# Detection of Chirality and Mutation of Knots and Links

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**Plan:**

(i) My research background

(ii) Chern-Simons Field Theory

(iii)Computation of Knot Invariants

(iv) Detection of Chirality.

(v) Mutations

# MY RESEARCH BACKGROUND

● **Knots,Links,Three-Manifold Invariants from Chern-Simons Field Theory**

*Work done with*

(a) Romesh Kaul and T.R. Govindarajan (IMSc):

*1) Nucl. Phys.* **402***(1993) 548*

*2) Nucl. Phys.* **422***(1994) 291*

● *3) Mod. Phys. Lett.***A9***(1994)3205*

● *4) Mod. Phys. Lett.***A10***(1995)1635*

(b) I.P. Ennes, A.V. Ramallo, J.M. Sanchez de Santos (Univ. of Santiago,Spain)

*5) Int. J. Mod. Phys.***A13***(1998) 2931*

(c) Swatee Naik (Math. Dept, Univ. of Nevada,USA)

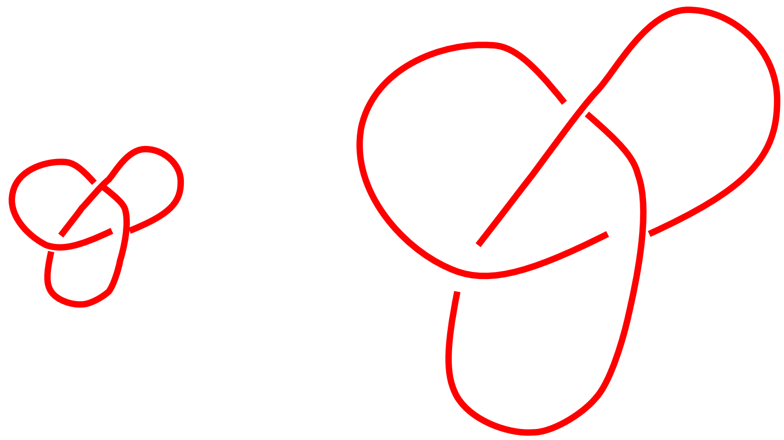
*6) Commun. Math. Phys.***209***(2000)29*

*(d) Romesh Kaul (IMSc, Chennai)*

*7) hep-th/0005096, Commun.Math.Phys.* **217***(2001) 295.*

● Last 9 years, I have been involved in the meaning of integers in the polynomial invariants in **topological string** stable states

# CHERN-SIMONS FIELD THEORY



Topologically equivalent Objects $C$

*(no notion of distance or size)*

**Hence the theory describing such objects must be** **metric independent** (Chern-Simons theory)

• Chern-Simons action $S$ on a three-manifold $M$ based on gauge group $G$:

$$S = \frac{k}{4\pi} \int_M \epsilon_{\mu\nu\lambda} \, d^3x \, Tr \left( A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda \right)$$

$k$ is the coupling constant, $A_\mu$'s are the gauge fields.

• Above objects $C$ carrying representation $R$ are described by **expectation value of Wilson loop operators** $W_R(C) = Tr[Pexp \oint A_\mu dx^\mu]$:

$$V_R[C] = \langle W_R(C) \rangle = \frac{\int_M [\mathcal{D}A] \, W_R(C) \, exp(iS)}{\mathcal{Z}[M]}$$

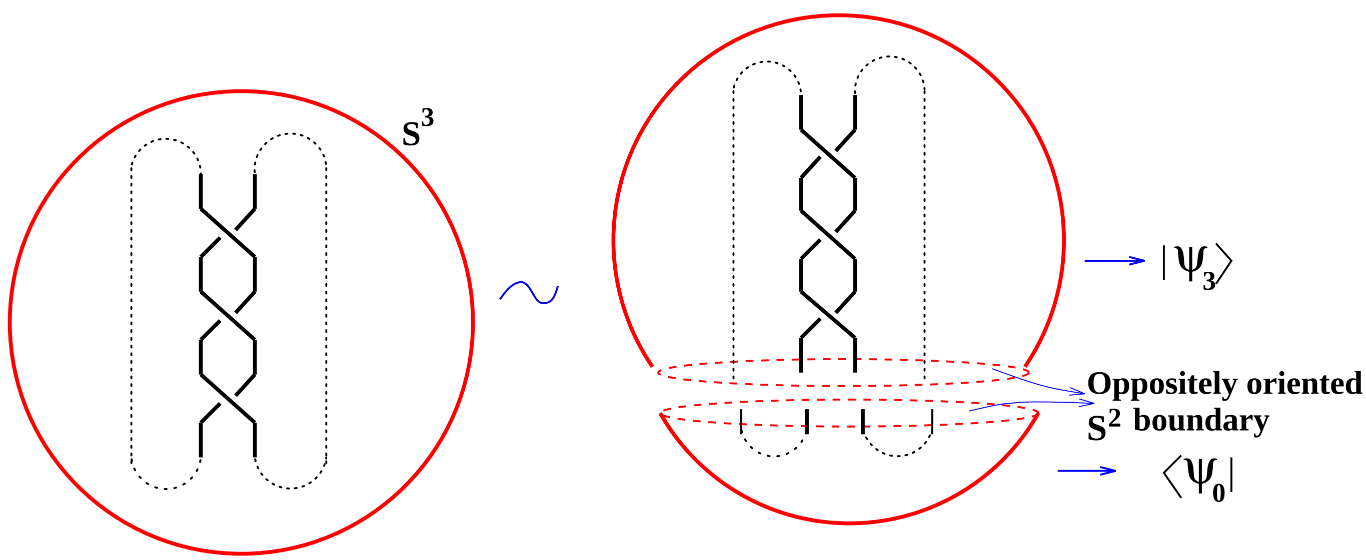
where $\mathcal{Z}[M] = \int_M [\mathcal{D}A] \, exp(iS)$ (partition function)

$V_R[C]$ are the **knot invariants**.

