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Three talks in Trieste
Fox coloring, Burnside groups, skein modules, Khovanov categorification of skein modules

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(after Cuautitlan notes)

Colorazioni di Fox, gruppi di Burnside, moduli skein, categorificazione di Khovanov di moduli skein

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Abstract Talk 1: Open problems in knot theory that everyone can try to solve.
Knot theory is more than two hundred years old; the first scientists who considered knots as mathematical objects were A.Vandermonde (1771) and C.F.Gauss (1794). However, despite the impressive grow of the theory, there are simply formulated but fundamental questions, to which we do not know answers. I will discuss today several such open problems, describing in detail the 30 year old Nakanishi 4-move conjecture. Our problems lead to sophisticated mathematical structures (I will describe some of them in tomorrows talks), but today's description will be absolutely elementary.
Talk 2: Applications of Burnside groups to knot equivalence. Counterexample to Montesinos Nakanishi 3-move conjecture (the closure of the 5-braid $(\sigma_1\sigma_2\sigma_3\sigma_4)^{10}$). If times allows I will desribe Lagrangian approximation of Fox p-colorings of tangles.
The introduction of the symplectic structure on the boundary of a tangle in such a way that tangles yields Lagrangians in the symplectic space.
Talk 3: Historical Introduction to Skein Modules.
I will discuss, in my last talk of the conference, skein modules, or as I prefer to say more generally, algebraic topology based on knots. I would like to discuss today, in more detail, skein modules related to the (deformations) of 3-moves and the Montesinos-Nakanishi conjecture but first I will give the general definition and I will make a short tour of the world of skein modules.

1 Talk 1: Open problems in knot theory that everyone can try to solve.

Knot theory is more than two hundred years old; the first scientists who considered knots as mathematical objects were A.Vandermonde (1771) and C.F.Gauss (1794). However, despite the impressive grow of the theory, there are simply formulated but fundamental questions, to which we do not know answers. I will discuss today several such open problems, describing in detail the 20 year old Montesinos-Nakanishi conjecture. Our problems lead to sophisticated mathematical structures (I will describe some of them in tomorrow's talks), but today's description will be absolutely elementary. Links are circles embedded in our space, R^3, up to topological deformation, that is two links are equivalent if one can be deformed into the other in space without cutting and pasting. We represent links on a plane using their diagrams (we follow the terminology of Lou Kauffman's talk).

First we should introduce the concept of an n move on a link. One should stress that the move is not topological, so that it can change the type of the link we deal with.

Definition 1.1 *An n-move on a link is a local change of the link illustrated in Figure 1.1.*

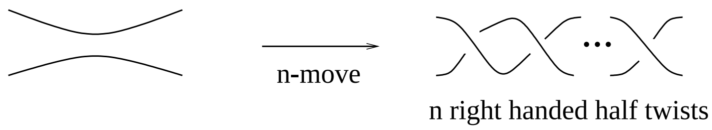
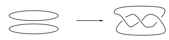


Fig. 1.1; n-move

In our convention the part of the link outside of the disk, in which the move takes part, is unchanged. For example  illustrates a 3-move.

Definition 1.2 *We say that two links, L_1, L_2 are n-move equivalent if one can go from one to the other by a finite number of n-moves and their inverses ($-n$ moves).*

If we work with diagrams of links then the topology of links is reflected by Reidemeister moves, that is two diagrams represent the same link in space if and only if one can go from one to the other by Reidemeister moves:

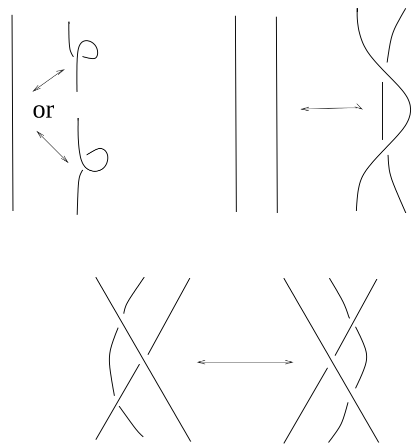
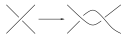


Fig. 1.2; Reidemeister moves

Thus we say that two diagrams, D_1 and D_2, are n-move equivalent if one can obtain one from the other by n-moves, their inverses and Reidemeister moves. To illustrate this let us notice that the move  is a consequence of a 3-move and a second Reidemeister move (Fig.1.3).

