The 2-variable Jones polynomial (homflypt) Braid equivalence in other set-ups

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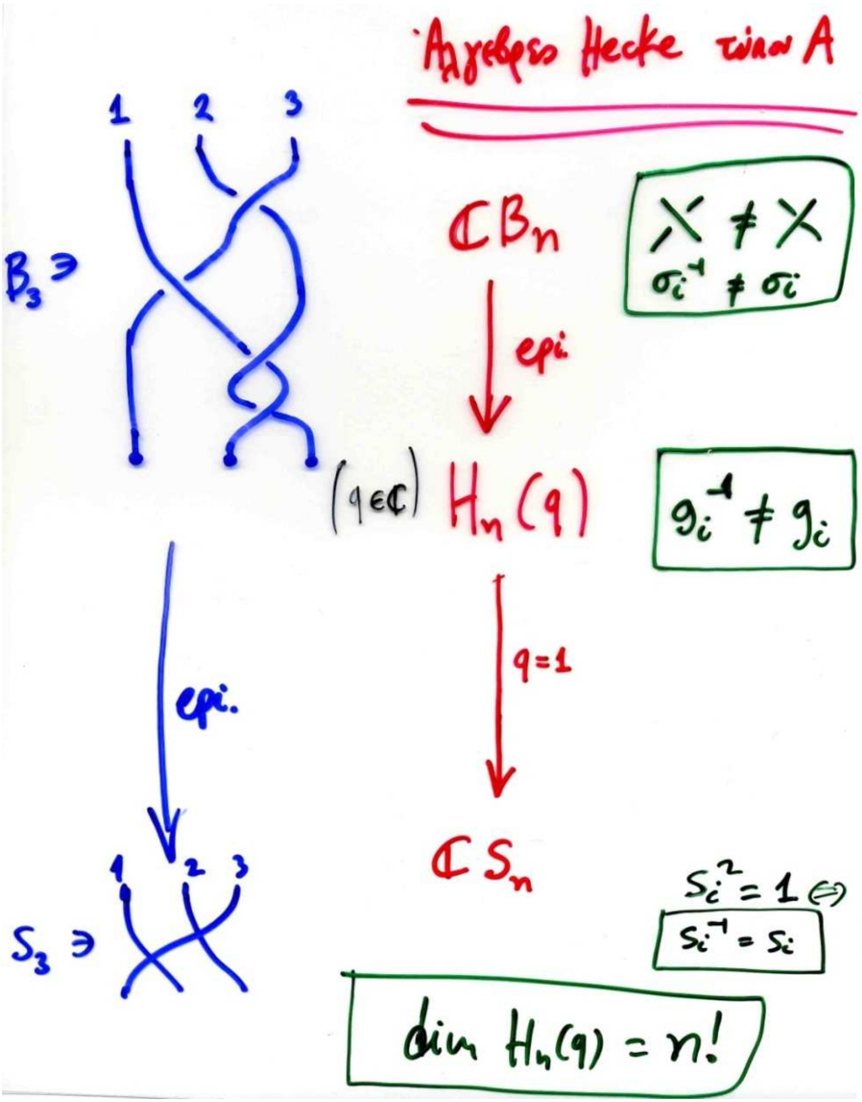
Braids, permutations and the Hecke algebra $H_n(q)$



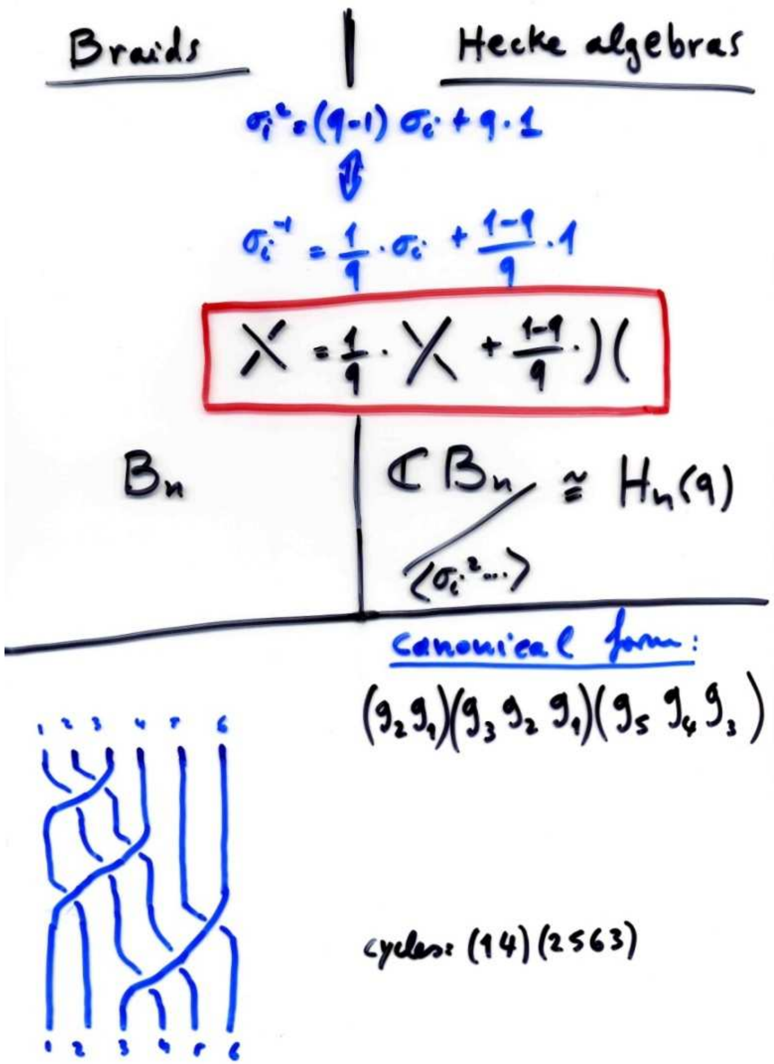
$$H_n(q) = CB_n / < \sigma_i^{\;2} = (q-1)\sigma_i + q \cdot 1 >$$

So, the generators g_i of the Hecke algebra satisfy the braid relations together with the quadratic relation:

$$g_i^{\;2} = (q-1)g_i + q \cdot 1$$

The 2-variable Jones polynomial (homflypt)

(it specializes to the Alexander and Jones polynomials)



A **linear basis** of the Hecke algebra $H_n(q)$ consists in monomials in the generators, with exponents +1, such that the highest index generator appears at most once.

