0$, $H_3^{Q,\rho}(\tilde{R}_3, \mathbb{Z}) \cong \mathbb{Z}$.

(ii) *The 3-chain* $c = (2,1,2) + (2,0,1) - (1,0,2) - (0,2,1) \in C_3^{Q,\rho}(\tilde{R}_{2n+1}, \mathbb{Z})$ *is a 3-cycle* ($c \in Z_3^{Q,\rho}(\tilde{R}_3, \mathbb{Z})$) *that represents a generator* $[c]$ *of* $H_3^{Q,\rho}(\tilde{R}_3, \mathbb{Z}) \cong \mathbb{Z}$.

(iii) *Any 3-cycle with less than 4 basis terms (triples) is null-homologous.*

(iv) *The 3-cochain* $\phi = A(0,1,0) + A(0,1,2) - A(0,2,1)$ *is a 3-cocycle* ($\phi \in Z^3_{Q,\rho}(\tilde{R}_3, \mathbb{Z})$) *that represents a generator of* $H^3_{Q,\rho}(\tilde{R}_3, \mathbb{Z}) \cong \mathbb{Z}$ *dual to* $[c]$, *that is:* $\phi([c]) = 1$.

(v) *The 3-cochain* $\phi' = A(0,1,0) + A(0,1,2) + A(0,2,0) - A(0,2,1) + A(1,0,1) - A(1,0,2) + A(1,2,0) + A(1,2,1) + A(2,0,1) + A(2,0,2) - A(2,1,0) + A(2,1,2)$ *is a 3-cocycle with* $\pm$ *monic terms such that* $\phi'([c]) = 4$.

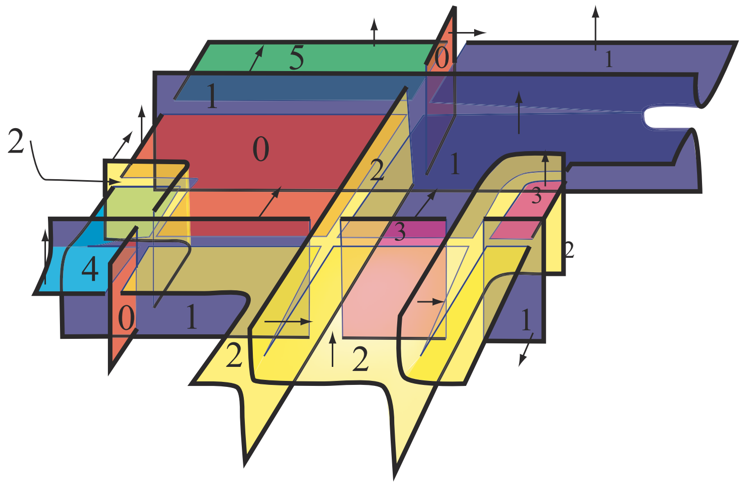


Figure 3: A diagram of the surface constructed

Theorem 4.2 *For any positive integer* N, *there is a closed 3-manifold* M *and a non-orientable surface-knot* F *in* $M \times [0,1]$ *such that* $t(F) > N$.

Proof. For the 3-cycle c of Lemma 4.1 (ii), make a colored triple point in a 3-ball for each basis term. The degenerating terms are capped by branch points. Connect them together to form a larger 3-ball B with all triple points and branch points included, see Fig. 3. The boundary ∂B contains a colored classical link diagram illustrated in Fig. 4. One can also obtain Fig. 4 from the formula for the 3-cycle c of Lemma 4.1 (ii) as follows: The 3-cycle c is represented by a colored diagram with region colors as depicted in Fig. 5. Take the "double" of Fig. 5 and extend, see Fig. 6. By smoothing the black dots that represent branch points, we obtain Fig. 4.

Then add 1-handles to connect double curves of the diagram. In Fig. 4 the attaching disks of 1-handles are indicated by dotted circles. The shapes of the circles, T-shaped, oval and circle, respectively, together with the colors of arcs indicate the pairs of the attaching regions. Note that the oval and circle ones must be rotated 180 degrees before identifying. This twist makes the surface