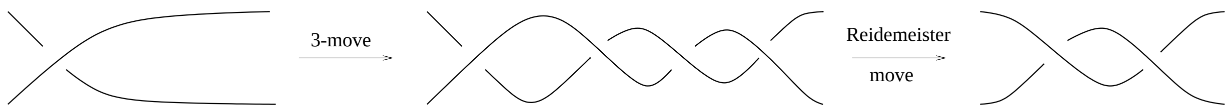


Fig. 1.3

Conjecture 1.3 (Montesinos-Nakanishi)
Every link is 3*-move equivalent to a trivial link.*

Yasutaka Nakanishi introduced the conjecture in 1981. José Montesinos analyzed 3-moves before, in connection with 3-fold dihedral branch coverings, and asked a related but different question.

Examples 1.4 *(i) Trefoil knots (left and right handed) are 3-move equivalent to the trivial link of two components:*

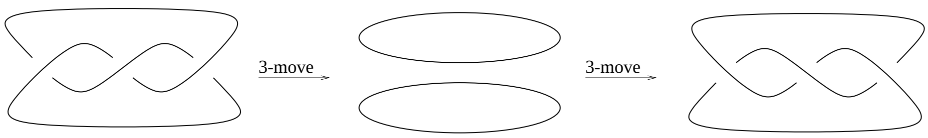


Fig. 1.4

(ii) The figure eight knot is 3-move equivalent to the trivial knot:

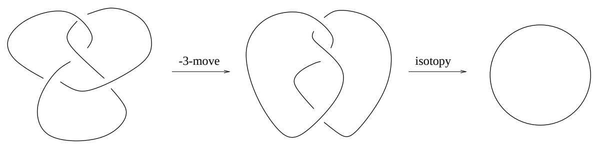


Fig. 1.5

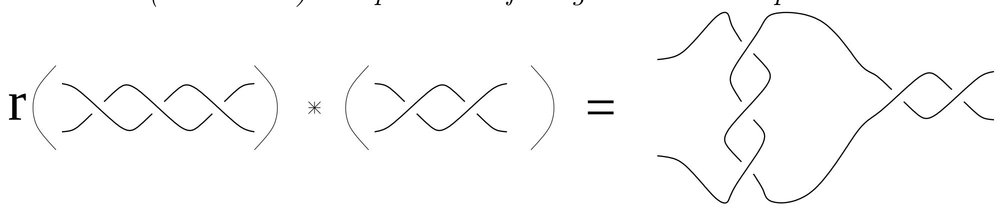
We will show later, in this talk, that different trivial links are not 3-move equivalent, however in order to achieve this conclusion we need an invariant of links preserved by 3-moves and different for different trivial links. Such invariant is the Fox 3-coloring. We will introduce it later (today in the simplest form and in the second lecture in a more general context of Fox n-colorings and Alexander-Burau-Fox colorings). Now let us present some other related conjectures.

Conjecture 1.5
Any 2-tangle is 3-move equivalent, to one of the four 2-tangles of Figure 1.6. We allow additionally trivial components in the tangles of Fig.1.6.

Fig. 1.6

Montesinos-Nakanishi conjecture follows from Conjecture 1.5. More generally if Conjecture 1.5 holds for some class of 2-tangles, then Conjecture 1.3 holds for a link obtained by closing any tangle from the class, without introducing any new crossing. The simplest interesting tangles for which Conjecture 1.5 holds are algebraic tangles in the sense of Conway (I will call them 2-algebraic tangles and present their generalization in the next talk). For 2-algebraic tangles Conjecture 1.5 holds by an induction and I will leave it as a pleasent exercise for you. The necessary definition is given below:

Definition 1.6 ([Co, B-S])
2-algebraic tangles are the smallest family of 2-tangles which satisfies:
(0) Any 2-tangle with 0 or 1 crossing is 2-algebraic.
*(1) If A and B are 2-algebraic tangles then $r^i(A) * r^j(B)$ is 2-algebraic; r denotes here the rotation of a tangle by 90^o angle along the z-axis, and * denotes the (horizontal) composition of tangles. For example*



A link is 2-algebraic if it is obtained from a 2-algebraic tangle by closing its ends without introducing any new crossings.

The Montesinos-Nakanishi conjecture was proven for many special families of links by my students Qi Chen and Tatsuya Tsukamoto [Che, Tsu, P-Ts]. In particular Chen proved that the conjecture holds for all five braid links except, possibly one family, represented by the square of the center of the 5-braid group, $(\sigma_1\sigma_2\sigma_3\sigma_4)^{10}$. He also found 5-braid link 3-move equivalent to it with 20 crossings. It is now the smallest known possible counterexample to Montesinos-Nakanishi conjecture, Fig.1.7.

