Topologically Ordered States and their Hamiltonians

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Introduction* -** The study of topologically ordered states with zero value of the local order parameters is important for classifying various phase states in low-dimensional systems, where the role of quantum fluctuations is significant. In this case, new types of ordering of strongly correlated spin degrees of freedom may be based on the employ of topological features of dynamics of excitations in low-dimensional systems; in particular, on the use of ***the effect of braiding of excitation world lines [1,2]. In spatially two-dimensional systems, braiding phenomena lead to fractional statistics of excitations and to the corresponding form of the topological order. Strongly correlated states form a certain part of the low-energy Hilbert space. When the considered excitations are spatially separated, a low-energy space turns to be degenerated, and its states are characterized by pure topological quantum numbers.

Analysis of the maps between Hamiltonians of exactly solvable quantum models [3,4], study of correlation functions [5], as well as the classification of the topological order [6] and quantum phase transitions make an incomplete list of problems in this field.

We consider a universal form for Hamiltonians of the systems, which are in the topologically ordered phase state. It is shown that in strongly correlated systems the Hamiltonian has a form of a sum of the projectors expressed by means of the Temperley-Lieb algebra operators. In the case of twice linked excitation world lines it has a form of a two-dimensional Bloch matrix. In the limit of the infinite value of the linking degree, **k** (the Kac-Moody algebra level), the system turns into the ordinary Heisenberg spin-1/2 model or into the biquadratic spin-1 one. Hamiltonians of the Fibonacci anyons [7] and their counterparts corresponding to the intermediate values of the linking degree are also considered.

Hamiltonians as Projectors

The enlarged symmetry algebras of low-dimensional models result directly from the fact that world lines of the particles never intersect.

How do low-energy effective Hamiltonians look like in this case?

Their form is as follows:

$$H = -\sum_{i,(l)} g_i^{(l)} P_i^{(l)}$$

$P^{(l)}_i$ is the projector ($P_i^2 = P_i$) onto spin-l representation for the pair spin states on i, i+1 sites; here g_i are coupling constants.

For example,

$$H_{HM} = \Sigma_i \vec{S}_i \vec{S}_{i+1} = 3\Sigma_i \left\{ P_i^{(1)} + \frac{1}{3} P_i^{(1/2)} - \frac{2}{3} \right\}$$

$$H_{AKLT} = \Sigma_i P_i^{(1)}$$

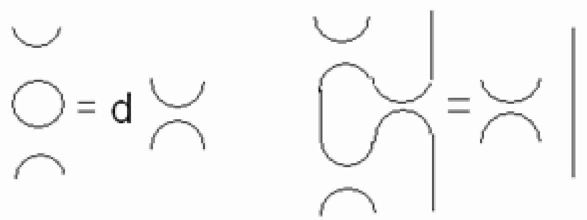
Temperley-Lieb algebra projectors

The generators e_i of the **TL algebra** are defined as follows

$$e_i^2 = d\, e_i\,,$$
$$e_i\, e_{i+1}\, e_i = e_i\,,$$
$$e_i\, e_k = e_k\, e_i \qquad (\,|k-i| \geq 2\,).$$

e_i acts non-trivially on the *i*th and (*i+1*)th particles.

Due to $e_i^2 = d\, e_i$, $(e_i/d)^2 = e_i/d$.
Therefore, effective Hamiltonians have a form of the sum of the Temperley-Lieb algebra projectors $H = -\Sigma_i\, e_i/d$ **in this loop representation**



where $d=q+q^{-1}=2\cos[\pi/(k+2)]$ is the Beraha number (a weight of the Wilson loop)

$$q = exp(\pi i/(k+2))$$

1. **The values of the parameter d lead to the finite-dimensional Hilbert spaces.**
2. **Besides, it turns out, that for the mentioned values of d the theory is unitary.**
3. **In the WZWN case, the colored generalization is given by** $e^l=\Sigma_{m=l}^{k}A(k,m,l)P^m$, **$(e_i^k)^2 = (k+1)e_i^k$. It means in particular that standard TL algebra corresponds in the continuous limit to theories which are defined on *coset* spaces.**