# COMPUTATION OF KNOT INVARIANTS

• These **knot invariants** ($V_R[C]$) can be directly evaluated using two inputs:
*1) Connection between Chern-Simons theory to Wess-Zumino conformal field theory.*
*2) Any knot can be obtained as a closure of braid*

**Take example** $C$



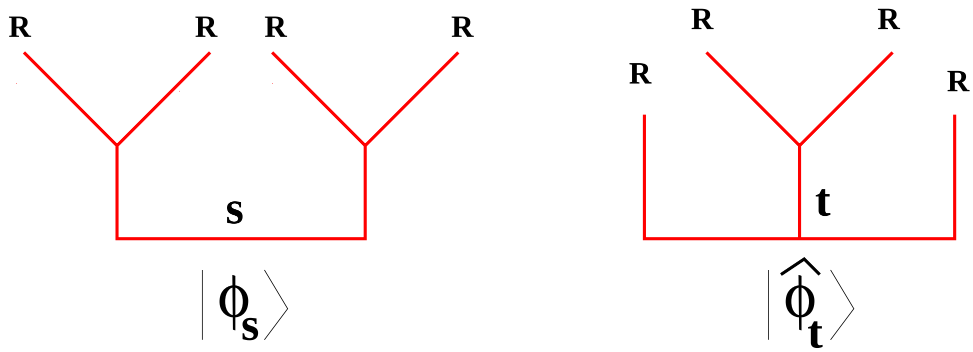

$$V_R[C] = \langle\psi_0|\psi_3\rangle = \langle\psi_0|\mathcal{B}^3|\psi_0\rangle$$

where $\mathcal{B}$ is the braiding operator.

To see the polynomial form of the invariant:

Expand the state $|\psi_0\rangle$ in a suitable basis in which $\mathcal{B}$ is diagonal.

For the four-punctured $S^2$ boundary, the bases are:




where $s, t \in R \otimes R$. These two basis are related by a duality matrix:

$$|\hat{\Phi}_t\rangle \;=\; a_{st}|\Phi_s\rangle$$

In this example, the braiding involves middle two strands,

$$|\Psi_0\rangle \;=\; \sum_t \mu_t \, |\hat{\Phi}_t\rangle$$

where $\mu_t \;=\; \sqrt{S_{0t}/S_{00}} \equiv \sqrt{dim_q t}$ (**unknot** normalisation).

$$V_R[C] \;=\; \langle \Psi_0 | \mathcal{B}^3 | \Psi_0 \rangle \;=\; \sum_t dim_q t (\lambda_t(R,R))^3$$

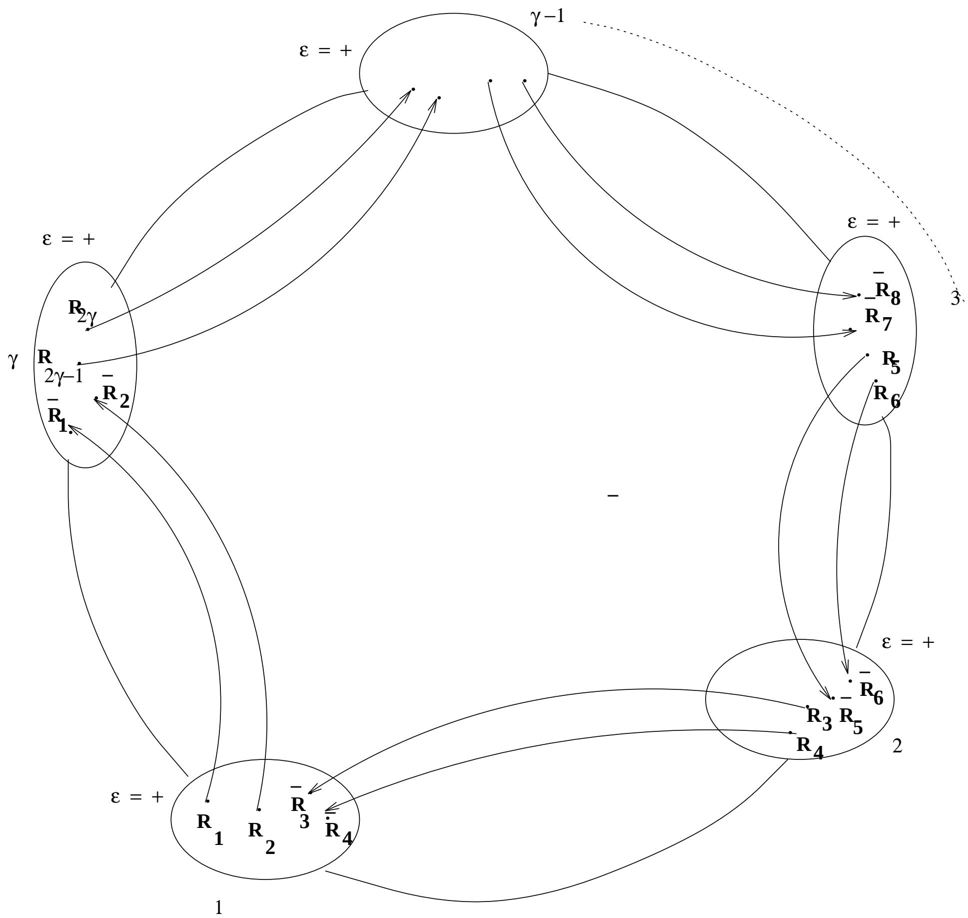
where braiding eigenvalue depends on the **framing** and also on the relative orientation on the two braiding strands.

