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Braid Order: History and Connection with Knots

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- An introduction to some of the many aspects of the standard braid order, with an emphasis on the few known connections with knot theory.

Plan :

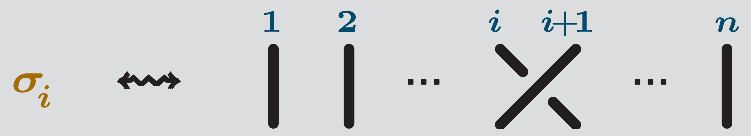
- The Braid Order in Antiquity
- The Braid Order in the Middle Ages
- The Braid Order in Modern Times (Knot Applications)

I. The Braid Order in Antiquity : 1985-95

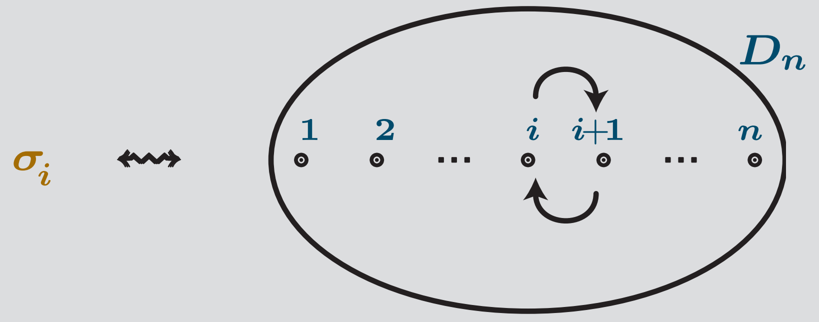
- The set-theoretical roots

- Artin's braid group B_n: $\left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle$.

$\simeq$ { braid diagrams }/ isotopy:



$\simeq$ mapping class group of D_n (disk with n punctures):



- **Theorem 1** (D., 1992): For β, β' in B_n, declare $\beta < \beta'$ if $\beta^{-1}\beta'$ has an expression in which the generator σ_i with minimal i appears only positively (no σ_i^{-1}).
Then $<$ is a left-invariant linear ordering on B_n.

$\uparrow$
$\beta < \beta'$ implies $\alpha\beta < \alpha\beta'$

- Example: $\beta = \sigma_1$, $\beta' = \sigma_2\sigma_1$. Then $\beta^{-1}\beta' = \sigma_1^{-1}\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2^{-1}$, so $\beta < \beta'$.

- **Question** : Where does Theorem 1 come from ?

- **Theorem 0** (D., 1986): If j is an elementary embedding of a self-similar rank, then the LD-structure of $\mathrm{Iter}(j)$ implies Π^1_1-determinacy.

???????

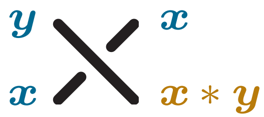
- At least, it should be clear that Theorem 1 relies on two non-trivial results:
  - Property **A**: a braid word containing σ_1 and not σ_1^{-1} cannot be trivial;
  - Property **C**: each braid has an expression with no σ_1 or no σ_1^{-1}.

- Braid diagram colorings: Start with a set S ("colours"), apply colours at the left ends of the strands in a braid diagram, propagate the colors to the right, and compare the initial and final colors.

- Option 1: Colors are preserved in crossings:

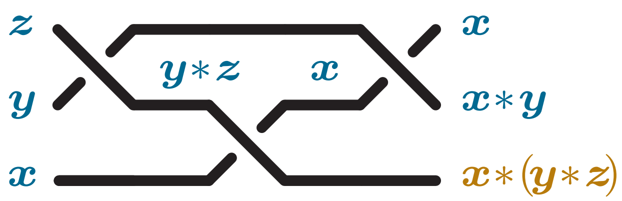
$\rightsquigarrow$ permutation of colors

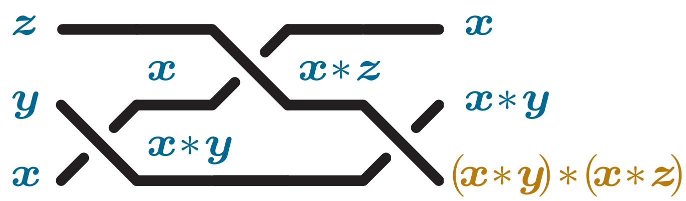
- Option 2: (Joyce, Matveev, Brieskorn, ...) Colors change under the rule



where $*$ is some binary operation on S.

- For an action of B_n on S^n, one needs compatibility with the braid relations:





- Hence: action of B_n iff $*$ satisfies the left self-distributivity law (LD):

$$x * (y * z) = (x * y) * (x * z).$$

- Classical examples of LD-systems:

  (= sets equipped with an operation satisfying the LD law)

  - $x * y = y$, leads to $B_n \twoheadrightarrow \mathfrak{S}_n$.
  - $x * y = xyx^{-1}$, leads to $B_n \to \mathrm{Aut}(F_n)$ (Artin)
  - $x * y = (1-t)x + ty$, leads to $B_n \to \mathrm{GL}_n(\mathbb{Z}[t, t^{-1}])$ (Burau)

- Note: in these examples, $x * x = x$ always holds.