Fig. 1.7

Previously (1994), Nakanishi suggested the 2-parallel of the Borromean rings (with 24 crossings) as a possible counterexample (Fig.1.8).

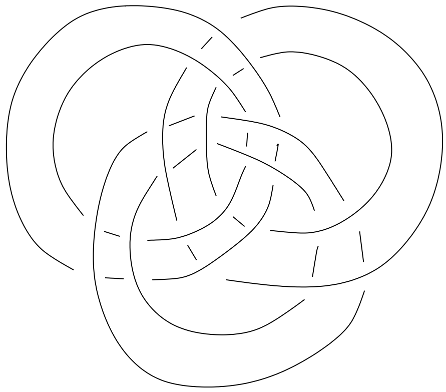


Fig. 1.8

We will go back tomorrow to theories motivated by 3-moves, now we will outline conjectures using other elementary moves.

Conjecture 1.7 (Nakanishi, 1979)
Every knot is 4*-move equivalent to the trivial knot.*

Examples 1.8 *Reduction of the trefoil and the figure eight knots is illustrated in Figure 1.9.*

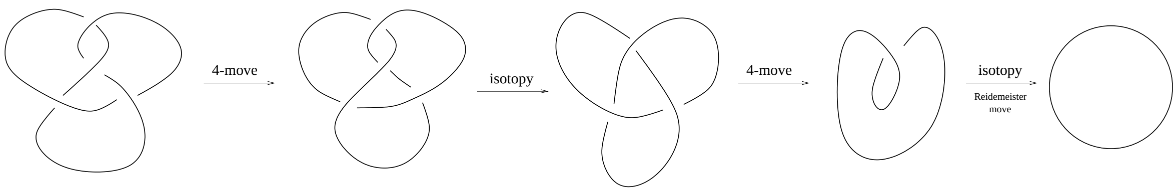
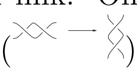
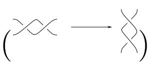


Figure 1.9

It is not true that every link is 5-move equivalent to a trivial link. One can show, using the Jones polynomial, that the figure eight knot is not 5-move equivalent to any trivial link. One can however introduce a more delicate move, called $(2,2)$-move (), such that the 5-move is a combination of a $(2,2)$-move and its mirror image $(-2,-2)$-move (), as illustrated in Figure 1.10 [H-U, P-3].

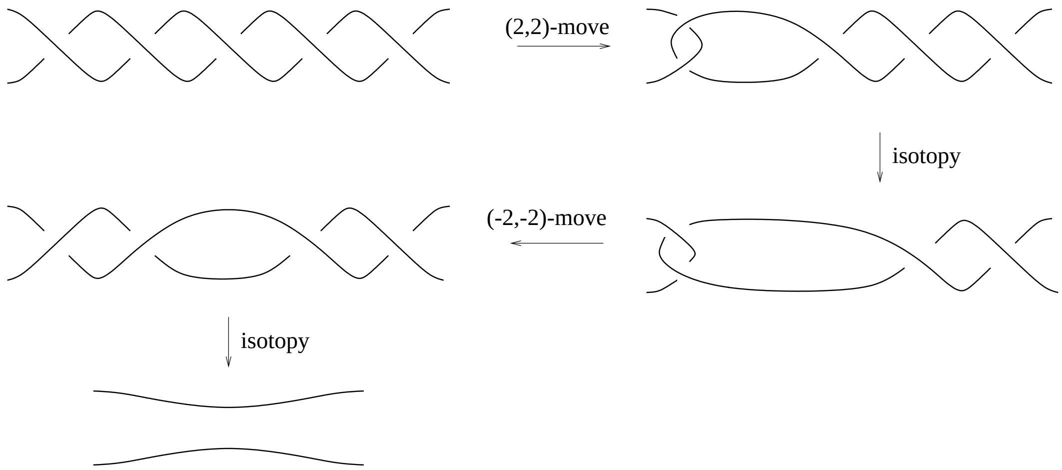


Fig. 1.10

Conjecture 1.9 (Harikae, Nakanishi, 1992) *Every link is* $(2,2)$*-move equivalent to a trivial link.*

As in the case of 3-moves, an elementary induction shows that the conjecture holds for 2-algebraic links (algebraic in the Conway's sense). It is also known that the conjecture holds for all links up to 8 crossings. The key element of the argument in the proof is the observation (going back to Conway [Co]) that any link up to 8 crossings (different than 8_{18}; compare the footnote 1) is 2-algebraic. The reduction of the 8_{18} knot to a trivial link of two components by my students, Jarek Buczyński and Mike Veve, is illustrated in Figure 1.11.

