An Introduction to the Theory of Knots and its Applications

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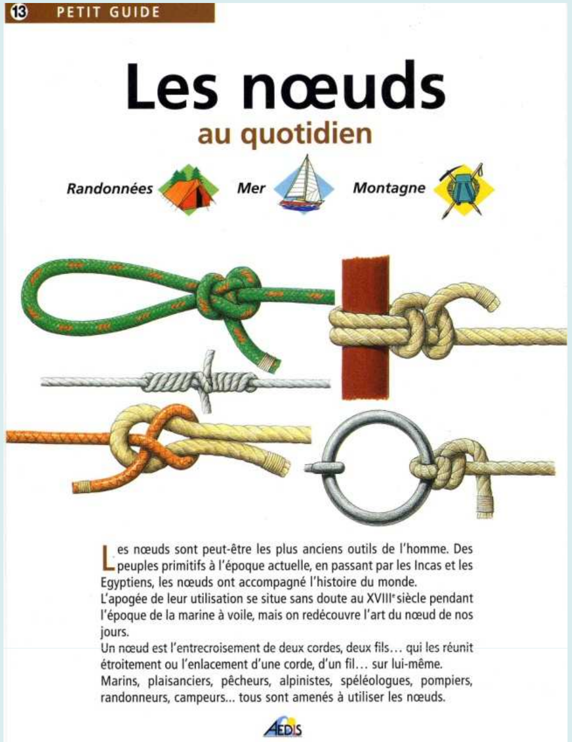
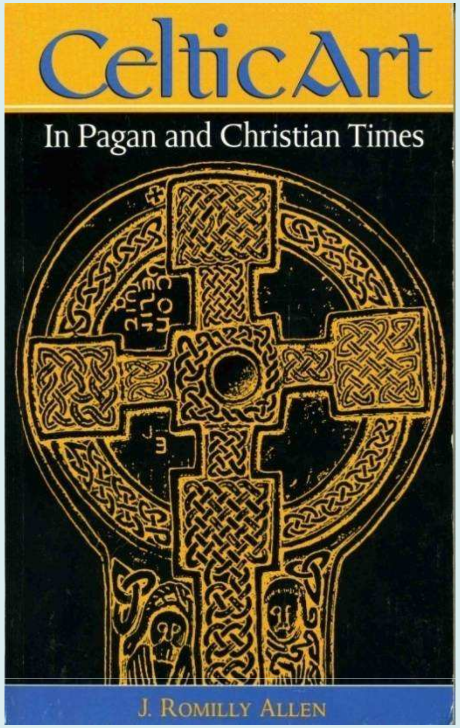
Advanced School on Knot Theory and its Applications to Physics and Biology

ICTP, 11-22 May 2009

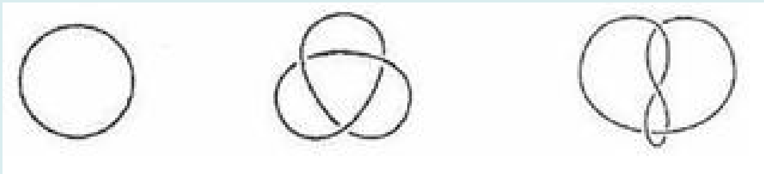

13 PETIT GUIDE
Les nœuds
au quotidien
Randonnées
Mer
Montagne
Les nœuds sont peut-être les plus anciens outils de l'homme. Des peuples primitifs à l'époque actuelle, en passant par les Incas et les Egyptiens, les nœuds ont accompagné l'histoire du monde.
L'apogée de leur utilisation se situe sans doute au XVIIIe siècle pendant l'époque de la marine à voile, mais on redécouvre l'art du nœud de nos jours.
Un nœud est l'entrecroisement de deux cordes, deux fils… qui les réunit étroitement ou l'enlacement d'une corde, d'un fil… sur lui-même.
Marins, plaisanciers, pêcheurs, alpinistes, spéléologues, pompiers, randonneurs, campeurs… tous sont amenés à utiliser les nœuds.
AEDIS


Celtic Art
In Pagan and Christian Times
J. Romilly Allen

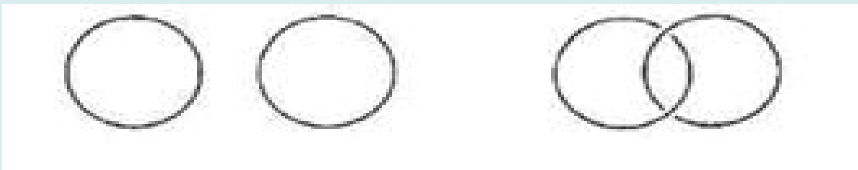
A **knot** is a smooth curve:

$$c: [0,1] \to R^3 \text{ ή } S^3$$
$$t \mapsto c(t)=(x(t),y(t),z(t))$$

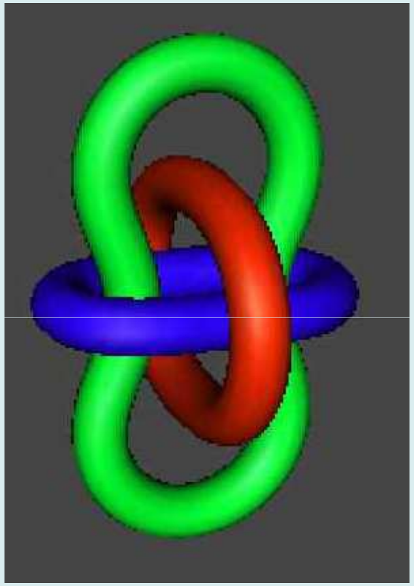
such that c is 1-1 and c(0)=c(1). Equivalently, a knot is a smooth embedding of the circle S^1 in R^3 or in S^3.

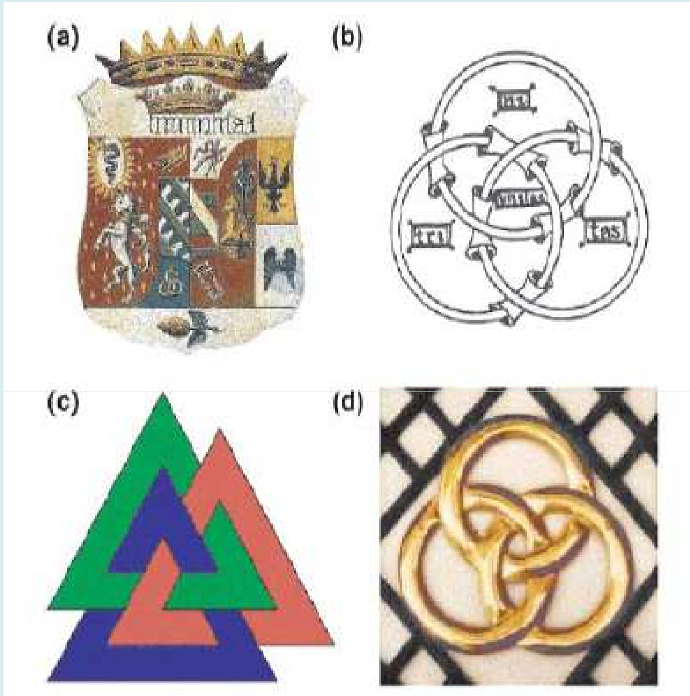


A **<u>link with k components</u>** is a smooth embedding of k copies of the circle in R^3 or in S^3.



The Borromean rings





- Carl Friedrich Gauss (1777-1855)

  "Zur mathematischen Theorie der elektrodynamischen Wirkungen", Werke Koenigl. Gessell. Wiss. Goettingen, Vol. 5, p. 605 (1833).

  Handwritten catalogue of 13 knots.
- Listing – student of Gauss
- Lord Kelvin (William Thomson, 1824-1907)

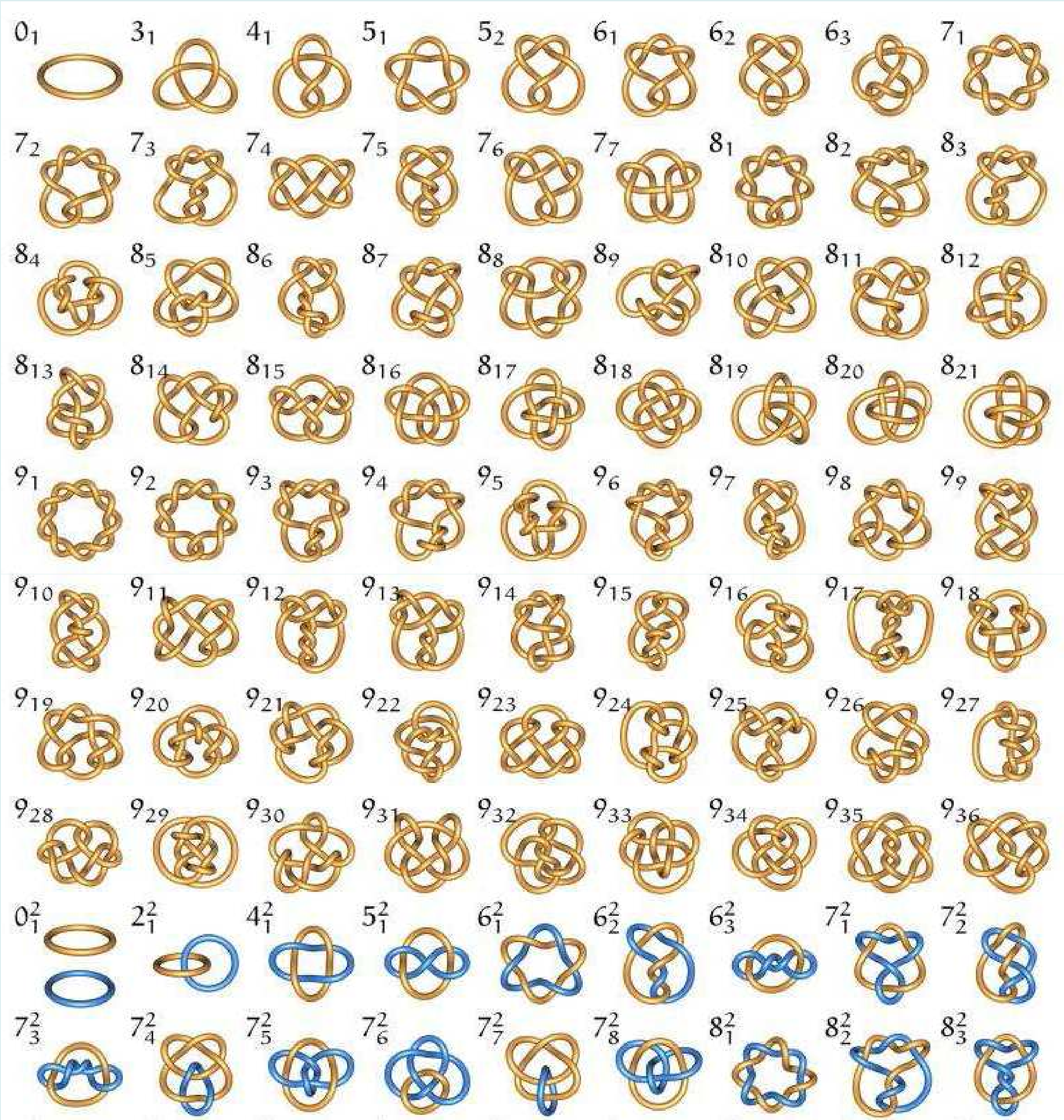
  "On vortex atoms", Philosophical Magazine, Vol. 34, pp. 15-24 (1867).

  «Atoms are knotted tubes of ether».
- Kirkman:

  By a knot of n *crossings, I understand a reticulation of any number of meshes of two or more edges, whose summits, all tessaraces, are each a single crossing, as when you cross your forefinges straight or slightly curved, so as not to link them, and such meshes that every thread is either seen, when the projection of the knot with its* n *crossings and no more is drawn in double lines, or conceived by the reader of its course,*

- Peter Guthrie Tait (1839-1901), Scotch physicist

  Catalogue up to 10 crossings (work of 20 years).
- Little: same catalogue almost at the same time.


0_1 3_1 4_1 5_1 5_2 6_1 6_2 6_3 7_1
7_2 7_3 7_4 7_5 7_6 7_7 8_1 8_2 8_3
8_4 8_5 8_6 8_7 8_8 8_9 8_{10} 8_{11} 8_{12}
8_{13} 8_{14} 8_{15} 8_{16} 8_{17} 8_{18} 8_{19} 8_{20} 8_{21}
9_1 9_2 9_3 9_4 9_5 9_6 9_7 9_8 9_9
9_{10} 9_{11} 9_{12} 9_{13} 9_{14} 9_{15} 9_{16} 9_{17} 9_{18}
9_{19} 9_{20} 9_{21} 9_{22} 9_{23} 9_{24} 9_{25} 9_{26} 9_{27}
9_{28} 9_{29} 9_{30} 9_{31} 9_{32} 9_{33} 9_{34} 9_{35} 9_{36}
0_1^2 2_1^2 4_1^2 5_1^2 6_1^2 6_2^2 6_3^2 7_1^2 7_2^2
7_3^2 7_4^2 7_5^2 7_6^2 7_7^2 7_8^2 8_1^2 8_2^2 8_3^2

No. of crossings	No. of knots
3	1
4	1
5	2
6	3
7	7
8	21
9	49
10	165

J. Navy Academy UK
Gauss (~1810)
Kirkman
Tait (1898)
Alexander& Briggs (1927)
Reidemeister (1932)

(2^n choices)

11	552
12 (1981)	2.176
13 (1982)	9.988
14	46.972
15	253.293
…	…

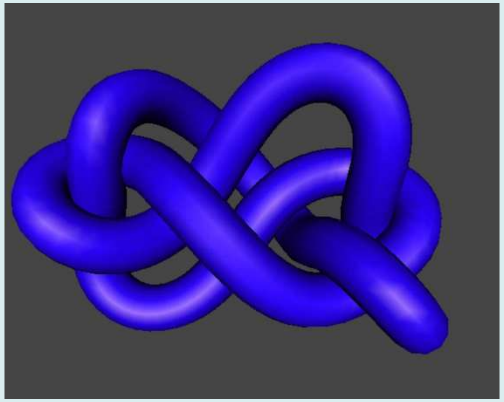
Little (1900)
Mary Haseman ('17)
J. Conway (1969)
Caudron (1978)
[V.F.R. JONES 1984]
Dowker (notation)
Thistlethwaite (HY)
Hoste & Weeks
Aneziris
…

Open Problem: How many different knot types are there?