- Definition : Say that an LD-system $(S, *)$ is orderable if there is a linear ordering $<$ on S satisfying $x < x * y$ for all x, y.

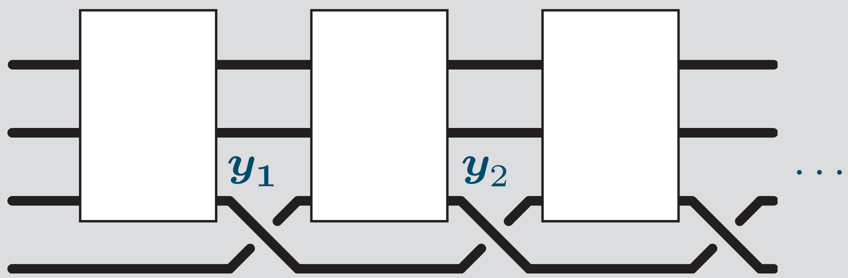
- Certainly orderable LD-systems are of a new flavour: $x < x * x \neq x$.

- Theorem 0.5 (D., 1991): There exist orderable LD-systems (namely: free LD-systems).

- Claim: Theorem 1 (braid order) directly comes from Theorem 0.5.

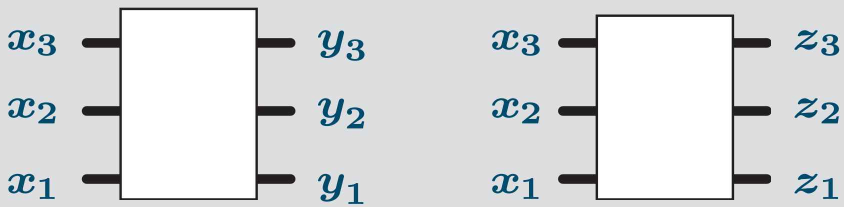
- **Claim**: Theorem 1 (braid order) directly comes from Theorem 0.5.

- Proof of Property A : A braid word with σ_1 and no σ_1^{-1} does not represent 1.



$$x \quad < \quad x*y_1 \quad < \quad (x*y_1)*y_2 \quad < \ldots \quad \neq x$$

- Proof of Property C : A linear ordering on braids:



$\rightsquigarrow$ Compare $(y_1, y_2, \ldots)$ and $(z_1, z_2, \ldots)$ lexicographically.

- **Question**: OK, but then, why to look for orderable LD-systems?

... because set theory told us

- Set theory studies infinity. By Gödel's theorem, every axiomatic system, e.g., the standard Zermelo-Fraenkel system ZF, is incomplete.

- Gödel's program: Complete ZF with axioms stating the existence of "hyper-infinite" sets ("large cardinals").

- Typically, strengthen

  "X is infinite iff $\exists j : X \to X$ injective non-surjective"

into

  "X is self-similar (= hyper-infinite) iff $\exists j : X \to X$ injective non-surjective and, moreover, j preserves everything that is definable from $\in$".

$\uparrow$

an elementary embedding

- Example: As $j : n \mapsto n + 1$ is injective non-surjective, $\mathbb{N}$ is infinite;
Now j preserves $<$, but not $+$: j is not an elementary embedding. In fact:
no (non-trivial) elementary embedding of $\mathbb{N}$ exists: $\mathbb{N}$ is infinite, but not self-similar.

- **Definition** : A rank is a set R such that $f : R \to R$ implies $f \in R$. (??)

- If R is a self-similar rank (= there exists an elem. emb. of R into itself)
  and i, j are elementary embeddings of R, then we can apply i to j.

- "Being an elem. emb." is definable from $\in$, so $i(j)$ is an elem. emb.:
  "application" is a binary operation on elementary embeddings of R.

- "Being the image under" is definable from $\in$, so $\ell{=}j(k)$ implies $i(\ell){=}i(j)(i(k))$,

$$i(j(k)) = i(j)(i(k)):$$

the "application" operation satisfies the LD law.

- **Proposition** : If j is an elementary embedding of a self-similar rank,
  then the iterates of j, make an LD-system $\mathrm{Iter}(j)$.

↑

closure of $\{j\}$ under application: $j(j)$, $j(j)(j)$...

- Remember the question: why to look for orderable LD-systems (Theorem 0.5)?

- **Theorem 0.2** (Laver, 1989): If j is an elem. embedding of a self-similar rank, then $\mathrm{Iter}(j)$ is an orderable LD-system.

- **Theorem 0.1** (D., 1989): If there exists at least one orderable LD-system, then the word problem of LD is solvable.

↑ deciding whether two terms are equal modulo LD

- **Corollary** : If there exists a self-similar rank, the word problem of LD is solvable.