Irreps of e_i's in the RSOS (height) representation

$$\mathbf{e}[i]|j_{i-1}j_ij_{i+1}\rangle \;=\; \sum_{j'_i}\left(\mathbf{e}[i]^{j_{i+1}}_{j_{i-1}}\right)^{j'_i}_{j_i}|j_{i-1}j'_ij_{i+1}\rangle$$

$$\left(\mathbf{e}[i]^{j_{i+1}}_{j_{i-1}}\right)^{j'_i}_{j_i} \;=\; \delta_{j_{i-1},j_{i+1}}\sqrt{\frac{S^0_{j_i}S^0_{j'_i}}{S^0_{j_{i-1}}S^0_{j_{i+1}}}}$$

$$S^{j'}_j := \sqrt{\frac{2}{(k+2)}}\,\sin[\pi\frac{(2j+1)(2j'+1)}{k+2}]$$

V. Jones, R.J. Baxter, V. Pasquier, H. Wenzl, A. Kuniba,
Y. Akutzu, M. Wadati, P. Fendley, 1984 - 2006

Some examples

1. In the case k=2, we have the transverse field Ising model:

$$H = -h\sum_j [\sigma_j^z \sigma_{j+1}^z + (g/h)\sigma_j^x]$$

The universal form of Hamiltonians in this case is as follows:

$$H = \sum_{\alpha,\mathbf{k}} \overline{\Psi}_{\mathbf{k}} h_\alpha(\mathbf{k}) \sigma^\alpha \Psi_{\mathbf{k}}, \qquad \sigma_\alpha = (\mathbb{I}, \sigma)$$

$$E_{\mathbf{k}} = \sqrt{h_3^2(\mathbf{k}) + |\Delta(\mathbf{k})|^2}, \; \Delta(\mathbf{k}) \equiv h_1(\mathbf{k}) + i h_2(\mathbf{k})$$

2. In the case k=3, irreps of the e_i's lead to the Hamiltonian of the Fibonacci anyons, where $\varphi=(1+\sqrt{5})/2$ is the golden ratio. This is the k=3 RSOS model which is a lattice version of the tricritical Ising model at its critical point [7].

$$H = \sum_i \left[(n_{i-1} + n_{i-1} - 1) - n_{i-1} n_{i+1} (\varphi^{-3/2} \sigma_i^x + \varphi^{-3} n_i + 1 + \varphi^{-2}) \right]$$

Intermediate k=4 case and k >> 1 limit

We have the following quantum dimensions and fusion rules of the primary fields

$$d_j = \frac{Sin\,[\pi(2j+1)/(k+2)]}{Sin\,[\pi/(k+2)]} \qquad \varphi_a \times \varphi_b = \sum_c N_{ab}^c \varphi_c \qquad N_{ab}^c = \sum_x \frac{S_{ax} S_{bx} S_{\bar{c}x}}{S_{1x}}$$

$$S_j^{j'} := \sqrt{\frac{2}{(k+2)}}\,\sin[\pi\frac{(2j+1)(2j'+1)}{k+2}]$$

In the case k=4, two species of interacting anyons (with quantum dimensions d_j, spins h_j and fusion rules) are given in table

(F.A. Bais, J.K. Slingerland, '08)

0	$d_0 = 1$	$h_0 = 0$
1	$d_1 = \sqrt{3}$	$h_1 = \frac{1}{8}$
2	$d_2 = 2$	$h_2 = \frac{1}{3}$
3	$d_3 = \sqrt{3}$	$h_3 = \frac{5}{8}$
4	$d_4 = 1$	$h_4 = 1$

$1 \times 1 = 0 + 2$			
$1 \times 2 = 1 + 3$	$2 \times 2 = 0 + 2 + 4$		
$1 \times 3 = 2 + 4$	$2 \times 3 = 1 + 3$	$3 \times 3 = 0 + 2$	
$1 \times 4 = 3$	$2 \times 4 = 2$	$3 \times 4 = 1$	$4 \times 4 = 0$

If k>>1, we get the XXX Heisenberg chain (for spin-1/2 case):

$$H = \sum_{i=1}^{N-1} E_i, \qquad E_i = \frac{\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y}{2} + \Delta_+ \frac{\sigma_i^z \sigma_{i+1}^z - 1}{2} + \Delta_- \frac{\sigma_i^z - \sigma_{i+1}^z}{4} \qquad \Delta_\pm = \frac{q \pm q^{-1}}{2}$$

$$q = exp(\pi i/(k+2))$$

and biquadratic Hamiltonians (for spin-1 models)

$$H = \sum_n H_{n,n+1} = \sum_n f_s(\vec{s}_n \cdot \vec{s}_{n+1}) \qquad f_1(x) = x - x^2$$

