# RiG, Racks and Nichols algebras
## A GAP package for racks and Nichols algebras

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Advanced School and Conference on Knot Theory and its Applications to Physics and Biology

# About GAP

The GAP Group, GAP - Groups, Algorithms, and Programming
Version 4.4.12; 2008
http://www.gap-system.org

GAP is a computer software for working with group theory

Some of the authors are
Frank Celler, Steve Linton, Frank Lübeck, Werner Nickel, Martin Schönert, Thomas Breuer, Alexander Hulpke and many others.

# About RiG

RiG is a GAP package for Racks and Nichols Algebras

Work in progress!

RiG is a GAP package for:

- Computing polynomial invariants of racks,
- Computing algebraic contructions from racks,
- Computing Rack (co)homology,
- Studying braided vector spaces,
- Studying Nichols algebras.

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# What is a rack?

**Definition**

A **rack** is a pair $(X, \triangleright)$, where $X$ is a non-empty set and $\triangleright : X \times X \to X$ is a function such that

$$x \mapsto i \triangleright x \text{ is bijective for all } i \in X,$$
$$i \triangleright (j \triangleright k) = (i \triangleright j) \triangleright (i \triangleright k) \text{ for all } i, j, k \in X.$$

**Remarks:**

- A rack $(X, \triangleright)$ such that $i \triangleright i = i$ is called a **quandle**.
- Equivalent definition/notation: $x \triangleright y = y^x = y * x$.

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# What is a rack?

Let $(X, \triangleright)$ be a rack. Assume that $X = \{x_1, x_2, \cdots, x_n\}$.
If we use the labelling

$$1 \equiv x_1,\ 2 \equiv x_2, \dots, n \equiv x_n,$$

the rack $(X, \triangleright)$ can be given by a matrix

$$M = (m_{ij}) = (i \triangleright j) \in \mathbb{Z}^{n \times n}.$$

# Playing with racks: examples

We will show some examples of racks:

**Definition (Dihedral rack)**

$(X,\triangleright)$ with $X=\{1,\cdots,n\}$ and $i\triangleright j=2i-j \pmod n$.

In RiG:

DihedralRack returns a dihedral rack of a given size.

**Example**

```
gap> DihedralRack(3);
[ [  1,  3,  2 ],
  [  3,  2,  1 ],
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# Playing with racks: examples

Let $p$ be a prime number and $q = p^m$ for $m \in \mathbb{N}$.
Let $\mathbb{F}_q$ be the finite field with $q$ elements.
Fix $0 \neq \zeta \in \mathbb{F}_q$.

**Definition (Affine rack)**

$(X, \triangleright)$ with $X = \mathbb{F}_q$ and $x \triangleright y = (1-\zeta)x + \zeta y$.

In `RiG`:
`AffineRack` returns an affine rack over $\mathbb{F}_q$.

**Example**

```
gap> AffineRack(GF(3), Z(3))=DihedralRack(3);
true
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Given a finite group $G$ and a conjugacy class $\mathcal{C}$

## Definition

$(X, \triangleright)$ with $X = \mathcal{C}$ and $g \triangleright h = ghg^{-1}$.

In RiG:
Rack returns the rack associated to the pair $(G, \mathcal{C})$.

## Example

```
gap> r := Rack(SymmetricGroup(3),(1,2));;
gap> s := DihedralRack(3);;
gap> r=s;
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# Playing with racks: isomorphisms

Rack equality is not always the answer. . .

# Playing with racks: isomorphisms

## Definition

Let $X, Y$ be racks. A bijective function $f : X \to Y$ such that $f(x_1 \triangleright x_2) = f(x_1) \triangleright f(x_2)$ is called an **isomorphism** of racks.

In `RiG`:

`IsomorphismRacks` returns (if it exists) some isomorphism between two given racks.

## Example

```
gap> r := Rack(AlternatingGroup(4),(1,2,3));;
gap> s := Rack(AlternatingGroup(4),(1,3,2));;
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(3,4)
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# Playing with racks: isomorphisms

This means:
The elements of $\mathtt{r}$ are: $(1,2,3)$, $(1,4,2)$, $(1,3,4)$, $(2,4,3)$.
The elements of $\mathtt{s}$ are: $(1,3,2)$, $(1,2,4)$, $(1,4,3)$, $(2,3,4)$.

And the isomorphism between $\mathtt{r}$ and $\mathtt{s}$ is given by:

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# Playing with racks: isomorphisms, another example

Let $G = \mathbb{S}_6$. Let $\mathcal{C}$ and $\mathcal{D}$ be the conjugacy classes of $(1,2)$ and $(1,2)(3,4)(5,6)$ in $G$.

**Proposition**

The racks of $\mathcal{C}$ and $\mathcal{D}$ are isomorphic.

Because, the non-inner automorphism of $\mathbb{S}_6$ applies $(1,2)$ in $(1,2)(3,4)(5,6)$.

**Example**

```
gap> c:=(1,2);;
gap> d:=(1,2)(3,4)(5,6);;
gap> r:=Rack(SymmetricGroup(6),c);;
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gap> IsomorphismRacks(r,s);
(2,5,13,15)(3,9,8,6,14,4,11,7,10)
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# Playing with racks: subracks

Let $(X, \triangleright)$ be a rack.

**Definition (Subrack)**

A subset $\emptyset \neq Y \subseteq X$ is a subrack iff $Y \triangleright Y \subseteq Y$.

In `RiG`:

`Subrack` returns the subrack generated by a given subset.

**Example**

```
gap> r := Rack(SymmetricGroup(4),(1,2));;
gap> r!.labels;
[ (1,2), (1,3), (1,4), (2,3), (2,4), (3,4) ]
gap> s := Subrack(r,[5,6]);;
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# (Co)Homology

Let $(X, \triangleright)$ be a rack. Let $C_n^R(X)$ be the free abelian group generated by $(x_1, x_2, \dots, x_n) \in X^n = X \times \cdots \times X$. Let $\partial^i : C_n^R(X) \to C_{n-1}^R(X)$ defined by

$$\partial^i(x_1, \dots, x_n) =$$

$$= (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) - (x_1, \dots, x_{i-1}, x_i \triangleright x_{i+1}, \dots, x_i \triangleright x_n).$$

We define $\partial : C_n^R(X) \to C_{n-1}^R(X)$ by

$$\partial(x_1, \dots, x_n) = \sum_{i=1}^{n} (-1)^{n+1} \partial^i(x_1, \dots, x_n).$$

**Lemma**

$(C_\bullet(X), \partial)$ *is a complex.*

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# (Co)Homology: examples

**Definition (Homology)**

$H_n(X,\mathbb{Z}) = H_n((C_\bullet(X),\partial),\mathbb{Z})$.

Some examples:

- $H_2(\mathbb{D}_3,\mathbb{Z}) = \mathbb{Z}$.
- $H_3(\mathbb{D}_5,\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_5$.

In `RiG`:

`RackHomology` returns the $n^{\text{th}}$-homology group.

**Example**

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gap> RackHomology(DihedralRack(3),2);
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Here is a small TODO list:

- Extend `RiG` to biracks and biquandles.
- Compute quandle (co)homology groups.
- Compute generators of the (co)homology groups.
- Study possible descompositions of a given racks.

Suggestions are welcome

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