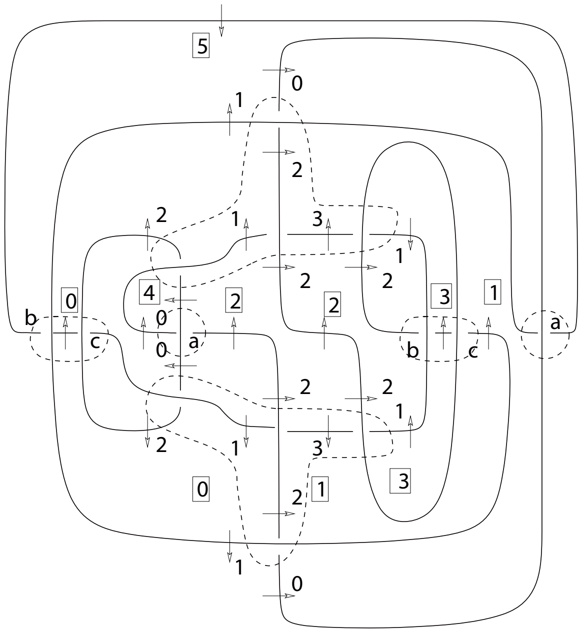


Figure 4: Adding 1-handles

non-orientable. A handlebody H of genus 3 results as an ambient manifold, and it has 5 closed curves on the boundary.

Attach 2-handles to H along the closed curves on the boundary. Let M_0' be the result, which is a compact 3-manifold with boundary. By capping off the boundary of M_0' by handlebodies, we obtain a closed orientable 3-manifold M_0 with a diagram D_0 in it, that represents c. Let m be an integer such that $4m > N$. Taking an m-fold knot connected sum, we have a connected closed 3-manifold $M = \#_m M_0$ and a connected, colored diagram $D = \#_m D$ in M which represents mc. By lifting D to $M \times [0, 1]$, we obtain the surface-knot F whose minimal triple point number is greater than N: Using the 3-cocycle ϕ' in Lemma 4.1, we have $t(F) \geq 4m$ by Lemma 2.5. Therefore we obtain the inequality $t(F) > N$. $\square$

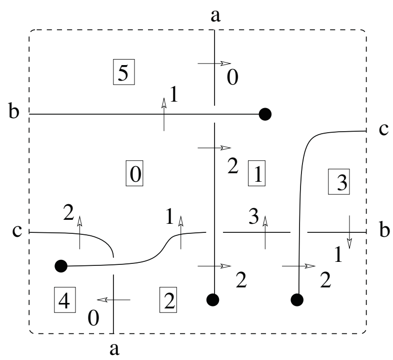


Figure 5: Representing the 3-cycle c

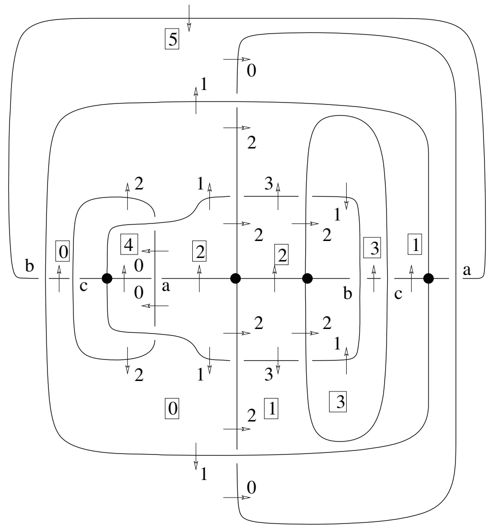


Figure 6: Assembling triple points

Note that using the 3-cocycle ϕ', we can also prove that the minimal triple point number of the constructed surface-knot F in the above proof is exactly $4m$.

The next result shows that homological conditions on the surface changes the triple point numbers.

Proposition 4.3 *Any surface-knot diagram colored with $\tilde{R}_3$ in a closed 3-manifold M that is null-homologous in $H_2(M;\mathbb{Z}_2)$ as an underlying generic surface, and with less than 8 triple points, is null-homologous in $H_3^{Q,\rho}(\tilde{R}_3,\mathbb{Z})$.*

For the proof, we need the following lemma, calculated by *Mathematica*. Let $Y=\{\alpha,\beta\}$, and let $\tilde{R}_3$ act on Y by $\alpha\cdot u=\beta$, $\beta\cdot u=\alpha$ for any $u\in\tilde{R}_3$. This provides cycles represented by colored diagrams with regions with checkerboard colorings. The map of deleting the first factor $\pi:(\alpha \text{ or } \beta, x_1,\ldots,x_n)\mapsto(x_1,\ldots,x_n)$ induces a chain map $\pi: C_n^{Q,\rho}(\tilde{R}_3,\mathbb{Z})_Y\to C_n^{Q,\rho}(\tilde{R}_3,\mathbb{Z})$.