Theorem (Ocneanu,1984): There is a unique **linear trace**:

$$tr: \bigcup_{n=1}^{\infty} H_n(q) \to \mathbb{Z}[q^{\pm 1}](z)$$

determined by the rules:

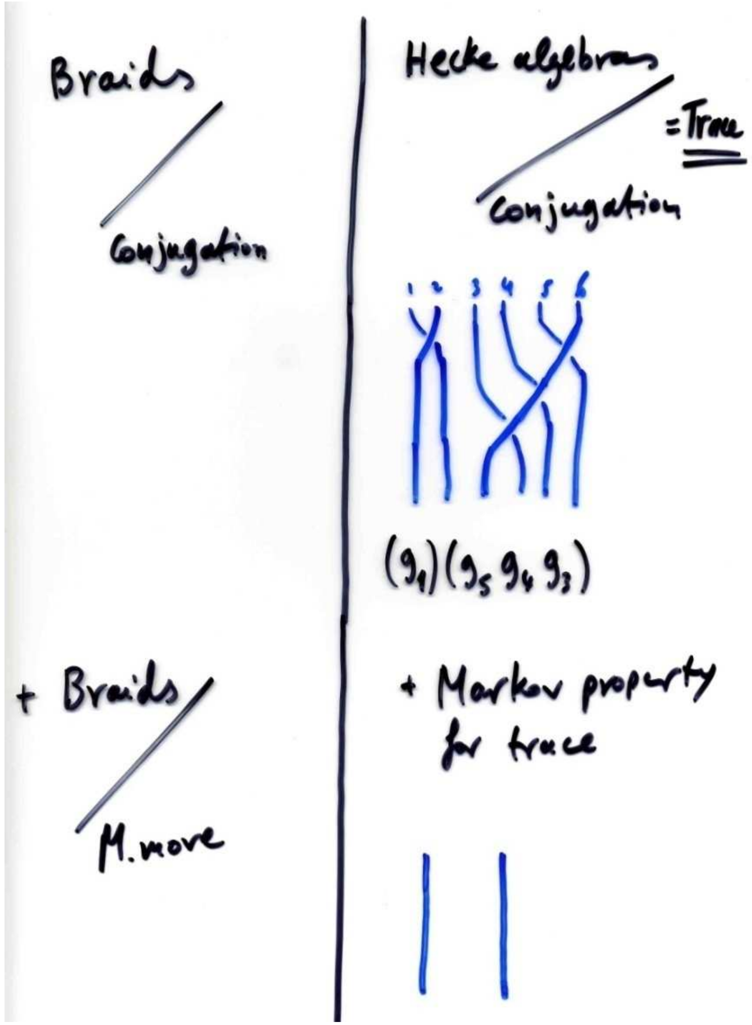
1) $tr(1) = 1$ for all n

2) $tr(ab) = tr(ba) \quad a, b \in H_n(q)$

3) $tr(ag_n) = z\, tr(a) \quad a \in H_n(q)$

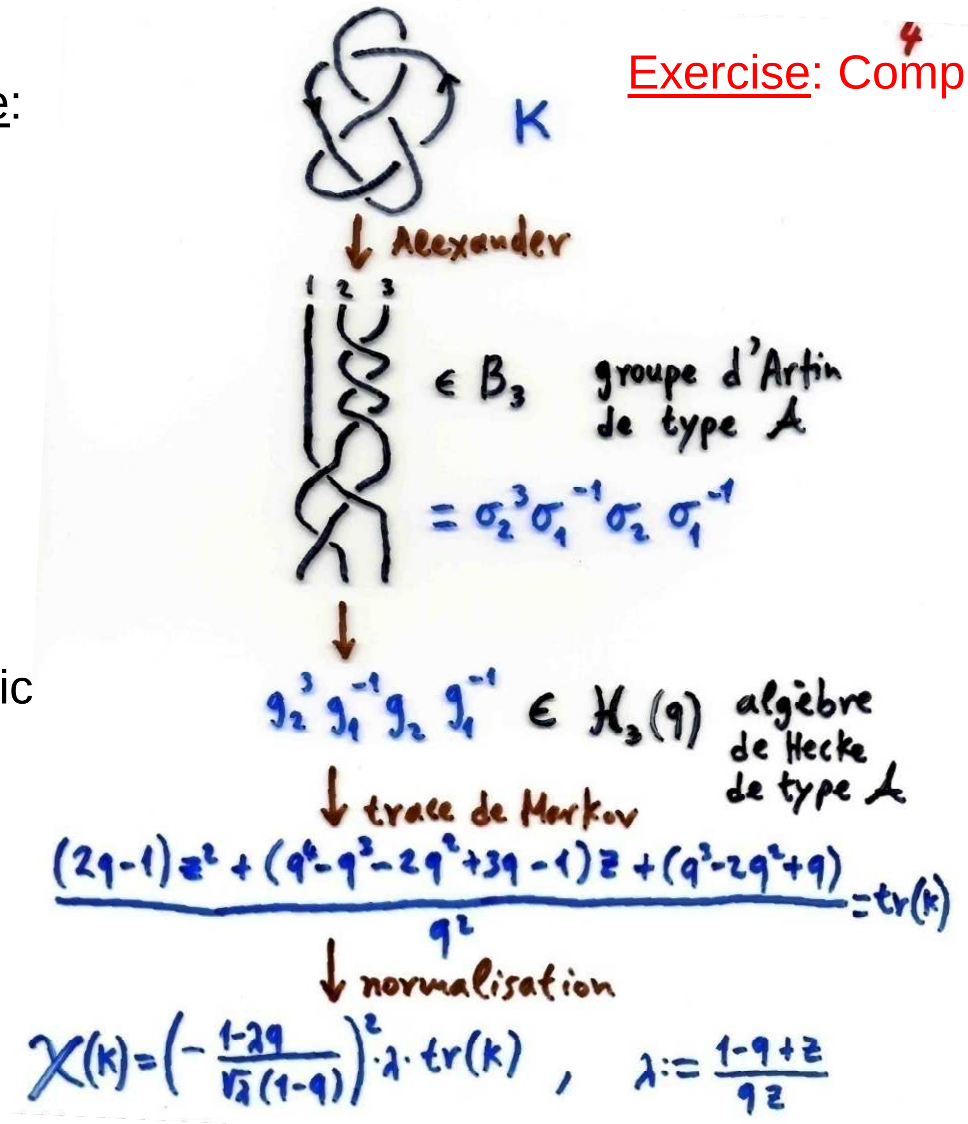
(Markov property)

V.F.R. Jones: In order to construct a knot invariant using the epimorphism of the braid group algebra onto the Hecke algebra, we need to take care of the equivalence moves in the Markov theorem. Conjugation is taken care of by rule 2). We only need to rescale the g_i and normalize the trace, so as to have the same values on the closures of the braids α, $\alpha\sigma_n$ and $\alpha\sigma_n^{-1}$, see [7,5,6]. Also [1,10].



An example:

Exercise: Compute $tr(g_1 g_2 g_3 g_2 g_3)$.



(Apply the quadratic relation and the relation for g_i^{-1} and use the fact that tr is linear.)

Knots and braids in other spaces

Q1. Can we imitate Jones's construction for knots in other 3-manifolds (handlebodies, knot complements, c.c.o. 3-manifolds, etc) or in other diagrammatic set-ups (virtual knot theory, welded knots, singular knots, etc)?

Q2. (V.F.R. Jones): Can we use other braid groups and other types of Hecke algebras?

Full answers when the manifold is a solid torus.