By `**knots**' we always mean `**knots or links**'.

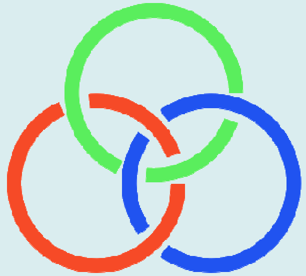
Two knots are of the same type, called **isotopic**, if there exists an orientation preserving homeomorphism of the 3-space that takes one to the other.

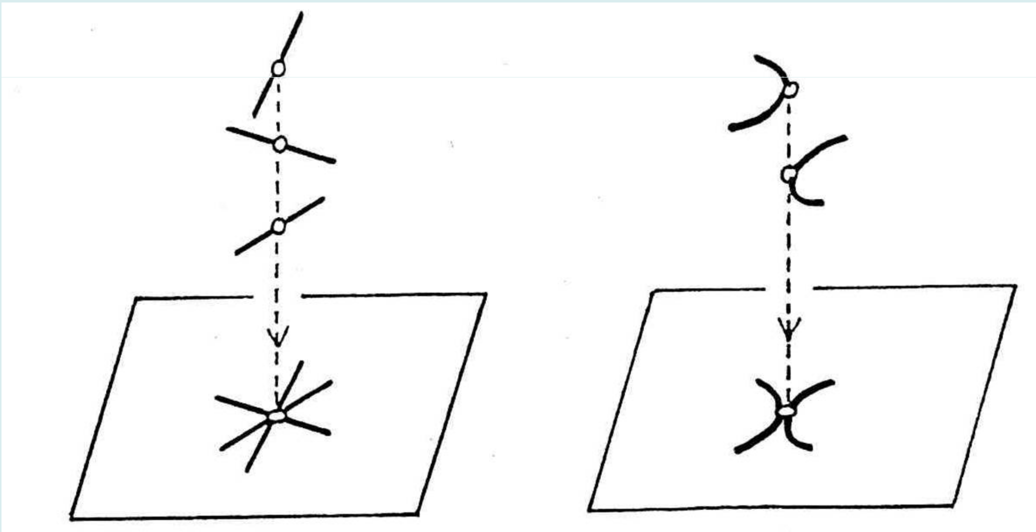
Let ***h***: A →B be a function. We say that ***h*** is a **homeomorphism** if ***h*** is continuous, and ***h*** has a continuous inverse.



~ O

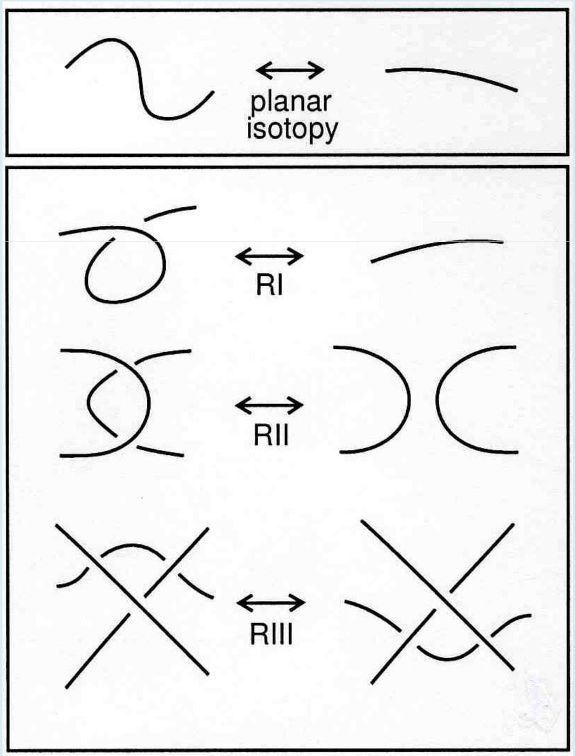
A **diagram** of a knot is a projection on a plane, with only finitely many double points, the crossings, with the extra information `under’ or `over’.



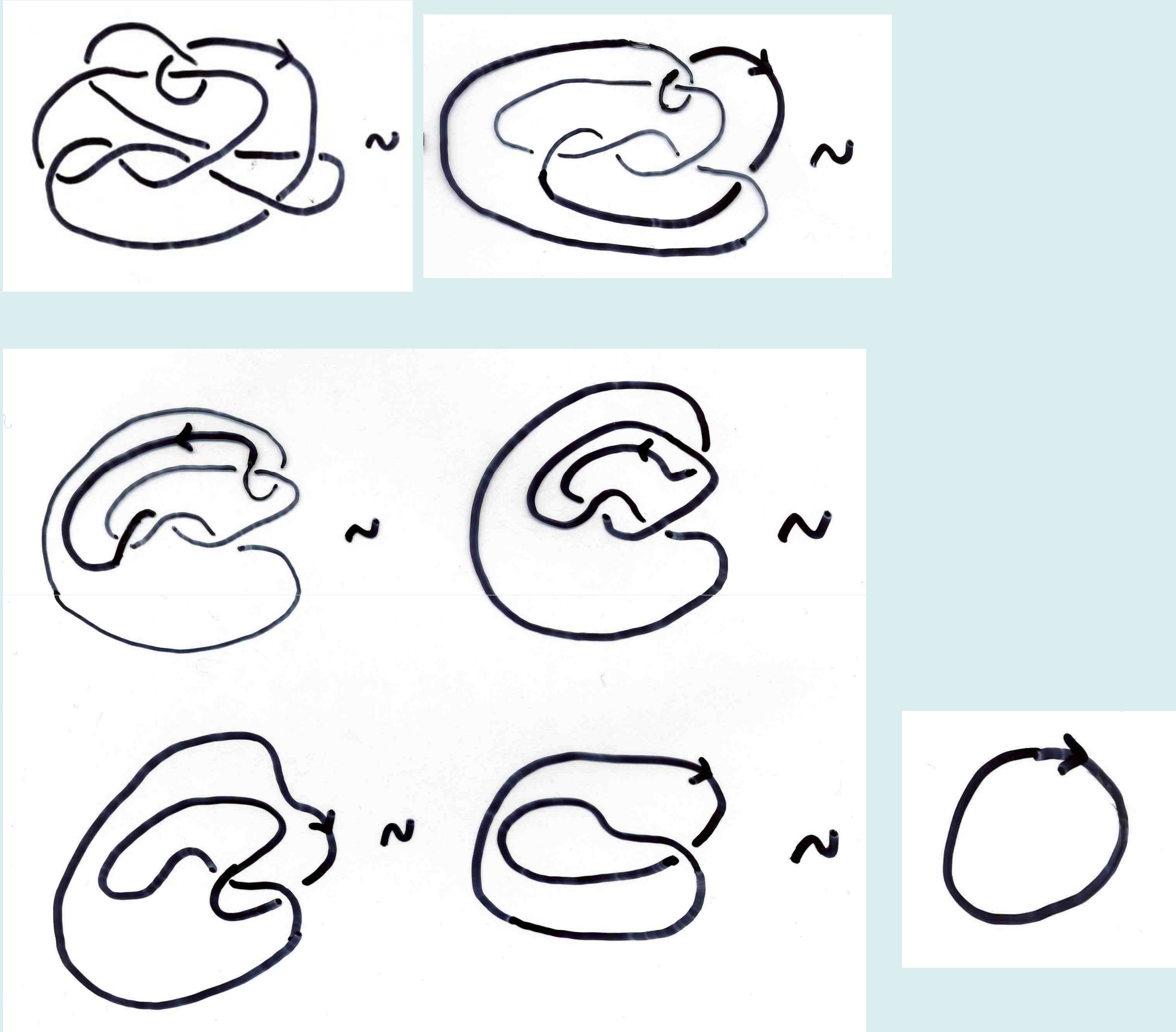


The above, together with cusps, form a set of projections of measure zero.

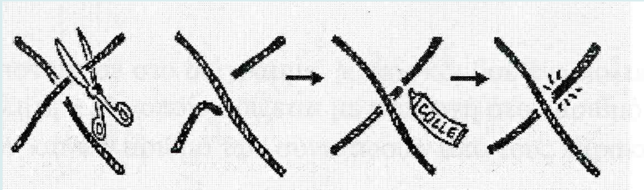
Reidemeister Theorem (1927): Two knots are isotopic iff any two diagrams of theirs differ by finitely many of the following moves:



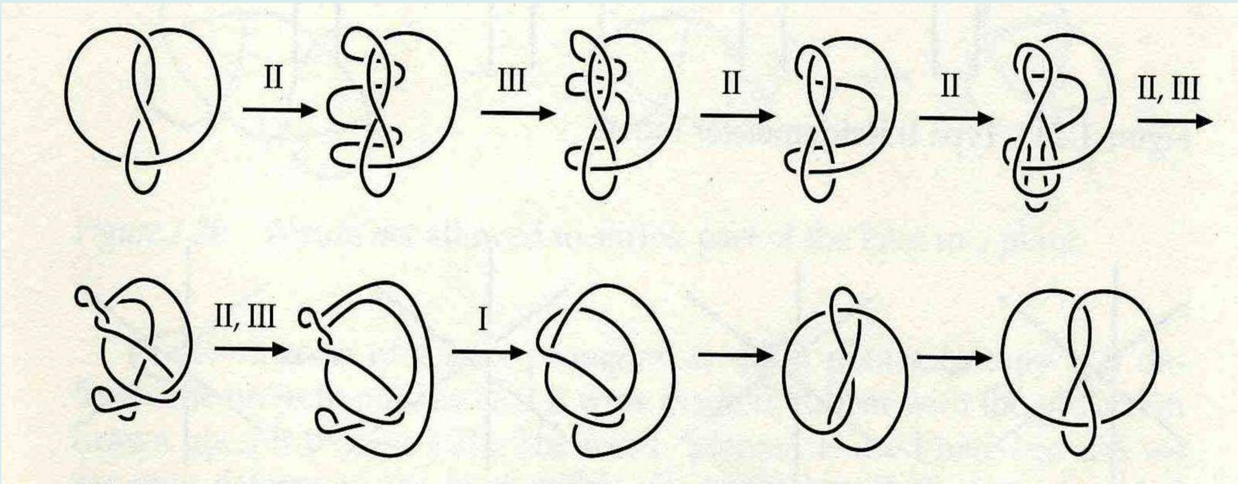
The Reidemeister moves



The **mirror image** –K of a knot K is obtained by switching all crossings.

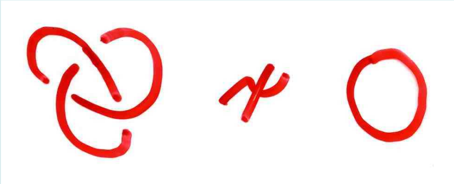
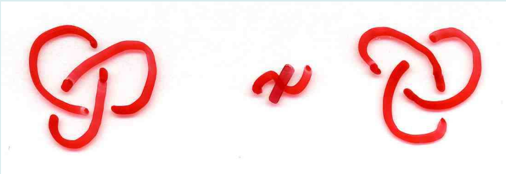


A knot that is isotopic to its mirror image is called `achiral’. E.g. the figure-8:



A knot that is not isotopic to its mirror image is called `chiral’. E.g. the trefoil.

How do we answer questions like the following?

- Why is  ?
- Why is  ?
- Why is  ?

Knot invariants

$\mathcal{L}$ ={all knots and links}

I: $\mathcal{L} \rightarrow$ S (e.g. numbers, polynomials,…)

is a <u>knot invariant</u> if

$K_1 \sim K_2 \Rightarrow I(K_1) = I(K_2)$

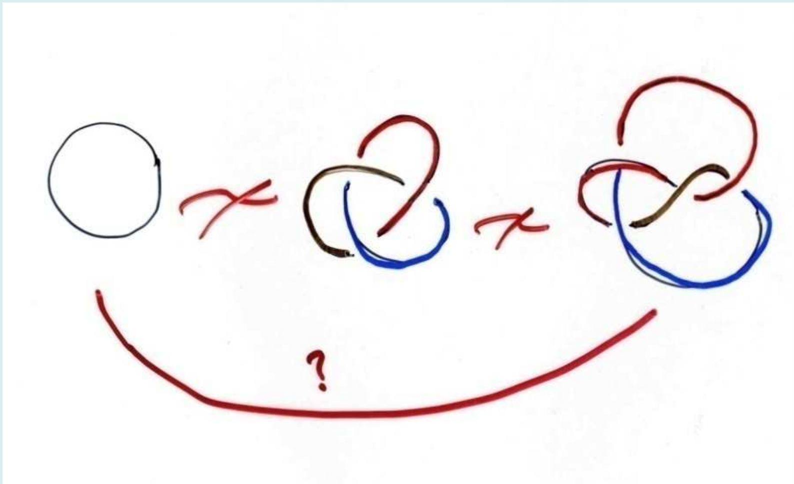
Equivalently:

$I(K_1) \neq I(K_2) \Rightarrow K_1$ not~ K_2

• 3-colourability

A knot K is said to be **3-colourable** if there is a projection P of K in which each arc in P can be coloured with one of three colours such that:
(i) at any crossing, either all three arcs are the same colour or they are all different,
(ii) at least two colours are used to colour the knot.

