The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA International Atomic Energy Agency

2034-2

Advanced School and Conference on Knot Theory and its Applications to Physics and Biology

11 - 29 May 2009

KNOTS
From combinatorics of knot diagrams to combinatorial topology based on knots

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From combinatorics of knot diagrams to combinatorial topology based on knots

Warszawa, November 30, 1984 – Bethesda, March 3, 2007

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e-print: http://arxiv.org/pdf/math.GT/0512630

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Introduction

This book is about classical Knot Theory, that is, about the position of a circle (a knot) or of a number of disjoint circles (a link) in the space R^3 or in the sphere S^3. We also venture into Knot Theory in general 3-dimensional manifolds.

The book has its predecessor in Lecture Notes on Knot Theory, which were published in Polish[1] in 1995 [P-18]. A rough translation of the Notes (by J.Wiśniewski) was ready by the summer of 1995. It differed from the Polish edition with the addition of the full proof of Reidemeister's theorem. While I couldn't find time to refine the translation and prepare the final manuscript, I was adding new material and rewriting existing chapters. In this way I created a new book based on the Polish Lecture Notes but expanded 3-fold. Only the first part of Chapter III (formerly Chapter II), on Conway's algebras is essentially unchanged from the Polish book and is based on preprints [P-1].

As to the origin of the Lecture Notes, I was teaching an advanced course in theory of 3-manifolds and Knot Theory at Warsaw University and it was only natural to write down my talks (see Introduction to (Polish) Lecture Notes).

...
SEE Introduction before CHAPTER I.

[1]The Polish edition was prepared for the "Knot Theory" mini-semester at the Stefan Banach Center, Warsaw, Poland, July-August, 1995.

Chapter II

History of Knot Theory

Abstract. Leibniz wrote in 1679: "I consider that we need yet another kind of analysis, ... which deals directly with position." He called it "geometry of position"(geometria situs). The first convincing example of geometria situs was Euler's solution to the bridges of Königsberg problem (1735). The first mathematical paper which mentions knots was written by A. T. Vandermonde in 1771: "Remarques sur les problemes de situation". We sketch in this chapter the history of knot theory from Vandermonde to Jones stressing the combinatorial aspect of the theory that is so visible in Jones type invariants. In the first section we outline some older developments leading to modern knot theory.

> *"When Alexander reached Gordium, he was seized with a longing to ascend to the acropolis, where the palace of Gordius and his son Midas was situated, and to see Gordius' waggon and the knot of the waggon's yoke.... Over and above this there was a legend about the waggon, that anyone who untied the knot of the yoke would rule Asia. The knot was of cornel bark, and you could not see where it began or ended. Alexander was unable to find how to untie the knot but unwilling to leave it tied, in case this caused a disturbance among the masses; some say that he struck it with his sword, cut the knot, and said it was now untied - but Aristobulus says that he took out the pole-pin, a bolt driven right through the pole, holding the knot together, and so removed the yoke from the pole. I cannot say with confidence what Alexander actually did about this knot, but he and his suite certainly left the*

> *waggon with the impression that the oracle about the undoing of the knot had been fulfilled, and in fact that night there was thunder and lightning, a further sign from heaven; so Alexander in thanksgiving offered sacrifice next day to whatever gods had shown the signs and the way to undo the knot."* [Lucius Flavius Arrianus, Anabasis Alexandri, Book II, c.150 A.D.,[Arr]]

Similar account, clearly based on the same primary sources is given by Plutarch of Chareonera (c. 46 - 122 A.D.). He writes in his "Lives" [Plu] (page 271):

> *"Next he marched into Pisidia where he subdued any resistance which he encountered, and then made himself master of Phyrgia. When he captured Gordium [in March 333 B.C.] which is reputed to have been the home of the ancient king Midas, he saw the celebrated chariot which was fastened to its yoke by the bark of the cornel-tree, and heard the legend which was believed by all barbarians, that the fates had decreed that the man who untied the knot was destined to become the ruler of the whole world. According to most writers the fastenings were so elaborately intertwined and coiled upon one another that their ends were hidden: in consequence Alexander did not know what to do, and in the end loosened the knot by cutting through it with his sword, whereupon the many ends sprang into view. But according to Aristobulus he unfastened it quite easily by removing the pin which secured the yoke to the pole of the chariot, and then pulling out the yoke itself."*

In this chapter we present the history of ideas which lead up to the development of modern knot theory. We are more detailed when pre-XX century history is reported. With more recent times we are more selective, stressing developments related to Jones type invariants of links. Additional historical information on specific topics of Knot Theory is given in other chapters of the book[1].

Knots have fascinated people from the dawn of the human history. We can wonder what caused a merchant living about 1700 BC. in Anatolia and exchanging goods with Mesopotamians, to choose braids and knots as his seal sign; Fig.1.1. We can guess however that stamps, cylinders and seals with knots and links as their motifs appeared before proper writing was invented about 3500 BC.

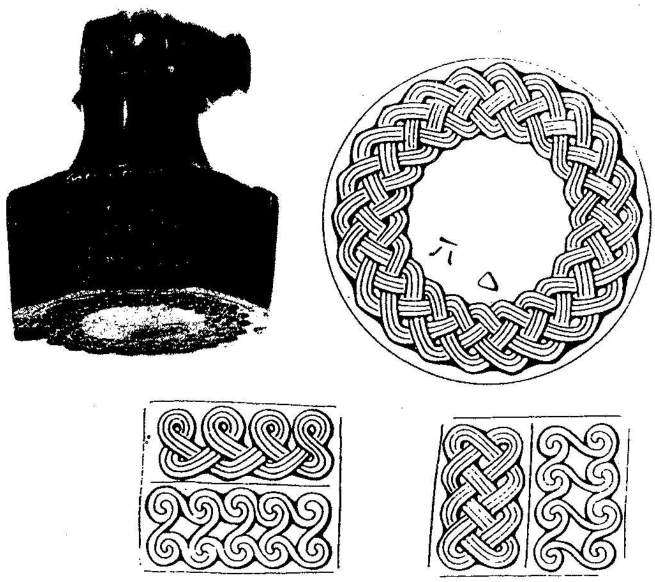

Figure 1.1 Stamp seal, about 1700 BC (the British Museum).

On the octagonal base [of hammer-handled haematite seal] are patterns surrounding a hieroglyphic inscription (largely erased). Four of the sides are blank and the other four are engraved with elaborate patterns typical of the period (and also popular in Syria) alternating with cult scenes...([Col], p.93).

[1]There are books which treat the history of topics related to knot theory [B-L-W, Ch-M, Crowe, Die, T-G]. J.Stillwell's textbook [Stil] contains very interesting historical digressions.

I am unaware of any pre-3500 BC examples but I will describe two finds from the third millennium BC.

The oldest examples outside Mesopotamia, that I am aware of, are from the pre-Hellenic Greece. Excavations at Lerna by the American School of Classical Studies under the direction of Professor J. L. Caskey (1952-1958) discovered two rich deposits of clay seal-impressions. The second deposit dated from about 2200 BC contains several impressions of knots and links[2] [Hig, Hea, Wie] (see Fig.1.2).

Fig. 1.2; A seal-impression from the House of the Tiles in Lerna [Hig].

[2]The early Bronze Age in Greece is divided, as in Crete and the Cyclades, into three phases. The second phase lasted from 2500 to 2200 BC, and was marked by a considerable increase in prosperity. There were palaces at Lerna, and Tiryns, and probably elsewhere, in contact with the Second City of Troy. The end of this phase (in the Peloponnese) was brought about by invasion and mass burnings. The invaders are thought to be the first speakers of the Greek language to arrive in Greece.

Even older example of cylinder seal impression (c. 2600-2500 B.C.) from Ur, Mesopotamia is described in the book Innana by Diane Wolkstein and Samuel Noah Kramer [Wo-Kr] (page 7); Figure 1.3, illustrating the text:

"Then a serpent who could not be charmed made its nest in the roots of the tree."

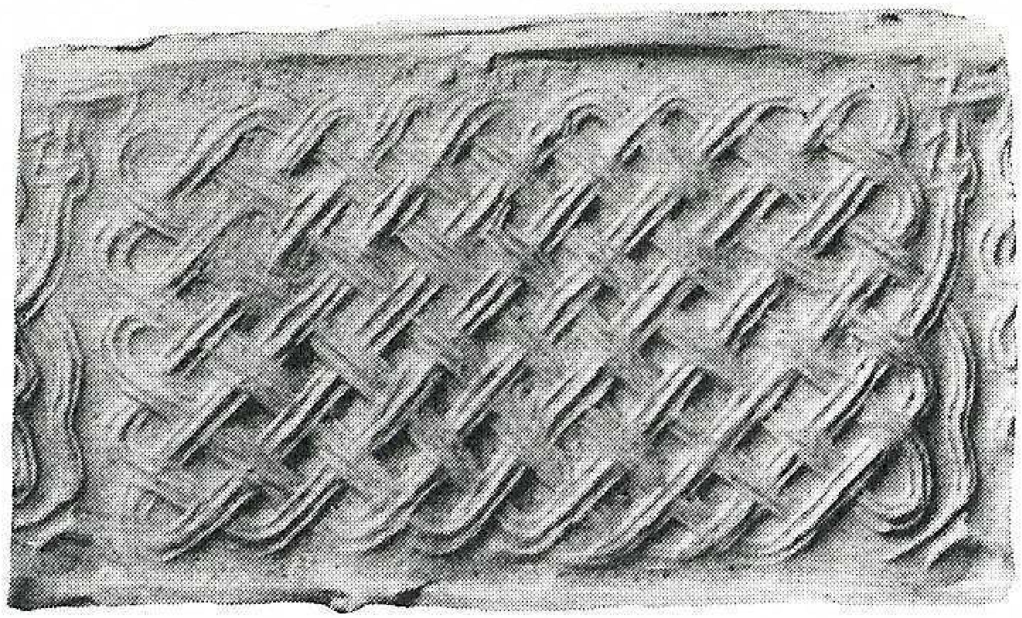

Fig. 1.3; *Snake with Interlacing Coil*

Cylinder seal. Ur, Mesopotamia. The Royal Cemetery, Early Dynastic period, c. 2600-2500 B.C. Lapis lazuli. Iraq Museum. Photograph courtesy of the British Museum, UI 9080, [Wo-Kr][3]

II.1 From Heraclas to Dürer

It is tempting to look for the origin of knot theory in Ancient Greek mathematics (if not earlier). There is some justification to do so: a Greek physician

[3]On the pages 179-180 they comment: The majority of the pictorial surface is covered with the inter- twined coils of a serpent, forming a lattice pattern. To the right its tail appears below the coils and its head above, with a bird perched upon it. Two snakes intertwined rather than one are shown on earlier representations of this motif. Snakes twist themselves together in this fashion when mating, suggesting this symbol's association with fertility.

named Heraklas, who lived during the first century A.D. and who was likely a pupil or associate of Heliodorus, wrote an essay on surgeon's slings[4][5]. Heraklas explains, giving step-by-step instructions, eighteen ways to tie orthopedic slings. His work survived because Oribasius of Pergamum (ca. 325-400; physician of the emperor Julian the Apostate) included it toward the end of the fourth century in his "Medical Collections". The oldest extant manuscript of "Medical Collections" was made in the tenth century by the Byzantine physician Nicetas. The Codex of Nicetas was brought to Italy in the fifteenth century by an eminent Greek scholar, J. Lascaris, a refugee from Constantinople. Heraklas' part of the Codex of Nicetas has no illustrations, and around 1500 an anonymous artist depicted Heraklas' knots in one of the Greek manuscripts of Oribasus "Medical Collections" (in Figure 2.1 we reproduce the drawing of the third Heraklas knot together with its original, Heraklas', description). Vidus Vidius (1500-1569), a Florentine who became physician to Francis I (king of France, 1515-1547) and professor of medicine in the Collège de France, translated the Codex of Nicetas into Latin; it contains also drawings of Heraklas' surgeon's slings by the Italian painter, sculptor and architect Francesco Primaticcio (1504-1570); [Da, Ra].

[4]Heliodorus, who lived at the time of Trajan (Roman Emperor 98–117 A.D.), also mentions in his work knots and loops [Sar]

[5]Hippocrates of Cos (c.460 - 375 B.C.) in his collection of notes: In the surgery; De officina medici; Cat' iētreion, deals with bandaging. Thessalos, Hippocrates' son, has been named also as the author. A commentary on the Hippocratic treatise on *Joints* was written by Apollonios of Citon (in Cypros), who flourished in Alexandria in the first half of the first century B.C. That commentary has obtained a great importance because of an accident in its transmition. A manuscript of it in Florence (Codex Laurentianus) is a Byzantine copy of the ninth century, including surgical illustrations (for example, with reference to reposition methods), which might go back to the time of Apollonios and even Hippocrates. Iconographic tradition of this kind are very rare, because the copying of figures was far more difficult than the writing of the text and was often abandoned [Sar]. The story of the illustrations to Apollonios' commentary is described in [Sar-2].

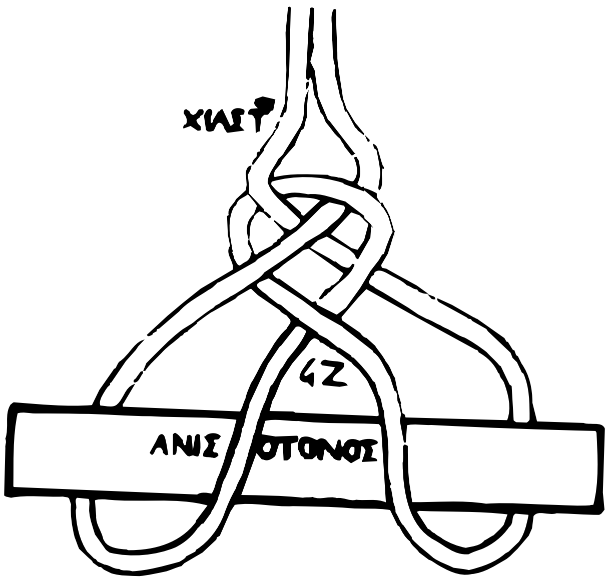

Figure 2.1; The crossed noose

"For the tying the crossed noose, a cord, folded double, is procured, and the ends of the cord are held in the left hand, and the loop is held in the right hand. Then the loop is twisted so that the slack parts of the cord crossed. Hence the noose is called crossed. After the slack parts of the cord have been crossed, the loop is placed on the crossing, and the lower slack part of the cord is pulled up through the middle of the loop. Thus the knot of the noose is in the middle, with a loop on one side and two ends on the other. This likewise, in function, is a noose of unequal tension"; [Da].

Heraklas' essay should be taken seriously as far as knot theory is concerned even if it is not knot theory proper but rather its application. The story of the survival of Heraklas' work; and efforts to reconstruct his knots in Renaissance is typical of all science disciplines and efforts to recover lost Greek books provided the important engine for development of modern science. This was true in Mathematics as well: the beginning of modern calculus in XVII century can be traced to efforts of reconstructing lost books of Archimedes and other ancient Greek mathematicians. It was only the work of Newton and Leibniz which went much farther than their Greek predecessors.

There are two enlighting examples of great Renaissance artists interest in knots: Engravings by Leonardo da Vinci[6] (1452-1519) [Mac]

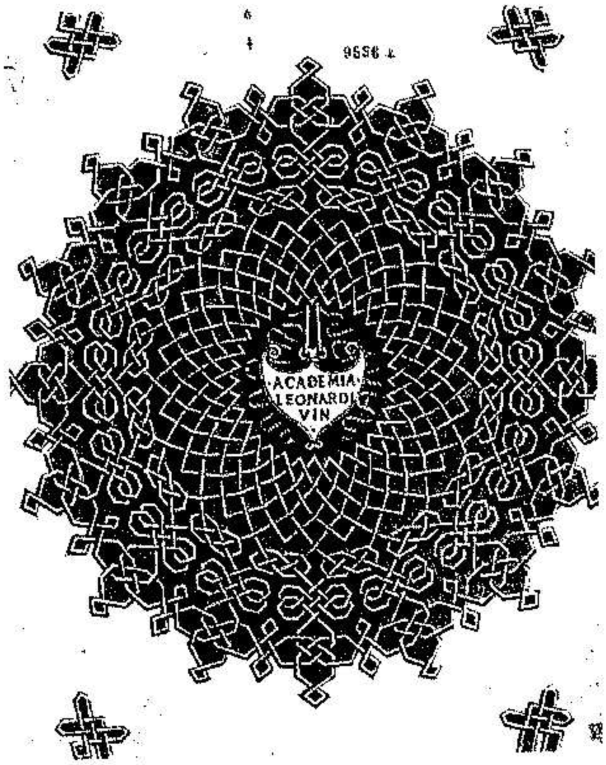

Fig. 2.2; A knot by Leonardo [Mac]; c. 1496

and woodcuts by Albrecht Dürer[7] (1471-1528) [Dur-1, Ha], Fig.2.3.

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[6]Giorgio Vasari writes in [Va]: "[Leonardo da Vinci] spent much time in making a regular design of a series of knots so that the cord may be traced from one end to the other, the whole filling a round space. There is a fine engraving of this most difficult design, and in the middle are the words: Leonardus Vinci Academia."

[7]"Another great artist with whose works Dürer now became acquainted was Leonardo da Vinci. It does not seem likely that the two artists ever met, but he may have been brought into relation with him through Luca Pacioli, the author of the book De Divina Proportione, which appeared at Venice in 1509, and an intimate friend of the great Leonardo. Dürer would naturally be deeply interested in the proportion theories of Leonardo and Pacioli. He was certainly acquainted with some engravings of Leonardo's school, representing a curious circle of concentric scrollwork on a black ground, one of them entitled Accademia Leonardi Vinci; for he himself executed six woodcuts in imitation, the Six Knots, as he calls them himself. Dürer was amused by and interested in all scientific or mathematical problems..." From: http://www.cwru.edu/edocs/7/258.pdf, compare [Dur-2].