- But the existence of a self-similar rank is an unprovable axiom, so the corollary does not give a solution for the word problem of LD.

  $\rightsquigarrow$ Construct a true orderable LD-system: Theorem 0.5 (orderable LD-systems) by investigating the "geometry group of LD".

  - As the latter group extends Artin's braid group: Theorem 1 (braid order).

- **Question**: Why care about $\mathrm{Iter}(j)$ and prove Theorems 0.1 and 0.2?

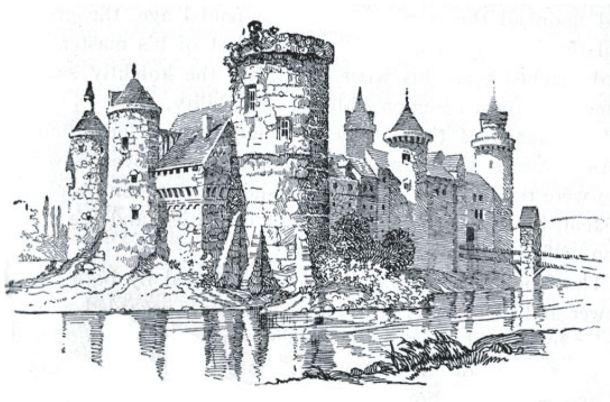
- **Theorem 0** (D., 1986): If j is an elem. embedding of a self-similar rank, then the LD-structure of $\mathrm{Iter}(j)$ implies Π^1_1-determinacy.

(" $\mathrm{Iter}(j)$ is **not** trivial ")

$\rightsquigarrow$ a continuous path from Theorem 0 (about sets) to Theorem 1 (about braids).

- Is the braid order an application of set theory?
  - Formally, no: braids appear when sets disappear.
  - In essence, yes: Orderable LD-systems have been investigated because set theory showed they might exist and be involved in deep phenomena.

- **An analogy**:
  - In physics: using physical intuition and/or evidence, guess some statement, then pass it to mathematicians for a formal proof.
  - Here: using logical intuition and/or evidence ($\exists$ self-similar rank), guess some statement ($\exists$ orderable LD-system), then pass it to mathematicians for a formal proof.