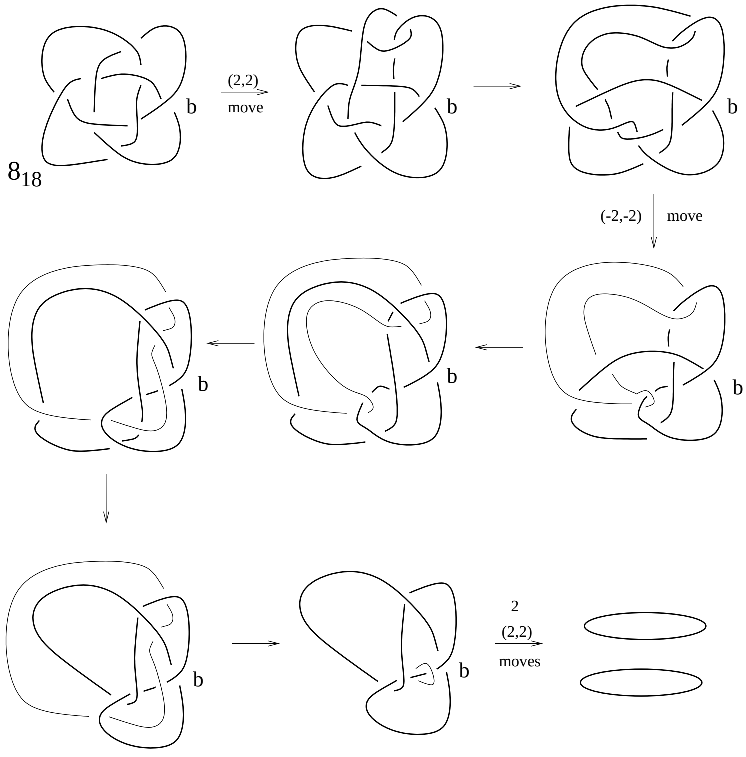


Fig. 1.11; Reduction of the 8_{18} knot

The smallest knot, not reduced yet, is the 9_{49} knot, Figure 1.12. Possibly you can reduce it!

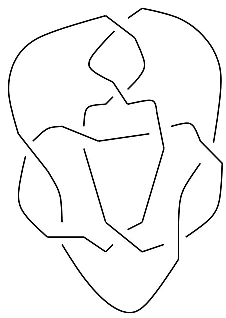


Figure 1.12

With respect to the next open question, I am much less convinced that the answer is positive, so I will not call it a "conjecture". First let define a (p, q) move as a local modification of a link as shown in Figure 1.13. We say that two links, L_1, L_2 are (p,q) equivalent if one can pass from one to the other by a finite number of (p,q),(q,p),$(-p,-q)$ and $(-q,-p)$-moves.

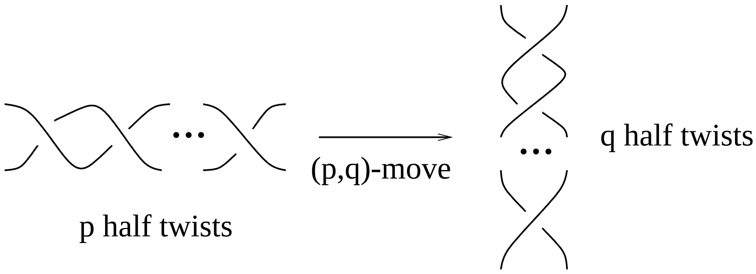


Fig. 1.13

Problem 1.10 ([Kir];Problem 1.59(7), 1995) *Is it true that any link is* $(2,3)$ *move equivalent to a trivial link?*

Example 1.11 *Reduction of the trefoil and the figure eight knots is illustrated in Fig. 1.14. Reduction of the Borromean rings is performed in Fig. 1.15.*