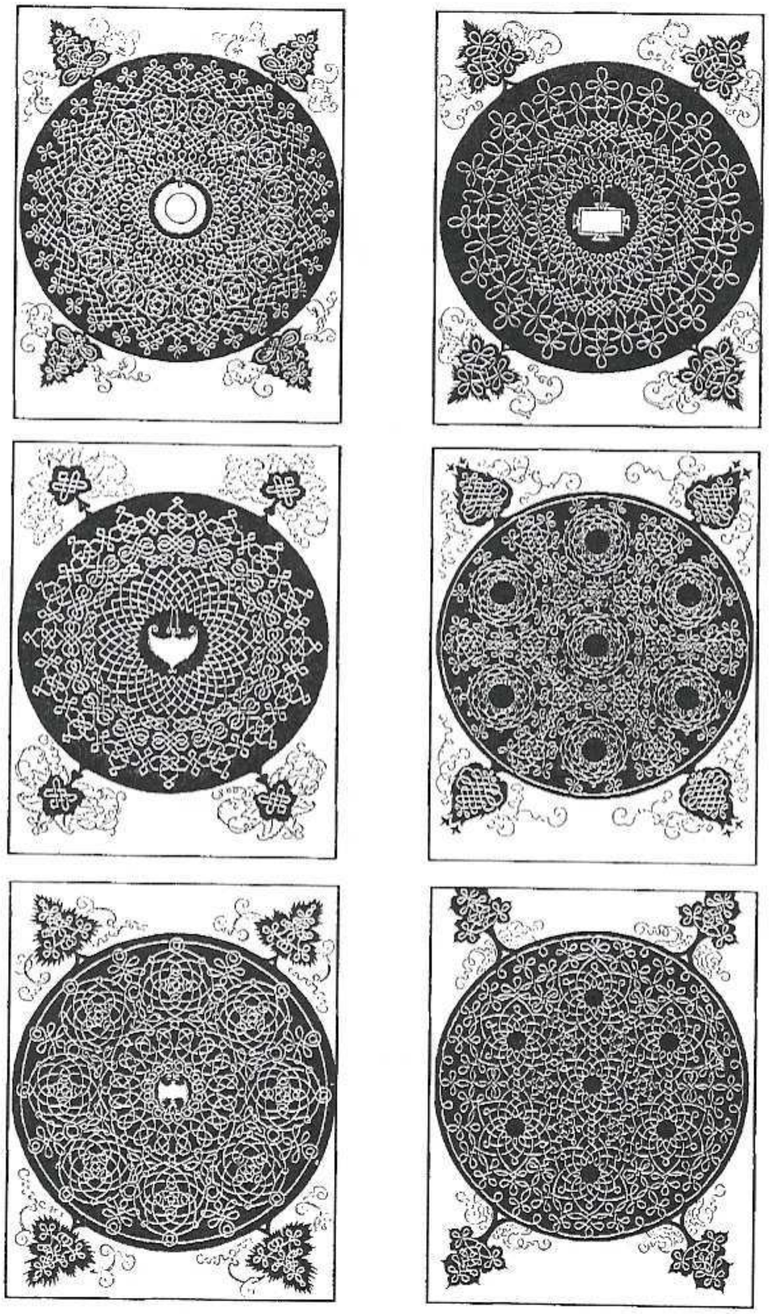


Fig. 2.3; Six knots by Dürer [Kur]; c. 1505-1507

II.2 Dawn of Knot Theory

We would argue, that modern knot theory has its roots with Gottfried Wilhelm Leibniz (1646-1716) speculation that aside from calculus and analytical geometry there should exist a "geometry of position" (geometria situs) which deals with relations depending on position alone (ignoring magnitudes). In a letter to Christian Huygens (1629-1695), written in 1679 [Lei], he declared: "I am not content with algebra, in that it yields neither the shortest proofs nor the most beautiful constructions of geometry. Consequently, in view of this, I consider that we need yet another kind of analysis, geometric or linear, which deals directly with position, as algebra deals with magnitude".

I do not know whether Leibniz had any convincing example of a problem belonging to the geometry of position. According to [Kli]:
"As far back as 1679 Leibniz, in his Characteristica Geometrica, tried to formulate basic geometric properties of geometrical figures, to use special symbols to represent them, and to combine these properties under operations so as to produce others. He called this study analysis situs or geometria situs... To the extent that he was at all clear, Leibniz envisioned what we now call combinatorial topology".

The first convincing example of geometria situs was studied by Leonard Euler (1707-1783). This concerns the bridges on the river Pregel at Königsberg (then in East Prussia)[8]. Euler solved (and generalized) the bridges of Königsberg problem and on August 26, 1735 presented his solution to the Russian Academy at St. Petersburg (it was published in 1736), [Eu]. With the Euler paper, graph theory and topology were born. Euler started his paper by remarking:
"The branch of geometry that deals with magnitudes has been zealously studied throughout the past, but there is another branch that has been almost unknown up to now; Leibniz spoke of it first, calling it the "geometry

[8]Euler never visited Königsberg. He was informed about the puzzle of bridges of Königsberg (and about a possible relation to Leibniz geometria situs) by future mayor of Danzig (Gdańsk) Carl Leonhard Gottlieb Ehler (1685-1753); there are 14 surviving letters from Ehler to Euler; the first is dated April 8, 1735. Ehler in turn was acting on behalf of Danzig mathematician Heinrich Kuhn (1690-1769) [H-W]. Kuhn was born in Königsberg, he studied at the Pedagogicum there, ... in 1733 he settled in Danzig. One should add that Kuhn was the first person to suggest geometric interpretation of complex numbers [Jan].

of position" (geometria situs). This branch of geometry deals with relations dependent on position; it does not take magnitudes into considerations, nor does it involve calculation with quantities. But as yet no satisfactory definition has been given of the problems that belong to this geometry of position or of the method to be used in solving them".

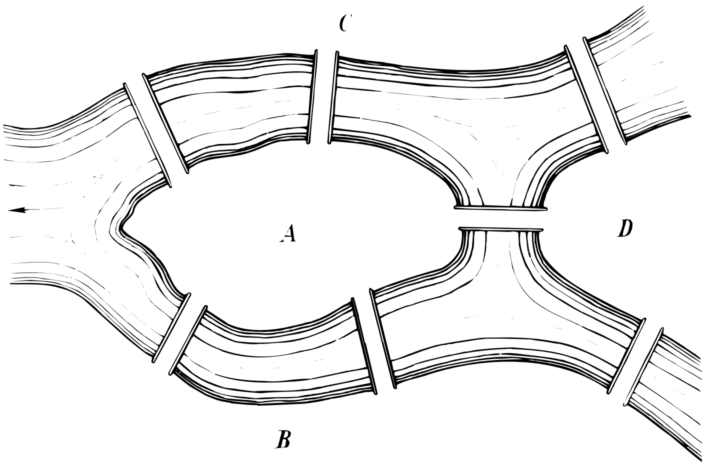


Figure 3.1; Bridges of Königsberg

For the birth of knot theory one had to wait another 35 years. In 1771 Alexandre-Theophile Vandermonde (1735-1796) wrote the paper: *Remarques sur les problèmes de situation* (Remarks on problems of positions) where he specifically places braids and knots as a subject of the geometry of position [Va]. In the first paragraph of the paper Vandermonde wrote:

Whatever the twists and turns of a system of threads in space, one can always obtain an expression for the calculation of its dimensions, but this expression will be of little use in practice. The craftsman who fashions a braid, a net, or some knots will be concerned, not with questions of measurement, but with those of position: what he sees there is the manner in which the threads are interlaced.

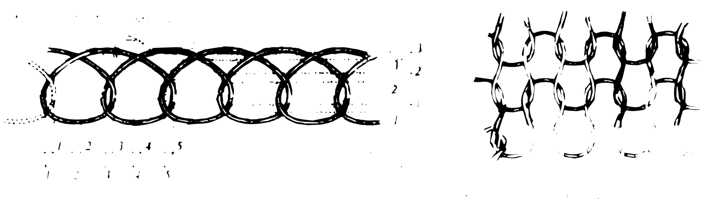


Figure 3.2; Knots of Vandermonde

In our search for the origin of knot theory, we arrive next at Carl Friedrich Gauss (1777-1855). According to [Stac, Dun] :

"One of the oldest notes by Gauss to be found among his papers is a sheet of paper with the date 1794. It bears the heading "A collection of knots" and contains thirteen neatly sketched views of knots with English names written beside them... With it are two additional pieces of paper with sketches of knots. One is dated 1819; the other is much later, ...". [9]

In July of 1995 I finally visited the old library in Göttingen, I looked at knots from 1794 - in fact not all of them are drawn - some only described; see Fig. 3.3 for one of the drawings[10].

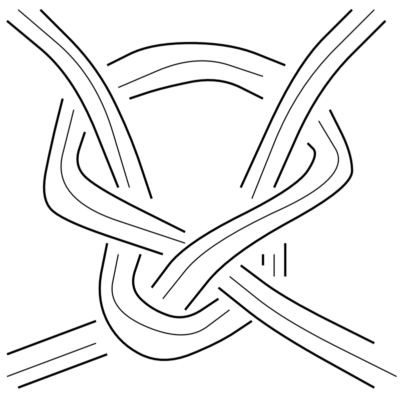


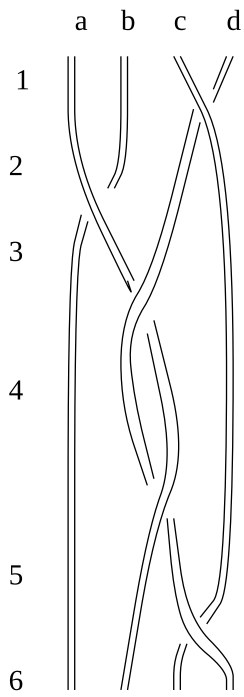
Figure 3.3; *Meshing knot*, 10'th knot of Gauss from 1794.

There are other fascinating drawings in Gauss' notebooks. For example, the drawing of a braid with complex coordinate description at each height (Figure 3.4; compare [Ep-1, P-21]), and the note that it is a good method of coding a knotting. It is difficult to date the drawing; one can say for sure that it was done between 1814 and 1830, I would guess closer to 1814[11].

[9] According to [Gr-2], the first English sailing book with pictures of knots appeared in 1769 [Falc].

[10] First eight drawings are reproduced in the preface to [T-G].

[11] As a curiosity one can add that of one of the notebooks (Handb. 3) in which Gauss had also drawn some knot diagrams has braids motives on its cover.



Veraindrungder Coordiniz

a	1	1	2+i	3+i	2+2i	2+2i
b	2	2	1	1	1	1
c	3	4	4	4	4	3
d	4	3+i	3+i	2+2i	3+2i	4+3i

Figure 3.4[12]

It is a good method of coding a knotting (from a Gauss' notebook (Handb.7)).

There is also the mysterious "framed tangle", see Fig.3.5 [Ga-1, P-29] whose interpretation is not yet convincingly given.

[12]Gauss coordinates are not always consistent; most of the time i is pointing downward but there are exceptions.

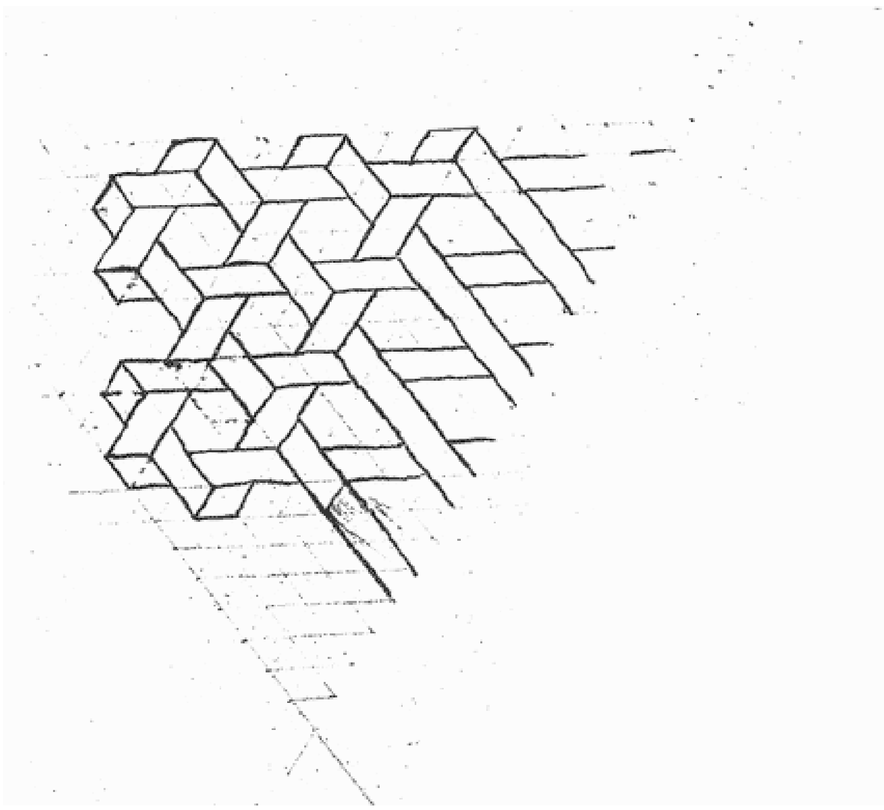
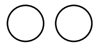
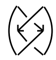


Fig. 3.5; Framed tangle from Gauss' notebook [Ga-1]

In his note (Jan. 22 1833) Gauss introduces the linking number of two knots[13]. Gauss' note presents the first deep incursion into knot theory; it establishes that the following two links are substantially different:
 ,  . Gauss' analytical method has been recently revitalized by Witten's approach to knot theory [Wit].

James Clerk Maxwell (1831-1879), in his fundamental book of 1873 "A treatise on electricity & magnetism" [Max] writes [14] : "It was the discovery by Gauss of this very integral, expressing the work done on a magnetic pole while describing a closed curve in presence of a closed electric current, and indicating the geometrical connection between the two closed curves, that led him to lament the small progress made in the Geometry of Position

[13]His method is analytical - the Gauss integral; in modern language Gauss integral computes analytically the degree of the map from a torus parameterizing a 2 component link to the unit 2-sphere.

[14]It was only six years after Gauss note was first published in his collected works in 1867.

since the time of Leibnitz, Euler and Vandermonde. We have now, however, some progress to report chiefly due to Riemann, Helmholtz and Listing."[15] Maxwell goes on to describe two closed curves which cannot be separated but for which the value of the Gauss integral is equal to zero; Fig.3.6.

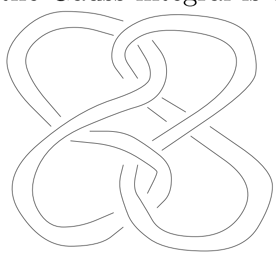


Figure 3.6; The link of Maxwell.

In 1876, O.Boeddicker observed that, in a certain sense, the linking number is the number of the crossing points of the second curve with a surface bounded by the first curve [Boe-1, Boe-2, Bog]. Hermann Karl Brunn[16] [Br] observed in 1892 that the linking number of a two-component link, considered by Gauss, can be read from a diagram of the link[17]. If the link has components K_1 and K_2, we consider any diagram of the link and count each point at which K_1 crosses under K_2 as $+1$ for  and -1 for  . The sum of these, over all crossings of K_1 under K_2, is the Gauss linking number.

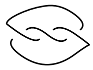
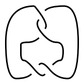
The year of 1847 was very important for the knot theory (graph theory and topology as well). On one hand, Gustav Robert Kirchhoff (1824-1887) published his fundamental paper on electrical circuits [Kir]. It has deep connections with knot theory, however the relations were discovered only about a hundred years later (e.g. the Kirchhoff complexity of a circuit corresponds to the determinant of the knot or link determined by the circuit). On the other hand, Johann Benedict Listing (1808-1882), a student of Gauss, published his monograph (Vorstudien zur Topologie, [Lis]). A considerable part of the monograph is devoted to knots. Even earlier, on April 1, 1836, Listing wrote a letter from Catania to "Herr Muller", his former school teacher[18], with the heading "Topology", concerning ... (2) winding paths of knots; and

[15] Gauss wrote in 1833, in the same note in which he introduced the linking number: "On the geometry of position, which Leibniz initiated and to which only two geometers, Euler and Vandermonde, have given a feeble glance, we know and possess, after a century and a half, very little more than nothing."

[16] Born August 1, 1862 Rome (Italy), died Sept. 20, 1939 (Munchen, Germany)[Bl].

[17] It is also noted by Tait in 1877 ([Ta], page 308).

[18] Johann Heinrich Müller (1787-1844) was the mathematics and astronomy master at *Musterschule* which Listing entered in 1816 [Bre-2].

(3) paths in a lattice [Bre-1, Bre-2, Stac]. Listing stated in particular that the right handed trefoil knot () and the left handed trefoil knot () are not equivalent. Later Listing showed that the figure eight knot  and its mirror image  are equivalent (we say that the figure eight knot, also called the Listing knot, is amphicheiral)[19]. Listing indebtness to Gauss is nicely described in the introduction to [Lis]: "Stimulated by by the greatest geometer of our days, who had been repeatedly turning my attention to the significance of this subject, during long time I did various attempts to analyze different cases related to the subject, given by natural sciences and their applications. And if now, when these reflections do not have a right yet to claim rigorous scientific form and method, I let myself to publish them as preliminary sketch of the new science, then I do this with the intention to turn attention to significance and potential of it, with help of collected here important information, examples and materials. I hope you let me use the name "Topology" for this kind of studies of spatial images, rather than suggested by Leibniz name "geometria situs", reminding of notion of "measure" and "measurement", playing here absolutely subordinate role and confiding with "géométrie de position" which is established for a different kind of geometrical studies. Therefore, by **Topology** we will mean the study of modal relations of spatial images, or of laws of connectedness, mutual disposition and traces of points, lines surfaces, bodies and their parts or their unions in space, independently of relations of measures and quantities. By means of the notion "trace", which is very close to the notion of movement, topology is related to mechanics, similarly as it is related to geometry. Of course, velocity, as well as mass, momentum, powers and moments of movement from the quantity point of view are not taken into consideration. Instead we consider only modal relations between moving or moved in space images. In order to reach the level of exact science, topology will have to translate facts of spatial contemplation into easier notion which, using corresponding symbols analogous to mathematical ones, we will be able to do corresponding operations following some simple rules." (Translated by M.Sokolov).