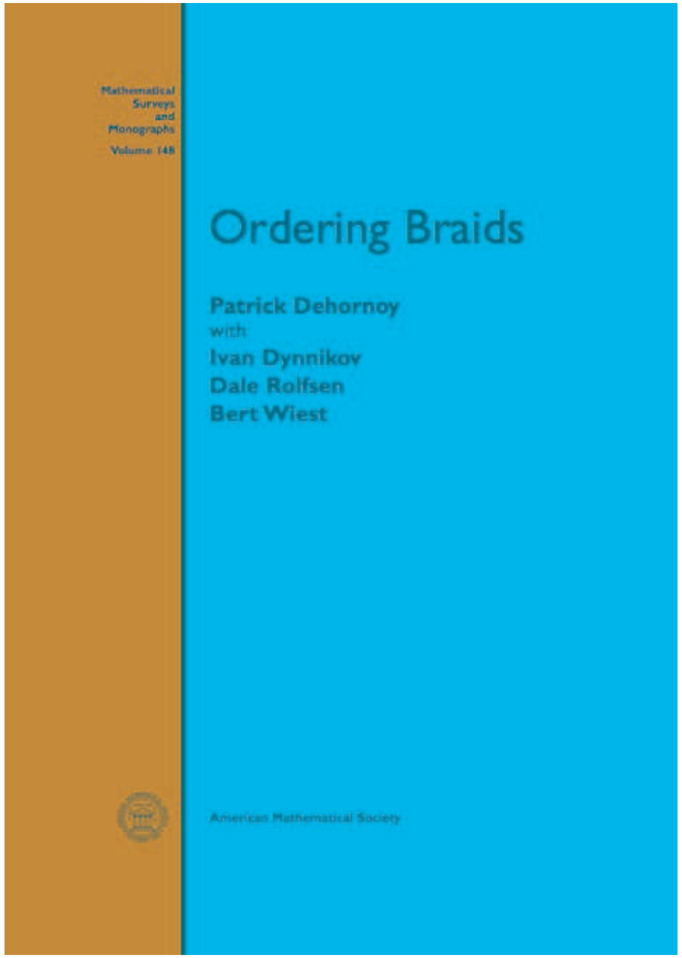


II. The Braid Order in the Middle Ages: 1995-2000

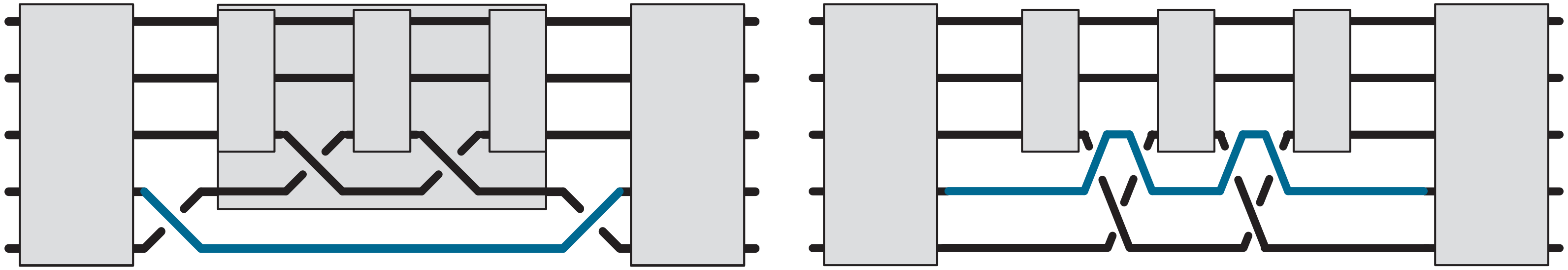
- Handle reduction
- Dynnikov's formulas

- **Theorems** (Burckel, D., Dynnikov, Fenn, Fromentin, Funk, Greene, Larue, Rolfsen, Rourke, Short, Wiest, ...):
  "Many different approaches
  lead to the same braid ordering".

- **Theorems** (Clay, Ito, Navas, Rolfsen, Short, Wiest, ...):
  "Many different braid orderings
  making an interesting space".



• Definition : A σ_i-handle is a braid word of the form $\sigma_i w \sigma_i^{-1}$ or $\sigma_i^{-1} w \sigma_i$ with w containing no $\sigma_j^{\pm 1}$ with $j \leqslant i$.



• Definition : Reducing a handle $\sigma_i^e w \sigma_i^{-e}$ means deleting the initial and final $\sigma_i^{\pm 1}$, and replacing each nested σ_{i+1} with $\sigma_{i+1}^{-e} \sigma_i \sigma_{i+1}^e$.

• Handle reduction is an isotopy; It extends free group reduction;
Irreducible words are: the empty word, σ-positive words, σ-negative words.

$\uparrow$
the σ_i with least i occurs positively only

• Theorem (D., 1995): A braid β satisfies $\beta = 1$ (resp. $\beta > 1$) iff some/any sequence of handle reductions from some/any braid word representing β finishes with the empty word (resp. with a σ-positive word).

- **Definition** : For x in $\mathbb{Z}$, put $x^+ = \max(0, x)$, $x^- = \min(x, 0)$, and

$$F^+(x_1, y_1, x_2, y_2) = (x_1 + y_1^+ + (y_2^+ - z_1)^+, y_2 - z_1^+, x_2 + y_2^- + (y_1^- + z_1)^-, y_1 + z_1^+),$$

$$F^-(x_1, y_1, x_2, y_2) = (x_1 - y_1^+ - (y_2^+ + z_2)^+, y_2 + z_2^-, x_2 - y_2^- - (y_1^- - z_2)^-, y_1 - z_2^-),$$

with $z_1 = x_1 - y_1^- - x_2 + y_2^+$ and $z_2 = x_1 + y_1^- - x_2 - y_2^+$.

- Let n-strand braid words act on $\mathbb{Z}^{2n}$ by

$$(a_1, b_1, \ldots, a_n, b_n) * \sigma_i^e = (a'_1, b'_1, \ldots, a'_n, b'_n)$$

with $a'_k = a_k$ et $b'_k = b_k$ for $k \neq i, i+1$, and

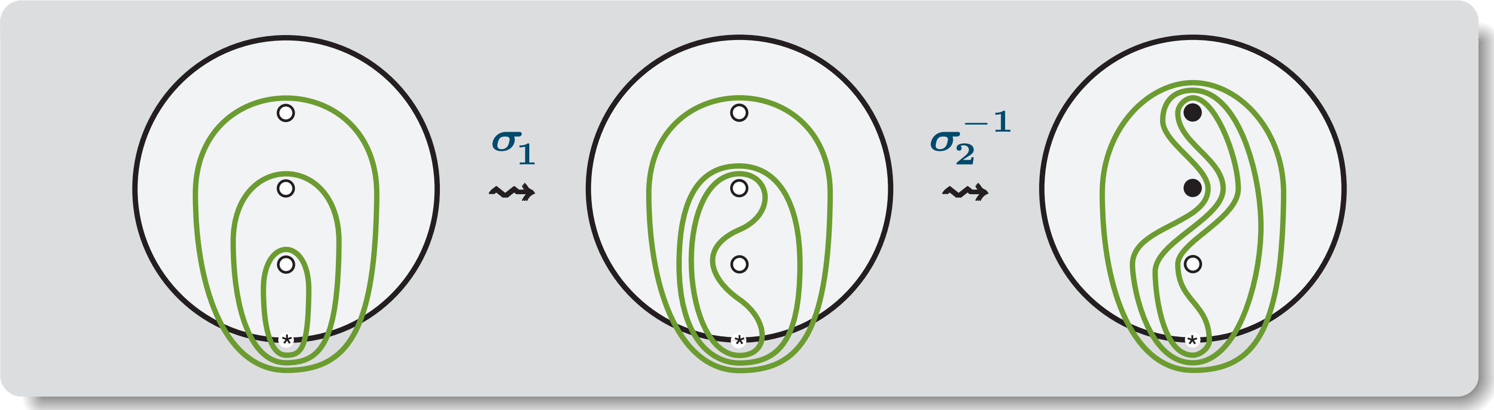
$$(a'_i, b'_i, a'_{i+1}, b'_{i+1}) = F^e(a_i, b_i, a_{i+1}, b_{i+1}).$$