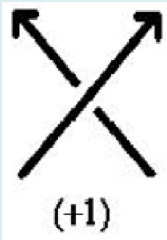
Theorem 1: A knot K has a projection P that can be 3-coloured iff every projection of K can be 3-coloured. Hence, 3-colourability is a knot invariant. (Show it!)

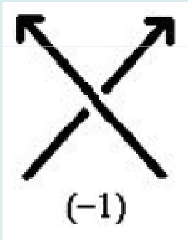


The unknot is not 3-colourable, the trefoil is, the figure-8 is not.

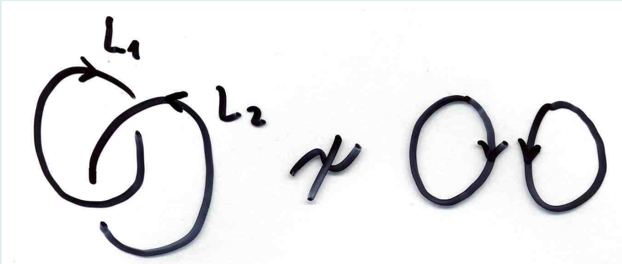
•<u>The linking number</u> (for oriented links)

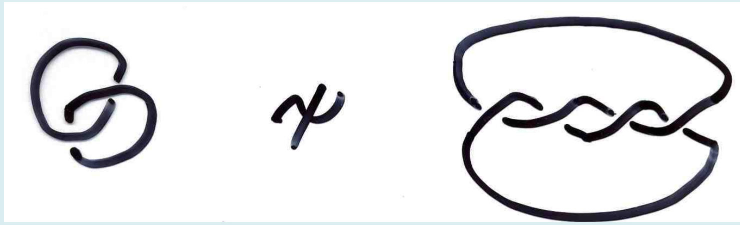
$$lk(L1,L2)=1/2\ \Sigma\ sign(L1,L2)$$





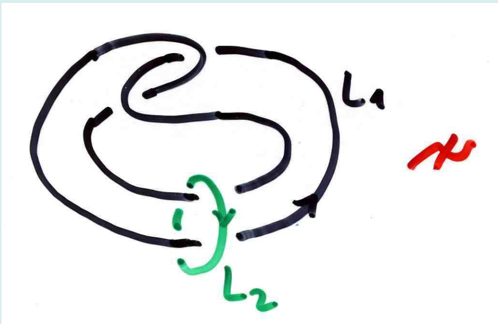
Theorem 3: The linking number is a link invariant. (Show it!)

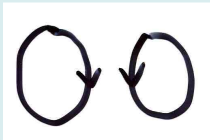


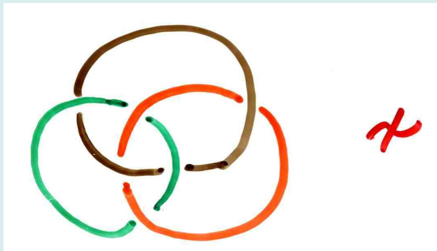


lk = 1 lk = 0 lk = 1 or -1 lk = 2 or – 2

But:

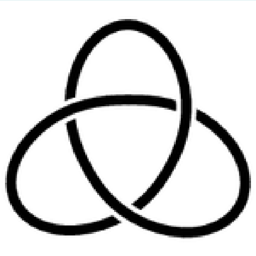




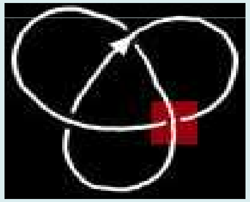


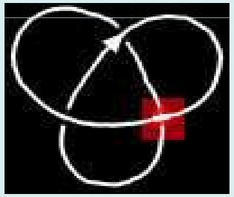
lk = 0 lk = 0 lk = 0 lk = 0

- The crossing number, c(K): the least number of crossings over all diagrams in the isotopy class of a knot K. E.g. c(trefoil) = 3.



- The unknotting number, u(K): the least number of crossing changes in order to get from a knot K the unknot. E.g. u(trefoil) = 1.





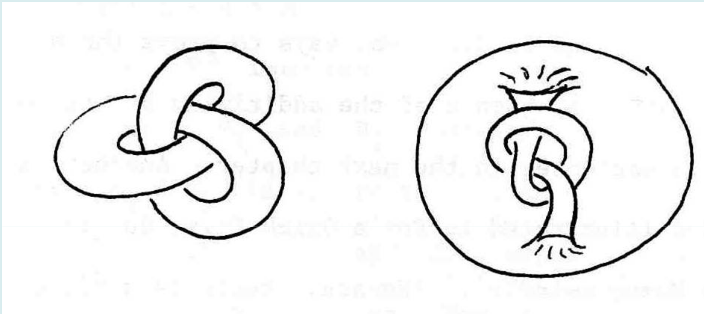
- The bridge number, b(K): the least number of bridges for the knot K. A bridge is considered to be an arc, a piece of a knot diagram between two undercrossings with no undercrossings in between, with at least one overcrossing. E.g. the usual knot diagram for the trefoil has three bridges, but b(trefoil) = 2 (show it!). Also, b(O) = 1.

Show that c(K), u(K) and b(K) are isotopy invariants.

- <u>The knot complement</u>

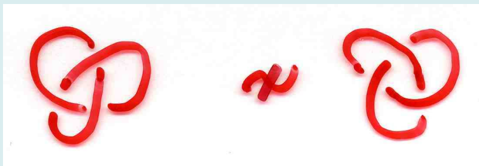
  $K_1 \sim K_2 \Rightarrow \; S^3 - K_1 \sim S^3 - K_2$

(hence also the fundamental group, the homology groups and the cohomology groups of the knot complement)



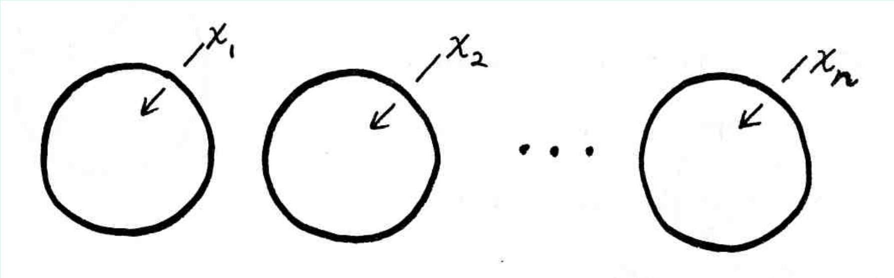
inside outside

Dehn (1914) distinguished the right-handed trefoil from the left-handed trefoil:



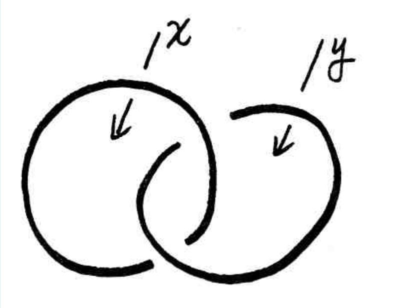
<u>Theorem</u> (Gordon&Luecke, 1989): $S^3 - K_1 \sim S^3 - K_2 \;\Rightarrow\; K_1 \sim K_2$.

(Not true for links).



Fundamental group of the n unlink = the free group on n generators (not abelian!).

Trivial knot Theorem: A knot is trivial iff its fundamental group = Z.



Fundamental group of the Hopf link = the free abelian group on 2 generators.

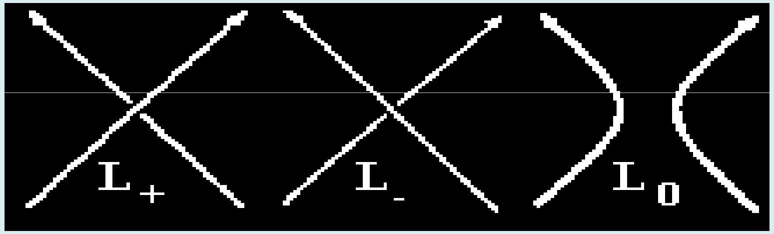
Find the fundamental group of the trefoil!

- <u>The Alexander polynomial</u> (Conway ~1960, Alexander 1928):

  For a given knot diagram it is defined inductively via the rules:

  Δ(O) = 1 and

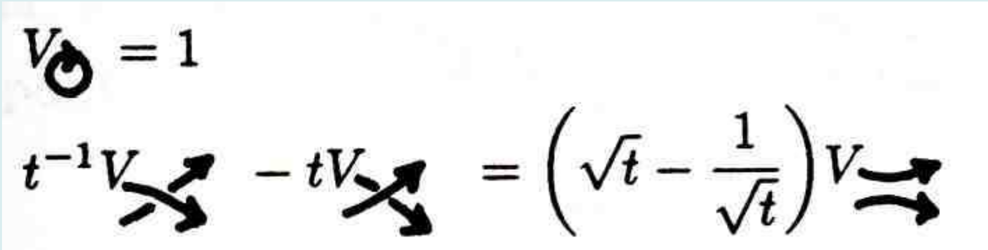
$$\Delta_{L_+} - \Delta_{L_-} = (t^{-1/2} - t^{1/2})\Delta_{L_0}.$$ (Skein relation)



<u>Note</u>: The Alexander polynomial does not distinguish between right-handed and left-handed trefoil (and generally mirror images).

But Δ(trefoil) ≠ Δ(figure-8).

The Jones polynomial, 1984

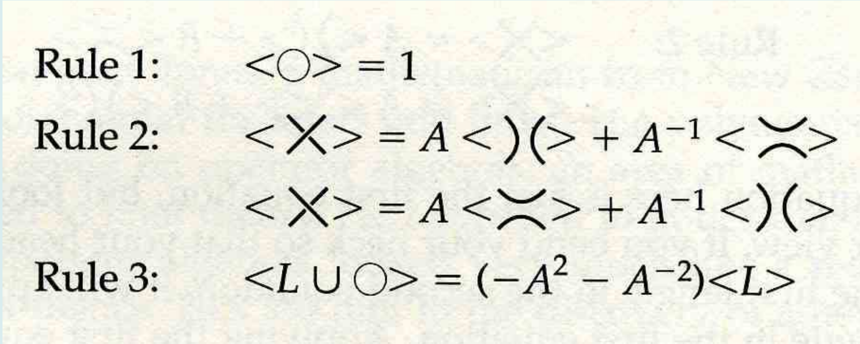


Show that V is an isotopy invariant.

L.H. Kauffman, 1984: The Jones polynomial is an invariant of non-oriented knots and it is equivalent to the Kauffman bracket:

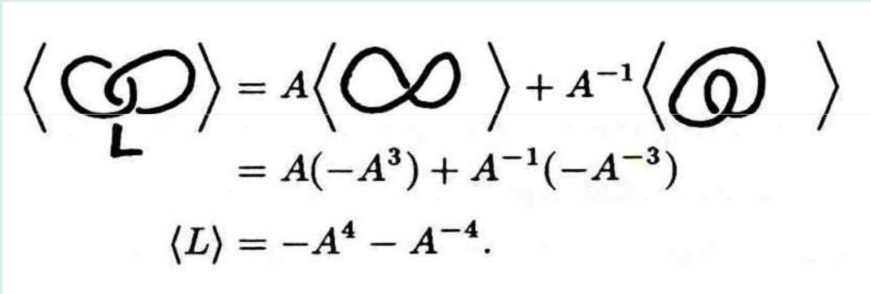
$\langle o \rangle = -A^{2} - A^{-2}$

$B = 1/A$

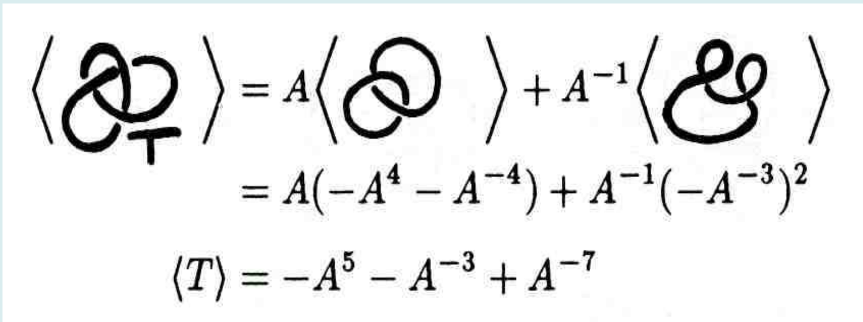


Show that the bracket is invariant under the moves RII and RIII.