As we mentioned before, Maxwell, in his study of electricity and magnetism, had some thoughts on knots and links (in particular motivated by the freshly published Gauss' collected works). The origin of modern knot

[19]This was observed in the note dated March 18, 1849 [Lit-1]

theory should be associated with four physicists: Hermann Von Helmholtz (1821-1894), William Thomson (Lord Kelvin) (1824-1907), Maxwell and Peter Guthrie Tait (1831-1901). We can quote after Tait's assistant in Edinburgh and later biographer, C.G.Knott [Kno]:

Tait was greatly impressed with Helmholtz's famous paper on vortex motion [[Helm]; 1858]... Early in 1867 he devised a simple but effective method of producing vortex smoke rings; and it was when viewing the behaviour of these in Tait's Class Room that Thomson was led to the conception of the vortex atom. In his first paper to the Royal Society of Edinburgh on February 18, 1867 [[Thoms]], Sir William Thomson refers... to the genesis of the conception. In turn Thomson's theory was Tait's motivation to understand the structure of knots. In Tait's words: *I was led to the consideration of the form of knots by Sir W. Thomson's Theory of Vortex Atoms, and consequently the point of view which, at least at first, I adopted was that of classifying knots by the number of their crossings... The enormous number of lines in the spectra of certain elementary substances show that, if Thomson's suggestion be correct, the form of the corresponding vortex atoms cannot be regarded as very simple[[Ta]].*

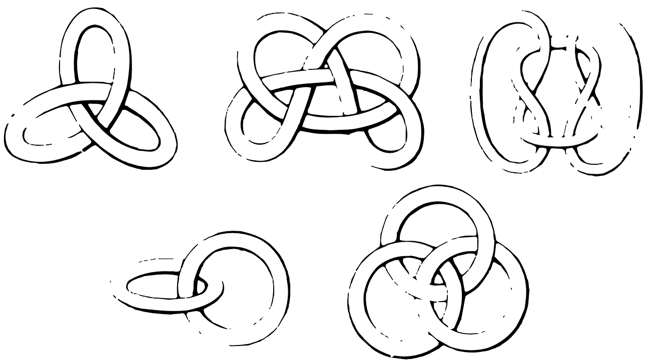


Figure 3.7. Knots and links of William Thomson (Kelvin) from 1867.

There is an interesting letter from Maxwell to Tait dated Nov. 13, 1867, which shows that Tait was sharing his ideas of knots with his friend [Kno, Lom]. In one of his rhymes Maxwell wrote (clearly referring to Tait):

Clear your coil of kinkings
Into perfect plaiting,
Locking loops and linkings
Interpenetrating.
[Kno]

Tait describes his work on knots in the following words ([Ta],1877): *When I commenced my investigations I was altogether unaware that anything had been written (from a scientific point of view) about knots. No one in Section A at the British Association of 1876, when I read a little paper[[Ta-0]] on the subject, could give me any reference; and it was not till after I had sent my second paper to this Society that I obtained, in consequence of a hint from Professor Clerk-Maxwell, a copy of the very remarkable Essay by Listing*[20]*, Vorstudien zur Topologie[[Lis]], of which (so far as it bears upon my present subject) I have given a full abstract in the Proceedings of the Society for Feb. 3, 1877. Here, as was to be expected, I found many of my results anticipated, but I also obtained one or two hints which, though of the briefest, have since been very useful to me. Listing does not enter upon the determination of the number of distinct form of knots with a given number of intersections, in fact he gives only a very few forms as examples, and they are curiously enough confined to three, five and seven crossings only; but he makes several very suggestive remarks about the representation of a particular class of "reduced" knots... This work of Listing's and an acute remark made by Gauss (which with some comments on it by Clerk-Maxwell, will be referred to later), seem to be all of any consequence that has been as yet written on the subject.* Tait's paper was revised May 11, 1877; he finishes the paper as follows: *After the papers, of which the foregoing is a digest, had been read, I obtained from Professor Listing*[21] *and Klein a few references to the literature of the subject of knots. It is very scanty, and has scarcely any bearing upon the main question which I have treated above. Considering that Listing's Essay was published thirty years ago, and that it seems to be pretty well known in Germany, this is a curious fact. From Listing's letter (Proc. R.S.E.. 1877, p.316), it is clear that he has published only a small part of the results of his investigations. Klein himself [Klein] has made the very singular discovery that in space of four dimensions there cannot be knots.*[22]

[20]In 1883 Tait wrote in *Nature* obituary after Listing death [Ta-1]: *One of the few remaining links that still continued to connect our time with that in which Gauss had made Göttingen one of the chief intellectual centres of the civilised world has just be broken by the death of Listing... This paper [Vorstudien zur Topologie], which is throughout elementary, deserves careful translation into English....* After more than a hundred years the paper is not translated (only Tait summary exists [Ta-2]) and one should repeat Tait appeal again: The paper very much deserves translation. One can add that in 1932 the paper was translated into Russian.

[21]Library of the University of California has a copy of *Vorstudien zur Topologie* which Listing sent to Tait with the dedication.

[22]Klein observation was noticed in non-mathematical circles and it became part of

The value of Gauss's integral has been discussed at considerable length by Boeddicker ... in an Inaugural Dissertation, with the title Beitrag zur Theorie des Winkels*, Göttingen, 1876.*

An inaugural Dissertation by Weith, Topologishe Untersuchung der Kurven-Verschlingung, Zürich, 1876 [Weith], is professedly based on Listing's Essay. It contains a proof that there is an infinite number of different forms of knots![23] The author points out what he (erroneously) supposes to be mistakes in Listing's Essay; and, in consequence, gives as something quite new an illustration of the obvious fact that there can be irreducible knots in which the crossing are not alternately over and under [24]. The rest of this paper is devoted to the relations of knots to Riemann's surfaces.

Tait, in collaboration with Reverend Thomas Penyngton Kirkman (1806-1895), and independently Charles Newton Little[25], made a considerable progress on the enumeration problem so that by 1900 there were in existence tables of (prime) knots up to ten crossings [Ta, Kirk-1, Lit-0, Lit-1]. These tables were partially extended in M.G. Haseman's doctoral dissertation of 1917/8[26], [Has-1]. Knots up to 11 crossings were enumerated by John H. Conway [Co-1] before 1969 [27].

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popular culture. For example, the American magician and medium Henry Slade was performing "magic tricks" claiming that he solves knots in fourth dimension. He was taken seriously by a German astrophysicist J.K.F.Zoellner who had with him a number of seances in 1877 and 1878.

[23] In fact it was proven only 20-30 years later and depended on the fundamental work of Poincaré on foundation of algebraic topology.

[24] It was proven only in 1930 by Bankwitz [Ban], using the determinant of a knot.

[25] Born Madura India, May 19, 1858. A.B., Nebraska 1879, A.M. 1884; Ph.D, Yale, 1885. Instructor math. and civil eng, Nebraska, 1880-84, assoc.prof.civil eng.84-90, prof, 90-93; visited Göttingen and Berlin, 1898-1899; math, Stanford, 1893-1901; civil eng, Moscow, Idaho, from 1901 , dean, col. eng, from 1911., died August 31, 1923 [Amer, Yale].

[26] Mary Gertrude Haseman, born March 6, 1889, Linton, Indiana, was the fifth doctoral student of C.A.Scott at Brynn Mawr College. She was teaching at University of Illinois, and died April 9, 1979.

[27] K.A.Perko, a student of Fox at Princeton, and later a lawyer from New York, observed a duplication in the tables: two 10-crossing diagrams represented the same knot, see Figure 3.8. Perko corrected also the Conway's eleven crossing tables: 4 knots were missed [Per-0, Per-2].

Figure 3.8; Knots of Perko.

Knots up to 13 crossings were enumerated by C.H.Dowker and M.B.Thistlethwaite [D-T, This-1], 1983[28]. For the further progress we can refer to [H-T] and [Hos-20]. The number of prime, unoriented, nonalternating knots per crossing number $7 \leq n \leq 16$ is: 0, 3, 8, 42, 185, 888, 5110, 27436, 168030, 1008906.
The number of prime, unoriented, alternating knots per crossing number $3 \leq n \leq 23$ is: 1, 1, 2, 3, 7, 18, 41, 123, 367, 1288, 4878, 19536, 85263, 379799, 1769979, 8400285, 40619385, 199631989, 990623857, 4976016485, 25182878921.[29] Knots and their mirror images are not counted separately.

To be able to make tables of knots, Tait introduced three basic principles (called now the Tait conjectures). All of them have been solved. The use of the Jones polynomial makes the solution of the first two Tait conjectures astonishingly easy [M-4, This-3, K-6] and the solution of the third Tait conjecture also uses essentially Jones type polynomials [M-T-1, M-T-2]. We formulate these conjectures below:

T1. An alternating diagram with no nugatory crossings, of an alternating link realizes the minimal number of crossings among all diagrams representing the link. A nugatory crossing is drawn (defined) in Figure 10(a).

T2. Two alternating diagrams, with no nugatory crossings, of the same oriented link have the same Tait (or writhe) number, i.e. the signed

[28]Before my book in Polish was published in 1995, I asked Jim Hoste about the actual status of knot's tabulation and he has informed me that he and, independently, M.B.Thistlethwaite are working on the extension up to 15 crossings of the existing knots' tables. They found already that there are 19 536 prime alternating, 14 crossing knots and 85 263 prime alternating, 15 crossing knots. Census for all knots (not necessary alternating) is not yet verified but suggests over 40 000 knots of 14 crossings and over 200 000 knots.

[29]Ortho Flint and Stuart Rankin, with coding by Peter de Vries, calculated alternating(23) = 25182878921 on a Compaq ES 45 in 228 hours, finishing on Mar 14, 2004 [EIS].

sum of all crossings of the diagram with the convention that  is +1 and  is −1.

T3. Two alternating diagrams, with no nugatory crossings, of the same link are related by a sequence of flypes (see Figure 3.9(b)).

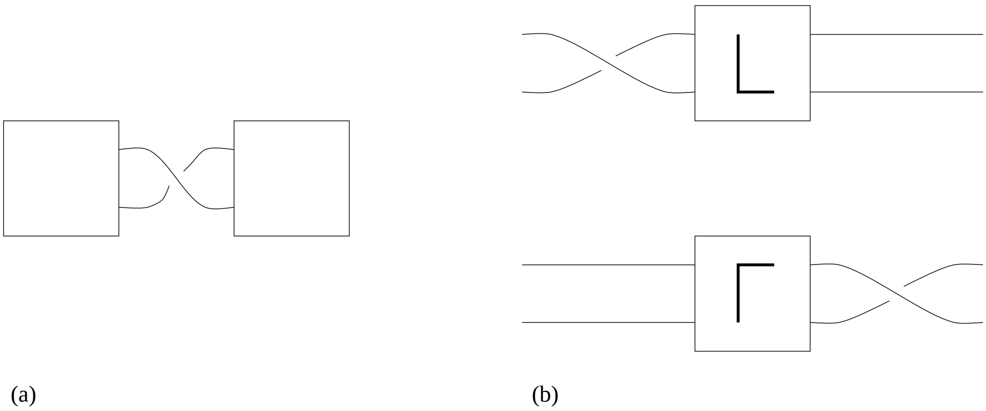


Figure 3.9

A very interesting survey on the developments in knot theory in XIX century can be found in the Dehn and Heegaard article in the Mathematical Encyclopedia [D-H], 1907[30]. In this context, the papers of Oscar Simony from Vienna [31] are of great interest [Sim, Ti-2]. Figure 3.10 describes torus knots of Simony. Simony was using continued fractions to describe torus knots [Sim, Grg] (in essence his method was analogous to that employed by J.H.Conway to describe 2-bridge knots [Co-1]).

[30]According to Dehn's wife, Mrs. Toni Dehn, *Dehn and Heegaard met at the third International Congress of Mathematicians at Heidelberg in 1904 and took to each other immediately. They left Heidelberg on the same train, Dehn going to Hamburg and Heegaard returning to Copenhagen. They decided on the train that an Encyclopedia article on topology would be desirable, that they would propose themselves as authors to the editors, and that Heegaard would take care of literature whereas Dehn would outline a systematic approach which would lay the foundations of the discipline [Mag].*

[31]Born April 23, 1852 in Vienna, died April 6, 1915 in Vienna [Pogg].

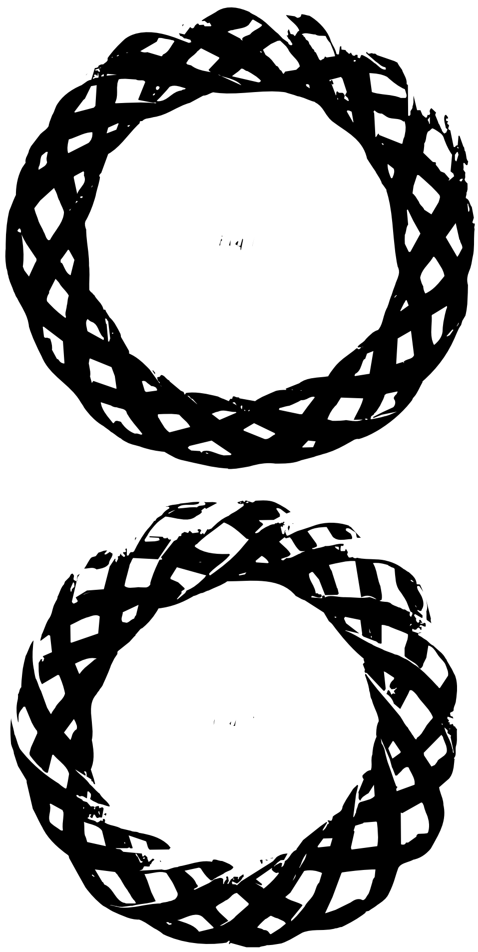
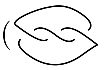


Figure 3.10; Torus knots of Simony from 1884.

II.3 Algebraic topology in Knot Theory

The fundamental problem in knot theory is to be able to distinguish non-equivalent knots. It was not achieved (even in the simple case of the unknot and the trefoil knot) until Jules Henri Poincaré (1854-1912) in his "Analysis Situs" paper ([Po-1] 1895) laid foundations for algebraic topology. Poul Heegaard (1871-1948) in his Copenhagen Dissertation of 1898 ([Heeg]) constructed the 2-fold branch cover of a trefoil knot and showed that it is the lens space, $L(3,1)$, in modern terminology. He also showed that the analogous branch cover of the unknot is S^3. He distinguished $L(3,1)$ from S^3

using the Betti numbers (more precisely he showed that the first homology group is nontrivial and he clearly understood that it is a 3-torsion group). He didn't state however that it can be used to distinguish the trefoil knot from the unknot; see [Stil] p.226.[32] Heinrich Tietze (1880-1964) used in 1908 the fundamental group of the exterior of a knot in R^3, called the knot group, to distinguish the unknot from the trefoil knot [Ti-1]. The fundamental group was first[33] introduced by Poincaré in his 1895 paper [Po-1].

Wilhelm Wirtinger (1865-1945) in his lecture delivered at a meeting of the German Mathematical Society in 1905 outlined a method of finding a knot group presentation (it is called now the Wirtinger presentation of a knot group) [Wir]. Max Dehn (1878-1952), in his 1910 paper [De-1] refined the notion of the knot group and in 1914 was able to distinguish the right handed trefoil knot () from its mirror image, the left handed trefoil knot (); that is Dehn showed that the trefoil knot is not amphicheiral [De-2] [34].

Tait was the first to notice the relation between knots and planar graphs. He colored the regions of the knot diagram alternately white and black (following Listing) and constructed the graph by placing a vertex inside each white region, and then connecting vertices by edges going through the crossing points of the diagram (see Figure 4.1)[D-H].

[32] For the English translation of the topological part of the Heegaard thesis see the appendix to [P-22].

[33] According to [Ch-M] Hurwitz' paper of 1891 [Hur] "may very well be interpreted as giving the first approach to the idea of a fundamental group of a space of arbitrarily many dimensions."

[34] In 1978, W.Magnus wrote [Mag]: *Today, it appears to be a hopeless task to assign priorities for the definition and the use of fundamental groups in the study of knots, particularly since Dehn had announced [De-0] one of the important results of his 1910 paper (the construction of Poincaré spaces with the help of knots) already in 1907.*

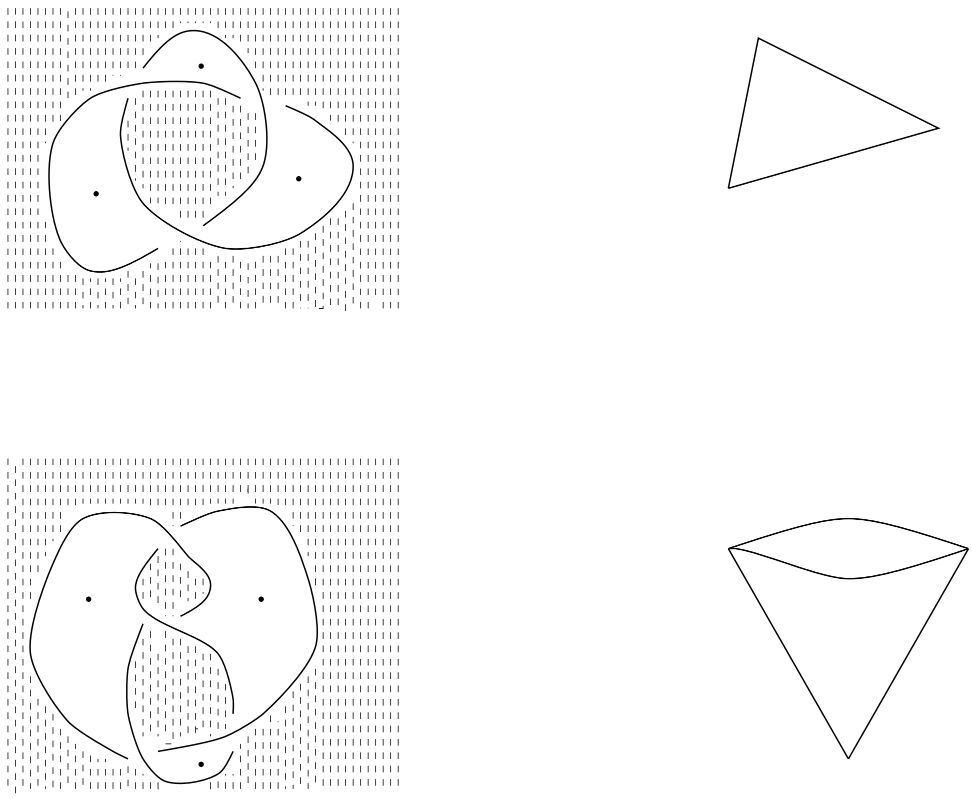


Figure 4.1

In 1912, George David Birkhoff (1884-1944) when trying to prove the four-color problem (formulated in 1852 by Francis Guthrie (1831-1899)), introduced the chromatic polynomial of a graph [Birk-1].