- Finally, define the **coordinates** of w to be $(0, 1, 0, 1, \ldots, 0, 1) * w$.

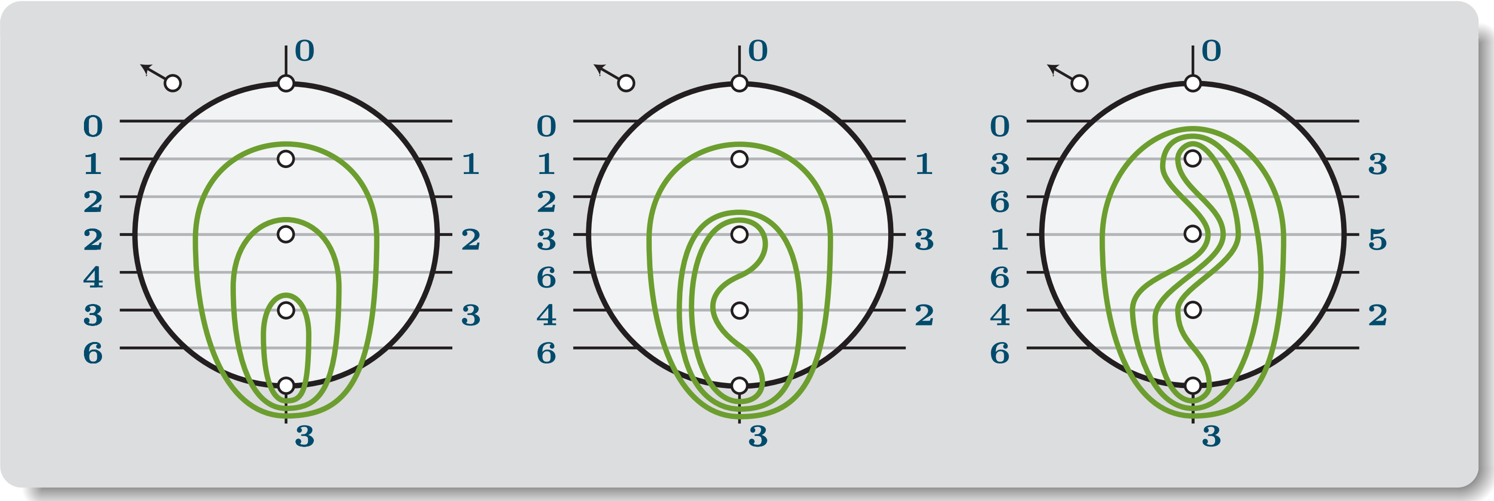
- Remark: Looks ackward, but actually very easy to implement.

- **Theorem** (Dynnikov, 2000): A braid β satisfies $\beta = 1$ (resp. $\beta > 1$) iff the coordinates of some/any braid word representing β are $(0, 1, 0, 1, \ldots, 0, 1)$ (resp. the first nonzero odd rank coordinate is positive).

- Braid = homeomorphism ⇝ braids act on curves drawn on D_n.



- Attribute coordinates by counting intersections with a fixed triangulation:



⇝ $3n+3$ numbers that determine the braid.

- Coordinates = half-differences between intersection numbers;

(going from $3n+3$ to $2n$ numbers)

- **Question** : What are the coordinates of $\beta\sigma_i$ in terms of those of β and of i ?

= compare the intersections of C and $\sigma_i(C)$ with the base triangulation T

$\uparrow$ $\uparrow$

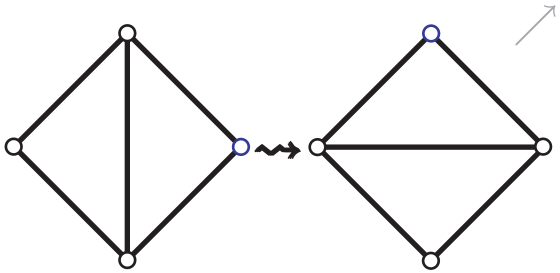
closed curve(s)