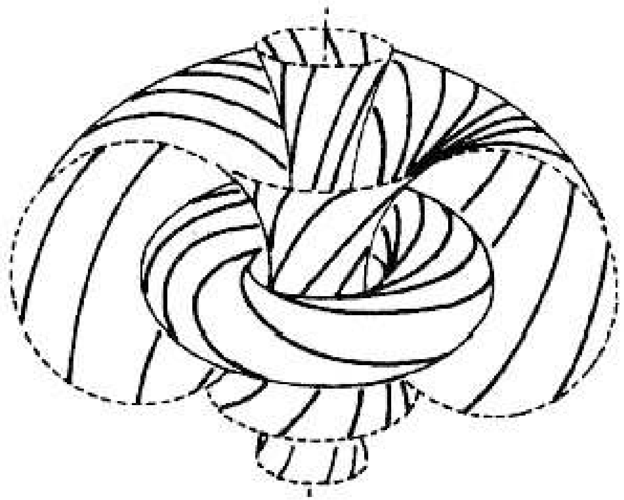
(S.L. 1994 & 1999, S.L.&Geck 1997)

Note: Throughout, by `knot' we mean `knot or link'.

Knots and braids in the solid torus

• The complement of a solid torus in the 3-sphere is another solid torus.

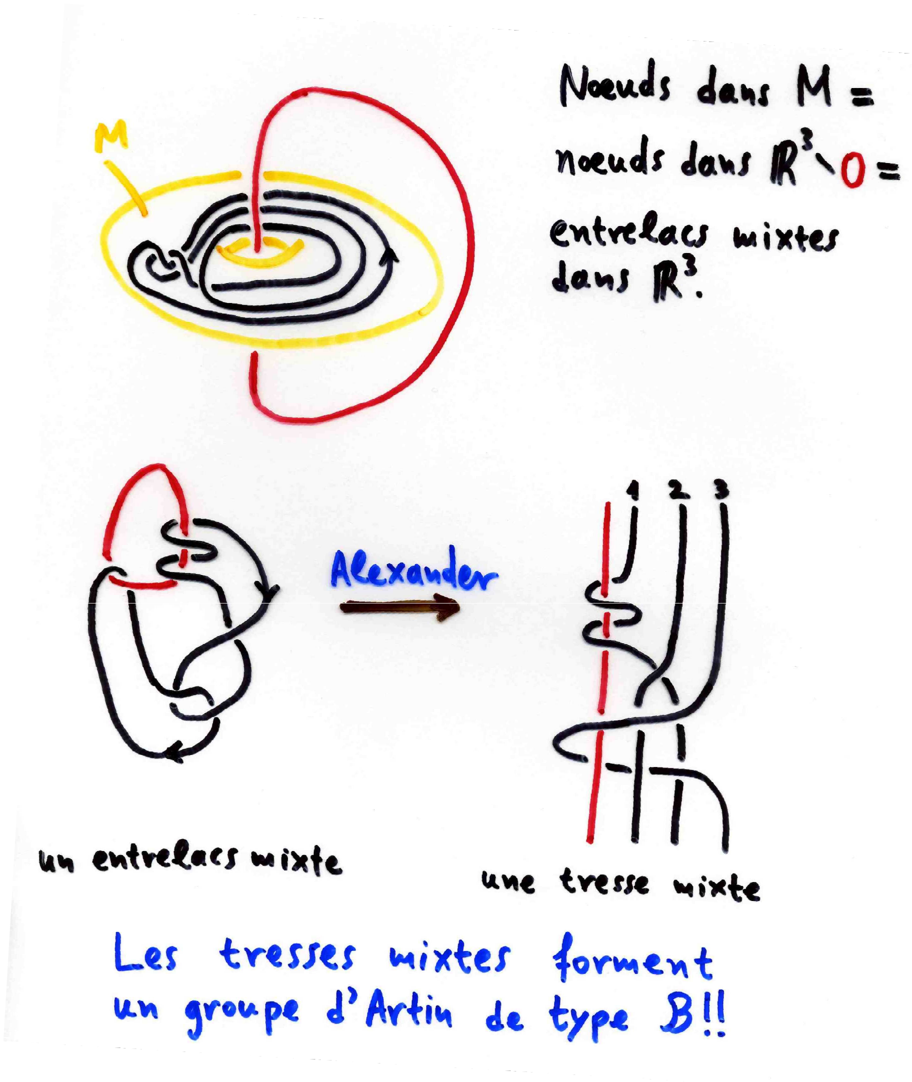
• So, a knot in a solid torus has to avoid the complement solid torus



Representing the complement solid torus by the red simple curve, our knot has to avoid the red curve.

So it can be represented by the **mixed link** containing the red closed curve.

Braiding the mixed link so that the fixed red strand lies always in the 1st position we obtain a **mixed braid**.

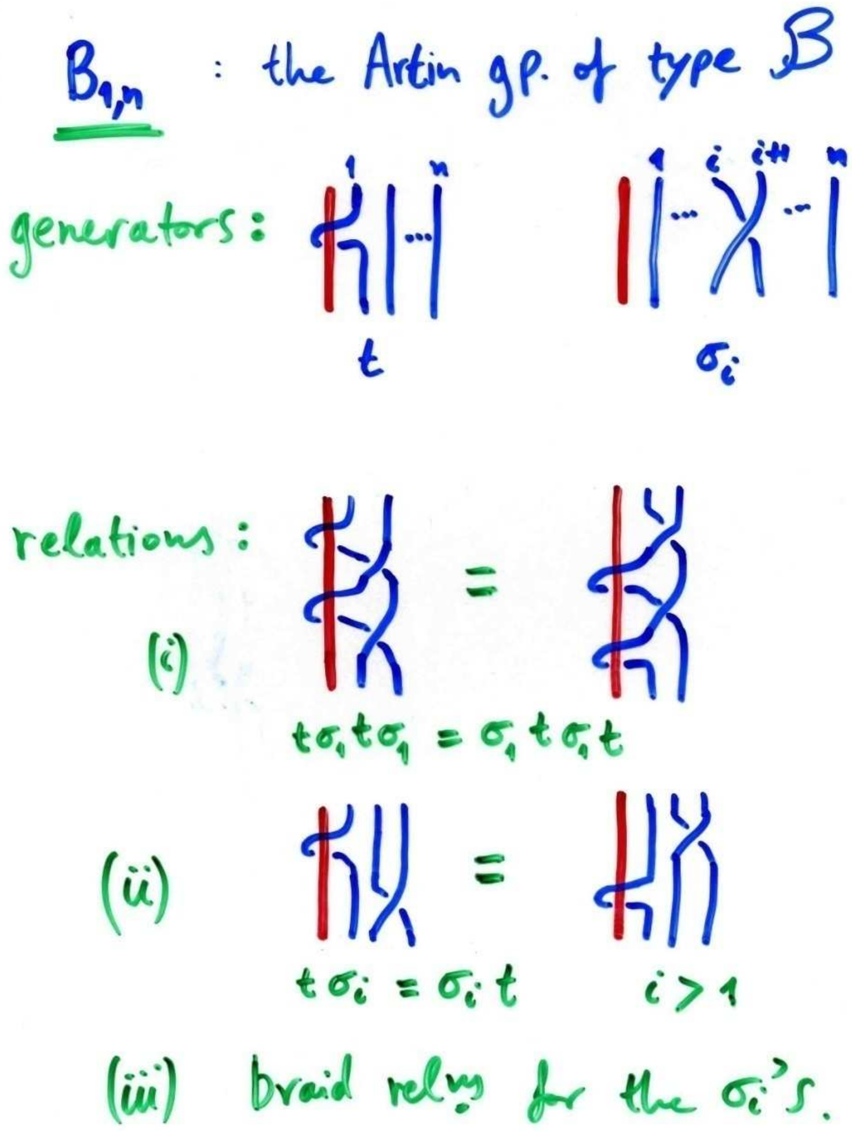


The set of mixed braids on n strands (not counting the red) forms a group. This is the **Artin group of type B**.