The breakthrough, from the point of view which focuses on the Jones type link invariants, was the invention by James Waddell Alexander (1888-1971) of a Laurent polynomial invariant of links ([Al-3], 1928)[35]. Alexander was a colleague of Birkhoff and we can conjecture that he knew about the chromatic polynomial.[36] We know for sure that when W. T. Tutte was generalizing the chromatic polynomial in 1947 [Tut-1], he was motivated by the

[35] Alexander described the numerical precursor to his polynomial, for the first time, in the letter to Oswald Veblen, 1919 [A-V].

[36] Birkhoff writes in [Birk-3] : "...Alexander, then [1911] a graduate student [at Princeton], began to be especially interested in the subject [analysis situs]. His well known "duality theorem," his contributions to the theory of knots, and various other results, have made him a particularly important worker in the field". We can also mention that in the fall of 1909 Birkhoff became a member of the faculty at Princeton and left for Harvard in 1912. His 1912 paper [Birk-1] ends with "Princeton University, May 4, 1912."

knot polynomial of Alexander. The Alexander polynomial can be derived from the group of the knot (or link). This point of view has been extensively developed since Alexander's discovery. More generally, the study of the fundamental group of a knot complement and the knot complement alone was the main topic of research in knot theory for the next fifty years, culminating in 1988 in the proof of Tietze [Ti-1] conjecture (that a knot is determined by its complement) by Gordon and Luecke [G-Lu]. We can refer to the survey articles by Ralph Hartzler Fox (1913 -1973) [F-1] and Gordon [Gor] or books [Bi-1, B-Z, K-3, Re-2, Ro-1] in this respect. However, Alexander observed also that if three oriented links, L_+, L_- and L_0, have diagrams which are identical except near one crossing where they look as in Figure 4.2, then their polynomials are linearly related [Al-3]. An analogous discovery about the chromatic polynomial of graphs was made by Ronald M. Foster in 1932 (see [Whit-1]; compare also [Birk-2] Formula (10)). In early 1960's, J. Conway rediscovered Alexander's formula and normalized the Alexander polynomial, $\Delta_L(t) \in Z[t^{\pm 1/2}]$, defining it recursively as follows ([Co-1]):

(i) $\Delta_o(t) = 1$, where o denotes a knot isotopic to a simple circle

(ii)

$$\Delta_{L_+} - \Delta_{L_-} = (\sqrt{t} - \frac{1}{\sqrt{t}})\Delta_{L_0}$$

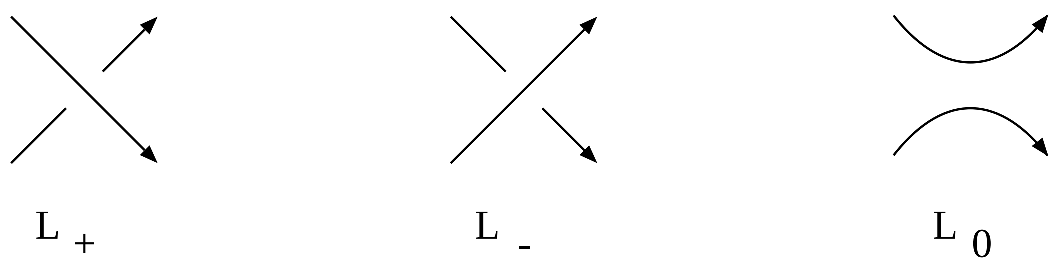


Figure 4.2

II.4 Jones revolution

In the spring of 1984, Vaughan Jones discovered his invariant of links, $V_L(t)$ [Jo-0, Jo-1, Jo-2][37], and still in May of 1984 he was trying various substitutions to the variable t, in particular $t = -1$. He observed that $V_L(-1)$ is equal to the classical knot invariant – determinant of a knot; however, initially he was unable to prove it. Soon he realized that his polynomial satisfies the local relation analogous to that discovered by Alexander and Conway and established the meaning of $t = -1$.[38] Thus the Jones polynomial is defined recursively as follows:

(i) $V_o = 1$,

(ii) $\frac{1}{t}V_{L_+}(t) - tV_{L_-}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})V_{L_0}(t)$.

In the summer and the fall of 1984, the Alexander and the Jones polynomials were generalized to the skein (named also Conway-Jones, Flypmoth, Homfly, Homflypt[39], generalized Jones, 2-variable Jones, Jones-Conway, Thomflyp, twisted Alexander) polynomial, $P_L \in Z[a^{\pm 1}, z^{\pm 1}]$, of oriented links [FYHLMO, P-T-1]. This polynomial is defined recursively as follows [FYHLMO, P-T-1]:

(i) $P_o = 1$;

(ii) $aP_{L_+} + a^{-1}P_{L_-} = zP_{L_0}$.

In particular $\Delta_L(t) = P_L(i, i(\sqrt{t} - \frac{1}{\sqrt{t}}))$, $V_L(t) = P_L(it^{-1}, i(\sqrt{t} - \frac{1}{\sqrt{t}}))$. In August 1985 L. Kauffman found another approach to the Jones polynomial[40]. It starts from an invariant, $< D > \in Z[\mu, A, B]$, of an unoriented

[37] Jones wrote in [Jo-6]: “It was a warm spring morning in May, 1984, when I took the uptown subway to Columbia University to meet with Joan S. Birman, a specialist in the theory of “braids”... In my work on von Neumann algebras, I had been astonished to discover expressions that bore a strong resemblance to the algebraic expression of certain topological relations among braids. I was hoping that the techniques I had been using would prove valuable in knot theory. Maybe I could even deduce some new facts about the Alexander polynomial. I went home somewhat depressed after a long day of discussions with Birman. It did not seem that my ideas were at all relevant to the Alexander polynomial or to anything else in knot theory. But one night the following week I found myself sitting up in bed and running off to do a few calculations. Success came with a much simpler approach than the one that I had been trying. I realized I had generated a polynomial invariant of knots.”

[38] The relation was also discovered independently in July 1984 by Lickorish and Millett.

[39] Homfly or Homflypt is the acronym after the initials of the inventors: Hoste, Ocneanu, Millett, Freyd, Lickorish, Yetter, Przytycki and Traczyk.

[40] First he thought that he produced a new knot polynomial and only analyzing the polynomial he realized that he found a variant of the Jones polynomial.

link diagram D called the Kauffman bracket [K-6]. The Kauffman bracket is defined recursively by:

(i)

$$< \underbrace{o \ldots o}_{i} >= \mu^{i-1}$$

(ii)

$$< L_+ >= A < L_0 > +B < L_\infty >$$

(iii)

$$< L_- >= B < L_0 > +A < L_\infty >$$

where L_+, L_-, L_0 and L_∞ denote four diagrams that are identical except near one crossing as shown in Figure 14, and $< \underbrace{o \ldots o}_{i} >$ denotes a diagram of i trivial components (i simple circles).

If we assign $B = A^{-1}$ and $\mu = -(A^2 + A^{-2})$ then the Kauffman bracket gives a variant of the Jones polynomial for oriented links. Namely, for $A = t^{-\frac{1}{4}}$ and D being an oriented diagram of L we have

$$V_L(t) = (-A^3)^{-w(D)} < D > \qquad \text{(II.1)}$$

where $w(D)$ is the *planar writhe* (*twist* or *Tait number*) of D equal to the algebraic sum of signs of crossings.

It should be noted, as first observed by Kauffman, that bracket $< >_{\mu,A,B}$ is an isotopy invariant of alternating links (and their connected sums) under the assumption that the third Tait conjecture (soon after proven by Menasco and Thistlethwaite [M-T-1, M-T-2]) holds.

In the summer of 1985 (two weeks before discovering the "bracket"), L. Kauffman invented another invariant of links [K-5], $F_L(a, z) \in Z[a^{\pm 1}, z^{\pm 1}]$, generalizing the polynomial discovered at the beginning of 1985 by Brandt, Lickorish, Millett and Ho [B-L-M, Ho]. To define the Kauffman polynomial we first introduce the polynomial invariant of link diagrams $\Lambda_D(a, z)$. It is defined recursively by:

(i) $\Lambda_o(a, z) = 1$,

(ii) Λ$(a, z) = a\Lambda_{|}(a, z)$; Λ$(a, z) = a^{-1}\Lambda_{|}(a, z)$,

(iii) $\Lambda_{D_+}(a, z) + \Lambda_{D_-}(a, z) = z(\Lambda_0(a, z) + \Lambda_{D_\infty}(a, z))$.

The Kauffman polynomial of oriented links is defined by

$$F_L(a,z) = a^{-w(D)}\Lambda_D(a,z)$$

where D is any diagram of an oriented link L.

Figure 5.1

Jones type invariants lead to invariants of three-dimensional manifolds [At, Wit, P-5, Tu-2, R-T-2, Tu-Vi, Tu-We, P-19]. We already mentioned that Jones type invariants of knots have been used to solve Tait conjectures. The Jones discovery, however, not only introduced a delicate method of analyzing knots in 3-manifolds but related knot theory to other disciplines of mathematics and theoretical physics, for example statistical mechanics, quantum field theory, operator algebra, graph theory and computational complexity. On the other hand the Jones polynomial gives a simple tool to recognize knots and as such is of great use for biologists (e.g. for analysis of DNA) and chemists (see for example [SCKSSWW]).

Biographical Notes

We give below a chronological list of selected people mentioned in the article (compare [B-L-W]):

Gottfried Wilhelm Leibniz (1646-1716)
Heinrich Kuhn (1690-1769)
Leonhard Euler (1707-1783)
Alexandre-Theophile Vandermonde (1735-1796)
Carl Friedrich Gauss (1777-1855)
Thomas Penyngton Kirkman (1806-1895)
Johann Benedict Listing (1808-1882)
Hermann Von Helmholtz (1821-1894)
Gustav Robert Kirchhoff (1824-1887)
William Thomson (Lord Kelvin) (1824-1907)
Peter Guthrie Tait (1831-1901)
James Clerk Maxwell (1831-1879)
Oscar Simony (1852-1915)
Jules Henri Poincaré (1854-1912)

Charles Newton Little (1858-1923)
Hermann Karl Brunn (1862- 1939)
Wilhelm Wirtinger (1865-1945)
Poul Heegaard (1871-1948)
Max Dehn (1878-1952)
Heinrich Tietze (1880-1964)
George David Birkhoff (1884-1944)
James Waddell Alexander (1888-1971)
Mary Gertrude Haseman (1889-1979)
Kurt W. F. Reidemeister (1893-1971)
Ronald M. Foster (1896-1998)
Hidetaka Terasaka (1904-1996)
Herbert Seifert (1907-1996)
Ralph Hartzler Fox (1913-1973).

Bibliography

[A-V] J. W. Alexander, Letter to Oswald Veblen, 1919, Papers of Oswald Veblen, 1881-1960 (bulk 1920-1960), Archival Manuscript Material (Collection), Library of Congress. (I would like to thank Jim Hoste for providing me with a copy of the letter).
To have a taste of Alexander letter, here is the quotation from the beginning of the interesting part: "When looking over Tait on knots among other things, He really doesn't get very far. He merely writes down all the plane projections of knots with a limited number of crossings, tries out a few transformations that he happen to think of and assumes without proof that if he is unable to reduce one knot to another with a reasonable number of tries, the two are distinct. His invariant, the generalization of the Gaussian invariant ... for links is an invariant merely of the particular projection of the knot that you are dealing with, - the very thing I kept running up against in trying to get an integral that would apply. The same is true of his "Beknottednes".

Here is a genuine and rather jolly invariant: take a plane projection of the knot and color alternate regions light blue (or if you prefer, baby pink). Walk all the way around the knot and ..."

[Al-3] J. W. Alexander, Topological invariants of knots and links, *Trans. Amer. Math. Soc.* 30, 1928, 275-306.

[Amer] American Men of Science, 1921, 3'rd edition, Edited by Catell, J.McKeen Brimhall, Dean R. Garrison, NY, The Science Press.

[Arch] T. Archibald, Connectivity and Smoke-Rings: Green's Second Identity in Its First Fifty Years, *Math.Magazine*, 62(2) 1989, 219-232.

[Arr] L. F. Arrian, *Anabasis Alexandri, Book II*, c.150 AD (English translation P.A. Brunt, Harvard University Press, 1976).

[At] M. Atiyah, *The geometry and physics of knots*, Cambridge Univ. Press, 1990, 75 pp.

[Ba-Wh] Bambach [Cappel], Carmen, and Lucy Whitaker. "The Lost Knots." Achademia Leonardi Vinci: Journal of Leonardo Studies and Bibliography of Vinciana 4 (1991), pp. 107+10, and unpaginated figures.

[Ban] C. Bankwitz, Über die Torsionszahlen der alternierenden Knoten, *Math. Ann.*, 103, 1930, 145-161.

[B-L-W] N. L. Biggs, E. K. Lloyd, R. J. Wilson, *Graph theory 1736-1936*, Clarendon Press, Oxford, 1986.

[Birk-1] G. D. Birkhoff, A determinant formula for the number of ways of coloring a map, *Ann. of Math.* (2)14, 1912, 42-46.

[Birk-2] G. D. Birkhoff, On the polynomial expressions for the number of ways of coloring a map, *Annali, Pisa, Scoula Normali Superiore*, s.2, v.3 (1934), 85-104. Also in George David Birkhoff Collected Mathematical Papers, Vol.3, AMS (1950), 29-47.

[Birk-3] G. D. Birkhoff, Fifty years of American Mathematics, Amer. Math. Soc., *Semicentennial Addresses*, v. 2, 1938, 270-315. Also in George David Birkhoff Collected Mathematical Papers, Vol.3, AMS (1950), 606-651.

[Bi-1] J. S. Birman, *Braids, links and mapping class group*, Ann. Math. Studies 82, Princeton N.J.: Princeton Univ. Press, 1974.

[Bl] W. Blaschke, Obituary: Hermann Brunn. (German) *Jber. Deutsch. Math. Verein.* 50, (1940). 163–166.

[Boe-1] O. Boeddicker, Beitrag zur Theorien des Winkels, Dissertation Göttingen 1876.

[Boe-2] O. Boeddicker, Erweiterung der Gausschen Theorie der Verschlingungen, Stuttgart, 1876.

[Bog] M. Bognár, *Fundations of linking theory*, Akadémiai Kiadó, Budapest 1992.

[B-L-M] R. D. Brandt, W. B. R.Lickorish, K. C. Millett, A polynomial invariant for unoriented knots and links, *Invent. Math.* 84, 1986, 563-573.

[Bre-1] E. Breitenberger, Gauss und Listing: Topologie und Freundschaft. (German) [Gauss and Listing: topology and friendship] Gauss-Ges. Gvttingen Mitt. No. 30 (1993), 3–56.

[Bre-2] E. Breitenberger, Johann Benedikt Listing, History of topology, 909–924, North-Holland, Amsterdam, 1999.

[Brill] A. Brill, *Math. Ann.* 18, 1881.

[Br] H. Brunn, Topologische Betrachtungen, *Zeitschrift für Mathematik und Physik*, 37, 1892, 106-116.

[B-Z] G. Burde, H. Zieschang, *Knots*, De Gruyter (1985).

[Ch-M] B. Chandler, W. Magnus, The history of Combinatorial group theory: A case study in the history of ideas, *Studies in the History of Mathematics and Physical Sciences* 9, Springer-Verlag, 1982.

[Col] D. Collon, *Ancient Near Eastern Art*, University of California Press, Berkeley, Los Angeles, 1995.

[Coom] Ananda Kentish Coomaraswamy, The iconography of Dürer's "Knots" and Leonardo's "Concatenation", Detroit Institute of Arts, 1944.

[Com] Comprehensive Dissertation Index, 1861-1972, Volume 5: Mathematics and Statistics, Xerox University Microfilms, Ann Arbor, Michigan, 1973.

[Co-1] J. H. Conway, An enumeration of knots and links, *Computational problems in abstract algebra* (ed. J.Leech), Pergamon Press (1969) 329 - 358.

[Cro] W. Crönert, *Sprachliches zu griechischen Ärzten, Archiv für Papyrus-forschungen und Vervandte Gebiete*, II (1902-1903), 474-482.

[Crowe] M. J. Crowe, *A history of vector analysis: The evolution of the idea of a vectorial system*, Dover Publications, 1994 (first publ. 1967).

[Da] C. L. Day, *Quipus and Witches' Knots, With a Translation and Analysis of "Oribasius De Laqueis"*, The University of Kansas Press, Lawrence 1967.

[D-W] R. Dedekind, H. Weber, Theorie der algebraischen Funktionen einer Veränderlichen, *J. de Crelle*, XCII (1882), 181-290. (Also in R.Dedekind, Ges.Math.Werke,t. I, p. 248-349).

[De-0] M. Dehn, *Jahresbericht der deutschen Mathematiker–Vereinigung*, Vol.16, 1907, p.573.

[De-1] M. Dehn, Über die Topologie des dreidimensionalen Raumes, *Math. Annalen.* 69 (1910), 137-168.

[De-2] M. Dehn, Die beiden Kleeblattschlingen, *Math.Annalen.* 75, 1914, 1-12.