- We have

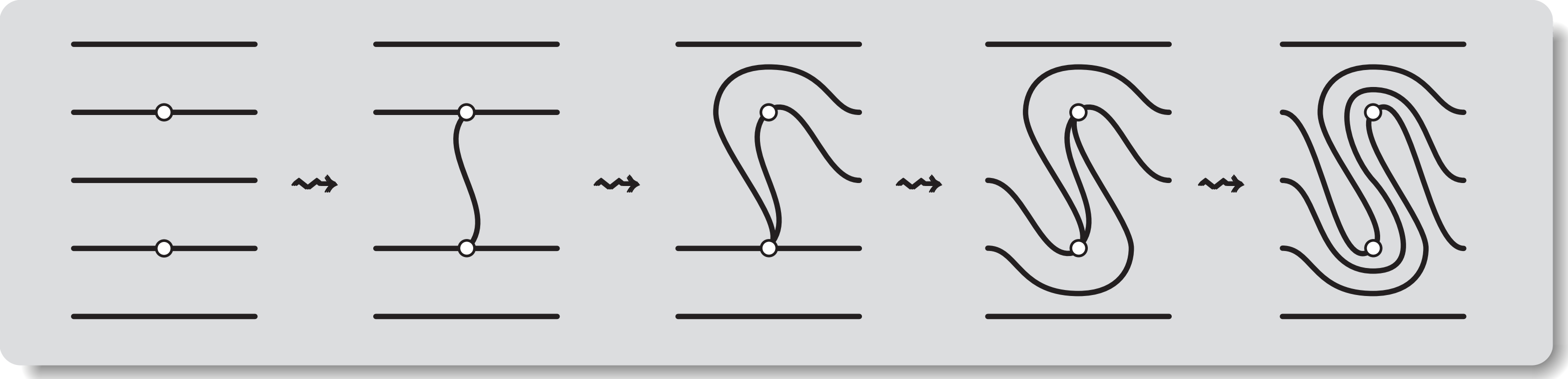
$$\#(\boldsymbol{\sigma_i}(\boldsymbol{C}) \cap \boldsymbol{T}) = \#(\boldsymbol{C} \cap \boldsymbol{\sigma_i}^{-1}(\boldsymbol{T}))$$

$\rightsquigarrow$ compare the intersections of C with T and $\sigma_i^{-1}(T)$.

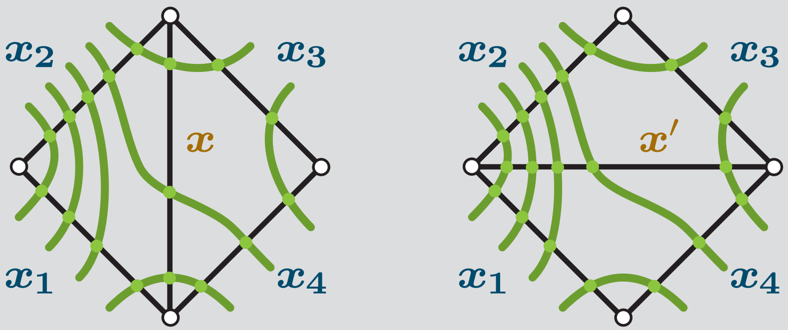
- **Proposition** : If T, T' are (singular) triangulations of a surface, one can go from T to T' by a finite sequence of flips.



- So it **must** be possible to go from T to $\sigma_i^{-1}(T)$ using a finite sequence of flips:



- For **one** flip, the formula is



$$x + x' = \max(x_1 + x_3, x_2 + x_4)$$

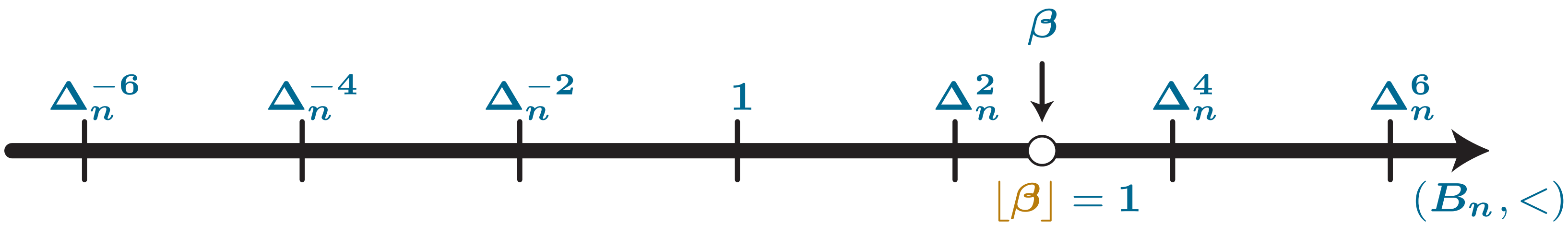
$\rightsquigarrow$ Dynnikov's formulae by a fourfold iteration.