Its main characteristic is a **degree 4 relation** between the loop generator t and the first crossing generator σ_1.

The **Hecke algebra of type B** can be defined as the quotient:

$$H_n(q,Q) = \frac{CB_{1,n}}{< \sigma_i^2 = (q-1)\sigma_i + q \cdot 1, \quad t^2 = (Q-1)t + Q \cdot 1 >}$$



Other Hecke-type algebras of type B

$$\frac{\mathbb{C}B_{1,n}}{\left\langle \begin{matrix}(t-u_1)(t-u_2)\cdots(t-u_r)=0 \\ \sigma_i^2=(q-1)\sigma_i+q\cdot 1\end{matrix} \right\rangle} = H_n(q,r)$$

$u_i \in \mathbb{C}$

The **cyclotomic Hecke algebra of type B** (Ariki&Koike, Broue&Malle). For r=2 it is isomorphic to the Hecke algebra of type B.

$$\frac{\mathbb{C}B_{1,n}}{\left\langle \sigma_i^2=(q-1)\sigma_i+q\cdot 1 \right\rangle} = H_n(q,\infty)$$

$$\left(\cong \underset{\text{tom Dieck}}{\tilde{H}_n(q)}\right)$$

Removing the relation for t we obtain an infinite dimensional algebra, which turns out to be isomorphic to the **affine Hecke algebra** of type A.

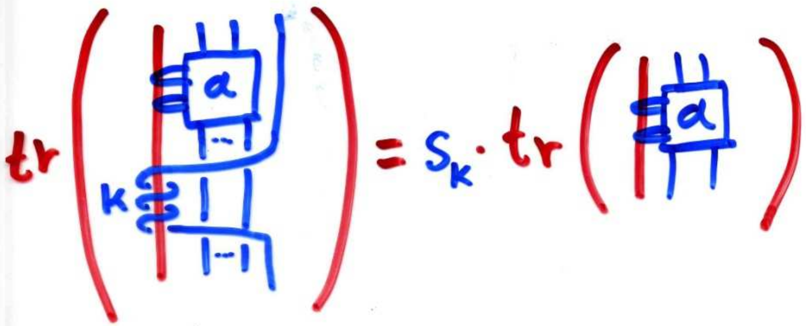
Setting $\mathcal{H}_n = H_n(q,r)$ or $H_n(q,\infty)$

and $\mathcal{H} := \bigcup_{\infty} \mathcal{H}_n$

Theorem (SL, 1999)

$\exists!\ tr : \mathcal{H} \to \mathbb{C}$

1) $tr(ab) = tr(ba)$
2) $tr(1) = 1$
3) $tr(a\sigma_n) = z \cdot tr(a)$
4) $tr(a\, t_n'^{\,k}) = s_k \cdot tr(a)$

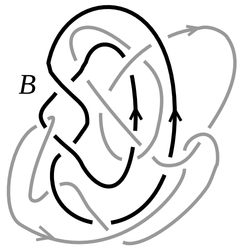


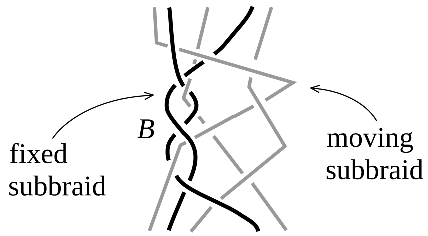
Rescaling and normalizing these traces we obtain all (infinitely many) different analogues of the homflypt polynomial for knots in the solid torus, see [8,2,9]. Also [11,4,3].

Knots and braids in knot complements, [17,18]

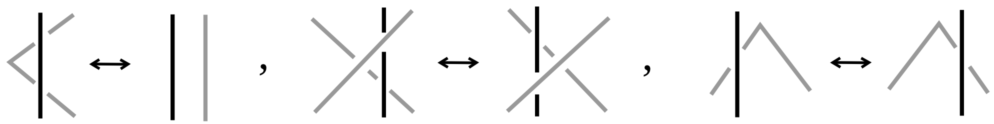
(See also [21,20])

Represent knots in the complement of the closure of B by **mixed links** in the 3-sphere.

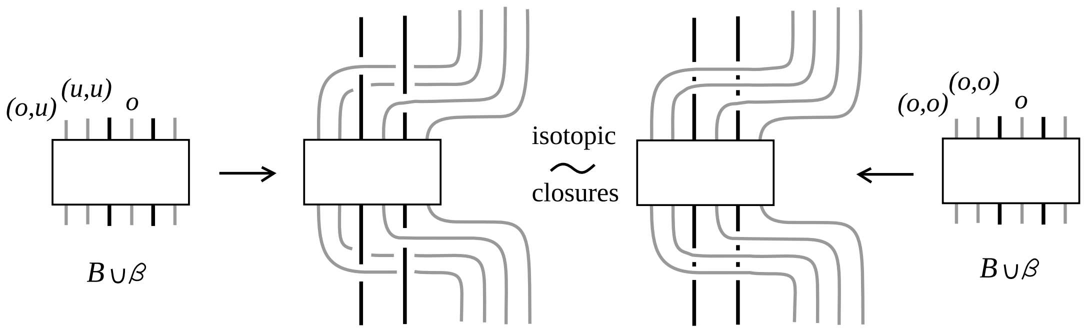




Isotopy is induced by the usual and the **mixed Reidemeister moves**.



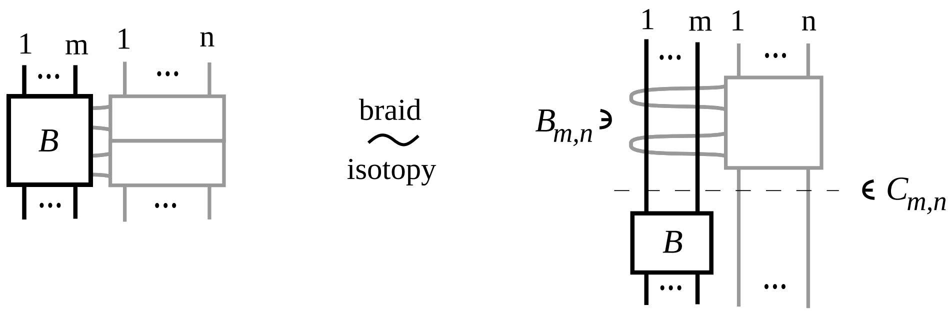
By Alexander's theorem mixed links are isotoped to **mixed braids**.