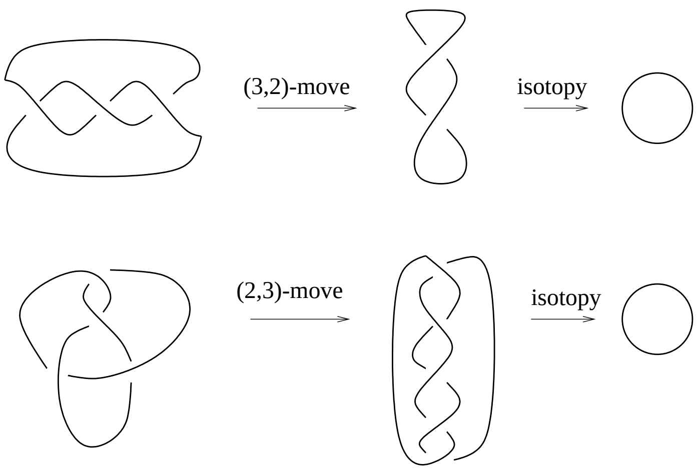


Figure 1.14

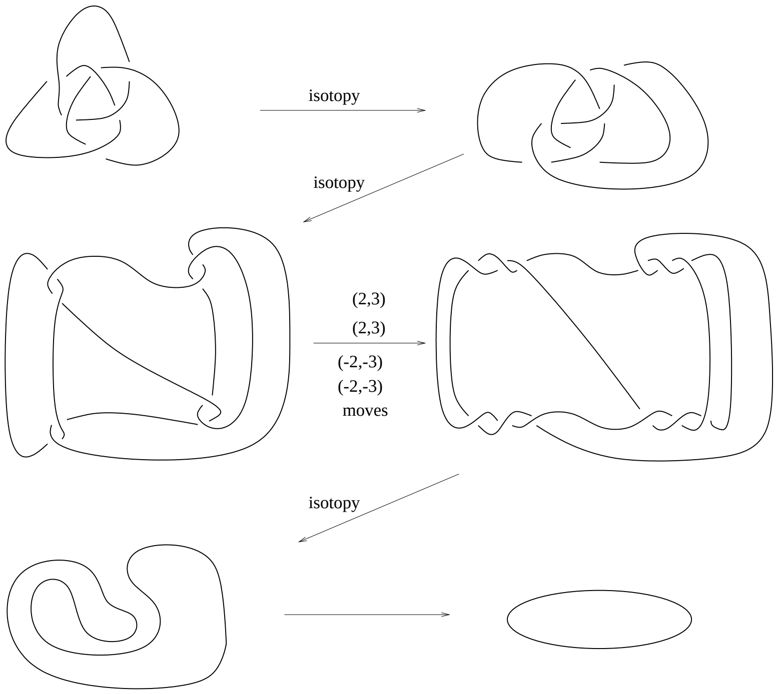


Figure 1.15

Generally, rather simple inductive argument shows that 2-algebraic links are $(2,3)$-move equivalent to trivial links. Figure 1.16 illustrates why the Borromean rings are 2-algebraic. By a proper filling of black dots one can also show that all links up to 8 crossings, but 8_{18}, are 2-algebraic. Thus, as in the case of $(2,2)$-equivalence, the only link, up to 8 crossings, which should be still checked is the 8_{18} knot[1]. Nobody really tried this seriously, so maybe somebody in the audience will try this puzzle.

[1]To prove that the knot 8_{18} is not 2-algebraic one considers the 2-fold branched cover of S^3 with this knot as the branching set, $M^{(2)}_{8_{18}}$. Montesinos proved that algebraic knots are covered by Waldhausen graph manifolds [Mo-1]. Bonahon and Siebenmann showed ([B-S], Chapter 5) that $M^{(2)}_{8_{18}}$ is a hyperbolic 3-manifold so cannot be a graph manifold. This manifold is interesting from the point of view of hyperbolic geometry because it is a closed manifold with its volume equal to the volume of the figure eight knot complement [M-V-1]. The knot 9_{49} of Fig.1.12 is not 2-algebraic as well because its 2-fold branched cover is a hyperbolic 3-manifold. In fact, it is the manifold I suspected from 1983 to have the smallest volume among oriented hyperbolic 3-manifolds [I-MPT, Kir, M-V-2].