III. The Braid Order in Modern Times: 2000-...

- Floor and closure
- Conjugacy via the μ function

• **Definition :** For β in B_n, the **floor** $\lfloor\beta\rfloor$ is the unique m satisfying

$$\Delta_n^{2m} \leqslant \beta < \Delta_n^{2m+2}.$$



• **Proposition** (Malyutin–Netsvetaev): The **stable floor** $\lfloor\beta\rfloor_s$ of β, defined to be $\lim_p \lfloor\beta^p\rfloor/p$, is a pseudo-character on B_n: one has $\lfloor\beta^p\rfloor_s = p\,\lfloor\beta\rfloor_s$, and

$$\big|\lfloor\beta_1\beta_2\rfloor_s - \lfloor\beta_1\rfloor_s - \lfloor\beta_2\rfloor_s\big| \leqslant 1.$$

(the only pseudo-character on B_n that is > 0 on braids > 1 and is 1 on Δ_n^2)

• **Principle :** Use the (absolute value of) the (stable) floor as a measure of complexity for a braid.

- Principle: If β is very small or very large in the braid ordering, then $\widehat{\beta}$ is simple.

- **Theorem** (Malyutin–Netsvetaev): If β satisfies $\beta < \Delta_n^{-4}$ or $\beta > \Delta_n^{4}$, then $\widehat{\beta}$ is prime and non-trivial.

- Proof: For χ a pseudo-character on B_n s.t. $\chi|_{B_{n-1}} = 0$, then $|\chi(\beta)| > \mathrm{def}(\chi)$ implies that $\widehat{\beta}$ is prime.
Apply to $\lfloor\ \rfloor_s$: both $\beta < \Delta_n^{-4}$ and $\beta > \Delta_n^{4}$ imply $|\lfloor\beta\rfloor_s| > 1$. $\square$

- **Theorem** (Malyutin–Netsvetaev): For each n, there exists $r(n)$ such that, if β in B_n satisfies $\beta < \Delta_n^{-2r(n)}$ or $\beta > \Delta_n^{2r(n)}$, then $\widehat{\beta}$ is represented by a unique conjugacy class in B_n.

$\widehat{\beta'} \approx \widehat{\beta}$ implies β' conjugated to β

- Proof: For each template move M, there exists r s.t. $|\lfloor\beta\rfloor| > r$ implies that $\widehat{\beta}$ is not eligible for M.
By the Birman-Menasco MTWS theory, $\exists$ finitely template moves for each n. $\square$

- (M.-N., 2000) $r(3) \leqslant 3$; (Matsuda, 2008) $r(4) \leqslant 4$; (Ito, 2009) $r(3) = 2$.

- **Conjecture** (Ito): $r(n) \leqslant n - 1$ for each n.

- Two more results connecting the complexity of a braid (floor) and the complexity of its closure:

- **Theorem** (Ito, 2008): If β satisfies $\beta < \Delta_n^{-2k-2}$ or $\beta > \Delta_n^{2k+2}$, then

$$4 \cdot \text{genus}(\widehat{\beta}) > k(n+2) - 2.$$

- **Theorem** (Ito, 2008): If β satisfies $\beta \leqslant \Delta_n^{-4}$ or $\beta \geqslant \Delta_n^{4}$ and $\widehat{\beta}$ is a knot, then
  - β is periodic iff $\widehat{\beta}$ is a torus knot,
  - β is reducible iff $\widehat{\beta}$ is a satellite knot,
  - β is pseudo-Anosov iff $\widehat{\beta}$ is hyperbolic.

- False for arbitrary braids:
  - σ_1^3 is periodic,
  - $\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2$ is reducible,
  - $\sigma_1^3\sigma_2^{-1}$ is pseudo-Anosov, whereas the three closures are a $(2,3)$-torus knot.

- **Theorem** (Laver, 1995): For every braid β and every i, one has $\beta^{-1}\sigma_i\beta > 1$.

- **Corollary** : The restriction of the braid order to B_n^+ is a well-ordering.

the submonoid of B_n generated by $\sigma_1, \dots, \sigma_{n-1}$

every nonempty subset has a minimal element

- **Definition** : For β in B_n^+, put

$$\mu(\beta) = \min\{\beta' \in B_n^+ \mid \beta' \text{ conjugate to } \beta\}.$$

$$\nu(\beta) = \min\{\beta' \in B_n^+ \mid \beta' \text{ Markov equivalent to } \beta\}.$$

- Are these definitions useful? Only if the functions can be computed ...

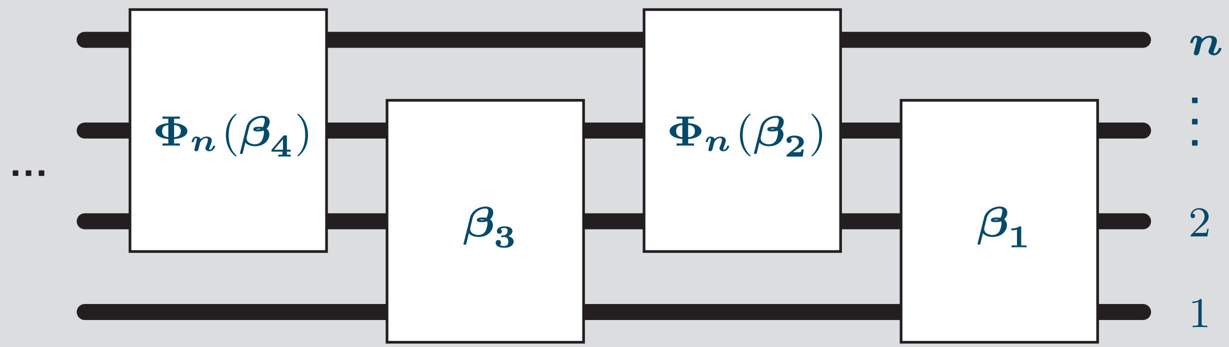
... so certainly not until recently.