For parallel right-handed braiding in standard framing,

$$\lambda_t^{(+)}(R,R) \;=\; (-1)^{\epsilon} q^{2C_R - C_t/2},\; q = e^{\frac{2\pi i}{k+C_v}}.$$

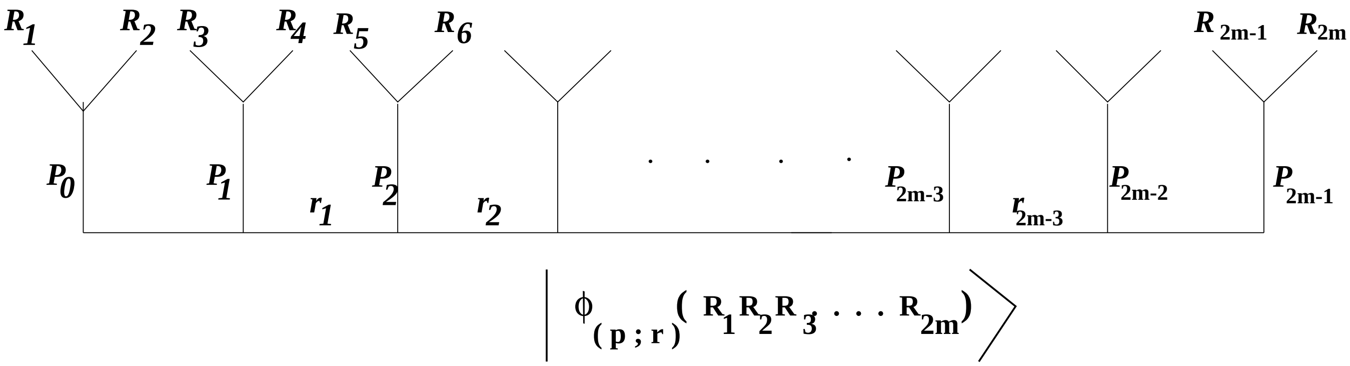
- Knot invariants are polynomials in the variable $q$

## The method for a four-punctured $S^2$ boundary is generalisable for $r$ such four-punctured $S^2$ boundaries




$$\nu_r = \sum_{R_s} \frac{|\phi^{(1)side}_{R_s}\rangle |\phi^{(2)side}_{R_s}\rangle \cdots |\phi^{(r)side}_{R_s}\rangle}{(dim_q R_s)^{\frac{r-2}{2}}} \,.$$

Also for the $S^2$ boundary with more than four-puctures, the basis-state will be

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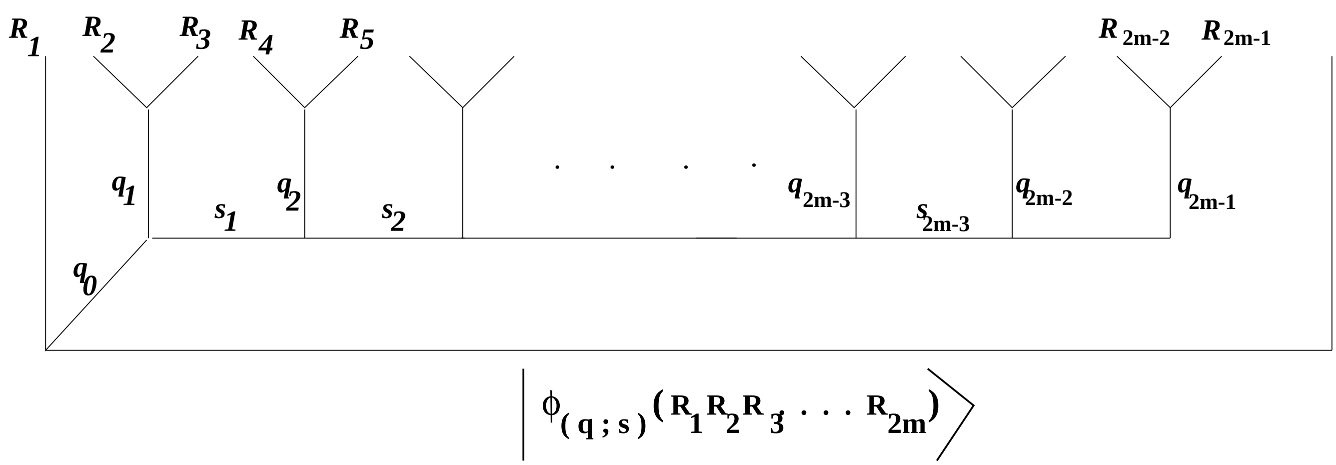



( b )

• The method enables evaluation of any knot/link directly without going through the recursive procedure

We can place any representation $R$ of any compact semi-simple gauge group on the knot and obtain **generalised knot invariants.**

**Special Cases: $R$=fundamental**

| Gauge Group | Polynomial |
|---|---|
| $SU(2)$ | Jones' |
| $SU(N)$ | Two-variable HOMFLY |
| $SO(N)$ | Two-variable Kauffman |