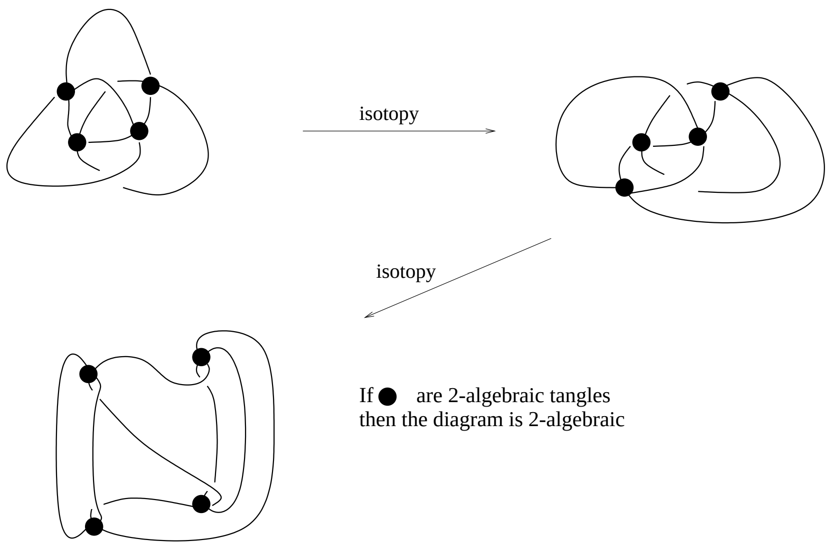


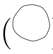
Figure 1.16

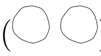
Fox colorings.
The 3-coloring invariant which we will use to show that different trivial links are not 3-move equivalent, was introduced by R.H. Fox[2] in about 1956 when explaining knot theory to undergraduate students at Haverford College ("in an attempt to make the subject accessible to everyone" [C-F]). It is a pleasant method of coding representations of the fundamental group of a link complement into the group of symmetries of a regular triangle, but this interpretation is not needed for the definition and most of applications of 3-colorings (compare [Cr, C-F, Fo-1, Fo-2]).

Definition 1.12 *(Fox 3-coloring of a link diagram).*
Consider a coloring of a link diagram using colors r (red), y (yellow) or b

[2]Ralph Hartzler Fox was born March 24, 1913. A native of Morrisville, Pa., he attended Swarthmore College for two years while studying piano at the Leefson Conservatory of Music in Philadelphia. He received his master's degree from the Johns Hopkins University and Ph.D. from the Princeton University in 1939 under the supervision of Solomon Lefschetz. Fox was married, when he was still a student, to Cynthia Atkinson. They had one son, Robin. After receiving his Princeton doctorate, he spent the following year at Institute for Advanced Study in Princeton. He taught at the University of Illinois and Syracuse University before returning to join the Princeton University faculty in 1945 and staying there until his death. He died December 23, 1973 in the University of Pennsylvania Graduate Hospital, where he had undergone open-hearth surgery [P-12].

(blue) in such a way that an arc of the diagram (from a tunnel to a tunnel) is colored by one color and at a crossing one uses one or all three colors. Such a coloring is called a Fox 3-coloring. If whole diagram is colored by just one color we say that we have a trivial coloring. Let $tri(D)$ denote the number of different Fox 3-colorings of D.

Example 1.13 *(i) $tri($$) = 3$ as the trivial diagram has only trivial colorings.*

(ii) $tri($$) = 9$, and more generally for a trivial link diagram of n components, U_n, one has $tri(U_n) = 3^n$.

(iii) For a standard diagram of a trefoil knot we have three trivial colorings and 6 nontrivial colorings, one of them is presented in Figure 1.17 (all other differ from this one by permutations of colors). Thus $tri($$)$ $= 3 + 6 = 9$

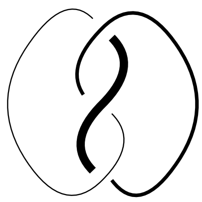


Fig. 1.17; Different colors are marked by lines of different thickness.

Fox 3-colorings were defined for link diagrams, they are however invariants of links. One needs only show that $tri(D)$ is unchanged by Reidemeister moves.

The invariance under R_1 and R_2 is illustrated in Fig.1.18 and the invariance under R_3 is illustrated in Fig.1.19.

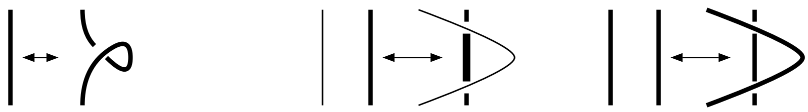


Fig. 1.18

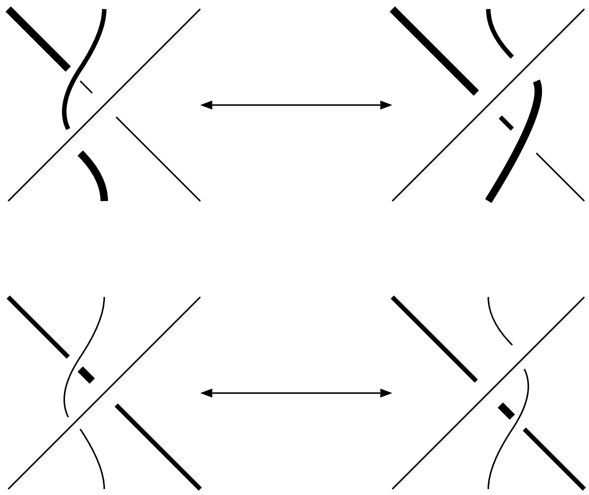


Fig. 1.19

The next property of Fox 3-colorings is the key in proving that different trivial links are not 3-move equivalent.