Lemma 4.4 *Let $\tilde{R}_3$, Y be as above.*

(i) $H_3^{Q,\rho}(\tilde{R}_3,\mathbb{Z})_Y\cong\mathbb{Z}\times\mathbb{Z}_3$.

(ii) *The 3-chain*

$$\begin{aligned}\gamma &= (\alpha,0,1,0)+(\alpha,0,1,2)+(\alpha,0,2,0)+(\alpha,1,2,0)\\ &\quad -(\alpha,2,1,0)+(\beta,0,2,0)+(\beta,1,2,0)+(\beta,2,0,1)\in C_3^{Q,\rho}(\tilde{R}_3,\mathbb{Z})_Y\end{aligned}$$

is a 3-cycle ($\gamma\in Z_3^{Q,\rho}(\tilde{R}_3,\mathbb{Z})_Y$) that represents a generator $[\gamma]$ of $\mathbb{Z}\subset H_3^{Q,\rho}(\tilde{R}_3,\mathbb{Z})_Y$.

(iii) *Any 3-cycle with less than 8 basis terms (quadruples) is null-homologous.*

(iv) *The 3-cochain $\phi''=A(0,1,0)+A(0,1,2)+A(0,2,0)-A(0,2,1)+A(1,0,1)-A(1,0,2)+A(1,2,0)+A(1,2,1)+A(2,0,1)+A(2,0,2)-A(2,1,0)+A(2,1,2)$ is a 3-cocycle ($\phi''\in Z_3^{Q,\rho}(\tilde{R}_3,\mathbb{Z})$) with $\pm$ monic terms such that $\phi''([\gamma])=8$.*

Lemma 4.5 *The induced map $\pi_* : H_3^{Q,\rho}(\tilde{R}_3, \mathbb{Z})_Y \to H_3^{Q,\rho}(\tilde{R}_3, \mathbb{Z})$ restricted to the $\mathbb{Z}$ factor multiplies the generator by 2.*

Proof. One computes $\pi_*([\gamma]) = \phi((0,1,0) + (0,1,2) + (0,2,0) + (1,2,0) - (2,1,0) + (0,2,0) + (1,2,0) + (2,0,1)) = 2$. □

Proof (of Proposition 4.3). Let D be a colored diagram in a closed 3-manifold M whose underlying generic surface represents a null-homologous class in $H_2(M;\mathbb{Z}_2)$, and that is non-trivial in $H_3^{Q,\rho}(\tilde{R}_3)$. In particular, we have $\phi(D) \neq 0$, where ϕ is a cocycle in Lemma 4.1 (iv). Then there is a checkerboard coloring for D as it is null homologous in $H_2(M;\mathbb{Z}_2)$, and let D' be the cycle in $Z_3^{Q,\rho}(\tilde{R}_3, \mathbb{Z})_Y$ represented by D with the checkerboard coloring. Since $\pi_*([D']) \neq 0$ in $H_3^{Q,\rho}(\tilde{R}_3, \mathbb{Z})$, by Lemma 4.5, $[D']$ is non-trivial in $H_3^{Q,\rho}(\tilde{R}_3, \mathbb{Z})_Y$. Then Lemma 4.4 (iii) implies that D must have at least 8 triple points. □

Concluding remarks. We conjecture that the extensions $\tilde{R}_m$ are connected for any odd positive integer m. Such a property is not only of an algebraic interest, but also desirable from point of view of using the quandles for coloring twist-spun knots for applications.

The most remarkable aspect of this quandle $X = \tilde{R}_3$ is its free part in H_3 despite its being connected (Lemma 4.1 (i)). It is known [17] that the ordinary quandle homology groups do not have free part if it is connected. This shows that the symmetric quandle homology is quite different from the original quandle homology, and this fact should be useful for non-orientable surfaces.

It is an interesting problem to compute the quandle (co)-homology of quandle extensions (which are given by surjective quandle homomorphisms) in terms of the homological information of the source, target, and fiber.

All classical 3-colorable knots we tested have non-trivial coloring by $\tilde{R}_3$, so we conjecture that it is always the case. We also conjecture that any surface-knot diagram in $\mathbb{R}^3$ colored by $\tilde{R}_3$ represents null-homologous class in $H_3^{Q,\rho}(\tilde{R}_3, \mathbb{Z})_Y$. It is an interesting fact that, from Proposition 4.3, that the homology class of the surface is related to the minimal triple point number.

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