Example 1:



Example 2:



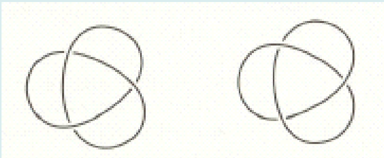
To make the bracket invariant under the move RI we consider the writhe of L, w(L), and define:

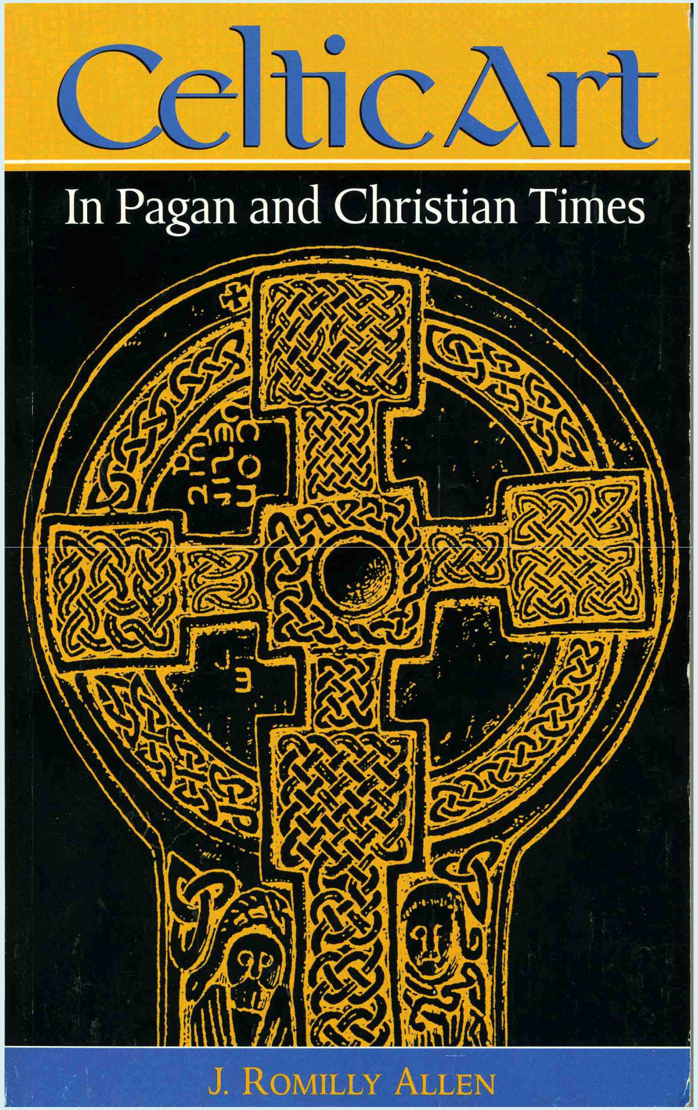
$$X(L) = (-A^3)^{-w(L)} <L>$$

Then X(L)(A) = $V_L(t)$

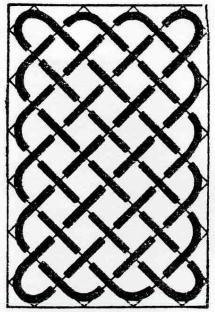
Show that: $V_K(t) = V_{-K}(t^{-1})$

Compute V for the right-handed and for the left-handed trefoil.



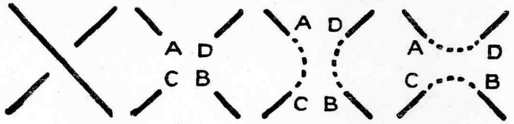


I now propose to explain how plaitwork is set out, and the method of making breaks in it. When it is required to fill in a rectangular panel with a plait the four sides of the panel are divided up into equal parts (except at the ends, where half a division is left), and the points thus found are joined, so as to form a network of diagonal lines. The plait is then drawn over these lines, in the manner shown on the accompanying diagram. The setting-out lines ought really to be double so as to define the width of the band composing the plait, but they are drawn single on the diagram in order to simplify the explanation.



Regular plaitwork without any break

If now we desire to make a break in the plait any two of the cords are cut asunder at the point where they cross each other, leaving four loose ends A, B, C, D. To make a break the loose ends are joined together in pairs. This can be done in two ways only: (1) A can be joined to C and D to B, forming a vertical



Method of making breaks in plaitwork

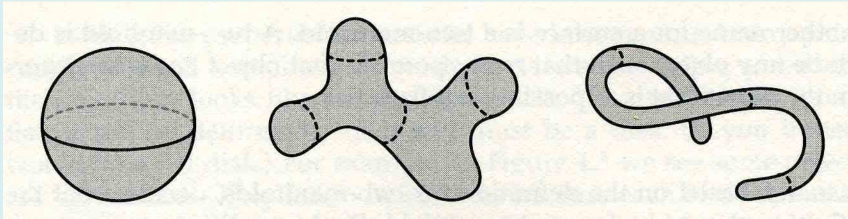
break; or (2) A can be joined to D and C to B, forming a horizontal break. The decorative effect of the plait is thus entirely altered by running two of the meshes

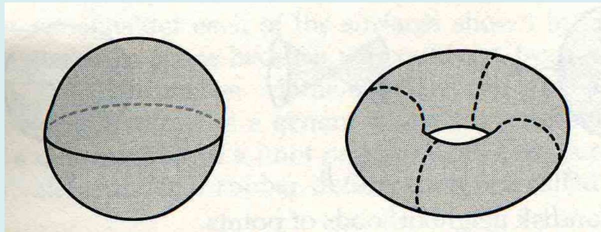
OPEN PROBLEM: Is there a non-trivial knot K with V(K)=1?

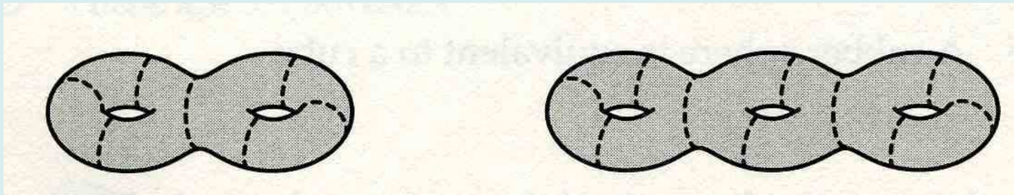
TABLE 3.1. Jones Polynomial Table

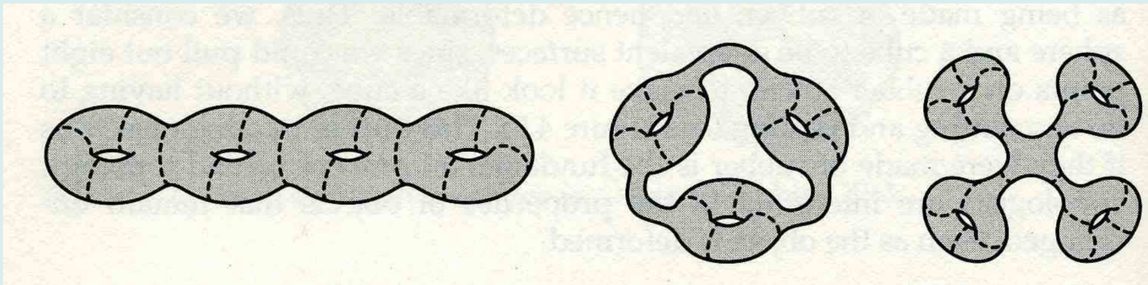
3_1	-1	1	0	1	**0**						
4_1	1	-1	**1**	-1	1						
5_1	-1	1	-1	1	0	1	0	**0**			
5_2	-1	1	-1	2	-1	1	**0**				
6_1	1	-1	1	-2	**2**	-1	1				
6_2	1	-2	2	-2	2	**-1**	1				
6_3	-1	2	-2	**3**	-2	2	-1				
7_1	-1	1	-1	1	-1	1	0	1	0	0	**0**
7_2	-1	1	-1	2	-2	2	-1	1	**0**		
7_3	**0**	0	1	-1	2	-2	3	-2	1	-1	
7_4	**0**	1	-2	3	-2	3	-2	1	-1		
7_5	-1	2	-3	3	-3	3	-1	1	0	**0**	
7_6	-1	2	-3	4	-3	3	**-2**	1			
7_7	-1	3	-3	**4**	-4	3	-2	1			
8_1	1	-1	1	-2	2	-2	**2**	-1	1		
8_2	1	-2	2	-3	3	-2	2	-1	**1**		
8_3	1	-1	2	-3	**3**	-3	2	-1	1		
8_4	1	-2	3	-3	3	**-3**	2	-1	1		
8_5	**1**	-1	3	-3	3	-4	3	-2	1		
8_6	1	-2	3	-4	4	-4	3	**-1**	1		
8_7	-1	2	**-2**	4	-4	4	-3	2	-1		
8_8	-1	2	-3	**5**	-4	4	-3	2	-1		
8_9	1	-2	3	-4	**5**	-4	3	-2	1		
8_{10}	-1	2	**-3**	5	-4	5	-4	2	-1		
8_{11}	1	-2	3	-5	5	-4	4	**-2**	1		
8_{12}	1	-2	4	-5	**5**	-5	4	-2	1		
8_{13}	-1	2	-3	5	-5	**5**	-4	3	-1		
8_{14}	1	-3	4	-5	6	-5	4	**-2**	1		
8_{15}	1	-3	4	-6	6	-5	5	-2	1	0	**0**
8_{16}	-1	3	-5	6	-6	6	**-4**	3	-1		
8_{17}	1	-3	5	-6	**7**	-6	5	-3	1		
8_{18}	1	-4	6	-7	**9**	-7	6	-4	1		
8_{19}	**0**	0	0	1	0	1	0	0	-1		
8_{20}	-1	1	-1	2	-1	**2**	-1				
8_{21}	1	-2	2	-3	3	-2	2	**0**			

Knots and surfaces







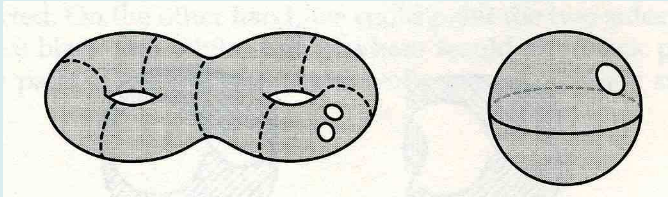


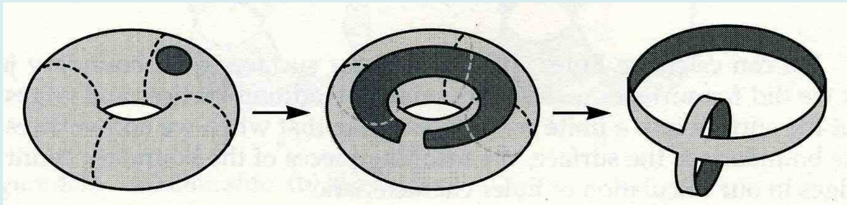
Theorem: The genus g (i.e. the number of holes) classifies the orientable compact, connected surfaces (up to homeomorphism).

$g = 1 - \chi/2$, where $\chi(\Sigma)$ = # vertices - # edges + # faces

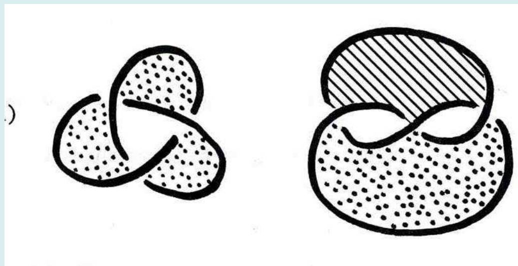
(the Euler characteristic) for any triangulation of the surface Σ.

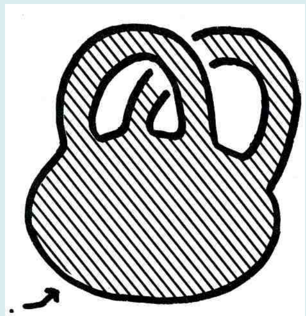
Surfaces with boundary



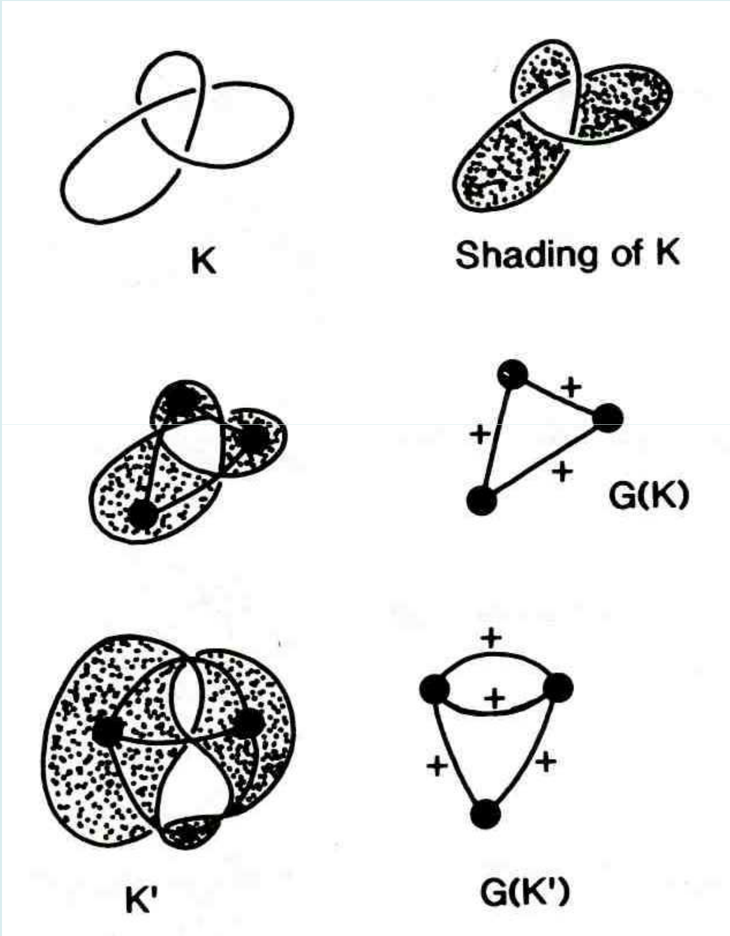


Seifert Theorem: Every knot is the boundary of an orientable surface.
(Give a proof!)