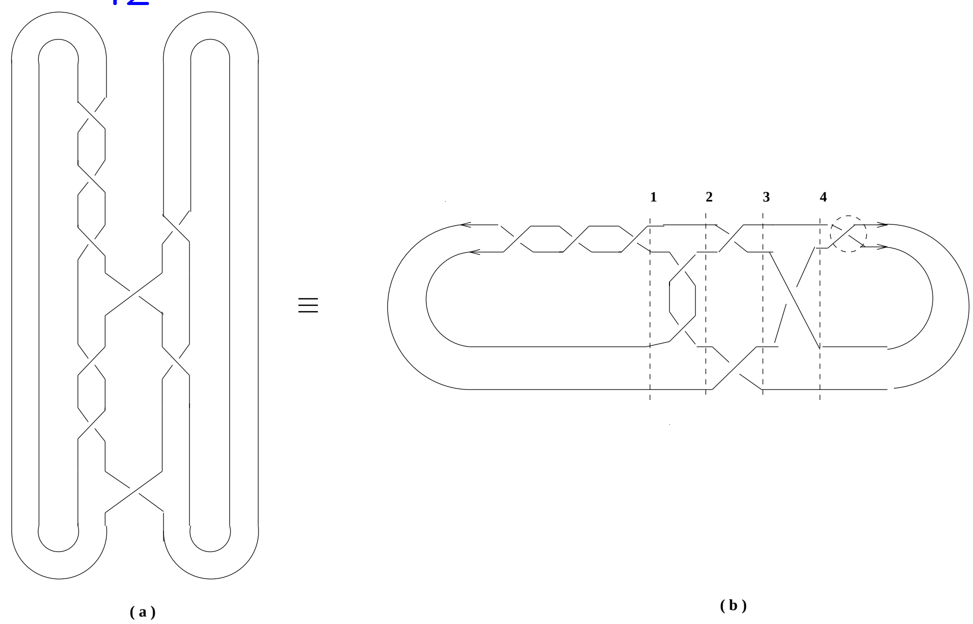
*Does generalised knot invariants solve the* **classification problem***??*
Knot Theory literature gives a list of Chiral knots and Mutant knots which are not distinguished by Jones', HOMFLY and Kauffman!

I must add the non-orientable knots like knot $8_{17}$ to this list

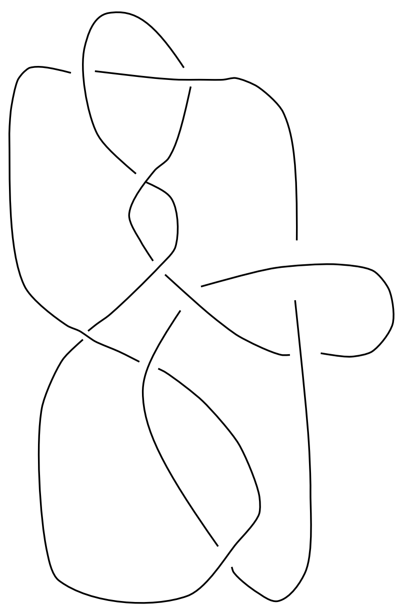
Use our method to check the powerfulness of generalised knot invariants to detect **chirality and mutations!**

# Examples of Chiral Knots:

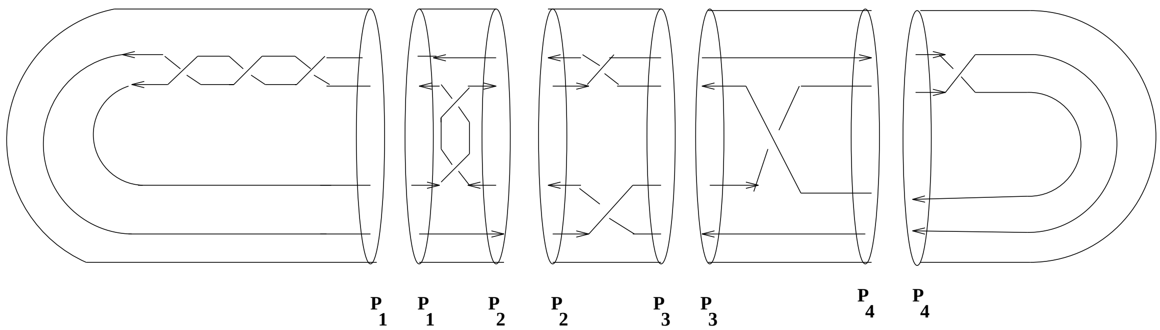
## 1) **Knot** $9_{42}$**:**




## 2) **Knot** $10_{71}$**:**



Using our method, the knot invariant for $9_{42}$ can be obtained as gluing of five building blocks as shown:




**For $G = SU(2)$, $R_n$ = $\overbrace{\square\square\square\square\square\square}^{n}$ (spin $n/2$ representation) placed on knot**

$$\nu_1(P_1) = \sum_{l_1=0} \sqrt{[2l+1]}(-1)^{3(n-l_1)} q^{-3/2[n(n+2)-l_1(l_1+1)]} |\phi^{(1)}_{l_1}\rangle$$

$$\nu_1(P_4) = \sum_{l_5=0} (-1)^{n-l_5} q^{-1/2[n(n+2)-l_5(l_5+1)]} |\phi^{(1)}_{l_5}\rangle$$

$$\nu_2(P_1;P_2) = \sum_{i_1,j_1,l_2,r=0} \frac{a_{l_1 r} a_{j_1 r} a_{l_2 r}\sqrt{[2l_2+1]}}{\sqrt{[2r+1]}} \times q^{n(n+2)-l_2(l_2+1)} |\phi^{(1)}_{i_1}\rangle |\phi^{(2)}_{j_1}\rangle$$

$$\nu_2(P_2;P_3) = \sum_{l_3=0} q^{l_3(l_3+1)} |\phi^{(1)}_{l_3}\rangle |\phi^{(2)}_{l_3}\rangle$$

$$\nu_2(P_3;P_4) = \sum_{i_2,j_2,l_4=0} (-1)^{l_4} q^{-l_4(l_4+1)/2} a_{l_4 i_2} a_{l_4 j_2} |\phi^{(1)}_{i_2}\rangle |\phi^{(2)}_{j_2}\rangle$$