- Remark: The braid order is quite bizarre: not Archimedian, not Conradian, ...

$\exists \beta, \beta' > 1\ \forall p\ (\beta^p < \beta')$ $\exists \beta, \beta' > 1\ \forall p\ (\beta < \beta' \beta^p)$

- Associate with every braid β in B_n^+
  a finite sequence $(..., \beta_3, \beta_2, \beta_1)$ of braids in B_{n-1}^+: the n-splitting of β.

n-flip (= horizontal symmetry)
$\downarrow$



- Iterate to obtain a unique normal form (= construct a tree for each braid)

- **Theorem** (D., 2007, building on Burckel, 1997):
  The order $<$ on B_n^+ is a ShortLex-extension of the order $<$ on B_{n-1}^+.

$\uparrow$

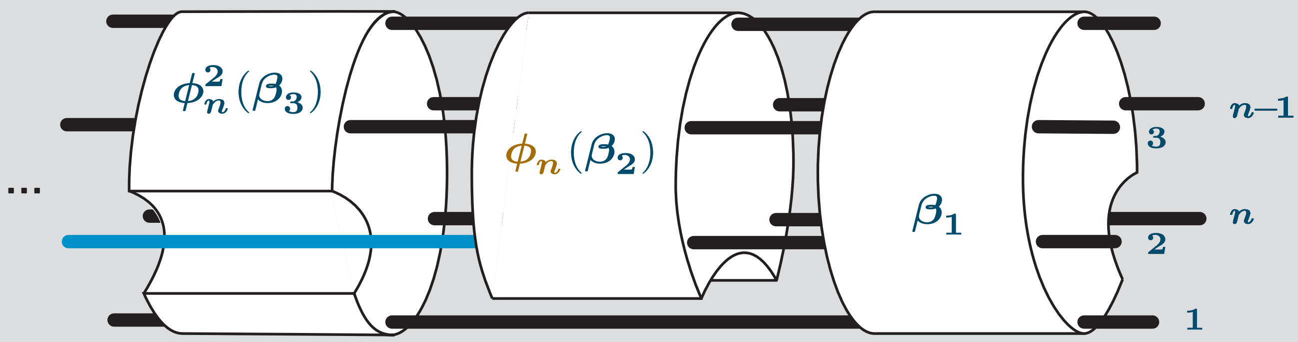
$\beta < \beta'$ holds iff the splitting of β is shorter than that of β' or
they have the same length and the splitting of β is lexicographically smaller

• **Definition** (Birman-Ko-Lee, 1997): The **dual** braid monoid B_n^{+*} is the submonoid of B_n generated by $(a_{i,j})_{1 \leqslant i < j \leqslant n}$ with

$$a_{i,j} = \sigma_{j-1}^{-1} \dots \sigma_{i+1}^{-1} \sigma_i \sigma_{i+1} \dots \sigma_{j-1}.$$

• Alternative Garside structure for B_n,
with the Cat_n non-crossing partitions replacing the $n!$ permutations.

• Then similar splitting of braids in B_n^{+*} into sequences of braids in B_{n-1}^{+*}.



• **Theorem** (Fromentin, 2008):
The order $<$ on B_n^{+*} is a ShortLex-extension of the order $<$ on B_{n-1}^{+*}.

- **Principle** : With the (alternating and) rotating normal form(s) of braids,
  we now have a practical way of controlling the braid order.

- Example: For β in B_n^+, one has $\lfloor\beta\rfloor \approx 2 \times ($length of the splitting of $\beta) - \mathbf{2}$.

- **Conjecture** (D., Fromentin, Gebhardt, 2009): For β in B_3^+,

$$\boldsymbol{\mu(\beta\Delta_3^2) = \sigma_1\sigma_2^2\sigma_1 \cdot \underset{\substack{\uparrow \\ \min\{\beta' \in B_n^+ \mid \beta' \text{ conjugate to } \beta\}}}{\mu(\beta)} \cdot \sigma_1^2}.$$

... more generally: a reasonable hope of computing the function μ.
(whence possibly solving the conjugacy problem in a completely new way).

- Then let's dream a little: What about $\underset{\substack{\uparrow \\ \text{similar to } \mu \text{ with Markov equivalence instead of conjugacy}}}{\nu}$?

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