Knots and planar graphs

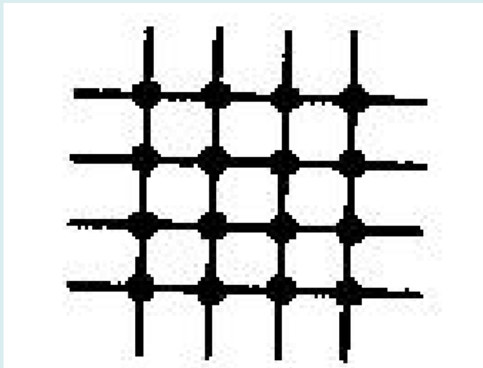


Knots and Statistical Mechanics

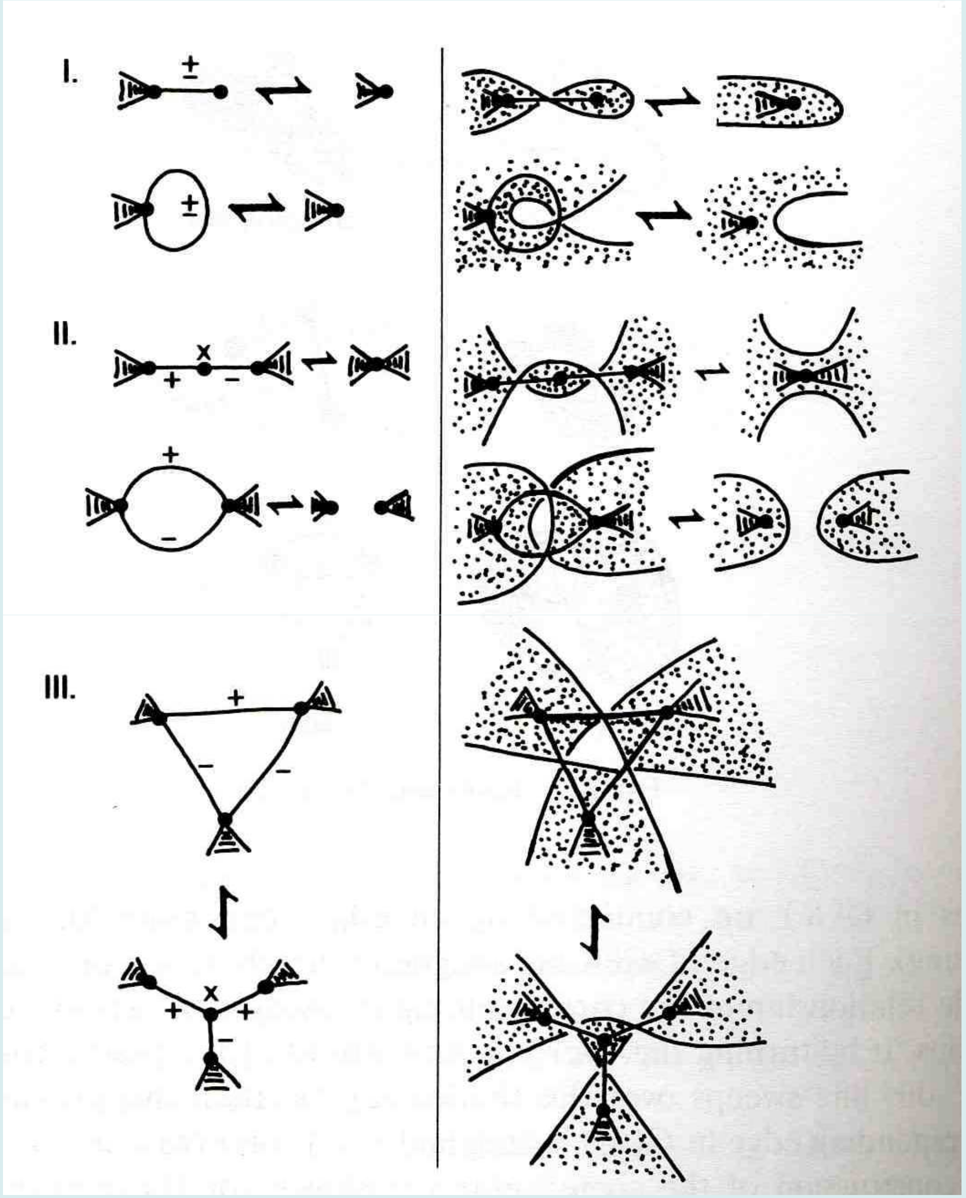
The **Potts model** (that explains the melting of the ice) has a partition function that corresponds to the Jones polynomial.

Planar graph of molecules.
Each molecule has a *spin*.

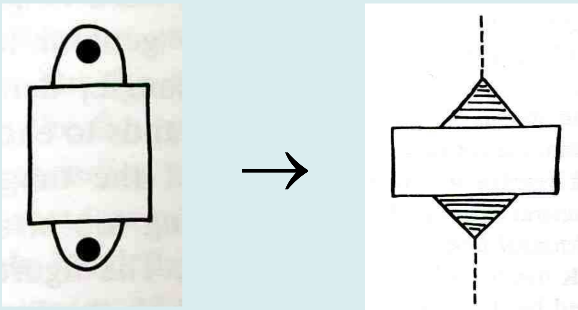
- $E(\sigma) = \Sigma\, \delta(\sigma_i,\sigma_j)$
  the energy of a state σ.
- $Z = \Sigma \exp (-E (\sigma) / KT)$
  the partition function.
- The function Z satisfies the rules of the dichromatic polynomial.

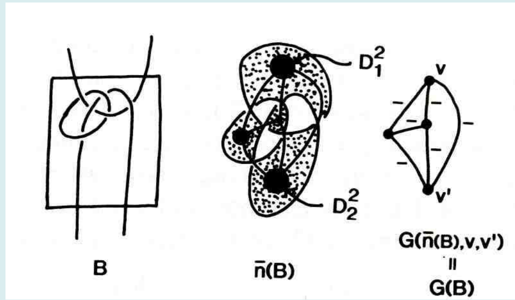


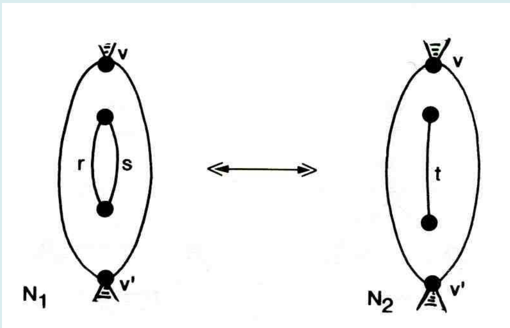
A state σ is a selection of spin for each vertex.

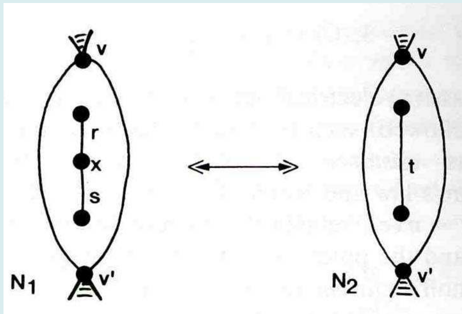

I.
±
±
II.
x
+
−
+
−
III.
+
−
−
x
+
+
−

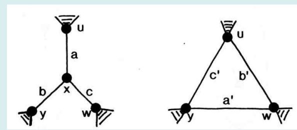
J. Goldman & L.H. Kauffman: The electrical conductance, c=1/R, is an isotopy invariant of an electrical network. (Show it!)











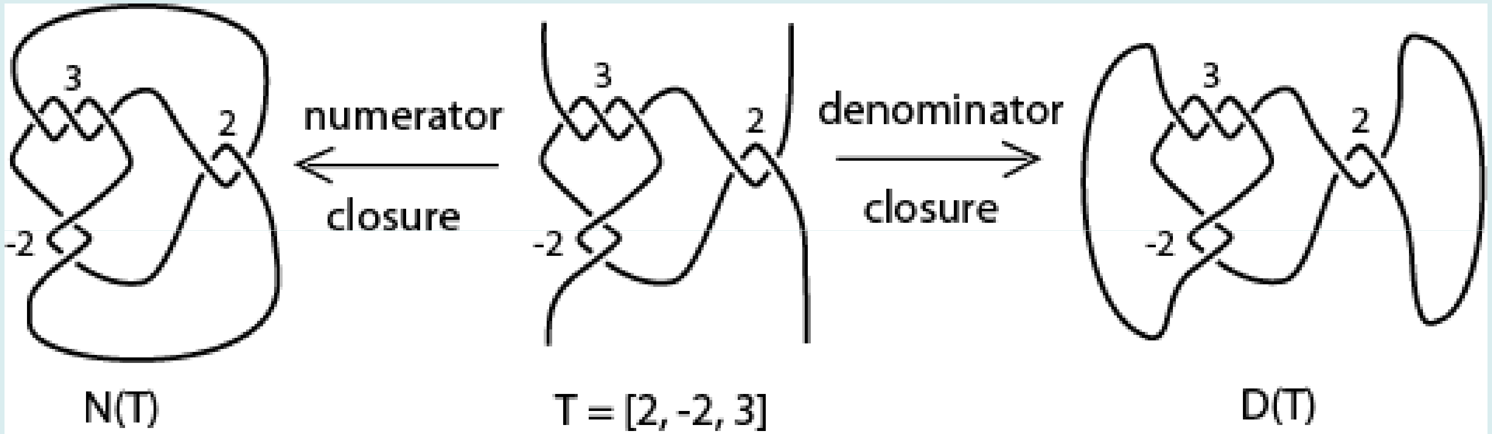
$t = r + s,$

$1/t = 1/r + 1/s$ or $t = rs/(r + s)$

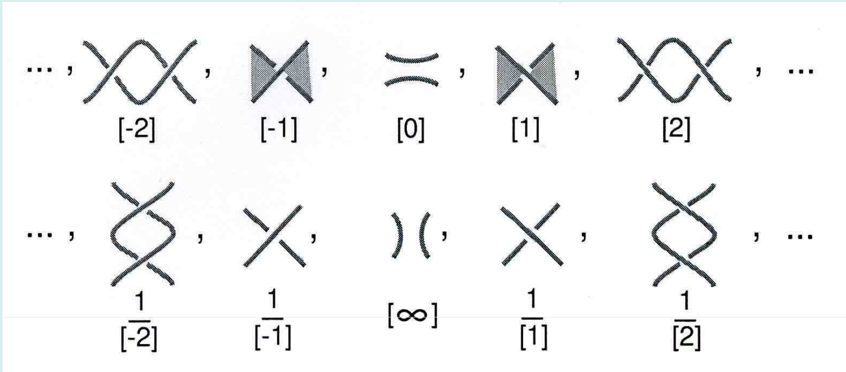
Connection in parallel connection

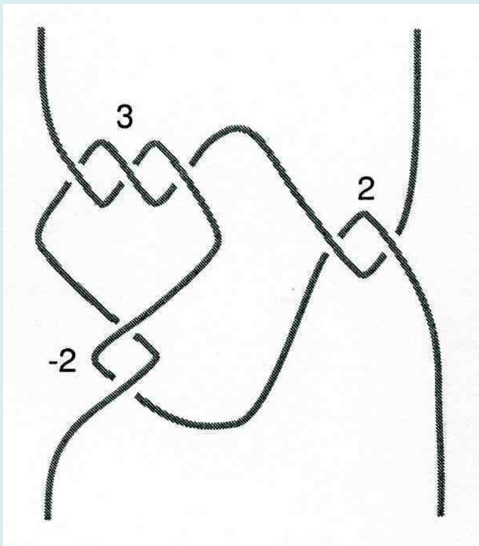
Serial

Rational tangles and rational knots

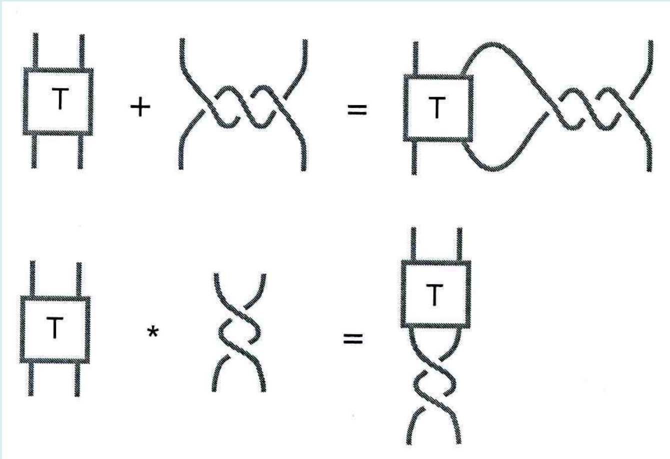


A **rational tangle** is an embedding of two arcs in a ball (with the endpoints on the boundary), which is homeomorphic to two trivial unlinked arcs. Equivalently, a rational tangle is obtained from two horizontal or two vertical arcs by consecutive twistings of neighbouring endpoints.