Using the above data, the knot invariant is

$$\begin{aligned} V_n[9_{42}] &= (-1)^n q^{\frac{-3}{2}[n(n+2)]} \sum_{r,l_1,l_2,j_1,j_2=0} \sqrt{[2l_1+1]} \times \\ & \sqrt{[2l_2+1]}\sqrt{[2j_2+1]} a_{l_1 r} a_{l_2 r} a_{j_1 r} a_{j_1 j_2} \times \\ & (-1)^{l_1} q^{\frac{3}{2}[l_1(l_1+1)]} q^{\frac{3}{2}[j_1(j_1+1)]} q^{-l_2(l_2+1)} q^{j_2(j_2+1)} \end{aligned}$$

$n = 1$ **gives Jones' polynomial**

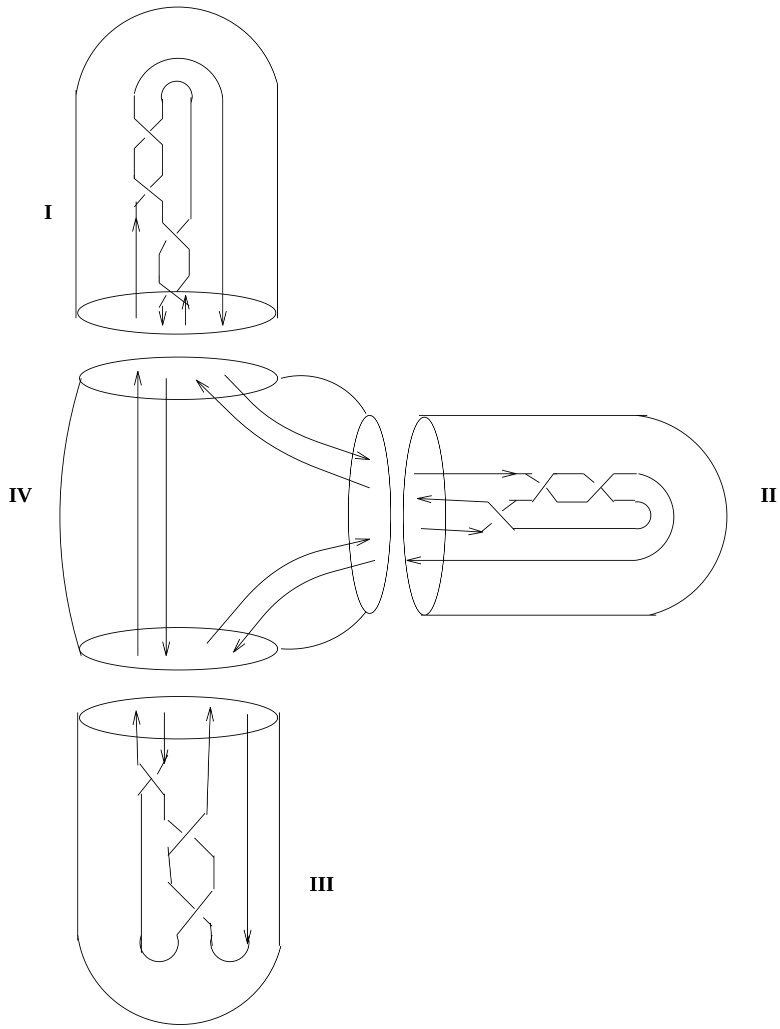
$n = 2$ **gives Akutsu-Wadati/Kauffman polynomial.**

For $n = 3$, the polynomial is $V_3[9_{42}] =$

$$\begin{aligned} & q^{45/2} - q^{41/2} - q^{39/2} + q^{35/2} + q^{23/2} + q^{21/2} - q^{19/2} \\ & -q^{17/2} + q^{13/2} - q^{9/2} + q^{5/2} + q^{3/2} + q^{-3/2} + q^{-5/2} \\ & -q^{-13/2} - q^{-15/2} + q^{-21/2} + 2q^{-23/2} - q^{-27/2} + 2q^{-31/2} \\ & -3q^{-35/2} - q^{-37/2} + q^{-39/2} + q^{-41/2} . \end{aligned}$$

**Obviously** $V_3[9_{42}](q) \neq V_3[9_{42}](q^{-1})$ **indicating that** $SU(2)$ **Chern-Simons spin** $3/2$ **invariant is powerful to detect chirality.**

## Similar exercise for knot $10_{71}$ by redrawing the diagram in the following way




$V_n[10_{71}] =$

$$(-1)^n q^{\frac{n(n+2)}{2}} \sum_{i,r,s,u,m=0} \sqrt{\frac{[2r+1][2s+1][2u+1]}{[2m+1]}}\, a_{im}$$

$$a_{ms} a_{rm} a_{iu} (-1)^s \; q^{-i(i+1)} q^{m(m+1)} q^{-r(r+1)} q^{u(u+1)} q^{\frac{3}{2}s(s+1)}$$