Lemma 1.14 ([P-1]) *$tri(D)$ is unchanged by a 3-move.*

The proof of the lemma is illustrated in Figure 1.20.

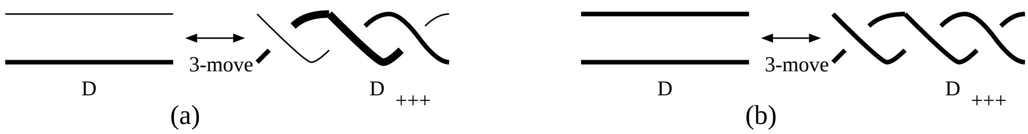


Fig. 1.20

The lemma also explains the fact that the trefoil has nontrivial Fox 3-colorings: the trefoil knot is 3-move equivalent to the trivial link of two components (Example 1.4(i)).

Tomorrow I will place the theory of Fox coloring in more general (sophisticated) context, and apply it to the analysis of 3-moves (and $(2,2)$ and $(2,3)$ moves) of n-tangles. Interpretation of tangle colorings as Lagrangians in symplectic spaces is our main (and new) tool. In the second lecture tomorrow, I will discuss another motivation for studying 3-moves: to understand skein modules based on their deformation.

2 Talk 2: Lagrangian approximation of Fox p-colorings of tangles.

We just have heard beautiful and elementary talk by Lou Kauffman. I hope to follow his example by having my talk elementary and deep at the same time. I will use several results introduced by Lou, like rational tangles and their classification, and will also build on my yesterday's talk. The talk will culminate by the introduction of the symplectic structure on the boundary of a tangle in such a way that tangles yields Lagrangians in the symplectic space. I could not dream of this connection a year ago; however now, after 10 month perspective, I see the symplectic structure as a natural development.

Let us start slowly from my personal perspective and motivation. In the spring of 1986, I was analyzing behavior of Jones type invariants of links when a link was modified by a k-move (or t_k, $\bar{t}_{2k}$ moves in the oriented case). My interest had its roots in the fundamental Conway's paper [Co]. In July of 1986, I gave a talk at the "Braids" conference in Santa Cruz. I was told by Murasugi and H.Nakanishi, after my talk, about the Nakanishi's 3-move conjecture. It was suggested to me by R.Campbell (Kirby's student in 1986) to consider the effect of 3-moves on Fox colorings. Only several years later, when writing [P-3] in 1993, I realized that Fox colorings can be successfully used to analyze moves on tangles by considering not only the space of colorings but also the induced coloring of boundary points. More of this later, now it is time to define Fox k-colorings.

Definition 2.1 *(i) We say that a link (or a tangle) diagram is k-colored if every arc is colored by one of the numbers $0, 1, ..., k-1$ (forming a group Z_k) in such a way that at each crossing the sum of the colors of the undercrossings is equal to twice the color of the overcrossing modulo k; see Fig.2.1.*

(ii) The set of k-colorings forms an abelian group, denoted by $Col_k(D)$. The cardinality of the group will be denoted by $col_k(D)$. For an n-tangle T each Fox k-coloring of T yields a coloring of boundary points of T and we have the homomorphism $\psi : Col_k(T) \to Z_k^{2n}$

.

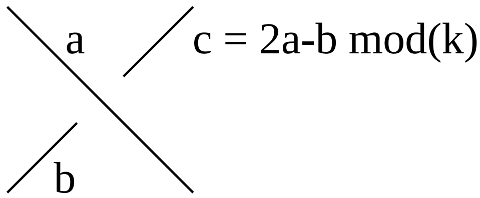


Fig. 2.1

It is a pleasant exercise to show that $Col_k(D)$ is unchanged by Reidemeister moves and by k-moves (Fig.2.2).