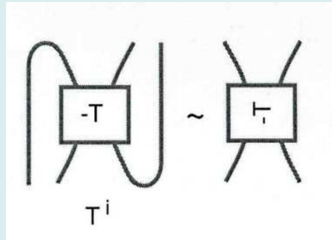


Operations:



(Addition)

(Multiplication)



(Invertion)

Lemma 1: $T * \frac{1}{[n]} = \frac{1}{[n] + \frac{1}{T}}$ *and* $\frac{1}{[n]} * T = \frac{1}{\frac{1}{T} + [n]}$.

Corollary 1: $T = [[a_1], [a_2], \ldots, [a_n]] := [a_1] + \cfrac{1}{[a_2] + \cdots + \cfrac{1}{[a_{n-1}] + \frac{1}{[a_n]}}}$

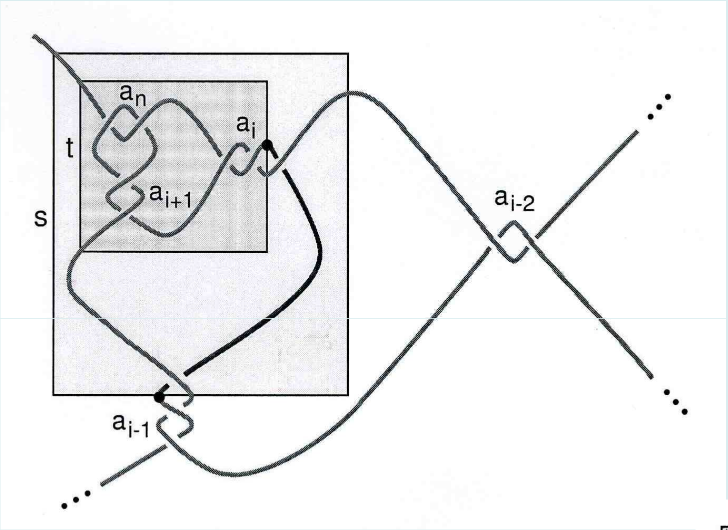
Lemma 2: Every rational number can be written (not uniquely) as a finite continued fraction. Moreover, formally, $\infty = 1/0$.

p/q= $[a_1, a_2, \ldots, a_n] := a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_{n-1} + \frac{1}{a_n}}}$

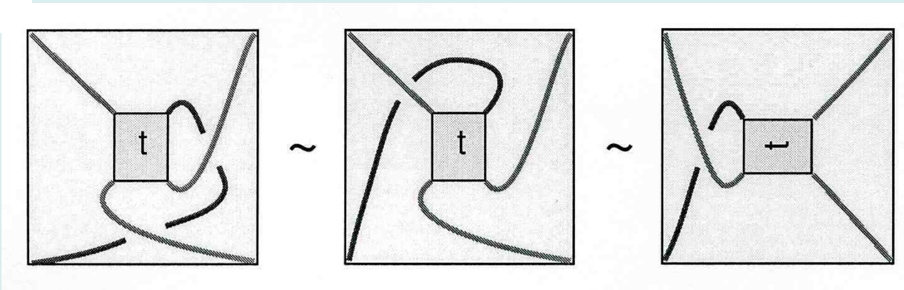
E.g. [4,-2,3] = 17/5 = [3,2,2]

Classification of rational tangles (Conway, 1970): Two rational tangles are isotopic iff they correspond to the same rational number or ∞.

Proofs: (Conway), Burde&Zieschang, Montesinos, Goldman&Kauffman (combl), S.L.&Kauffman (combl).

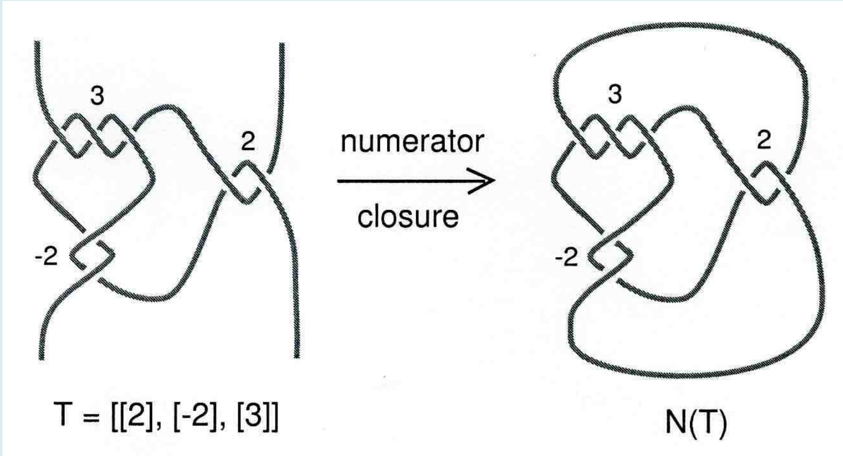


Lemma 3: Every rational tangle is alternating.



Find an alternating and a non-alternating representation for the rational tangle [15/4].

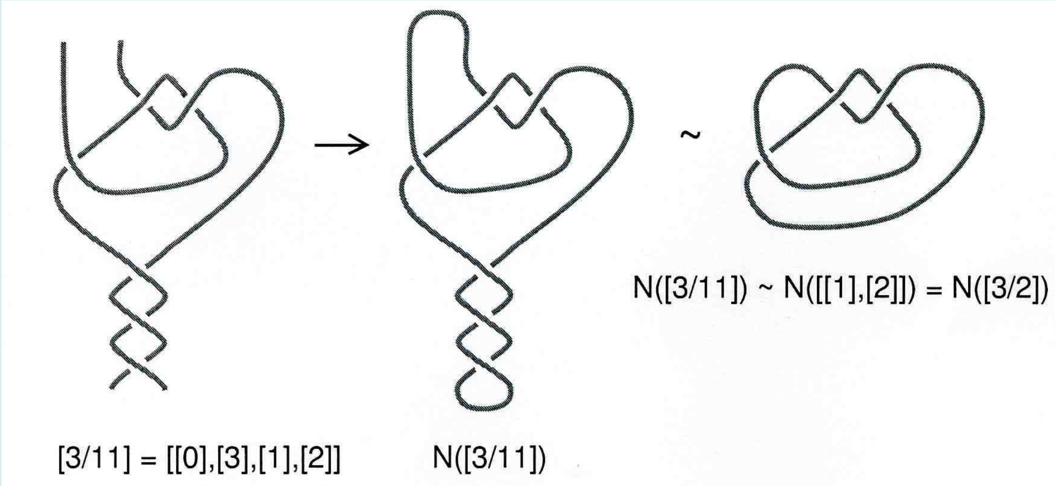
Rational knots

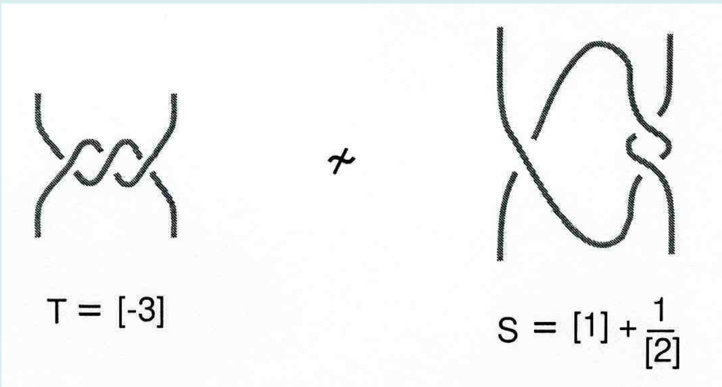


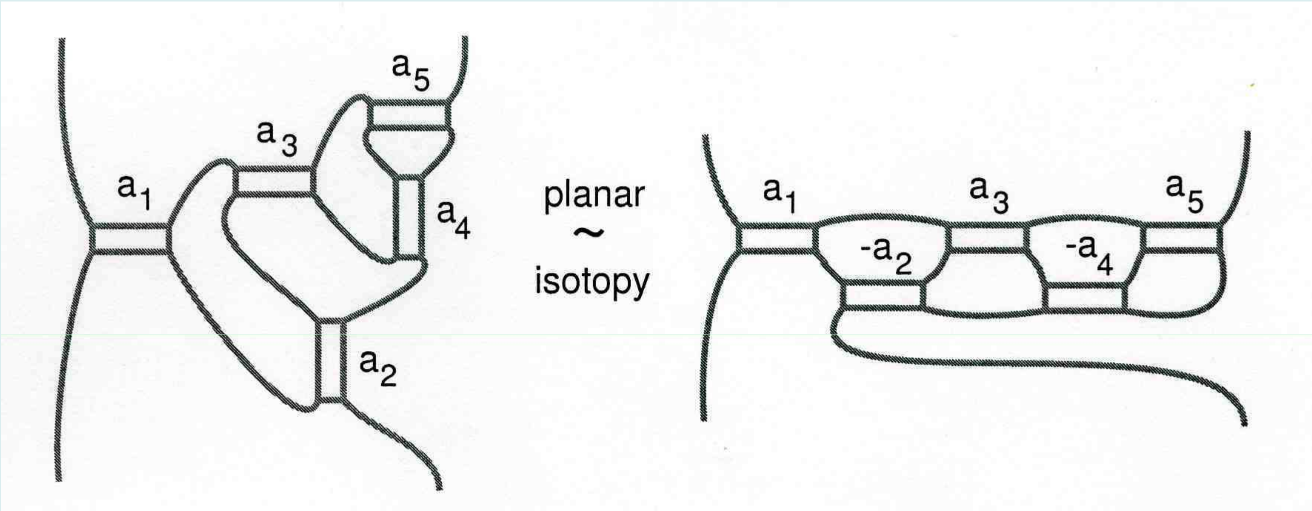
Theorem 2 (Schubert, 1956) *Suppose that rational tangles with fractions $\frac{p}{q}$ and $\frac{p'}{q'}$ are given (p and q are relatively prime. Similarly for p' and q'.) If $K(\frac{p}{q})$ and $K(\frac{p'}{q'})$ denote the corresponding rational knots obtained by taking numerator closures of these tangles, then $K(\frac{p}{q})$ and $K(\frac{p'}{q'})$ are isotopic if and only if*

1. $p = p'$ *and*
2. *either* $q \equiv q' \bmod p$ *or* $qq' \equiv 1 \bmod p$.

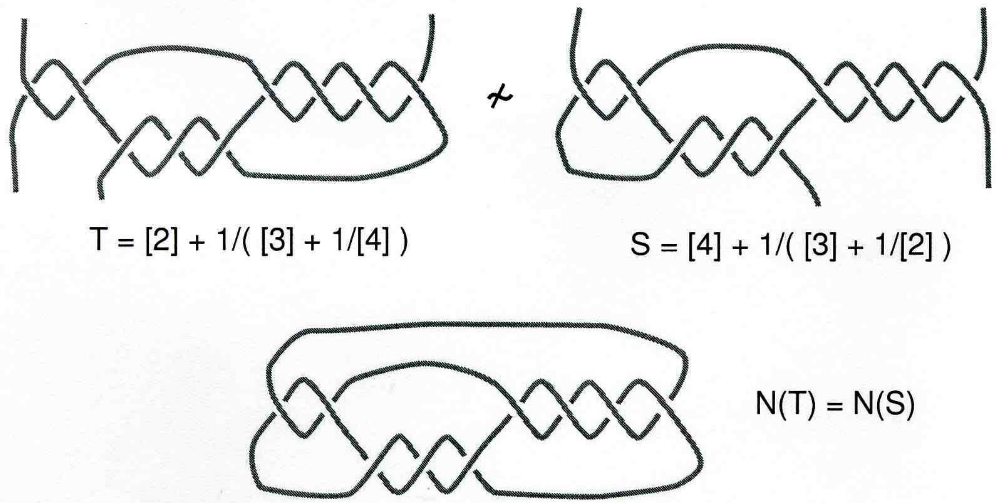
Proofs: Schubert, Burde, S.L.&Kauffman (combl).