For $n = 3$, we have checked that

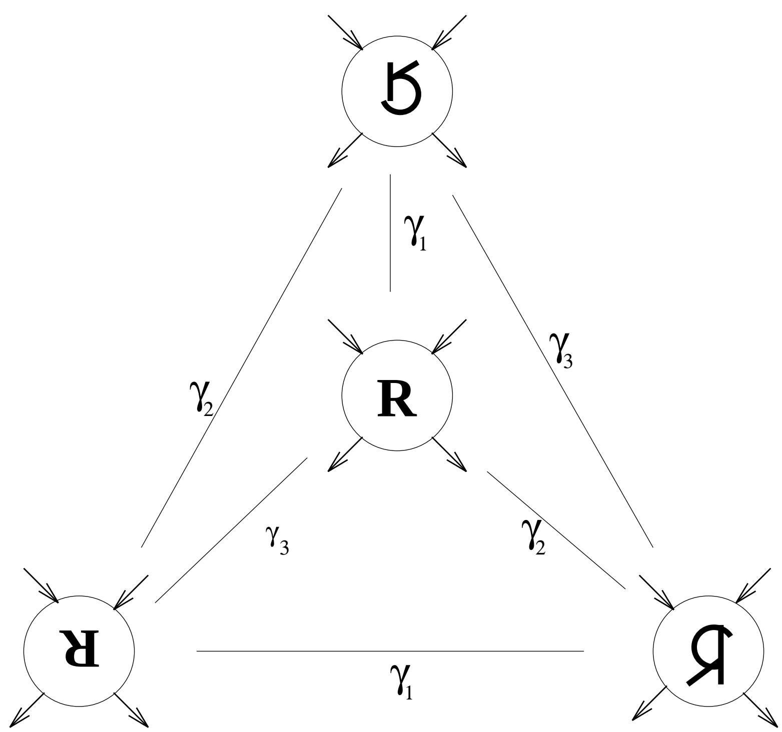
$$V_3[10_{71}](q) \neq V_3[10_{71}](q^{-1})$$

confirming the powerfulness of generalised Chern-Simons invariant in detecting chirality!

**Next question**

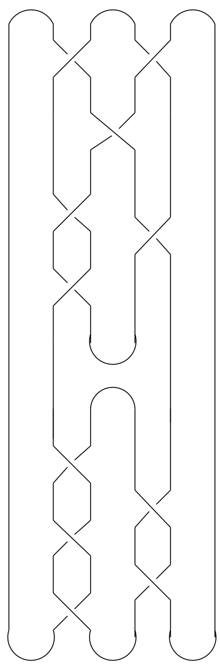
**What is mutation and mutant knots?**

**Can the generalised invariants detect mutation?**



**Example:**

Eleven-crossing Kinoshita-Terasaka and Conway knots are mutants



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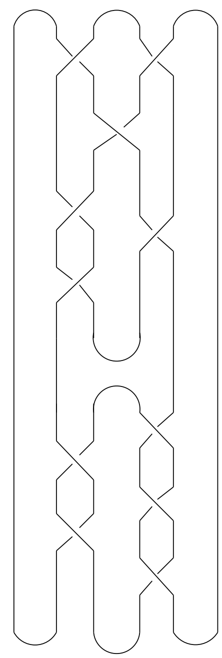

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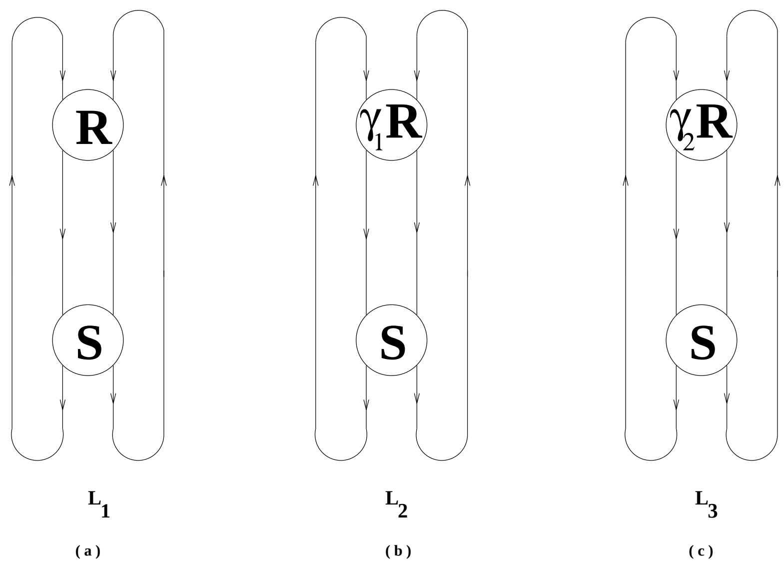
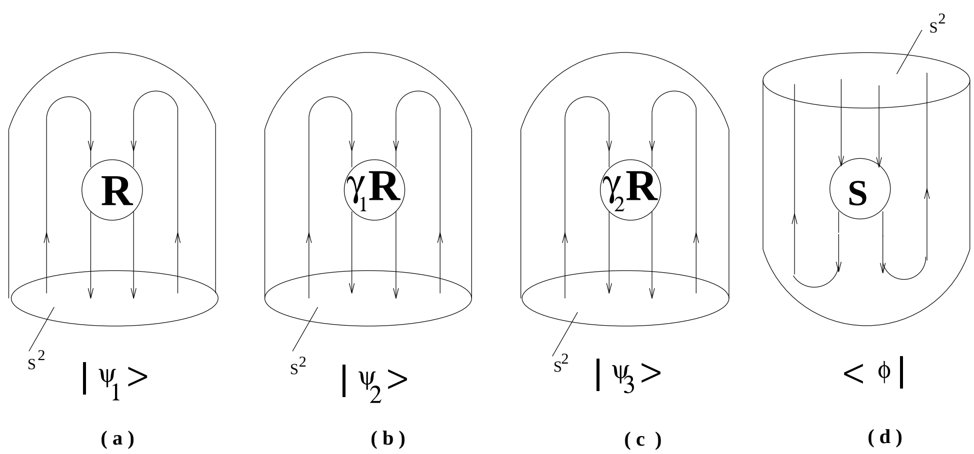
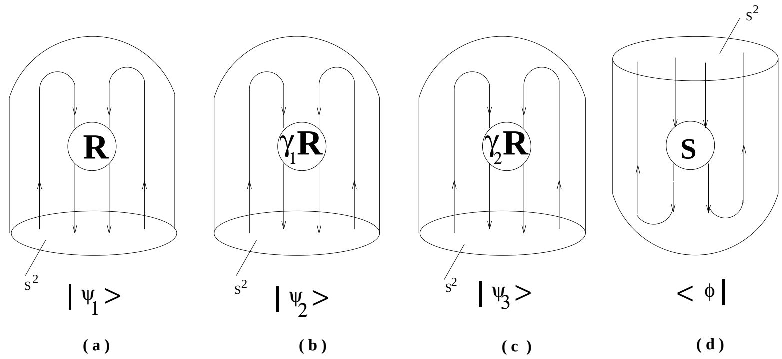
# Mutant knots