[De-3] M. Dehn, The mentality of the mathematician. A characterization, *Math. Intelligencer*, 5(2), 1983, 18–26.

[De-4] M. Dehn, Papers on group theory and topology. (English) Translated from the German and with introductions and an appendix by John Stillwell. With an appendix by Otto Schreier. Springer-Verlag, New York-Berlin, 1987. viii+396 pp

[D-H] M. Dehn, P. Heegaard, Analysis situs, *Encykl. Math. Wiss.*, vol. III AB3 Leipzig, 1907, 153-220.

[Die] J. A. Dieudonné, *A history of algebraic and differential topology 1900-1960*, Birkhäuser, Boston, Basel, 1989.

[Dil] Dilgendey, *Topologische Studien*, Leipzig, 1890.

[D-T] C. H. Dowker, M. B. Thistlethwaite, Classification of knot projections, *Topology and its applications* 16, 1983, 19-31.

[Dun] G. W. Dunnington, *Carl Friedrich Gauss, Titan of Science*, Hafner Publishing Co., New York, 1955.

[Dur-1] Albrecht Durer Master Printmaker, Department of Prints and Drawings, Boston Museum of Fine Arts, Hacker Art Books, New York, 1988.

[Dur-2] Albrecht Durer, Diary of his Journey to the Netherlands, 1520-1521, translated by P.Trou, edited by J.A.Goris and G.Marlier, London, Lund Humphries, 1971.

[EIS] Encyclopedia of Integer Sequences (on-line), http://www.research.att.com/ njas/sequences/A002864

[Ep] M. Epple, Geometric aspects in the development of knot theory, History of topology (ed. I.M.James) , 301–357, North-Holland, Amsterdam, 1999.

[Ep-1] M. Epple, Die Entstehung der Knotentheorie (Braunschweig, 1999).

[Eu] L. Euler, Solutio problematis ad geometriam situs pertinentis, *Commentarii Academiae Scientiarum Imperialis Petropolitanae* 8, 1736, 128-140.

[Falc] W. Falconer, *A universal dictionary of the marine*, London 1769.

[F-1] R. H. Fox, A quick trip through knot theory, *In: Top. 3-manifolds*, Proc. 1961 Top.Inst.Univ. Georgia (ed. M.K.Fort, jr), 120-167. Englewood Cliffs. N.J.: Princeton-Hall, 1962.

[FYHLMO] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett, A. Ocneanu, A new polynomial invariant of knots and links, *Bull. Amer. Math. Soc.*, 12, 1985, 239-249.

[Ga-1] C. F. Gauss, Notebooks, Old library in Göttingen.

[Ga-2] K. F. Gauss, Zur mathematischen Theorie der electrodynamischen Wirkungen, 1833, *Werke*, Königlichen Gesellschaft der Wissinchaften zu Gottingen, 5, 1877, 602-629.

[Gor] C. McA. Gordon, Some aspects of classical knot theory, *In: Knot theory*, L.N.M. 685, 1978, 1-60.

[Gor-1] C. McA. Gordon, 3-dimensional topology up to 1960. History of topology (ed. I.M.James), 449–489, North-Holland, Amsterdam, 1999.

[G-Lu] C. McA. Gordon, J. Luecke, Knots are Determined by their Complements, *Bulletin of the A.M.S*, 20, 1989, 83-87.

[Grg] E. Grgeta, Simony knots, and continued fractions, Senior Thesis, George Washington University, 1988.

[Gr-1] P. van de Griend, The history of knot theory, preprint 18, Aarhus University, April 1992.

[Gr-2] P. van de Griend, Notes on Knots, Århus, March 1993.

[Gr-3] P. van de Griend, A history of topological knot theory; History and science of knots, 205–260, Ser. Knots Everything, Vol. 11, World Sci. Publishing, River Edge, NJ, 1996.

[T-G] J. C. Turner, P. van de Griend, Ser.Knots Everything, Vol. 11, World Sci. Publishing, River Edge, NJ, 1996.

[G-S] B. Grünbaum, G. C. Shephard, Interlace patterns in Islamic and Moorish art, "The Visual Mind - Art and Mathematics". Edited by Michele Emmer, MIT Press (1993); Chapter 22, 147-155. (Articles from "Leonardo").

[Ha] F. Harary, The knots and links of Albrecht Dürer, *Atti Accad. Pontaniana (N.S.)* 34, 1985, 97-106.

[H-K-W] P. de la Harpe, M. Kervaire, C. Weber, On the Jones polynomial, *L'enseignement Mathématique*, 32 (1986), 271-335.

[Has-1] M. G. Haseman, On knots, with a census of the amphicheirals with twelve crossings, *Trans. Roy. Soc. Edinburgh*, 52 (1917), 235-255 (also Ph.D thesis, Bryn Mawr College, 1918 [Com]).

[Has-2] M. G. Haseman, Amphicheiral knots, *Trans. Roy. Soc. Edinburgh* 52, 1918 597-602.

[Has-3] M. G. Haseman, Obituary, *Linton Daily Citizen Newspaper*, April 10, 1979.

[Hea] M. C. Heath, "Early Helladic Clay Sealings from the House of the Tiles at Lerna," *Hesperia* 27, 1958, 81-120.

[Heeg] P. Heegaard, Forstudier til en Topologisk Teori for de algebraiske Fladers Sammenhæng, København, 1898, Filosofiske Doktorgrad; French translation: Sur l'Analysis situs, *Soc. Math. France Bull.*, 44 (1916), 161-242.

[Helm] H. Helmholtz, Über Integrale der hydrodynamischen Gleichungen, Welche den Wirbelbewegungen entsprechen, *Crelle's Journal für Mathematik*, 55, 1858, 25-55; English translation by P.G.Tait: *Philosophical Magazine*, June 1867, 485-511.

[Hemion] G. Hemion, *The classification of knots and 3-dimensional spaces*, Oxford University Press, 1992.

[He] Heraklas, first century A.D., included in Oribasius "Medical collections", fourth century A.D.

[Hig] R. Higgins, Minoan and Mycean art, Thames and Hudson, 1997 (Revised edition).

[Hind] A. M. Hind, Two Unpublished Plates of the Series of Six "Knots" Engraved after designs by Leonardo Da Vinci, *The Burlington Magazine for Connoisseurs*, Vol. 12, No. 55, October 1907, 41-43.

[Ho] C. F. Ho, A new polynomial for knots and links; preliminary report, *Abstracts AMS* 6(4), 1985, p. 300.

[H-W] B. Hopkins, R. J. Wilson, The Truth about Konigsberg, T̂he College Mathematics Journal, May 2004, 198-207.

[Hos-2] J. Hoste, e-mails, 2 and 7 Sept. 1993, and 9 June 1994.

[Hos-20] J. Hoste, The enumeration and classification of knots and links, in W. Menasco and M. Thistlethwaite (eds.) Handbook of Knot Theory; Amsterdam: Elsevier, 2005, 209-232.

[H-T] J. Hoste, M. Thistlethwaite, J. Weeks, The First 1701936 Knots, *Math. Intell.*, 20(4), 1998, 33-48

[Hur] A. Hurwitz, Über Riemannsche Flächen mit gegebenen Verzweigung spunkten, *Math. Annalen*, 39, 1891, 1-61.

[Jan] A. Januszajtis, Scientists in Old Gdansk: 17th and 18th Centuries (From the history of science in ancient Gdansk), *Task Quarterly* 5(3), 2001, 389-399.
http://www.pg.gda.pl/archiwalne/januszajtis/2000/III.PDF

[Jo-0] V. F. R. Jones. Letter to J.Birman, May 31, 1984.

[Jo-1] V. F. R. Jones. A polynomial invariant for knots via Von Neuman algebras, *Bull. Amer. Math. Soc* 12 (1985) 103-111.

[Jo-2] V. F. R. Jones. Hecke algebra representations of braid groups and link polynomials, *Ann. of Math.* 126(2), 1987, 335-388.

[Jo-6] V. F. R. Jones, Knot theory and statistical mechanics, *Scientific American*, 263, November, 1990, no. 5, 4, 98–103.

[K-3] L. H. Kauffman, *On Knots*, Princeton University Press, 1987.

[K-5] L. H. Kauffman, An invariant of regular isotopy, *Trans. Amer. Math. Soc.*, 318(2), 1990, 417–471.

[K-6] L. H. Kauffman, State models and the Jones polynomial, *Topology* 26, 1987, 395-407.

[K-8] L. H.Kauffman, *Formal knot theory*, Mathematical Notes 30, Princeton University Press, 1983.

[Ken] A. E. Kennelly, The equivalence of triangles and three-pointed stars in conducting networks, *Electrical World and Engineer*, 34, 1899, 413-414.

[Kir] G. R. Kirchhoff, Über die Auflosung der Gleichungen, auf welche man bei der Untersuchung der linearen Vertheilung galvanischer strome geführt wird, *Annalen d. Physik und Chemie* 72, 1847, 497-508.

[Kirk-1] T. P. Kirkman, The enumeration description and construction of knots with fewer than 10 crossing, *Trans. R. Soc. Edinburgh*, 32, 1885, 281-309.

[Klein] F. Klein, *Mathematische Annalen*, IX, 478.

[Kli] M. Kline, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, 1972.

[Kno] C. G. Knott, *Life and Scientific work of Peter Guthrie Tait*, Supplementing the two volumes of scientific papers published in 1898 and 1900, Cambridge University Press, 1911.

[Kur] The complete woodcuts of Albrecht Dürer, Edited by Dr. Willi Kurth, Dover publications, Inc. New York, 1983 (translated from 1927 German edition).

[Leh] D. M. Lehmer, Obiturary of Charles Newton Little, *Science*, Vol. LVIII, No. 1508, October 1923, N.Y. City, the Science Press.

[Lei] G. W. Leibniz, *Math. Schriften,* 1 Abt., Vol 2, 1850, 19-20.

[Lis] J. B. Listing, *Vorstudien zur Topologie*, Göttinger Studien (Abtheilung 1) 1, 1847, 811-875.

[Lit-0] C. N. Little, On knots, with a census for order 10, *Trans. Connecticut Acad. Sci.*, 18, 1885, 374-378 (also Ph.D. thesis, 1885, Yale University [Com]).

[Lit-1] C. N. Little, Non-Alternate ± Knots, *Trans. Royal. Soc.Edinburgh*, XXXIX (part III(No.30) - for sessions 1898-99), 1900, 771-778. (Communicated by Professor Tait. (With three plates). (Read July 3, 1899).

[Lit-2] C. N. Little, Non-Alternate knots of order eight and nine, (1889), *Trans. Royal. Soc.Edinburgh*, XXXV 1890,

[Lit-3] C. N. Little, Alternate ± Knots of order eleven, (read 21st July; revised December 1890), *Trans. Royal. Soc.Edinburgh*, XXXVI 1892, 235-255.

[Lom] S. L. Lomonaco, The modern legacies of Thomson's atomic vortex theory in classical electrodynamics, Proceedings of Symposia in Applied Mathematics, Vol. 51, 1996, 145-166.

[Mac] E. MacCurdy, *The Notebooks of Leonardo*, New York: Reynal and Hitchcook, 1938, vol.2, p.588.

[Mag] W. Magnus, Max Dehn. *Math. Intelligencer* 1 (1978), 132–143. (Also in: Wilhelm Magnus Collected Paper, Edited by G.Baumslag and B.Chandler, Springer-Verlag, 1984).

[Max] J. C. Maxwell, *A treatise on electricity and magnetism*, Oxford 2 (1873).

[M-T-1] W. M. Menasco, M.B.Thistlethwaite, The Tait Flyping Conjecture, *Bull. Amer. Math. Soc.* 25 (2), 1991 403-412.

[M-T-2] W. M. Menasco, M.B.Thistlethwaite, The classification of alternating links, *Annals of Math.*, 138, 1993, 113-171.

[Mey] Fr. Meyer, Dissertation, München 1878. Also *Edinb. Proc.*, 13, 1886.

[Mi] L. G. Miller, The earliest(?) description of a string figure, *The American Anthropologist*, XLVII (1945), 461-462.

[M-M] E. S. Munkholm, H. J. Munkholm, Poul Heegaard (ed. I.M.James), History of topology, 925–946, North-Holland, Amsterdam, 1999.

[M-4] K. Murasugi, Jones polynomial and classical conjectures in knot theory, *Topology*, 26(2), 1987, 187-194.

[Ol] L. Olesko (ed.), Science in Germany, Osiris (2nd series) 5 (1989).

[Per-0] Perko, Kenneth A., Jr. On the classification of knots, *Proc. Amer. Math. Soc.*, 45 (1974), 262–266.

[Per-1] K. A. Perko, *NAMS* 25(6), 1978.

[Per-2] K. A. Perko, Primality of certain knots, *Topology Proceedings*, vol. 7, 1982, 109-118.

[Per-3] K. A. Perko, Invariants of 11-crossing knots, Publications Math. d'Orsay, 1980.

[Plu] Plutarch of Chareonera, The age of Alexander, translated by I. Scott-Kilvert, Penguin Books, 1982.

[Pogg] Poggendorff's Biographisch Literarisches Handwörterbuch, Vol. 3, 1858-83 (A-Z), 1898.

[Po-1] H. Poincaré, Analysis Situs (&12), *Journal d'Ecole Polytechnique Normale*, 1 (1895), 1-121.

[Po-2] H. Poincaré, First Complément à l'analysis situs, 1899; in Œuvres, vol VI, Gauthier-Villars, Paris, 1953.

[P-1] J. H. Przytycki, Survey on recent invariants in classical knot theory, Uniwersytet Warszawski, Preprinty 6,8,9; 1986. (the third chapter of this book is based on [P-1]).

[P-5] J. H. Przytycki, Skein modules of 3-manifolds, *Bull. Ac. Pol.: Math.*; 39(1-2), 1991, 91-100.

[P-13] J. H. Przytycki, History of the knot theory from Vandermonde to Jones, *Aportaciones Matemáticas Comunicaciones*, 11, 1992, 173-185.

[P-18] J. H. Przytycki, *Teoria węzłów: podejście kombinatoryczne*, (Knots: combinatorial approach to the knot theory), Script, Warsaw, August 1995, 240+XLVIIIpp.

[P-19] J. H. Przytycki, Algebraic topology based on knots: an introduction, *Knots 96*, Proceedings of the Fifth International Research Institute of MSJ, edited by Shin'ichi Suzuki, World Scientific Publishing Co., 1997, 279-297.

[P-21] J. H. Przytycki, Classical roots of Knot Theory, *Chaos, Solitons and Fractals*, Vol. 9 (No. 4-5), 1998, 531-545.

[P-22] J. H. Przytycki, Knot theory from Vandermonde to Jones, with the translation of the topological part of Poul Heegaard Dissertation: Forstudier til en topologisk teori for de algebraiske flader sammenh[ae]ng, København 1898, Filosofisk Doktorgrad (Preliminary studies towards the topological theory of connectivity of algebraic surfaces), by A. H. Przybyszewska, Preprint 43, Odense University 1993.

[P-29] J. H. Przytycki, From 3-moves to Lagrangian tangles and cubic skein modules, *Advances in Topological Quantum Field Theory*, Proceedings of the NATO ARW on New Techniques in Topological Quantum Field Theory, Kananaskis Village, Canada from 22to 26 August 2001; Ed. John M. Bryden,

October 2004,
e-print: `http://front.math.ucdavis.edu/math.GT/0405248`

[P-35] J. H. Przytycki, The interrelation of the Development of Mathematical Topology in Japan, Poland and USA: Notes to the early history of the Knot theory in Japan, *Annals of the Institute for Comparative Studies of Culture, TWCU*, Vol. 63, 2002, 61-86;
e-print: `http://front.math.ucdavis.edu/math.HO/0108072`

[P-T-1] J. H. Przytycki, P. Traczyk, Invariants of links of Conway type, *Kobe J. Math.* 4 (1987) 115-139.

[Ra] J. Raeder, *Oribassi Collectionum Medicarum Reliquiae*, 4 volumes, Leipzig and Berlin, 1928-1933.

[Re-2] K. Reidemeister, *Knotentheorie.* Ergebn. Math. Grenzgeb., Bd.1; Berlin: Springer-Verlag (1932) (English translation: Knot theory, BSC Associates, Moscow, Idaho, USA, 1983).

[R-T-2] N. Y. Reshetikhin, V.Turaev, Invariants of three manifolds via link polynomials and quantum groups. *Invent. Math.* 103 (1991) 547-597.

[Ro-1] D. Rolfsen, *Knots and links*, Publish or Perish, 1976 (second edition, 1990).

[SSW] H. Sachs, M. Stiebitz, R. J. Wilson, An Historical Note - Euler Konigsberg Letters, *Journal of Graph Theory*, 12 (1), 1988, 133-139.

[Sar] G. Sarton, Ancient Science through the Golden Age of Greece, Dover Pub., 1993 (first edition: Harvard Univ. Press, 1952).

[Sar-2] G. Sarton, Introduction to the history of science, Baltimore: Williams and Wilkins, 1927-1948; (quotation from Vol. 1 p.216).

[Sha] H. I. Sharlin, T.Sharlin, *Lord Kelvin, the dynamic Victorian*, Penn. State University Press, 1979.

[She] R. B. Sher, Max Dehn and Black Mountain College, *Math. Intelligencer* 16(1), 1994, 54–55.

[Sil] R. H. Silliman, William Thomson: smoke rings and nineteenth-century atomism, *Isis*, 54, December 1963, 472-.

[Sim] O. Simony, Über eine Reihe neuer Thatsachen aus dem Gebiete der Topologie, *Math. Annalen*, 19 (1882), 110-120, and 24 (1884), 253-280.

[Stac] P. Stäckel, Gyuss als Geometer, in the X'th Volume of Gauss' Collected Works..D.

[Stil] J. Stillwell, Classical Topology and Combinatorial Group Theory, Grad. Texts in Math. 72, Springer-Verlag 1980.

[Str] D. J. Struik, *A concise history of mathematics*, Dover Publications, Inc. New York, 1967.