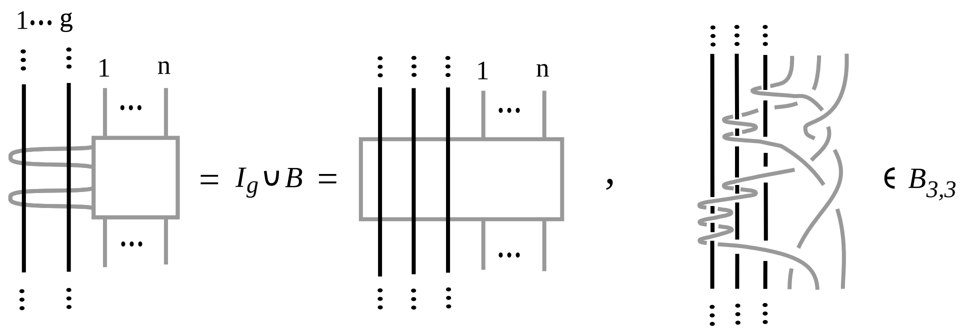
Parting mixed braids

Braid structures in knot complements, c.c.o. 3-mfds, handlebodies [15]

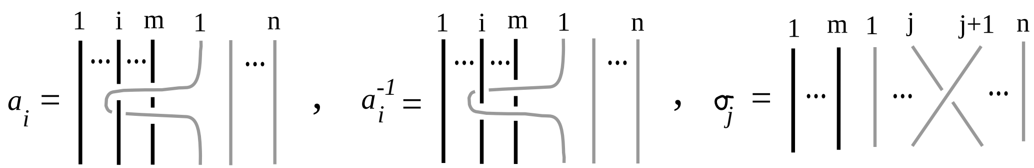
Using Artin's combing for pure braids we can separate the knotting in the 3-manifold from the braid B that represents the 3-manifold.



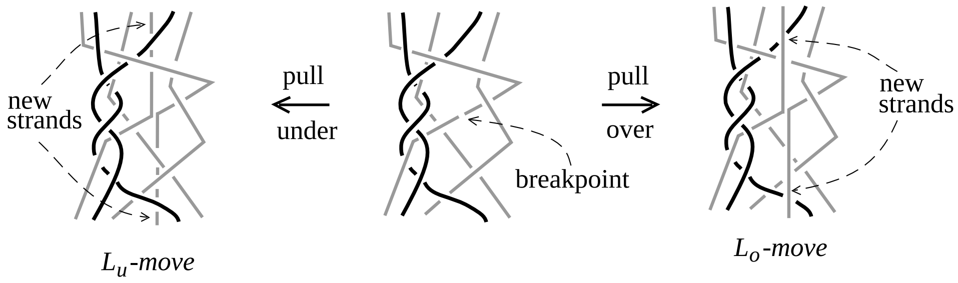
Braids of the top part form the **braid groups** $B_{m,n}$



which have nice presentations, with generators:
See [].

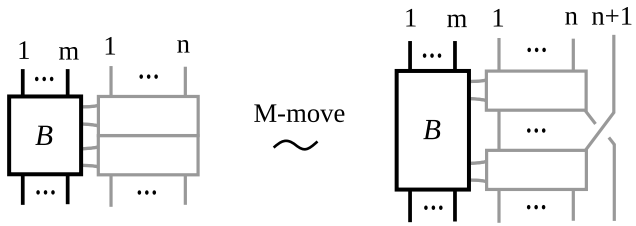


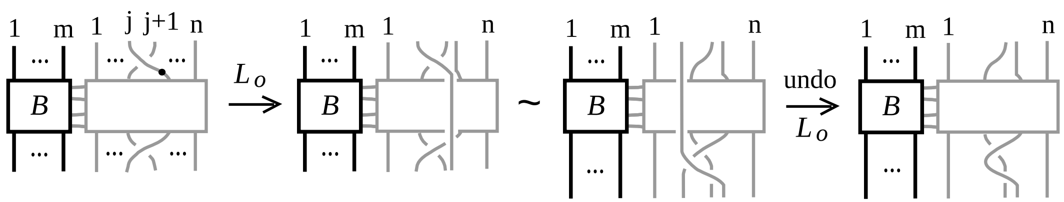
Braid equivalence moves in knot complements [17,18]



- L-moves that do not affect the fixed manifold braid

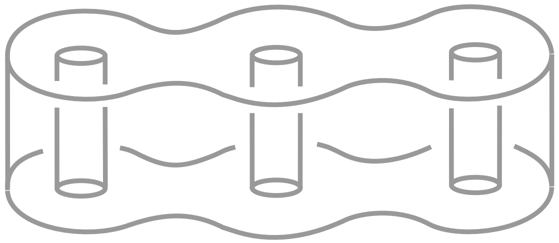
- Stabilization moves

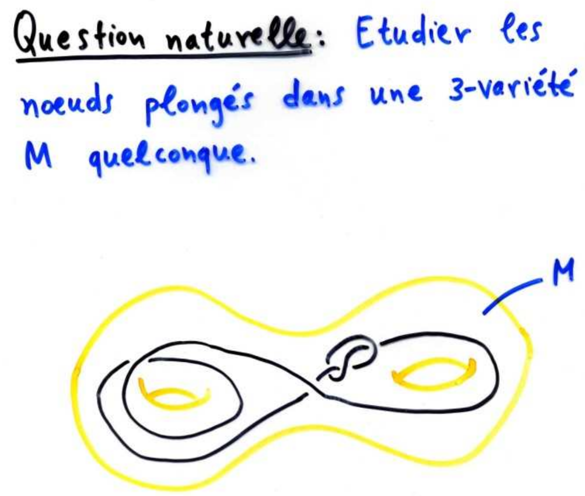




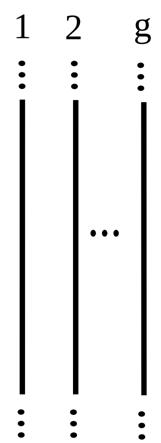
- All conjugations are allowed.

Knots and braids in a handlebody, [12]

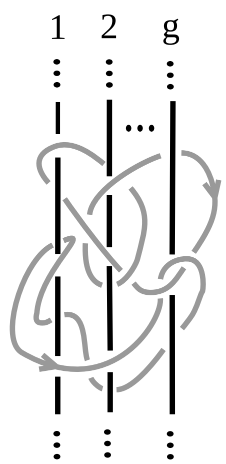




Represent the handlebody of genus g by an infinite identity braid on g strands.