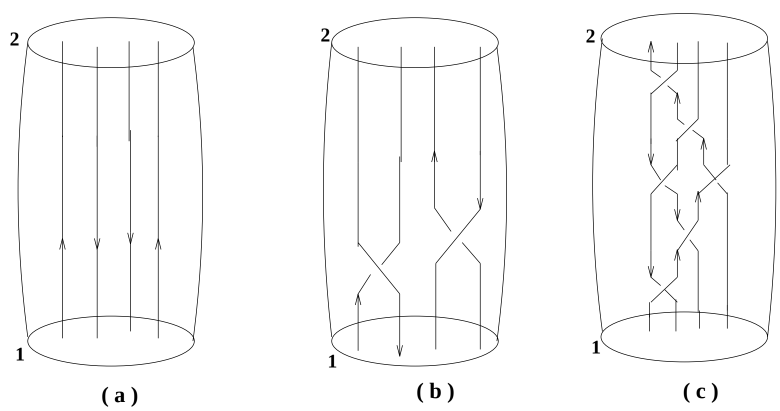

These mutant knots correspond to gluing state (d) with any of (a),(b),(c) states:



The states (a), (b), (c) can be obtained by gluing state (a)

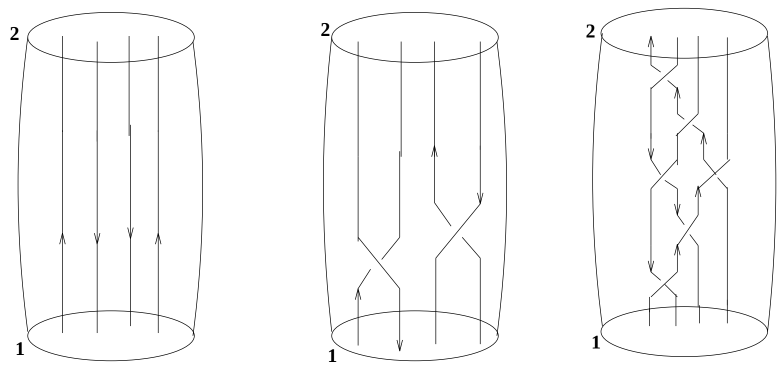



with two-boundary states in the following way:




Denote the state (a) as $\nu_1$ , (b) as $\nu_2$ and state (c) as $\nu_3$ .

Clearly, $\nu_2$ and $\nu_3$ represent mutation $\gamma_1$ and $\gamma_2$ respectively



( a ) ( b ) ( c )

**As braid words, (a), (b), (c) are different but the states are same, we will see!**

$$\nu_2 = \sum_l |\phi_l^{side(1)}\rangle b_1 b_3^{-1} |\phi_l^{side(2)}\rangle = \mathcal{C}\nu_1 .$$

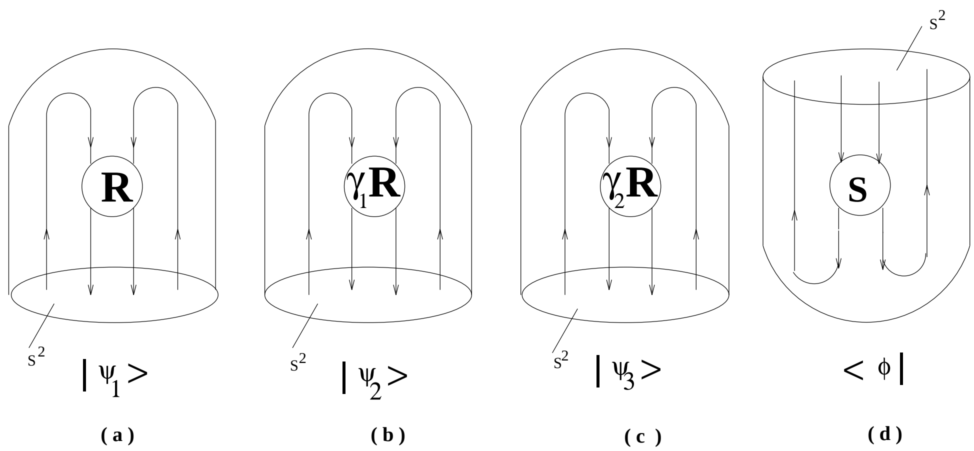
where we have used

$$b_1|\phi_l^{side}\rangle = b_3|\phi_l^{side}\rangle = \lambda_l^{(-)}(R,\bar{R})|\phi_l^{side}\rangle ,$$

and the operator $\mathcal{C}$ interchanges the representations on the first and second, the third and fourth punctures in that basis.