[3/11] = [[0],[3],[1],[2]]
N([3/11])
N([3/11]) ~ N([[1],[2]]) = N([3/2])


T = [-3]
S = [1] + 1/[2]



a5
a3
a1
a4
a2
planar
~
isotopy
a1
a3
a5
-a2
-a4

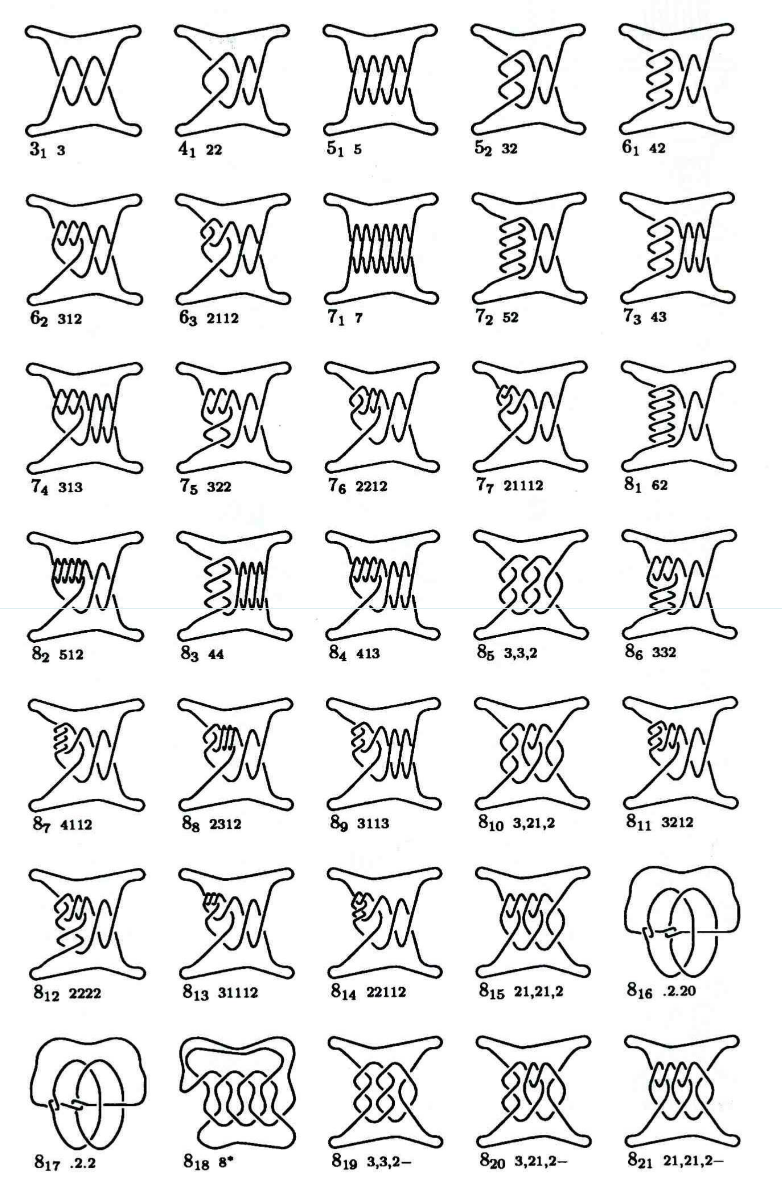


$$F(T) = 2 + \frac{1}{3+\frac{1}{4}} = \frac{30}{13} \quad \text{and} \quad F(S) = 4 + \frac{1}{3+\frac{1}{2}} = \frac{30}{7}.$$

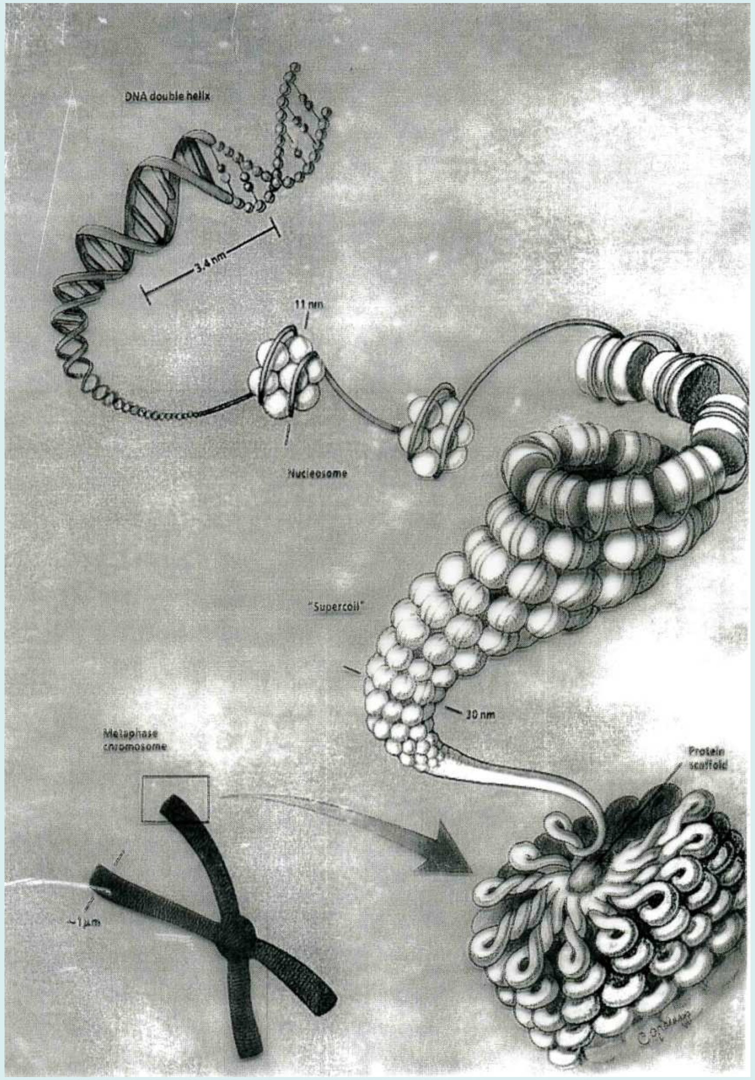
$$7 \cdot 13 \equiv 1 \, mod \, 30$$

Theorem 4 (Palindrome Theorem) *Let $\{a_1, a_2, \ldots, a_n\}$ be a collection of n non-zero integers, and let $\frac{P}{Q} = [a_1, a_2, \ldots, a_n]$ and $\frac{P'}{Q'} = [a_n, a_{n-1}, \ldots, a_1]$. Then $P = P'$ and $QQ' \equiv (-1)^{n+1} \, mod \, P$.*

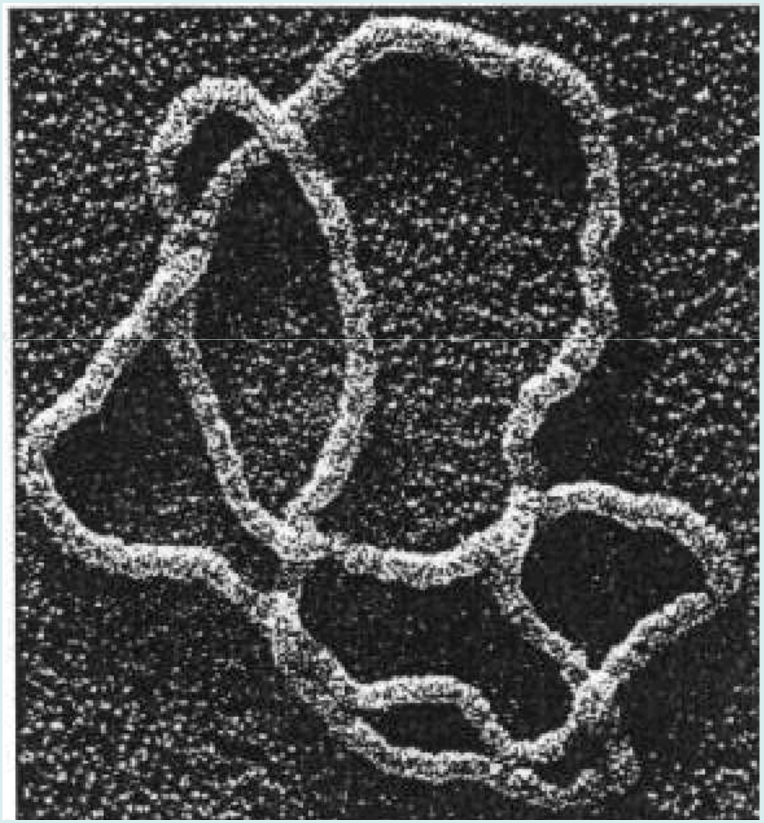
The first rational knots:



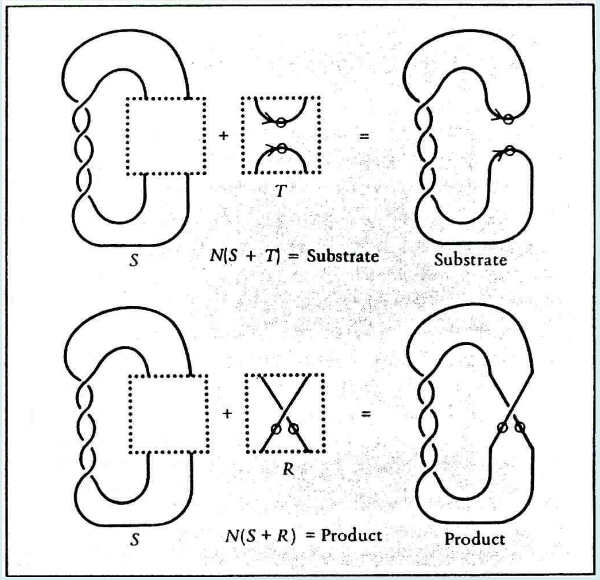
DNA Recombination



Knotted DNA



Site-specific recombination. Enzymes: TopoisomerasesTn3 Resolvase and Recombinase. We would like to understand how exactly and where exactly the enzymes act.



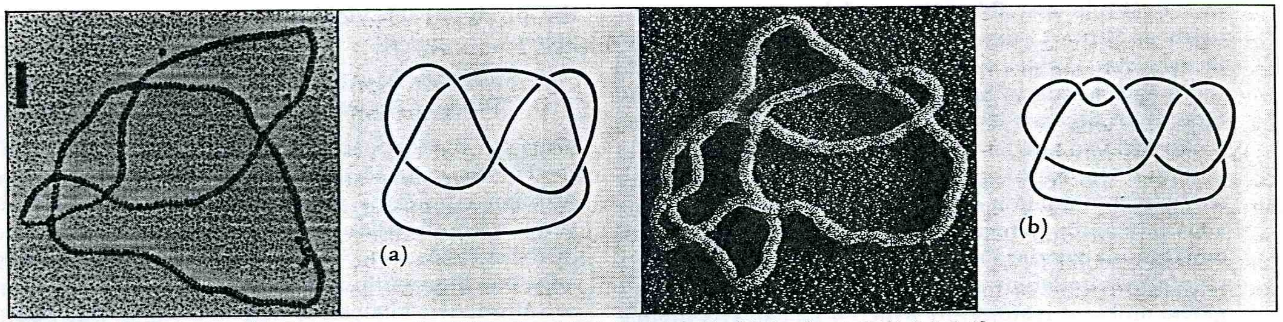
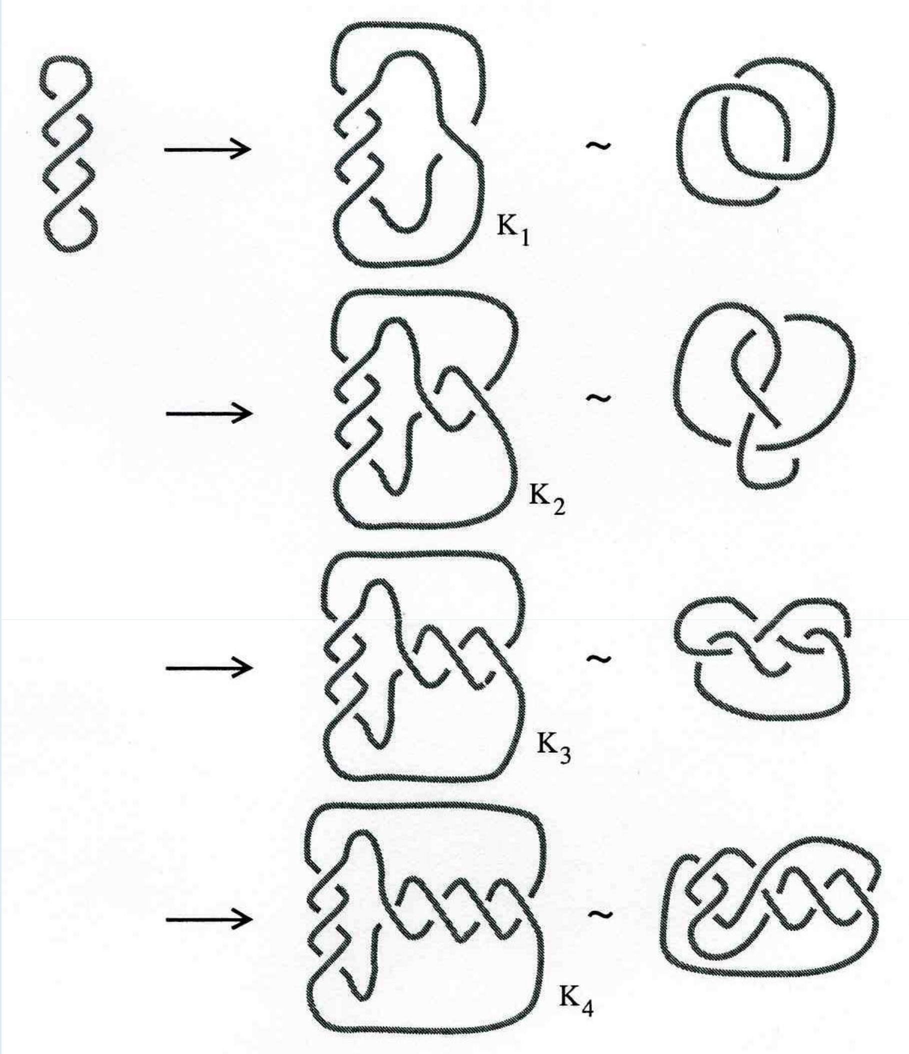


Figure 8. DNA 4-plats (Tn3) (a) shows the Whitehead link [1,1,1,1,1]; (b) the knot 6_2 [1,2,1,1,1].