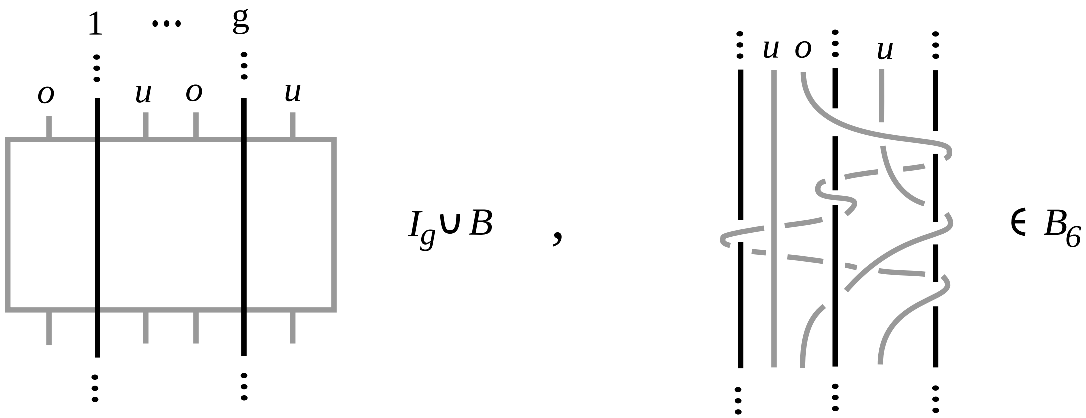


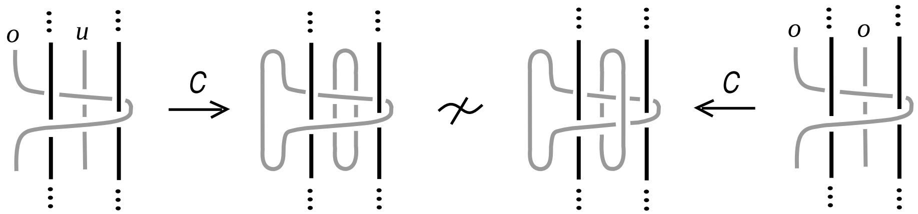
I_g ,



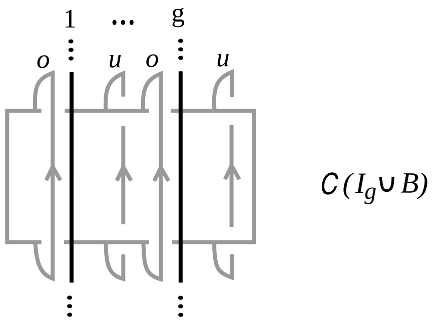
$I_g \cup L$

The closure for braids in a handlebody, [12]



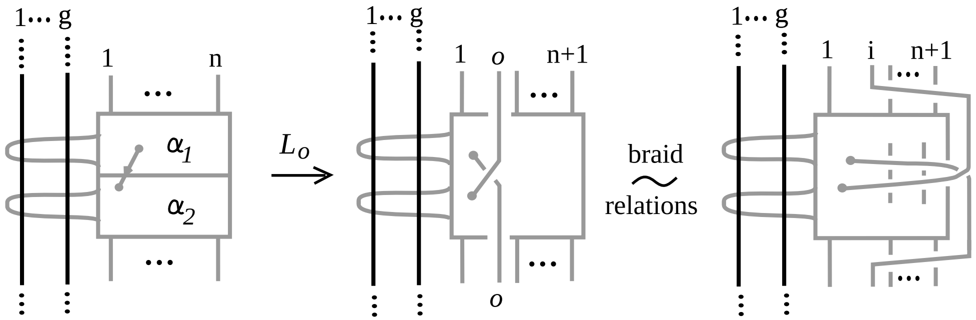


We must specify whether we close a pair of corresponding strands by an arc in the front or in the back of the mixed braid.

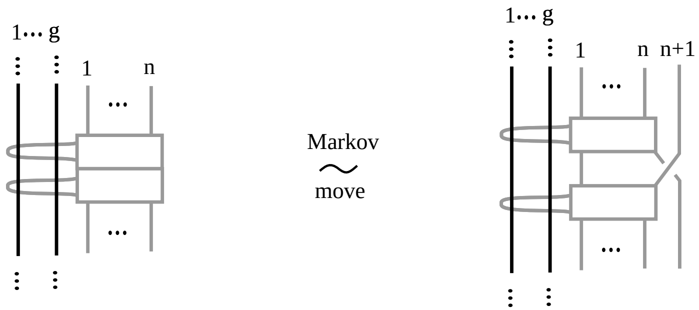


Braid equivalence moves in handlebodies, [12]

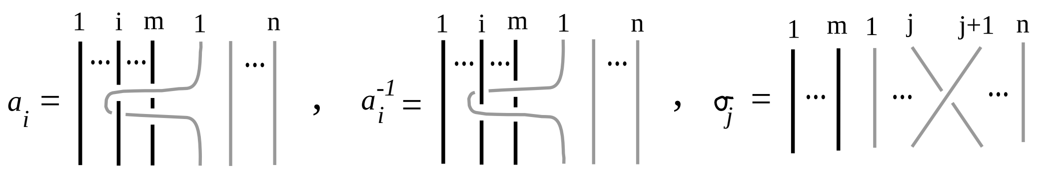
L-moves



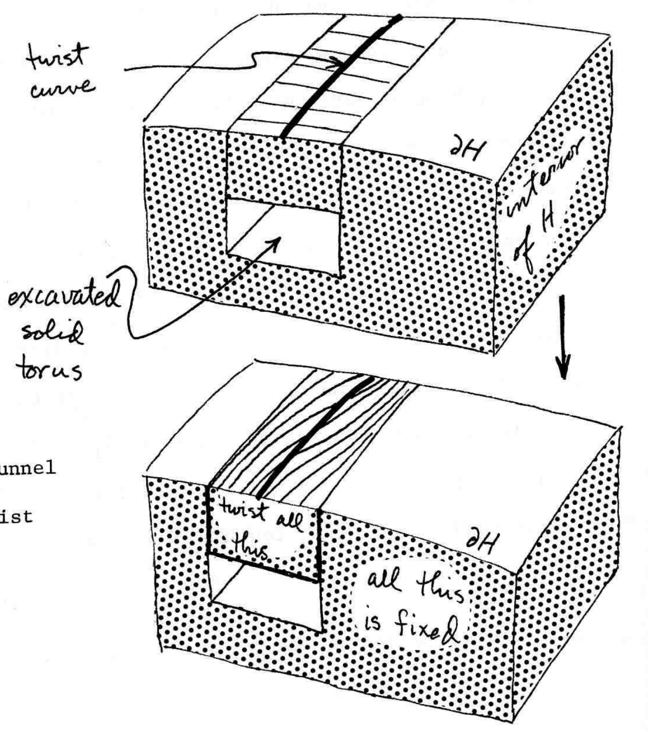
Stabilization moves



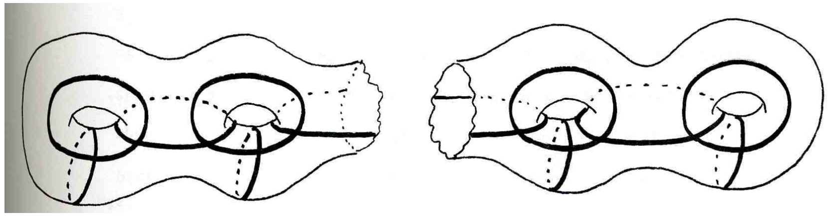
As before, conjugation by crossings can be realized by L-moves.
But conjugations by the loop generators are not allowed!

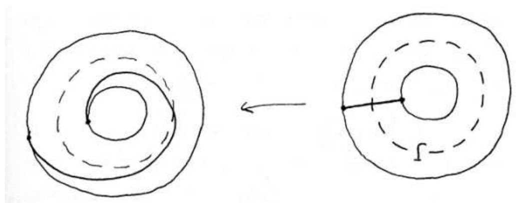


Topological surgery in 3-D



Excavating a tunnel to extend a twist homeomorphism.