Similarly, we can show

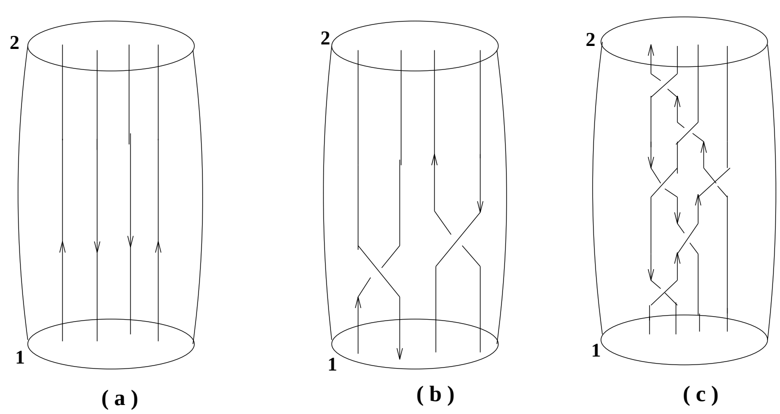
$$\nu_3 = \sum_l |\phi_l^{side(1)}\rangle b_1 b_2 b_1 b_3 b_2 b_1 |\phi_l^{side(2)}\rangle = \mathcal{C}\nu_1 .$$




So, the **generalised invariants** of the mutants $L_1, L_2, L_3$ obtained by gluing (a),(b) and (c) with (d) are **same**

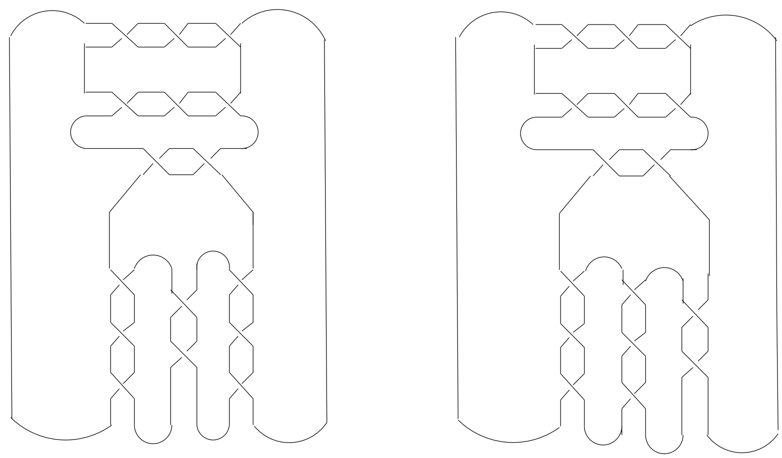
**What is the reason??**

**Four-point conformal block** plays a crucial role to make $\nu_2$ and $\nu_3$ to be same as $\nu_1$ (identity braid)!




**Strategy: Go beyond 4-pt conformal block by taking composite braiding**

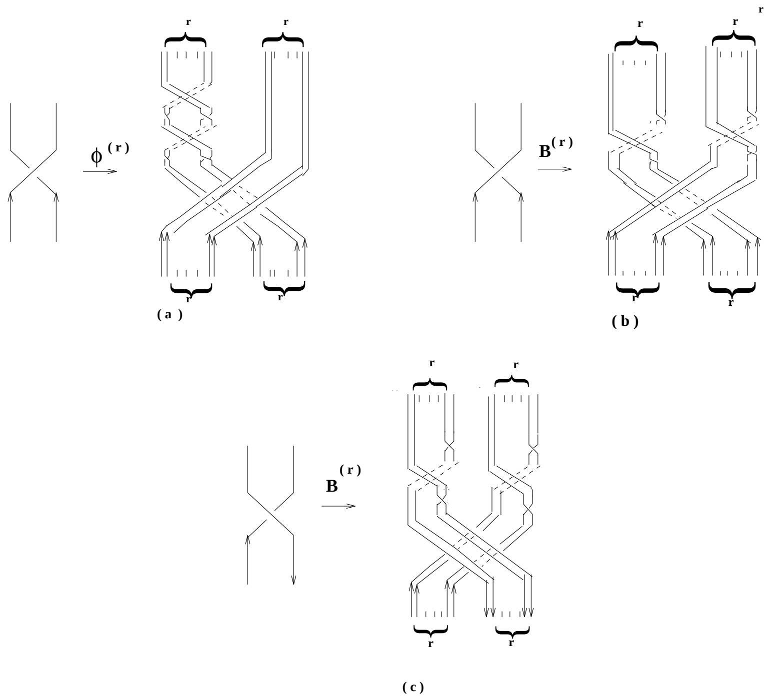
**Incidentally, two 16-crossing mutant knots where one is chiral and the other is achiral!**



( a ) chiral ( b ) achiral

Clearly, chirality of the chiral 16-crossing knot is not detected by our **generalised invariants**

## $r$-Composite Braiding




We studied braid-group representation of composite braids and obtained eigen-basis and eigenvalues

We showed that the composite invariant for knots are sum of the generalised knot invariants which implies

$\boxed{\text{Composite invariants cannot detect mutations in knots.}}$

However, some mutant links can be distinguished by composite invariants.

## Data from the Knot Theory School and Conference

• From Morton, I learnt that some specific representations of $SU(3)_q$ and $SU(4)_q$ can distinguish 11-crossing mutant knots!
Should check out the 16-crossing mutant knots with this method.

So far, I thought the Chern-Simons invariants must coincide with quantum-group invariants but **appears** to be not true for **mixed representations** placed on the knot.

• We checked and found that the Khovanov homology invariants does not distinguish 11-crossing mutant knots.
M. Khovanov says that 'there is a proof that the $Z_2$ homology invariants cannot detect mutations!'

• Talking to P.Ozsvath, I see that the genus computation in Floer homology gives different results for the two 11-crossing mutants. Must check for 16-crossing mutants and prove in general!

**Thank You**