Processive Recombination



Tangle Model: Ernst & Sumners, 1989

Tangle Model: Ernst & Sumners, 1989

THEOREM 1: *Suppose that tangles S, T, and R satisfy the following equations: (i) N(S + T) = [1] (the unknot); (ii) N(S + R) = [2] (the Hopf link); (iii) N(S + R+ R) = [2,1,1] (the figure-8 knot). Then {S,R} = {(−3,0),(1)}, {(3,0),(−1)}, {(−2,−3,−1),(1)}, or {(2,3,1),(−1)}.*

THEOREM 2: *Suppose that tangles S, T, and R satisfy the following equations: (i) N(S + T) = [1] (the unknot); (ii) N(S + R) = [2] (the Hopf link); (iii) N(S + R + R) = [2,1,1] (the figure-8 knot); (iv) N(S + R + R + R) = [1,1,1,1,1] (the Whitehead link). Then, S = (−3,0), R = (1), and N(S + R + R + R + R) = [1,2,1,1,1].*

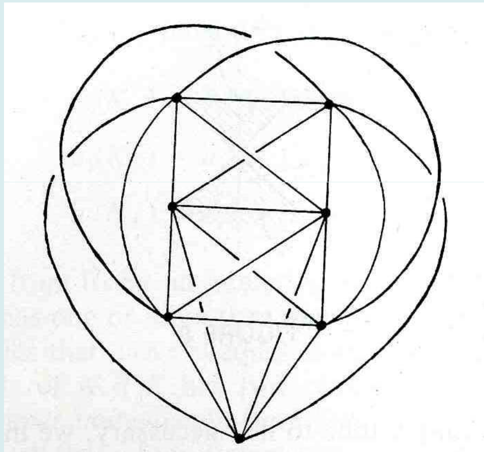
For such experiments Ernst and Sumners [12] used the classification of rational knots in the unoriented case, as well as results of Culler, Gordon, Luecke and Shalen [9] on Dehn surgery to prove that the solutions $S + nR$ must be *rational tangles*. These results of Culler, Gordon, Luecke and Shalen tell the topologist under what circumstances a three-manifold with cyclic fundamental group must be a lens space. By showing when the 2-fold branched covers of the DNA knots must be lens spaces, the recombination problems are reduced to the consideration of rational knots. This is a deep application of the three-manifold approach to rational knots and their generalizations.

Knots and graphs in space

Let K_n be the complete graph on n vertices.

An **embedding** of K_n in 3-space is a homeomorphic image.

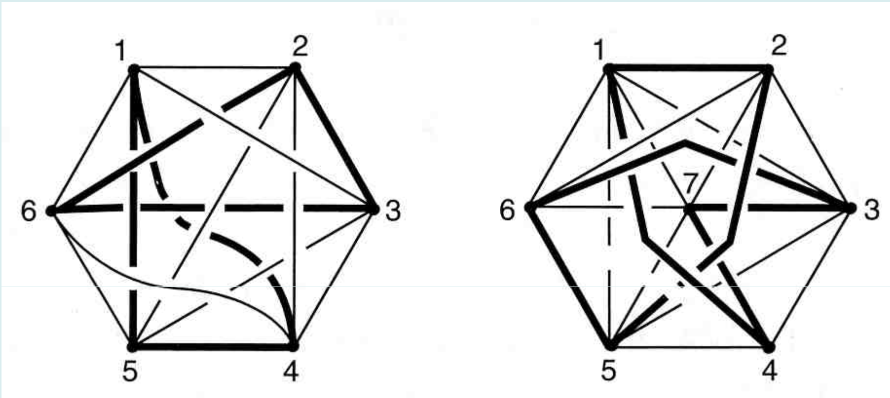
E.g. for n=7:



Theorem (Conway & Gordon, 1977, 1983):

i) Every embedding of K_6 contains a non-trivial link.

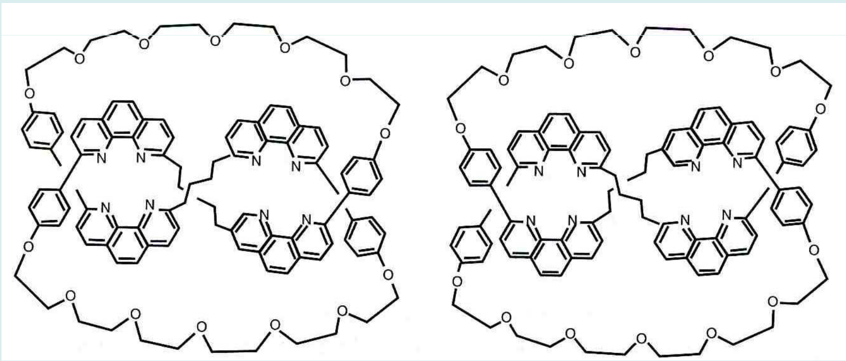
ii) Every embedding of K_7 contains a non-trivial knot (a Hamilton cycle).



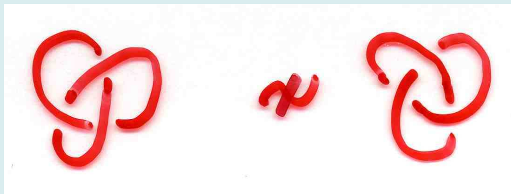
Idea: For (i) use the linking number. For (ii): Two embeddings of a graph Γ will differ by isotopy and possibly by crossing switches. So, we are looking for a knot invariant that does not change under crossing switches. Such an example is the **arf invariant**. We then show that there exists an embedding of K^7 with non-trivial sum of arf invariants, summing over all cycles in the graph.

Knots in Chemistry (Polymers)

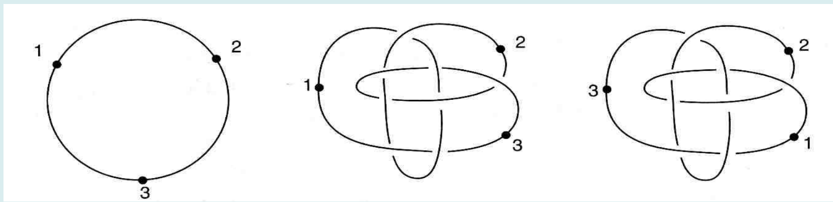
For many molecules chemists are interested in constructing in the lab other molecules, the structure of which is related to the structure of the original molecule, but has better physical properties. **Topological isomers** comprise such an example. These are molecules with the same abstract graph but, as embedded graphs, one cannot be isotoped to the other. So, it is important to decide theoretically if two molecules with the same abstract graph are not topologically equivalent.



A pair of topological isomers



Another example:



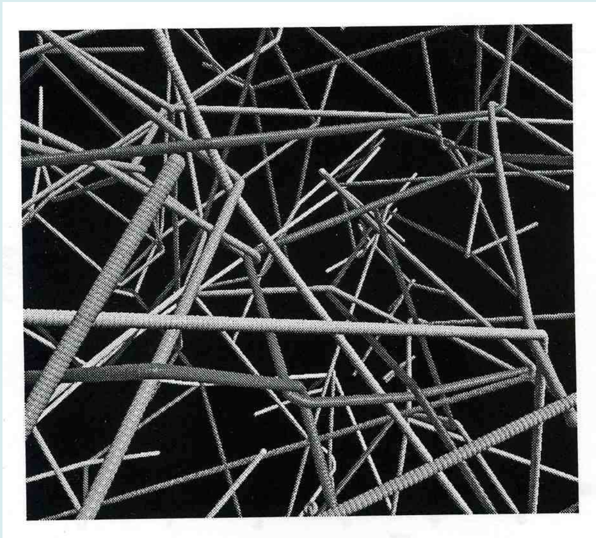
E. Flapan, J. Simon,….

Future applications

Topological Analysis of Linear Polymer Melts

Christos Tzoumanekas* and Doros N. Theodorou†
Department of Materials Science and Engineering, School of Chemical Engineering, National Technical University of Athens, Zografou Campus, 15780 Athens, Greece and Dutch Polymer Institute (DPI), The Netherlands
(Dated: 21st September 2005)

We introduce an algorithm for the reduction of computer generated atomistic polymer samples to networks of primitive paths. By examining network ensembles of Polyethylene and cis-1,4 Polybutadiene melts, we quantify the underlying topologies through the radial distribution function of entanglements and the distribution of the number of monomers between entanglements. A suitable scaling of acquired data leads to a unifying microscopic topological description of both melts.



Quantum Gravity and the Standard Model

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Fotini Markopoulou† and Lee Smolin‡
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and
Department of Physics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada
(Dated: February 28, 2006)

We show that a class of background independent models of quantum spacetime have local excitations that can be mapped to the first generation fermions of the standard model of particle physics. These states propagate coherently as they can be shown to be noiseless subsystems of the microscopic quantum dynamics[1]. These are identified in terms of certain patterns of braiding of graphs, thus giving a quantum gravitational foundation for the topological preon model proposed in [12].

These results apply to a large class of theories in which the Hilbert space has a basis of states given by ribbon graphs embedded in a three-dimensional manifold up to diffeomorphisms, and the dynamics is given by local moves on the graphs, such as arise in the representation theory of quantum groups. For such models, matter appears to be already included in the microscopic kinematics and dynamics.

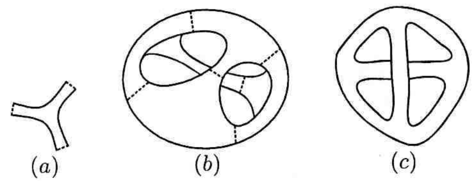


FIG. 1: (a) A trinion. (b) A trinion decomposition of a 2-surface $\mathcal{S}$ is a ribbon graph Γ.

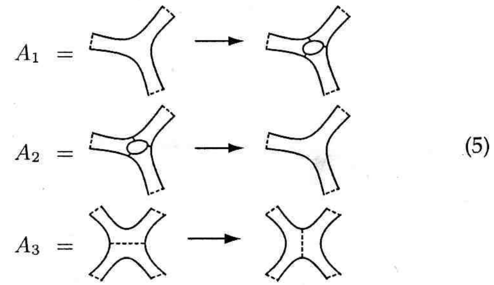


FIG. 2: The three generators of evolution on the ribbon graph space $\mathcal{H}$. They are called expansion, contraction and exchange moves.

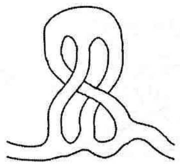
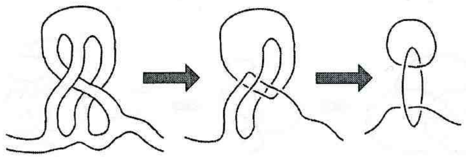


FIG. 5: A simple braid inside a ribbon graph



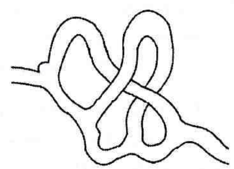
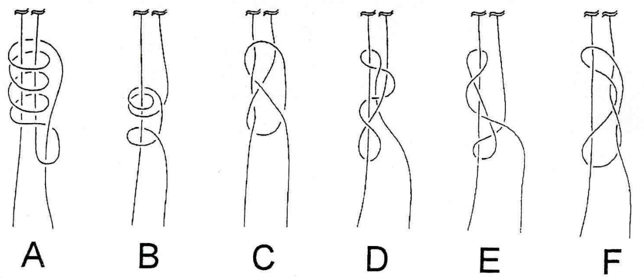


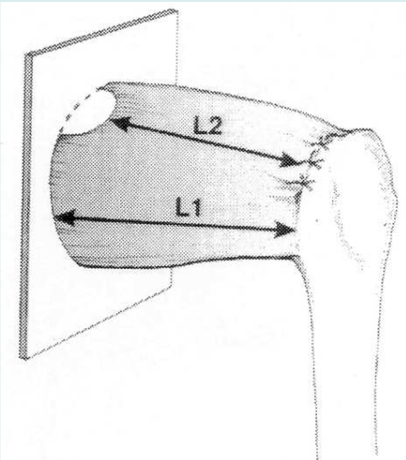
FIG. 7: A possible evolution of the braid under an exchange move.

Arthroscopic Knots: Determining the Optimal Balance of Loop Security and Knot Security

Ian K. Y. Lo, M.D., F.R.C.S.C., Stephen S. Burkhart, M.D., K. Casey Chan, M.D., and Kyriacos Athanasiou, Ph.D., P.E.



'IGURE 1. Sliding knot configurations evaluated. (A) Duncan loop. (B) Nicky's knot. (C) Tennessee slider. (D) Roeder knot. (E) SMC knot. F) Weston knot.



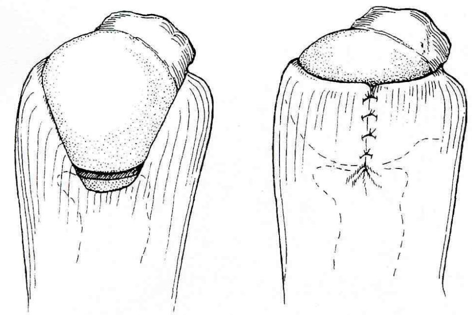


FIG 3. (A) U-shaped rotator cuff tear. (B) Partial side-to-side repair causes a "margin convergence" of the tear toward the greater tuberosity. This increases the cross-sectional area and decreases the length of the tear, thereby decreasing strain (since elongation is directly proportional to the length of the tear and inversely proportional to the cross-sectional area).

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