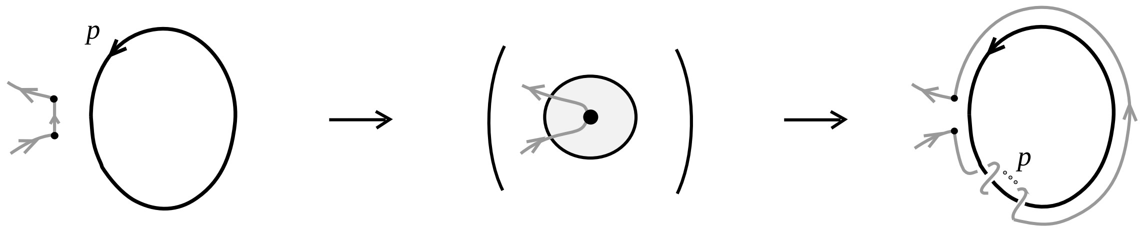




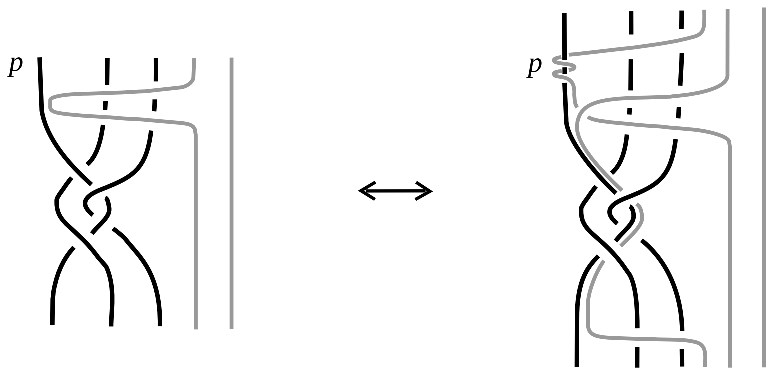
<u>Theorem</u> (Lickorish–Wallace): Every c.c.o. 3-manifold can be obtained from the 3-sphere by (integral) surgery along a knot or link.
See [19,23,22,14].

Knots and braids in c.c.o. 3-manifolds, [17,18]

A knot in the 3-manifold has the following extra freedom
(beyond the isotopy in the complemement of the surgery knot):



On the mixed braid level we have the extra `**band move**':



Braid equivalence moves in c.c.o. 3-manifolds [17,18]

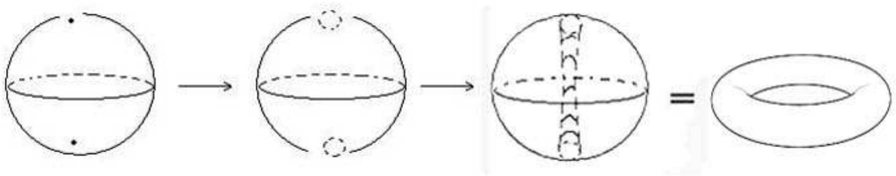
- L-moves that do not affect the fixed manifold braid
- Stabilization moves and all conjugations are allowed
- The algebraic band moves

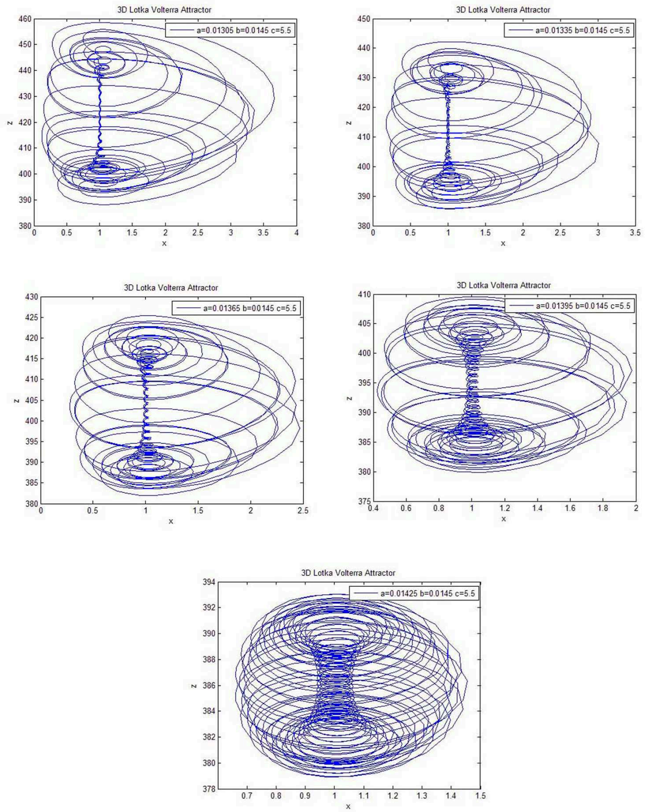
(See also [21,20])

Open Problems

- Find finite dimensional quadratic quotient algebras of the braid groups $B_{m,n}$, (start with m=2).
- Find `nice' linear bases for these algebras.
- Construct Markov traces on these algebras.
- Normalize the traces to define Jones-type invariants for knots in related manifolds.