Yang-Baxter and Pentagon Relations

The general approach to the Hamiltonian dynamics of nontrivial models describing topologically ordered states is the relation between YB and PR's (R. Kashaev, '95)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \quad : \; V^{\otimes 3} \circlearrowleft \qquad\qquad R : V \otimes V \circlearrowleft$$

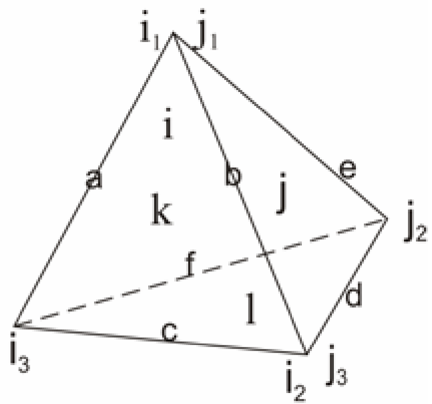
$$F_{12}F_{13}F_{23} = F_{23}F_{12} : \; V^{\otimes 3} \circlearrowleft \qquad\qquad V : \; V \otimes V \circlearrowleft$$

$$R_{12,34} = \left(F_{14}^{t_4}\right)^{-1} F_{13} F_{24}^{t} \left(F_{23}^{-1}\right)^{t_2}, \; : W \otimes W , \quad W = V \otimes V^{*}$$

The F-matrix indices here are corresponded to four faces of the tetrahedron; t and t_i's means the total and partial transpositions respectively.

Tetrahedral (2+1)D Anzatz

A renormalization group fixed point is given by the following tetrahedral anzatz



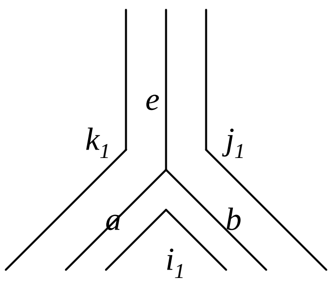
$$T_{[i][j][k][l]} = F^{abc}{}_{def} \cdot\lambda(i_1 j_1 k_1) \cdot\lambda(i_2 j_2 l_2) \cdot\lambda(i_3 k_3 l_3) \cdot\lambda(j_4 k_4 l_4).$$

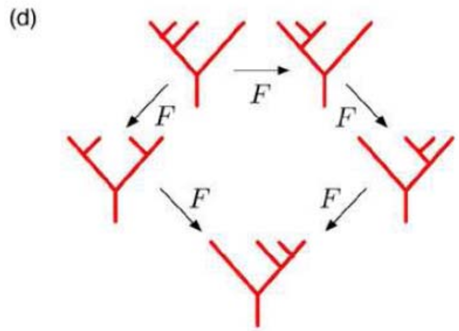
T-matrices define a partition function

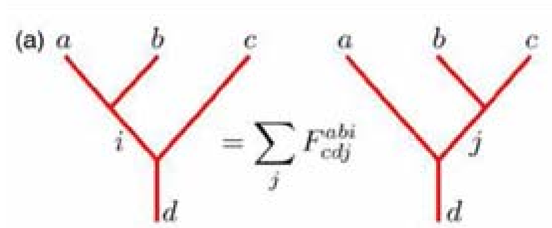
$$Z = Tr\, e^{-\beta H} = \sum_{ijkl} T_{jfei} T_{hgjk} T_{qklr} T_{lits} \cdots = tTr \otimes_i T$$

and F-objects obey the pentagon equation

$$\Sigma_n F^{mlq}{}_{kpn} F^{jipm}{}_{ns} F^{jsn}{}_{lkr} = F^{jip}{}_{qkr} F^{riq}{}_{mls}$$







Sites (i_1, etc.), links (a, etc.) and faces (i,j,k,l) are labeled by indices corresponding to boundaries of the tetrahedron.

$$F^{abi}_{cdj} = \left\{ \begin{array}{ccc} a & b & j \\ c & d & i \end{array} \right\}_q$$

V. Turaev, N. Reshetikhin, A. Kirillov, O. Viro, L. Kauffman, '89 – '92

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