[SCKSSWW] D. W. Sumners (Editor), N. R. Cozzarelli, L. H. Kauffman, J. Simon, D. W. Sumners, J. H. White, S. G. Whittington, *New Scientific Applications of Geometry and Topology*, Proceedings of Symposia in Applied Mathematics, Vol. 45, AMS 1992.

[Ta-0] P. G. Tait, Some elementary properties of closed plane curves, *Messenger of Mathematics*, New Series, No.69, 1877, (communicated at the 1876 Meeting of the British Association). Reprinted in Scientific Papers of P. G. Tait, 270-272.

[Ta] P. G. Tait, On knots I, II, III, Scientific Papers, Cambridge University Press, 1898-1900. Including: *Trans. R. Soc. Edin.*, 28, 1877, 35-79. Reprinted by Amphion Press, Washington D.C., 1993.

[Ta-1] P. G. Tait, Johann Benedict Listing, *Nature*, February 1, 1883. Reprinted in Scientific Papers of P.G.Tait.

[Ta-2] P. G. Tait, Listing's Topologie (Introductory address to the Edinbutgh Mathematical Society, November 9, 1883), *Philosophical Magazine*, January, 1884. Reprinted in Scientific Papers of P.G.Tait.

[This-1] M. B. Thistlethwaite, Knot tabulations and related topics, *Aspects of topology, LMS Lecture Notes Series*, 93 (1985), 1-76.

[This-3] M. B. Thistlethwaite, A spanning tree expansion of the Jones polynomial, *Topology*, 26 (1987), 297-309.

[Thomp] S. P. Thompson, The life of William Thomson: Baron Kelvin of Largs, 2 vols., London: Macmillan, 1910.

[Thomps] D'A. W. Thompson, On Growth and Form, Dover Publications, 1992 (republication of "A New Edition" by Cambridge University Press, 1942; "I wrote this book in wartime, and its revision has employed me during another war".)

[Thom] J. J. Thomson, A treatise on the motion of vortex rings, The Adams prize in 1882, (published in 1883).

[Thoms] W. H. Thomson (Kelvin), On vortex motion, *Trans. R. Soc. Edin.*, 25 (1867), 217-260. See also *Proceedings of the Royal Society of Edinburgh*, February 1867.

[Ti-1] H. Tietze, Über die topologischen Invarianten mehrdimensionaler Mannigfaltigkeiten, *Monatshefte für Mathematik and Physik* 19, 1908, 1-118.

[Ti-2] H. Tietze, Über Simony-Knoten und Simony-Ketten mit vorgeschriebenen singulären Primzahlen für die Figur und für ihr Spiegelbild, *Math. Zeitschrift*, 49, 1943, 351-369.

[Tu-2] V. G. Turaev, The Conway and Kauffman modules of the solid torus, *Zap. Nauchn. Sem. Lomi* 167 (1988), 79-89. English translation: *J. Soviet Math.* 52, 1990, 2799-2805.

[Tu-Vi] V. G. Turaev, O.Y.Viro, State sum invariants of 3-manifolds and quantum 6j-symbols, *Topology*, 31 (1992), 865-902.

[Tu-We] V. G. Turaev, H.Wenzl, Quantum invariants of 3-manifolds associated with classical simple Lie algebras, *Internat. J. Math.*, 4(2), 1993,323-358.

[Tut-1] W. T. Tutte, A ring in graph theory, *Proc. Cambridge Phil. Soc.*, 43, 1947, 26-40.

[Va] A. T. Vandermonde, Remarques sur les problèmes de situation, *Mémoires de l'Académie Royale des Sciences* (Paris) (1771), 566-574.

[Weil] A. Weil, Riemann, Betti and the birth of topology, *Arch. for Hist. Exact Sci.*, 16, 1979, 91-96.

[Web] C. Weber, A stroll through knot theory, *New. J. Chem.*, 17 (1993), 627-644.

[Weith] H. Weith, *Topologische Untersuchung der Kurven-Verschlingung*, Inaugural Dissertation, Zürich 1876.

[Whit-1] H. Whitney, The coloring of graphs, *Ann. of Math.*, 33, 1932, 688-718.

[Wh] W. Whitten, Knot complements and groups, *Topology*, 26, 1987, 41-44.

[Wie] M. H. Wiencke, "Further Seals and Sealings from Lerna," *Hesperia* 38, 1969, 500-521.

[Wir] W. Wirtinger, Über die Verzweigungen bei Funktionen von zwei Veränderlichen, *Jahresbericht d. Deutschen Mathematiker Vereinigung*, 14 (1905), 517. (The title of the talk supposedly given at September 26 1905 at the annual meeting of the German Mathematical Society in Meran).

[Wit] E. Witten, Quantum field theory and the Jones polynomial, *Comm. Math. Phys.*, 121, 1989, 351-399.

[Wo-Kr] D. Wolkstein, S. N. Kramer, Inanna: Queen of heaven and earth, Harper Collins Publisher, 1983.

[Yale] Yale University, Obituary Record of graduates deceased during the year ending July 1, 1924, New Haven (Published by the University), 1924, p.1211.

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II.5 Poul Heegaard thesis of 1898

Poul Heegaard (1871-1948) is well known to topologists because of his theorem about decomposition of any 3-manifold into two handlebodies. His achievements in knot theory are much less known.[41] One of the reasons for this was the fact that his 1898 Doctoral Dissertation [Heeg] was written in Danish. It was translated into French only 18 years later. I learned of the Heegaard work on knots only from the book by J. C. Stillwell [Stil]. When visiting Odense University (1992-1994), I had a privilege to have every day access to a copy of Heegaard dissertation and I was lucky to know very gifted high school student, Agata Przybyszewska, who kindly agreed to translate parts of the dissertation. We present here the Przybyszewska's translation (from Danish) of the Preface, and the sections 6-12 of Heegaard's Dissertation. The second, topological part of the dissertation is translated by Przybyszewska in [P-22] (references in the main part of the chapter). One can hope that the whole Dissertation will be translated soon. As to importance of Heegaard results we can best quote from the Stillwell book [Stil]:
"Heegaard's results lay dormant (although noted by Tietze 1908) until the publication of the French translation of his thesis in 1916. The translation was checked for mathematical soundness by J.W.Alexander, fresh from his work on homology groups, and we may surmise that the collision of these ideas led to the fruitful discoveries which were to follow. Alexander must also have read Tietze 1908 at this time, because in short order he disposed of two of the most important of Tietze's conjectures: Alexander 1919a[42] shows that there are nonhomeomorphic lens spaces with the same group, while Alexander 1919b[43] proves that any orientable 3-manifold is a branched cover of S^3. Later in 1920 he finally took the cue from Heegaard's example and looked for torsion in cyclic covers of S^3 branched over various knots."

[41] The only, known to me, research papers on topology by Heegaard are [D-H, Heeg]. For other Heegaard's work see his bibliography (Brjan Toft found the box donated to Odense Math. Dept. by Niels Erik Nørlund (1885-1981) with Heegaard papers (as in the Heegaard's bibliography except [D-H] and only French version of [Heeg]).

[42] Note on two three-dimensional manifolds with the same group, *Trans. Amer. Math. Soc.*, 20 (1919), 339-342.

[43] Note on Riemann spaces, *Bull. Amer. Math. Soc.*, 26 (1919), 370-372.

Preliminary studies towards the topological theory of connectivity of algebraic surfaces.

Poul Heegaard
Copenhagen, 1898

1. Preface

It is commonly known, which development took the theory of functions of one independent variable, when the imaginary values of the independent variable were considered and the theory became linked with a geometrical presentation of the imaginary numbers.

When we shall construct a theory of functions with two independent variables, it will therefore be natural to look for a similar presentation. The fact that such a long time has lapsed before the work on this quite natural generalization was started can, among other reasons, be explained by the fact that the examinations with two independent variables are a lot more difficult than with one. The variety of possibilities produces conditions, for which there is no analogy in the theory of one independent variable. Picard observes thus:

" On voit, par ce qui précède, les différences profondes qui separent la théorie des fonctions algébrique d'une variable de la théorie de fonctions algébriques de deux variables indépendantes. L'analogie qui souvent est un guide excellent, peut devenir ici bien trompeuse."[44]

Another difficulty on the way towards a lucid presentation is of course the fact that the geometrical structures that should play the part of the Riemann surfaces are 4-dimensional.

Even if the analogy can be misleading in our examination of the details, then a survey of the methods employed in the theory of one independent variable will give us a good working program for such a study. Let us then present such a survey.

The theory of functions with one independent variable is very closely connected with the theory of the algebraic curves. The geometry of such a curve becomes therefore of fundamental importance.

The examinations are set up from quite a different points of view. The most important made use of:

1. the elementary algebraic theorems. Among these we can mention

   (a) Examination of the adjoint polynomials, created by Brill and Nöther in the paper: Ueber die algebraischen Functionen (M.A. vol. 7, 1873).

   (b) The examinations of the linear groups of points made by Italians. They have tried to liberate the former theory from its projective form, so they

[44]We see from the proceeding the important differences that separate the theory of algebraic functions of one variable from the theory of algebraic functions of two independent variables. The analogy that often is an excellent guide, can just as well be faulty.

could develop it independently of such concepts as degree, class e.t.c., cf. a summary by Castelnuovo and Euriques (Sur quelques recent resultantas dans la th'eorie des surfaces algébriques. M.A. vol. 48, p.242, 1897)

(c) Number-geometrical examinations, especially a number of papers by Zeuthen (e.g. M.A. vol.3 and vol. 9)

2. the examinations of the algebraic curves which belong to the transcendental functions. These examinations originate from Riemann's pioneer work (1857) are well known, so there is no reason to mention them any further.
3. the topological examinations of the Riemann surfaces that represent the algebraic curve. Here once again two different approaches were chosen:

(a) We can either determine the connectivity number of surfaces using a theorem which we can regard as a generalization of Euler's theorem about polyhedra (Riemann, Neumann).

(b) Or we can puncture the Riemann surface and next bring it by continuous deformation into a normal form. As far as we can tell this approach was only implemented by Jul. Petersen (Foreleasninger over Funktionsteori, chapter IV). Listing uses indeed such a procedure in shaping the "diagram" of a spatial figure in hereby arriving to a generalization of Euler's theorem (Census räumlicher Complexe, 1862), and Betti (1871) uses such a consideration in his examinations of the connectivity number of a n-dimensional space in generality, but both of these papers seem to have been unnoticed for a long time. This method gives us a remarkably lucid presentation of the discussed situation.

The transformations of algebraic surfaces play an analogous part to the theory of functions of two variables. There already exist an amount of works - especially from recent times - in which the problem is treated from points of view that correspond to these enumerated here.
1. Elementary algebraic examinations.

(a) Adjoint polynomials. This theory originates from Clebsch (C.R. Dec. 1868) an Nöther (Zur aindeutigen Entsprechen... I & II, M.A. vol.2, 1869 and vol.8, 19874)

(b) Linear systems of curves. The Italians have created a theory of linear systems of curves on surfaces that is analogous with the before mentioned theory of groups of points on curves. By means of this the invariants of surfaces can be determined (cf. the mentioned summery by Castelnuevo and Enriques).

(c) Surface transformations have also been examined by Zeuthen from the number geometrical point of view (Études géom'etriques..., M.A. vol.4).

2. Examinations by transcendent functions. Already in his paper in M.A. vol. 2 Nöther considers the integrals of the form:

$$\int\int \frac{Q(xyz)dxdy}{f'_x}$$

Picard introduces the integrals of the form:

$$\int_{x_0 y_0 z_0}^{xyz} Pdx + Qdy$$

Where P and Q satisfy the condition of integrability (Liouville Journ. 1885&1886). Finally, Picard has in his prized paper (Memoire sur la theórie des fonctions algébriques de deux variables (Liouv. Journ. series 4. vol. 5, 1889) and in the recently printed book on the same subject (Picard at Simart: Théorie des fonctions algébriques de deux variables indépendantes, 1987)) given a coherent presentation of the whole theory.
Poincaré's paper "Sur les residus..." (Acta mathematica, vol. 9, 1887) should also be mentioned here.

3. Topological examinations. This direction it is almost empty. In the works of Picard there is a lot, but none of it is carried through, as he whenever possible prefers the analytical presentation. The whole trouble is that the Riemann-Betti theory of connectivity numbers is very insufficient and difficult to use when we speak of manifolds of more than 2 dimensions. Poincaré has tried to complete it (Analysis Situs, Journ, de l'école politéchnique, series 2. Cah. 1. 1895) without succeeding, according to our opinion. Later Picard has given his share: W. Dyck has already before worked on the problem (Beitriäge sur Analysis situs, M.A. vol. 32 and 37), but a quite satisfying theory can not be found anywhere.

Previous to the examination in this direction, we therefore need to have a theory of topological correspondence of manifolds of dimension greater than 2. What already does exist in this direction, can be compared to that what is mentioned in 3(b).

The following pages contain no unified totality, only a study of the problem; its difficulty can be used as an excuse therefore. To let the examinations be considered as they should be, it will perhaps be a good idea to advance a line of thought that have been the guide of my examinations.

By accident I have noticed that the Zeuthen-Halphen generalization of the genus theorem could be proved by a pure topological argument; namely, for any two Riemann surfaces, for which we assume a $\mu - \nu$ value-correspondence between their points, we can construct new Riemann surfaces, which correspond bijectively to each other. The equation, which says that two constructed surfaces have equal connectivity number, states exactly the mentioned theorem. The fact, which is so important to the enumerative geometry, that the generalized genus theorem can be used without infinitesimal examinations of the coincident points (or their substitutes), while these examinations are necessary when using the other formulas of correspondence (cf. Acta mathem., vol. 1, p. 171) let us see these circumstances in a new light. As I have found later the proof in a paper written by Hurwitz (M.A. vol. 39), I shall not go into details of the proof here.

My thought was that it could be possible to construct a theorem similar to that for the algebraic surfaces, but before that a lot of work ought to be done: there should be created something for algebraic surfaces that could correspond to the Riemann surfaces of algebraic curves; next topological criteria for their unique correspondence should be advanced. And, as it is already mentioned, the material for such an examination that already existed was either insufficient or full of mistakes. It was in trying to correct these mistakes, and to fill up the gaps, that the contents of the following pages originated.

Second part
On topological connectivity numbers

6 Topology.

Descartes' analytical method was presumably a universal method for solving geometrical problems, but as a rule it gives constructions that are far behind the Greeks in simplicity and elegance. During his attempts of penetrating the principles of the curious geometrical analysis, Leibniz formulated a number of observations, which he called *Analysis situs* or *Geometria situs*. Analysis situs or Geometria situs. (Leibnizens gesammelte Werke herausg. von G. H. Pertz,1858, mathematische Schriften vol. 1, p. 178: De analysi situs). Sometimes in connection with this, Leibniz has been named the father of the modern topology: i.e. the branch of mathematics, which aims at the qualitative properties of the objects analyzed without occupying itself with the quantities and the metrics. Leibniz notices somewhere on the occasion of his theory:[45]

"Figura in universum praeter quantitatem continet qualitatem seu formam," but in reality his theory has no points of similarity with what we today understand as topology. In the recent times the mathematicians became interested in a lot of topological problems. It should be enough to remind oneself of

- the studies of topological knots, nets, and the like by Tait or Simony;
- the graphs;
- the appearance of the graphical curves; the problem of colorings, and hereby separating, the countries on a map by means of 4 different colors;
- the problem of folding a stamp
- and "last but not least" of the extensive attempts to make a theory of the connectivity numbers of the n-dimensional manifolds, and of generalizing Euler's theorem of polyhedra – 2 attempts that are closely connected.

(A good survey of the literature can be found in (W.Dyck: Beiträge zur Analysis

[45] A figure generally contains besides an extent (quantitas) a nature (qualitas) or form.

situs, M.A. vol. 32)). The usage is somewhat staggering, but it is most fair to reserve the name 'analysis situs' for the last mentioned kind of studies, and against it, to use the name 'topology' about all research of qualitative nature, as Listing has proposed it (Vorstudien zur Topologie).

7 Analysis situs.

Presumably, it will be very difficult to formulate basic theorems about the connectivity in n-dimensional manifolds; anyhow it will probably be a good idea, if we did study the concrete cases in more details, than it has been done so far, instead of going at the whole matter in its abstract generality. In theory, the logic should be enough to ensure the development of mathematics, but practice shows us, what a mighty lever is the sense of tact that develops when a theory is applied to a large class of the concrete cases; even if a theory appears to be very logical and plausible, it can very easily contain mistakes that will only be revealed when it is additionally controlled by the understanding.