An aside: Topological surgery in 2-D

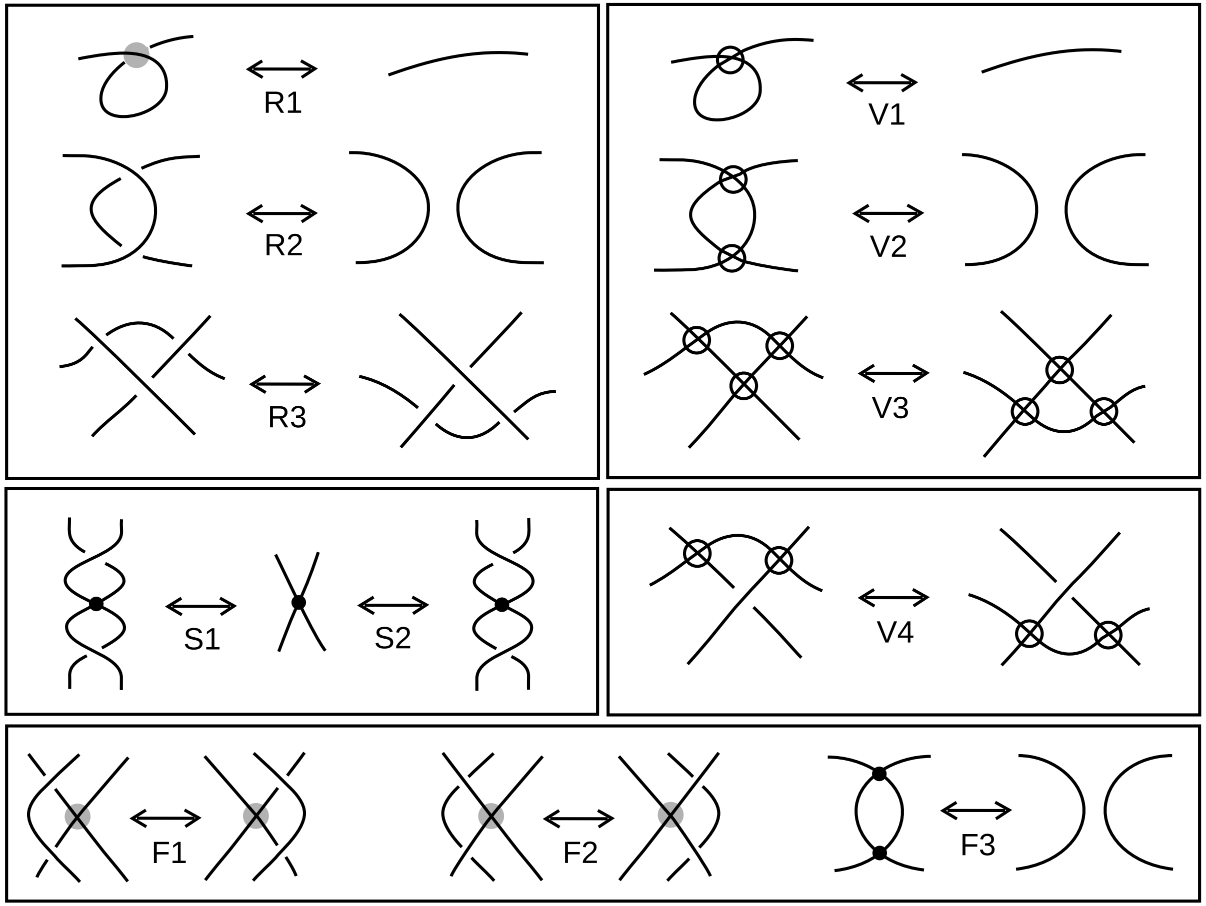




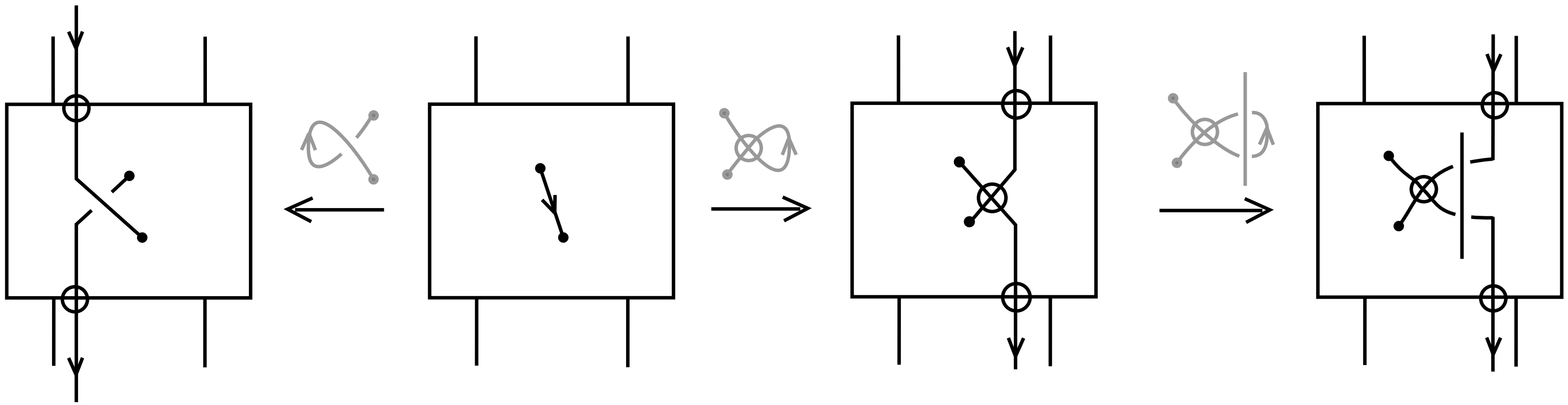
There is a 2-predator, 1-pray dynamical system with chaotic attractor solution that realizes 2-D topological surgery by changing a parameter (observation by S.L.)

See arXiv:0812.2367v1 [math.DS] 12 Dec 2008

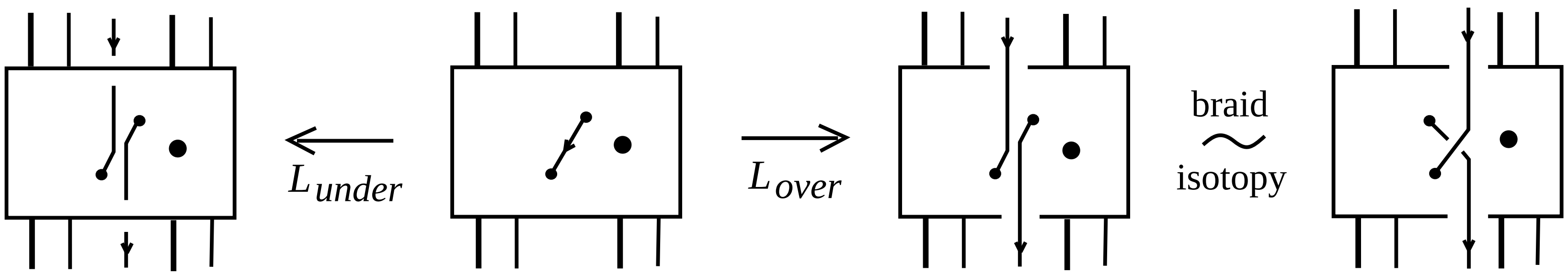
Knots in other diagrammatic set-ups, [16,13]



Virtual knots, welded knots, singular knots



L-moves for virtual braids



L-moves for singular braids

Some References on 1st part

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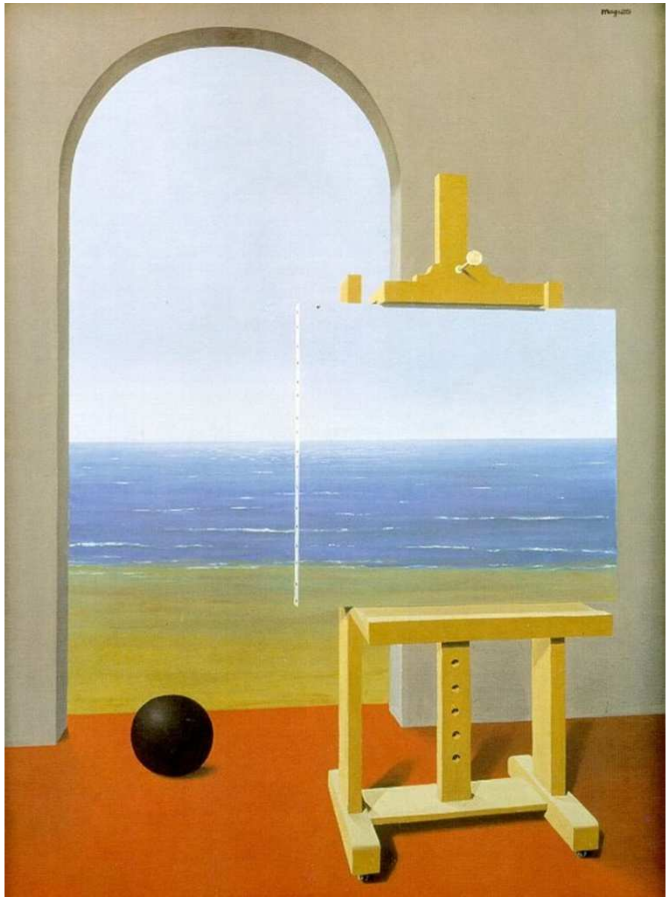

THANK YOU!!!