Riemann and Betti are the first who have tried to generalize the theory of connectivity of surfaces, which Riemann himself with great luck has applied to the theory of Abelian integrals. After Riemann's death Betti put forward a paper on the mentioned subject (Sugli spazi di un numero qualanque di dimensioni, Ann. di matem. series 2, vol. 4, 1871). He does not speak of any collaboration between himself and Riemann; but we judge from the fragments of the theory, which Weber has collected together from notes that Riemann wrote down (Riemann: Gesammelte mathematische Werke, 2. Aufl. fragment XXIX), that Riemann has contributed essentially during his stay in Italy (Ges. Werke, p.555) to the thoughts that are found in Betti's paper. It is where the definition can be found of the connectivity numbers of the manifolds of dimensions greater than 2. Later, different people have tried to improve and complete the theory:

- Dyck in "Beitr"age zur Analysis situs"(M.A. vol. 32 and vol.37),
- Poincaré in the paper "Analysis Situs", and
- Picard in his studies of functions of two variables (see p.5)

Even before I knew the last mentioned works, I have decided to try another way than the one of Riemann-Betti that is to say, an attempt to generalize *Jul. Petersen's* puncture method which I recalled from lectures. (I became acquainted with Listing's "Diagram" and "Trema" a long time after, and the same goes for the studies by Betti, which are related to this subject, and which I until lately only have known from the short summary in "Fortshritteder Mathematik"). Among other things it seemed awkward to me that the n connectivity numbers were not adequate for a topological characterization of a manifold when $n > 2$. When I became acquainted with the papers that I'll mention later, especially that one of Poincaré, I started to doubt about the rightness of my choice, as I compared the elegant methods that I met there, with the somewhat clumsy and awkward theory that I myself was working on; but then I thought to have discovered that the patch, which I have chosen, would illuminate some circumstances that would not appear clearly by means of the other method, and because I held a method in my hands to find sufficient criteria for the equivalence of n-dimensional manifolds, so I decided to continue despite of the great difficulties, which I encountered. The fact that two manifolds are equivalent does mean that they correspond to each other, point to point, in such a way that two arbitrary points in one of them are close to coincide when the two corresponding points in the other manifold are. (We will later return to the connection between Poincaré's concept of homémorphisme, Analysis Situs &2, and the concept of equivalence that has been defined here). The question that arises is, how we shall cut a closed manifold (variéte fermée; Poincaré 1.c. &1) to make it simply connected. To solve this problem we use the following procedure: the manifold is punctured: i.e. the elementary manifold, which constitutes the neighborhood of a point, is removed. From the puncture arises a boundary, and it is extended by the means of continuous deformation in such a way that we remove more and more of the given manifold. We continue like this, until a part of the boundary encounters other parts of itself; in such places we stop the deformation, when the distance between the parts that are encountering has become infinitely small. If we continue in this way, then we end up with a diagram, which is made from a system of manifolds of lower dimensions than the original one[46] – or rather: it is made from the closest neighborhood of this system, that is, a manifold that is infinitely small in the n^{th} dimension. We call the system of manifolds of a lower dimension, which is the boundary of the diagram, its nucleus. This diagram has two meanings. First, it presents the manifold which is equivalent with the given one after the puncture. If we succeed in proposing normal forms into which the diagrams could be reduced, then the fact that the normal forms of the diagrams are identical will be the necessary and sufficient condition for two diagrams to be equivalent. – But the nucleus of the diagram indicates the cuts that shall be done to make the given manifold simply connected; if we make the discussed extending in reverse order, then the manifold, which remains after performing the cuts determined by the diagram, will contract into that elementary manifold, which is removed by the puncture.

We will later return to this comparison of results that are found along this path, and the Riemann-Betti theory of connectivity numbers.

8. The diagram of the Riemann surfaces

We will here shortly outline the development of this theory, although the theory of connectivity numbers of the Riemann surfaces is well known, and it is even found discussed in a way very similar to the one that is presented here in *Jul. Petersen* (Theory of functions, chap. IV). We will especially show that it can be build on the basis of the diagram *only* that is to say without using the Euler's generalized

[46]It is called a spine of the manifold in the modern terminology [translator's remark]

theorem (the theorem of the invariant number of surfaces, $t - f$).

We imagine the Riemann surface carried by a sphere; we place arcs of a circle from a point a to n points, which carry the branching points of the surface; we assume that they have the following orders: $k_1 - 1, k_2 - 1, \ldots, k_f - 1$. Somewhere, we puncture *all* the n leaves, and by extending the puncture, their edges are led to the branching points and to a along the drawn arcs of main circle.What remains from the surface, after having removed some bands with dead ends, is the following:

1. f elementary surfaces around the branching points,
2. n elementary surfaces above a, or
3. $k_1 + k_2 + \ldots k_f$ bands, which connect the surfaces mentioned at the beginning with the the surfaces mentioned last.

We cut the $n - 1$ bands, whose boundaries belong to the different edge curves, and the scraps are removed. Now, the surface represents a diagram, which corresponds to 1 puncture. It contains $f + n$ elementary surfaces in all, which are connected with $\Sigma(k) - n + 1$ bands; we use the $f + n - 1$ of these bands to connect the $f + n$ elementary surfaces in such a way that they form an elementary surface. From its edge originates

$$\sum_1^f (k) - n + 1 - f - n + 1 = \sum_1^f (k - 1) - 2n + 2$$

handles that all can easily be matched into double handles. Let m be the degree of a manifold on an arbitrary complete curve on the algebraic curve which corresponds to the surface; then we have:

$$\Sigma(m - 1) + n' - 2n + 2 = 2p.$$

We have therefore obtained a normal form with p double handles, and the necessary and sufficient condition for two Riemann surfaces to be topologically equivalent is therefore that they have equal p's.

If we would like to have a normal form for the *closed* Riemann surface, then we slide the two mouths of one of the handles from each double handle to the center of the other one, and thereafter by continuously widening its mouths absorb the edges of the other handle. Thereafter, we obtain Klein's normal form by closing the surface with an elementary manifold: a sphere with p tubular handles.

What was presented here can serve as a programme of the research that will be instituted. We will mention here yet another way of processing the matter, which is suitable for being generalized to algebraic surfaces. We assume that the algebraic curve φ_n has no singular points. (How the situation looks when such point exist, can be seen by passing to the limit from the curve already considered). If the curve is changed continuously, without introducing singular points, and the degree remains unchanged, then the result will be an equivalent Riemann surface. We will let it turn into a curve ψ_n, which is infinitely close to being dissolved into

n arbitrarily placed, straight lines. The fact that this can always be done, can be realized by e.g. considering the linear bundle $\varphi_n + \lambda\psi_n = 0$; the curve will only have singular points for a specific, finite number of values of λ, and we can therefore let λ go from 0 to ∞ without passing through them. The straight lines are produced from n concentric spheres with equal radii; as we go to ψ_n by dissolving the $\frac{1}{2}n(n-1)$ double points, there will be two branching points from each double point; branching points matched in this way are infinitely close one to another, and are connected by the branching lines.

We cut them out of the surface by the means of two circles around two branching points, each on its own leaf, that are matched in this way; the part that was cut out is now transformed into a tube; this is repeated for all the other pairs, and after having moved the n spheres away from each other, the $\frac{1}{2}n(n-1)$ tubes are added in the right way to their holes; $n-1$ tubes are used to transform the spheres into 1 sphere, and from that one there originates $\frac{1}{2}n(n-1) - n - 1 = \frac{1}{2}(n-1)(n-2) = p$ tubular handles: we have once again obtained Klein's normal form.

If a closed surface is one-sided, then its diagram contains, besides the double handles, an amount of single handles, each with 1 twist.

9. The diagram of the 3-dimensional manifolds

Let a closed, 3-dimensional manifold be defined by

$$x_i = \theta_i(y_1, y_2, y_3) \ (i = 1, 2, \ldots, n)$$

and by an amount of inequalities of the form

$$\psi_i(y_1, y_2, y_3) > 0$$

and by the analytical extension of this space (cf. Poincaré, Analysis Situs §3), or, in keeping it more with the nature of topology: let it be defined as what we obtain from joining elementary spaces, whose boundaries are split into areas by a network of lines, which [the elementary surfaces] are joined together one with another in a specific way; it happens so that the neighborhood of every point in the manifold that comes from the lines in the network becomes an elementary space. If the manifold defined as above is supposed to be connected, then, to begin with, it is possible to join these spaces into 1 elementary space, whose surface satisfies the same conditions. There are no difficulties in the following extension of a puncture.

The nucleus of the diagram will be formed from pieces of surfaces that meets at pieces of curve; they meet in points, the junctions of the nucleus of the diagram. We could imagine the [pieces of] surfaces being simply connected, as it is possible in the opposite case to place the pieces of curve with their extreme points onto the pieces of curves of the diagram, so that they cut the pieces of surface into simply connected parts; the pieces of curves, which are placed in this way, can be counted among the other ones.

Now, we proceed to consider the diagram itself. Every junction is surrounded by a elementary manifold, which e.g. we can shape as a sphere; the pieces of curve, which connect them are surrounded by string-formed spaces that are fastened with small elementary surfaces to the spheres just discussed. Let us call these spaces *strings*; they can be described by an elementary surface, which moves from one of the spheres to another (this can by the way happen in two essential ways; see §11). The space that surrounds one of the pieces of elementary surface of the diagram is a plate-formed space that is bounded by two elementary surfaces, which make the two faces of the plate, and by a band of surface of the connectivity of the annulus, which makes the side of the plate; this space, which we will call a *plate*, is fastened with its edge along a piece of surface on the surface of the body, which is formed from the junctions and the strings. Let the diagram have a junction spheres, b strings and c plates. a spheres are now connected by $a-1$ strings into 1 elementary space, which makes a spherical form, and from this central sphere originates $b-a+1=p$ strings; on the surface, whose connectivity number is $2p+1$ runs c closed curves, along which the edges of the plates is $2p+1$ runs c closed curves, along which the edges of the plates should be fastened.

The necessary and sufficient condition for having such a system of strings with fastening curves for plates to produce the diagram of a closed space is that the fastening curves are formed from p annular cuts, which do not cut up the surface of the string system.

Namely, the necessary and sufficient condition is that the surface is equivalent, after the addition of the plates, to the surface x of the elementary space after that is removed by the puncture: i.e. it has the same connectivity as a sphere.

It the diagram really corresponds to a closed manifold, then the two faces of an arbitrarily chosen plate will correspond to two simplyconnected areas on x. If the plate is removed from the diagram, then the connectivity of its surface is changed, as the two faces of the plate are replaced by that band along which the plate was fastened, and which is equivalent to an annulus. To have x to go through the same change of connectivity, the two corresponding areas are cut out and connected with a tube. If we continue in this way until all the plates are removed, then x is replaced by a sphere, which has as many tubular handles, as there were plates; this surface has to be of the same connectivity as the surface of the string system. (Hereby we *do not* say that the space, which it bounds, has to be of the same connectivity as the string system itself). We can conclude from the above that *the number of the plates has to be p.* The connection between the two surfaces is such that the lines, which run through one of the on p fastened bands, correspond to the lines that on the other plane cut through one of the p handles (meridian curves). The surface of the system of strings is therefore not disconnected if we remove the fastening bands.

The fact, that the mentioned condition is sufficient, can be seen by noticing that the surfaces, which are equivalent, are the surfaces that remain, after we have placed arbitrary systems from the biggest number of annular cuts that do not disconnect [the surface] in two equivalent, closed surfaces. The surface that remains, when there are removed p fastening bands from the surface of the string system, is therefore equivalent with a sphere with $2p$ holes. The surface of the body that we get by fastening the plates is therefore equivalent with that sphere, which we can get by capping off the $2p$ holes with elementary surfaces.

10. Oriented manifolds.

Indikatrix. Before we continue the studies of the diagram, it is necessary to add some remarks. Three points a, b and c on a closed curve determine a positive direction on it; in a surface the positive orientation can be determined by determining a positive direction on a small, closed curve that does not cut itself; hereby once again it is possible to determine a positive and a negative side of the surface in that part of space, which closely surrounds the small, closed curve. If we extend this [determination of the direction] to the whole surface, then it can be seen that it is possible to distinguish between *one-sided* and *two-sided* surfaces. It can be seen that here is a field for research, which may be extended to include manifolds of higher dimensions; the orientations on the positive and negative sides of curves and surfaces are fundamental concepts of topology.

The neighborhood of a point on a curve is a little line segment, whose extreme points we will call 1 and 2, thus that 12 indicates the positive direction of the line segment; we will call 12 an *indikatrix of the* 1^{st} *order* (cf. *Dyck*, M.A, vol. 32, p. 473), and we say that a curve provided with an indikatrix is *oriented.* — The neighborhood of a point in a surface form a small, closed curve; if it is oriented, then the piece of a surface that hereby is bounded is called *indikatrix of the* 2^{nd} *order.* — In a space, which is defined in the discussed way, the neighborhood of a point are a sphere-like surface; when provided with an indikatrix of the 2^{nd} order then we obtain the *indikatrix of the* 3^{rd} *order* of the space. — In a very similar way we could make the *indikatrix of the* 4^{th} *order* in **T** or in a 4-dimensional manifold, which we get from joining together elementary manifolds in **T**. The definition can be extended, so that it analytically defines the manifolds of any dimension, but it will be difficult to append it in this state when the manifold is not easy to grasp. Therefore, only if the dimension number is smaller than 5, then this study will be of any importance to us. *An indikatrix of the* n^{th} *order* contains a number of successive indikatrix of the orders $n-1, n-2, \ldots, 1$; the last one is the piece of line 12. The indikatrix can be displaced in the manifold; we can let it return to its original position, so that the successive indikatrix, until the one of the 2^{nd} order, cover each other; then there are two possibilities: *either* 12 can be put back into its original position *or* it can be put back in the position 21. If the first case holds for all possible displacements in the manifold, then we say it is *two-sided*, otherwise it is *one-sided.* A two-sided manifold can be oriented in two different ways; such two manifolds are called *opposites.*

I have already been working on these definitions when I saw Poincaré's paper (Analysis Situs, §4 and §8); however I continued using my own ones as these of Poincaré are obviously designed for topological studies in analytical shape. I have noticed that the definition, which can be found in Picard's and Simart's book (p. 23), agrees with mine according to the contents, but I have preferred to follow my original idea, which give the imagination something more tangible to lean against.

We would like to orient *the boundary* of a two-sided and oriented n-dimensional manifold $\mathbf{M}$ by the indikatrix of the manifold itself. It can be displaced in such a way that the part of its boundary, which is made by the indikatrix of the $(n-1)$ order will be in the boundary of $\mathbf{M}$, and this can (along with all them of a lower order, which it contains) be used for orienting the boundary. Conversely, the manifold itself can be oriented in a similar way by the means of the indikatrix of the boundary. Below, we will still demand that the boundary is oriented in agreement with the indikatrix of the manifold.

(The usual definition of a positive and a negative side of a surface is based on its positive orientation and assumes that the indikatrix of the *space* or its substitute is already given — the right hand, a watch or something like that).

Oriented corners. Let an m-dimensional manifold be defined to be the collection of points

$$(y) \equiv (y_1, y_2, \ldots, y_m),$$

where the real values of y are not subjected to any condition as for their variation. We denote the point $(0, 0, \ldots, 0)$ by o. The half-lines $A, B, \ldots, K$ (i.e. that part of the straight lines that goes form o in a positive direction) are drawn through this point and every one of the points $(a), (b), \ldots, (k)$. We assume that the points have such a general position that the determinant $\Delta =\mid a, b, \ldots, k \mid$ is different from 0. As p denotes an integer between 1 and m, one can be sure that it will be never possible to place a manifold of $(p-1)^{th}$ dimension through q of the lines. A plane manifold of the $(m-1)^{th}$ order, which goes through the remaining half-lines, corresponds to each of the half-lines. These m plane manifolds form, in connection with the half-lines, what we will call an m-dimensional corner together with o as its vertex and the half-lines as its edges; we call it oriented when the succession [in which we name the edges] is determined. Again, $m-1$ of the half-lines form, in the plane manifold, in which they are, an $(m-1)$-dimensional corner, and so on; in general: p arbitrary edges are in a p-dimensional manifold and form a p-dimensional corner, in the sense above.

We could easily determine the connection between the ordering of the edges and the determination of indikatrix in the manifold (y). As boundary of the indikatrix we can choose for example the spherical manifold $\Sigma(y^2) = r^2$. This cuts the edge A at the point a'; the neighborhood of a' is used as indikatrix of the $(m-1)^{th}$ order, and we imagine that it bounds the common part of $\Sigma(y^2) = r^2$ and the plane manifold of $(m-1)^{th}$ dimension through $B, \ldots, K$. This manifold is treated in the same way: the neighborhood of the point, b', at which it cuts the spherical manifold, is used as indikatrix of the $(m-2)^{th}$ order, and is bounded by [the parts of] the plane manifold through $C, \ldots, K$. Continuing in this way, following the order of edges which is determined by the orientation of the corner, we end up with a closed curve (the intersection between $\Sigma(y^2) = r^2$ and the plane through the last two corners I and K). The intersection of I we call i', and the one of K we call 1; the negative extension of K cuts $[\Sigma(y^2) = r^2]$ in a point, which we will call 2. We let the line segment $1i'2$ be the indikatrix of the 1^{st} order, and hereby the indikatrix of the n^{th} order is determined. The connection between it and the corner can shortly be described as follows: the first edge of the corner meets the boundary of the indikatrix at a point of the indikatrix of the $(m-1)^{th}$ order, its other edge at a point of the indikatrix of the $(m-2)^{th}$ order e.t.c., its $(m-1)^{th}$ edge at a point of the indikatrix of the first order, and finally its m^{th} edge at the point 1, and its negative extension at the point 2.

By continuous displacement of the corner, we still maintain the condition that no p edges may ever be in the same plane manifold of $(p-1)^{th}$ dimension. If the vertex of the corner is moved to the point (t), and if the edges of the corner are determined by the points $(a'), (b'), \ldots, (k')$, then the determinant

$$\Delta = \mid a'-t, b'-t, \ldots, k'-t \mid$$

will have the same sign as Δ at the starting point.

The corner can be moved in such a way that the $m-1$ first edges covers the corresponding edges in a given m-dimensional corner; the corners are said to be of the same or opposite kind depending on whether the m^{th} edge can be covered by the m^{th} [one] or not.

We imagine an indikatrix in (y) chosen in agreement with the co-ordinate corner: i.e. the corner, whose vertex is in o, and whose edges go through the points

$$\begin{array}{c}(1,0,\ldots,0)\\(0,1,\ldots,0)\\\ldots\ldots\ldots\ldots\\(0,0,\ldots,1)\end{array}$$

Here $\Delta = 1$; consequently Δ' will be positive for all the corners that are oriented in agreement with the indikatrix of (y), and vice verse.

Comparison with the theory of Poincaré. (Analysis situs §8). An elementary manifold determined by (Anal. sit. §3)

$$y_i = \theta_i(y_1, y_2, \ldots, y_m)\ (i = 1, 2, \ldots, m)$$

together with inequalities of the form

$$\psi(y_1, y_2, \ldots, y_m) > 0$$

can be considered as an image of an elementary manifold in (y), determined by the inequalities $\psi(y) > 0$. We orient it by the means of an indikatrix, which is the image of the one in (y).

Now we will outline, how it can be seen that our presentation agrees with that of Poincaré. The problem is as follows:

There are given two elementary manifolds of the m^{th} order, υ_1 is determined by

$$x_i = \theta_i(y_1, y_2, \ldots, y_m),$$
$$\mid y_k \mid < \beta_k;$$

and υ_2 is determined by

$$x_i = \theta_i'(z_1, z_2, \ldots, z_m),$$

$$| z_k |< \gamma_k .$$

It is assumed that v_1 and v_2 have an elementary manifold v' in common. *Poincaré* defines the sequence of the parameters z by means of the sequence of the parameters y — or, what significantly is the same thing: the sequence of the y's by the sequence of the z's — as he demands that the function determinant [Jacobian]

$$\frac{\partial(y_1, y_2, \ldots, y_m)}{\partial(z_1, z_2, \ldots, z_m)}$$

is positive for the points in v'.

We define the order of the z's by demanding that the two indikatrices in v', which correspond to the coordinate corners in (y) and (z), agree with each other.

Now, we will show that these two definitions yield the same result. Let

$$(y^0) \equiv (y_1^0, y_2^0, \ldots, y_m^0)$$

and

$$(z^0) \equiv (z_1^0, z_2^0, \ldots, z_m^0)$$

be the two points from (y) and (z), which correspond to the same point in v'. We consider the corner in (z), which has its vertex in (z^0), and whose edges go through the points

$$\begin{array}{llll} (z_1^0 + dr & , z_2^0 & , \ldots & , z_m^0 \quad), \\ (z_1^0 & , z_2^0 + dr & , \ldots & , z_m^0 \quad), \end{array}$$

................................,

$$(z_1^0 \quad , z_2^0 \quad , \ldots \quad , z_m^0 + dr \quad),$$

where dr is a positive, infinitely small quantity. As the determinant Δ' becomes here $(dr)^m$, thus this corner is of the same kind as the coordinate corner in (z); the indikatrix that it determines is therefore in agreement with the orientation of (z). According to *our* definition the indikatrix in (y) can be determined, because we form the indikatrix in (y) using the equations (1) and (2) make the image in (y) of that part of (z), which corresponds to v', and at the same time it forms the image of the indikatrix of (z). Now, the important thing is to show that the corner, which corresponds to the indikatrix, which is constructed in this way, is of the same kind as the coordinate corner, when that one [the corner] is oriented according to the *Poincaré* definition. That corner is however defined as having its vertex at (y^0) and its edges going through the points

$$\left(y_1^0 + \left(\frac{\partial y_1}{\partial z_1}\right)_0 dr, y_2^0 + \left(\frac{\partial y_2}{\partial z_1}\right)_0 dr, \ldots, y_m^0 + \left(\frac{\partial y_m}{\partial z_1}\right)_0 dr, \right),$$
$$\left(y_1^0 + \left(\frac{\partial y_1}{\partial z_2}\right)_0 dr, y_2^0 + \left(\frac{\partial y_2}{\partial z_2}\right)_0 dr, \ldots, y_m^0 + \left(\frac{\delta y_m}{\partial z_2}\right)_0 dr, \right),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\left(y_1^0 + \left(\frac{\partial y_1}{\partial z_m}\right)_0 dr, y_2^0 + \left(\frac{\partial y_2}{\partial z_m}\right)_0 dr, \ldots, y_m^0 + \left(\frac{\partial y_m}{\partial z_m}\right)_0 dr, \right),$$

where the marks after the partial derivatives suggest that their values are to be taken at the point (z_0). Δ' has in this corner the value

$$\frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(z_1, z_2, \ldots, z_n)} \cdot (dr)^n,$$

which, according to the Poincaré definition, is positive, *q.e.d.*

An interchange of two of the edges of the corner, or an exchange of the positive direction of one edge, will change the type of the corner.

11. The continuation of the studies of diagrams of 3-dimensional manifolds.

We return to the diagram. The problem that is to be solved, is to reduce it to a normal form; I did not succeed in finding such a form, but I will here provide some remarks concerning the solution of the problem.

First, what is the condition that the closed, 3-dimensional manifold, which is defined by the diagram, is two-sided? First, we can make every closed curve in it run fully in the diagram, and its course in it can be limited again into the string system. Therefore, the necessary and sufficient condition is that it is possible to return to the central sphere with an unchanged indikatrix after having ran along a line along an *arbitrary* string. The two mouths of the string are elementary surfaces on the surface of the central sphere. We determine the positive direction on one of their edges, e.g. *clockwise* (as seen from the outside). If we move the closed curve along the surface of the string, until it covers the edge of the other mouth, then we have in this way determined a positive direction on it, which is either clockwise or anti-clockwise. (We still assume that the central sphere is placed in our space. If the first case holds then the attaching of the strings to the central sphere cannot actually be carried out in our space, but presumably in **T**). *In the first case, the manifold is evidently one-sided; if the second case holds for all the strings, then the manifold is two-sided.*

It is always possible to imagine the string system of a two-sided manifold as being embedded in our space; by cutting a string, suitably twisting it, and thereafter correctly joining its ends, it is possible to achieve that the fastening bands run along the surface of the string without twisting around it.

The diagram can be transferred into equivalent forms by continuous deformation; we especially mention

1. The fastening bands could be moved arbitrarily on the surface of the string system, if only the different bands never touch one another.

2. Each of the mouths of the string could be moved along that surface, which is made by the central sphere and the $p-1$ strings, assuming it does not pass any band during its movement; the bands, which run towards the discussed mouth of the string, must follow it, and additionally are moved in agreement with 1).

3. The fastening bands could, in addition to moves 1) and 2), be submitted to a move along the one face of a plate. The result of the displacement is made most easily by letting the band form a bay towards a point of the band, which corresponds to the plate, thereafter breaking the first band in a point, which is close to the other one, and inserting a twist along that other band, before closing again.

A lot of simplifications can be done by the means of these moves to the diagram. For example, if a band returns from the mouth of a band to the same one, without having passed any other strings, then the bay that the band has made hereby can be removed. If a band restricts itself to have only one branch running on one of the strings (that it itself runs across), then it is possible by the moves 2) and 3) to have that the piece of band will run once across the string and thereafter back into itself without having rung across other strings, and it is besides possible to have that the other pieces of band are removed from the string. When that has happened, it [the string] can apparently be contracted into the central sphere by the means of the plate, which belong to the band, and is, from now on, of no interest to us. I suppose that the mentioned condition for having a band removed from the diagram, is also a necessary condition, but I have not succeeded in giving a quite irreproachable proof of it.

One could be tempted to think that it could be possible by continuous deformation to have the p bands separated, so that each of them would run on its string from the p strings. It can however be proved that it is impossible, and therefore the problem of reducing the diagram into a normal form is probably very difficult. We will restrict ourselves to consider only such cases, in which the separation can be carried through, but before that we will shortly discuss *the different classes of annular cuts that are not disconnecting*, which exist on a torus surface (placed in our space), as we include in a class all such curves that by continuous deformation can be transferred one into another.

The first class includes the *meridian curves*, which run around the torus surface; they could form the boundary of simply connected pieces of surface, which all are in the space inside the surface; none of the other considered curves have this property. Another class is made of the *longitude curves*, which are defined by the fact that they make the boundary of simply connected pieces of surface, which are in the space outside of the torus surface. If we draw a meridian curve, λ, and a longitude curve, β, (provided with positive orientations) then it is possible to construct a new class of annular cuts, e.g. by going n times around in the positive direction along β, and thereafter closing the curve with a line along λ either in the positive or the negative direction; let us call such a curve $[n\beta \pm \lambda]$. In a similar way $[\beta \pm n\lambda]$ makes a class of curves, which is represented by one that goes once around β, but then makes n circulations along λ before it is closed. If the torus is cut through along a meridian curve, and if then the boundary curves are twisted n times in the suitable direction, and if the body is joined together again in the correct way, then the curve will be reduced into a longitude curve; (the original longitude curve will of course cease being such a curve). The complete classification is quite difficult, and therefore there is no reason for further pursuing the matter here.

A string on the diagram can be transformed into a torus, which is connected with the central sphere by a cylinder. The corresponding fastening band runs on its surface; it follows one of the curves just mentioned. According to our assumption each band can be made to run alone, on its own string. If the band is a *longitude curve*, then, as earlier noticed, both the torus and the plate can be contracted into the central sphere and *plays therefore no role*. The case, where all the bands run everywhere as *meridian curves*, is especially easy, and the situation is quite analogous to what we met during our studies of the surfaces. If we imagine the diagram in our space, and that one again being in **T**, it is easy to form a closed space, which has the given one as its diagram. If a plate is fastened to a band, then we place it in **T** above our space, P, and in such a way that its projection on P will be in one of the pieces of elementary surface, in the diagram that the band is bounding. The edge of the surface is *bent down* into our space and fastened to the diagram along the band. Now, we imagine once again that every torus is transformed into a string. Thereafter we let the plate become thicker, in such a way that the fastening bands take up the whole surface of the band. The manifold, which is formed in this way, is closed by an elementary manifold, whose projection is the central sphere. Hereby a form is obtained, which well corresponds to the *Klein's* normal form of surfaces. The contour on P of the closed 3-dimensional manifold is just made by such a surface, and besides, the manifold consists of two parts, whose projections both form a body in P that the contour is bounding, e.g. the one part being above, and the other one below P. It can be said that it originates from placing a certain amount of *handles on a sphere*. Such a handle is made by removing two elementary manifolds from the sphere; the boundary of one of them is moved from the sphere, and thereafter moved along the sphere in such a way that it covers the boundary of the other one, of course in such a way that the manifold becomes two-sided. The neighborhood of a closed curve in **T** is bounded by a space of the kind that is described below.

Every closed surface in **T** is surrounded by a 4-dimensional manifold, which is bounded by one that is 3-dimensional, which we will call the *hull* of the surface. (In a similar way the hull of any closed manifold or point that is placed in a given manifold is determined). Let us now determine the diagram of the hull of a torus surface in **T**. It consists of three strings; each of the three bands runs on two of the strings and with two branches in the opposite directions one to another. The easiest way of illustrating this appearance to oneself is to let the central sphere be placed with its center in a right-angled system of co-ordinates in the space, and to let the strings follow the three axis, as they are imagined to continue through the points at infinity. Then the lines of intersection with the three co-ordinate planes indicate the course of the bands. Here is an example of a diagram where the bands could not be made to run on each of its own string. This diagram illustrates the example, which is mentioned in *Picard's* and *Simart's* book, chap. II, No. 18. Besides, similar diagrams can be obtained for the hulls of the closed surfaces in T. However, these examples have not exhausted the variety of diagrams.[47]

Earlier, we considered a diagram made by a torus, on whose surface the fastening line ran as meridian curve; but it can however be running in infinitely many ways. We merely considered an example, which we later will refer to, that is to say where the *fastening line is a curve* $[2\beta + \lambda]$; if the torus is squeezed flat in such a way that the band becomes its edge, then we get *Möebius'* well known one-sided piece of surface.

As already mentioned earlier the nucleus of the diagram indicates, what kind of cuts (line segments and surface) have to be performed to make the punctured manifold simply connected. We could also go another way around to determine these cuts; the diagram is equivalent with the punctured manifold; but to make it simply connected p plates can be pierced with a channel to begin with; we could cut the strings through with cutting surfaces that have meridian curves as their boundaries, and to have these [meridian curves] become cuts in the diagram, then it is necessary at each point of intersection between a meridian curve and a piece of band to let the cutting surface send off a tongue into the plate and towards the channel, which pierces the plate. Consequently, by such a point of intersection the edge of the cutting surface runs from the meridian curve, onto the one side of the plate, to the channel, along its surface, and back to the meridian curve along the other side of the surface. Of course we can inversely regard this system of cuts as a diagram of the original manifold, just when we imagine that these parts will remain, which before were removed. Then the channels become strings originating from that manifold, which was removed by the original puncture; the surface cuts become plates. There is a sort of dual connection between the two diagrams: the strings in one of them correspond to the plates in the other one, and vice verse; just as many bands run across a string, as there are pieces of bands that are bounding the matching plate; just as many plates that send of bands across a particular string, just as many strings have bands from a particular plate e.t.c. The latter method of making diagrams has been applied by *Betti*.

If the diagram is closed by connecting its surface with the surface of a sphere, then, as mentioned, we get a manifold, which is equivalent with the given one. Besides, the parts could have been disassembled in somewhat another way, as the plate instead of being added to the string system could have been added to the surface of the sphere with their two sides. In this way a body with p handles, which is quite similar to the string system, is obtained. The surfaces should be joined; it is sufficient to know the system of not disconnecting annular cuts on the one of them,

[47] In modern terminology a circle bundle over a surface was constructed [translator's remark].

which corresponds to the curves β on the string-surface of the other one, and the system, which corresponds to the curves λ. For example two tori, whose boundaries are joined in such a way that the meridian curves cover the longitude curves and vice verse, correspond to a diagram with one string, whose band is a longitude curve. The manifold itself is equivalent with a sphere. This also agrees with the fact that the space inside a torus surface can be transformed into the space outside the torus surface, in such a way that longitude curves on the surface will become meridian curves and vice verse — everything with the assumption that the point at infinity of the space is considered as being equivalent with the neighborhood of a point, in such a way that everything that is outside a certain sphere, can be brought inside it by an inversion. As another example we consider two tori, whose boundaries are joined, in such a way that the longitude curves cover the longitude curves, and the meridian curves cover the meridian curves; consequently we get a manifold, which is equivalent with the hull of a closed curve in **T**. Finally we choose two tori, where the longitude and meridian curves on the one side correspond to the curves $[2\beta + \lambda]$ and β on the other side; this corresponds to the before mentioned diagram with the fastening band $[2\beta + \lambda]$.

Yet, we still have to mention a circumstance that apparently causes that the theory becomes more difficult when the number of dimension exceeds 2. If a handle of the diagram of a closed, two-sided surface is cut through, then afterwards the joining can essentially happen in only 1 way, if the surface is supposed to continue being two-sided, and by the joining one will return safely to a diagram that is equivalent with the first one. On the other hand if one considers a 3-dimensional manifold, such as the one that make the boundary of the neighborhood of a torus surface in **T**, then this manifold can be cut through by a torus surface (which corresponds to a meridian curve on a surface, whose hull is being studied). *However the joining can happen in infinitely many mainly different ways:* β and λ from the one boundary can be joined with any two of the earlier discussed annular cuts, when they merely cut each other at one point.

The whole *Second part* is translated in [P-22].

Poul Heegaard's bibliography:

[Heeg] P. Heegaard, Forstudier til en Topologisk Teori for de algebraiske Fladers Sammenhæng, København, 1898, Filosofiske Doktorgrad; French translation: Sur l'Analysis situs, *Soc. Math. France Bull.*, 44 (1916), 161-242.

[D-H] M. Dehn, P. Heegaard, Analysid situs, *Encykl. Math. Wiss.*, vol. III AB3 Leipzig, 1907, 153-220.

[H-1] P. Heegaard, Der Mathematikunterricht in Dänemark, *Internationale Matchematische Unterrichtskommission I.M.U.K*, København 1912. (Teaching of Mathematics in Denmark).

[H-2] P. Heegaard, Bidrag til Grafernes Theori, Tredie Skandinaviske Matematikerkomgres i Kristiania, 1913, 1-6.

[H-3] P. Heegaard, Omriss av kronologiens historie i Europa (med sœrlig henblikk pá den norske almanakk av 1644, ????? 51-96.

[H-4] P. Heegaard, Hamiltons Dodekaederspil, *Nyt Tidsskrift Matematik*, 29(4), 1919, 81-88.

[H-5] P. Heegaard, Mindetale, oer prof..dr.H.G.Zeuthen, Særtrykk av Videnskapsselskapets Forhandlinger 1920, 1-3.

[H-6] P. Heegaard, Hovedlinjer i H.G.Zeuthens videnskabelige production, Foredrag i Norsk Matematisk Forening, *Særtrykk av Norsk Matematisk Tidsskrift*, 1920, 1-5.

[H-7] P. Heegaard, Eudoksos' Hippopede. Et foredrag holdt i Norsk Matematisk Forening, *Særtrykk av Norsk Matematisk Tidsskrift*, 1921, 1-10.

[H-8] P. Heegaard, Felix Klein, Et foredrag holdt i Norsk Matematisk Forening 25. mars 1926, *Særtrykk av Norsk Matematisk Tidsskrift*, 1926, 1-7.

[H-9] P. Heegaard, Om en generalisasjon til rummet av Sophus Lies fremstilling av imaginære elementer i planet, Den syvende Skandinaviske Matematikerkongress, I Oslo 19-22 August 1929, Oslo 1930, 92-95.

[H-10] P. Heegaard, Scandinavie. Les modifications essentielles de l'enseignement mathématique dans les principaux pays depuis 1910, *L'Enseignement mathématique*, XXIX(4-6), 1930, 307-314.

[H-11] P. Heegaard, Appendix P.Osl 6. The date of the Horoscope of Philoe (in "Papyri Osloenses" by S.Eitrem, L.Amundsen), Oslo, on commission by Jacob Dybwad, 1931, 146-151.

[H-12] P. Heegaard, Über die Heawoodschen Kongruenzen, *Særtrykk av Norsk Matematisk, Forenings Skrifter*,Serie II, Nr. 1-12, 1933, 47-54.

[H-13] P. Heegaard, Norvège. La préparation théorique et pratique des professeurs de Mathématiques de l'enseignement secondaire, *De L'Enseignement mathématique*, 5-6, 1933, 360-364.

[E-H] F. Engel, P. Heegaard, Sophus Lies Samlede Avhandlinger. Blind I, *Særtrykk av Norsk Matematisk Tidsskrift*, 1934, 1-9.

[H-14] P. Heegaard, Et brev fra Abel til Degen, *Særtrykk av Norsk Matematisk Tidsskrift*, 2. hefte 1935, 1-6.

[H-15] P. Heegaard, Bemerkungen zum Vierfarbenproblem, *Mat. Sb.* (New Ser.), 1(43):5, (1936), 685-693.

[H-16] P. Heegaard, La représentation des points imaginaires de Sophus Lie at sa valeur didactique, Atti del Congresso Internazionale dei Matematici, Bologna, 3-10 settembre 1938 - VI, 421-423.

[H-17] P. Heegaard, Die Topologie unddie Theorie der algebraischen Funktionen mit zwei komplexen Variabeln, Congrès des Mathématiques à Helsingfors 1938, 1-8.