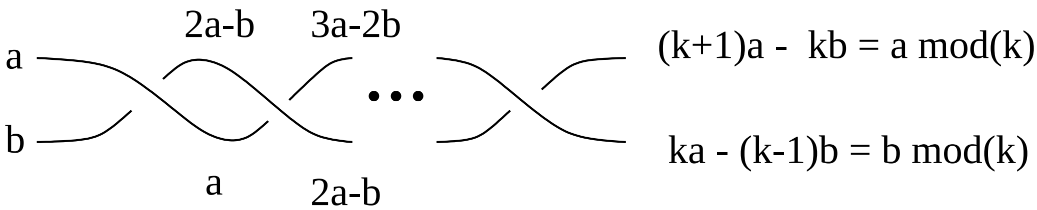


Fig. 2.2

Let us look closer at the observation that a k-move preserves the space of Fox k-colorings and at the unlinking conjectures described till now. We discussed the 3-move conjecture and the 4-move conjecture for knots (it does not hold for links[3]). As I mentioned yesterday, not every link can be simplified using 5-moves, but the 5-move is a combination of $(2,2)$ moves and these moves might be sufficient to reduce every link. Similarly not every link can be reduced via 7-moves, but the 7-move is a combination of $(2,3)$-moves[4] which might be sufficient for reduction. We stopped at this point yesterday, but what can one use instead of a general k-move? Let us consider the case of a p-move where p is a prime number. I suggest (and state publicly for the first time) that one should consider *rational moves*, that is, a rational $\frac{p}{q}$-tangle of Conway is substituted in place of the identity tangle[5]. The important observation for us is that $Col_p(D)$ is preserved by $\frac{p}{q}$-moves. Fig.2.3 illustrates the fact that $Col_{13}(D)$ is unchanged by a $\frac{13}{5}$-move.

[3]Kawauchi suggested that for links one should conjecture that two link-homotopic links are 4-move equivalent .

[4]To be precise, a 7-move is a combination of a $(-3,-2)$ and $(2,3)$ moves; compare Fig.1.10.

[5]The move was first considered by J.M.Montesinos[Mo-2]; compare also Y.Uchida [Uch].

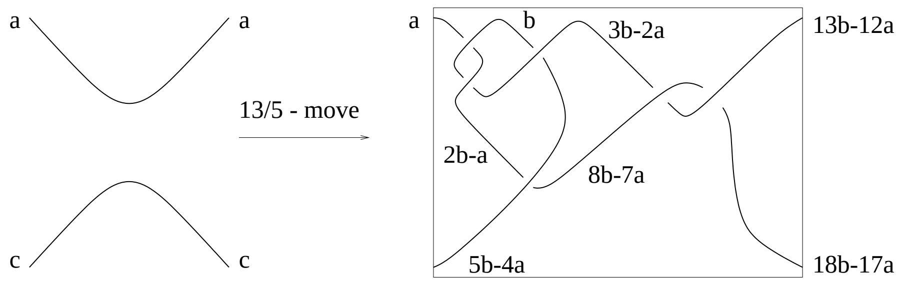


Fig. 2.3

We should also note that (m, q)-moves are equivalent to $\frac{mq+1}{q}$-moves (Fig.2.4) so the space of Fox $(mq + 1)$-colorings is preserved too.

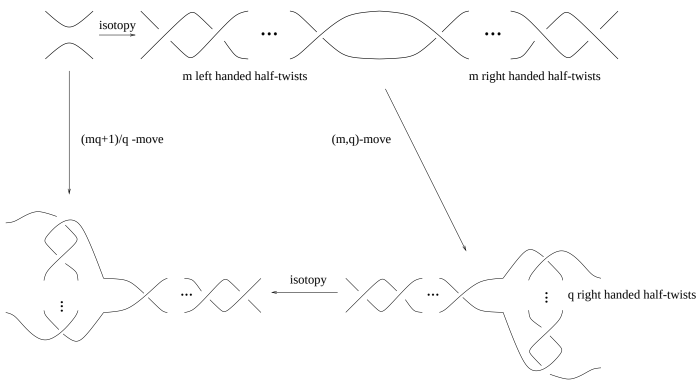


Fig. 2.4

We just have heard about the Conway's classification of rational tangles at the Lou's talk, so I will briefly sketch definitions and notation. The 2-tangles shown in Figure 2.5 are called rational tangles with Conway's notation $T(a_1, a_2, ..., a_n)$. A rational tangle is the $\frac{p}{q}$-tangle if $\frac{p}{q} = a_n + \frac{1}{a_{n-1}+...+\frac{1}{a_1}}$.[6] Conway proved that two rational tangles are ambient isotopic (with boundary fixed) if and only if their slopes are equal (compare [Kaw]).

[6] $\frac{p}{q}$ is called the slope of the tangle and can be easily identified with the slope of the meridian disk of the solid torus being the branched double cover of the rational tangle.

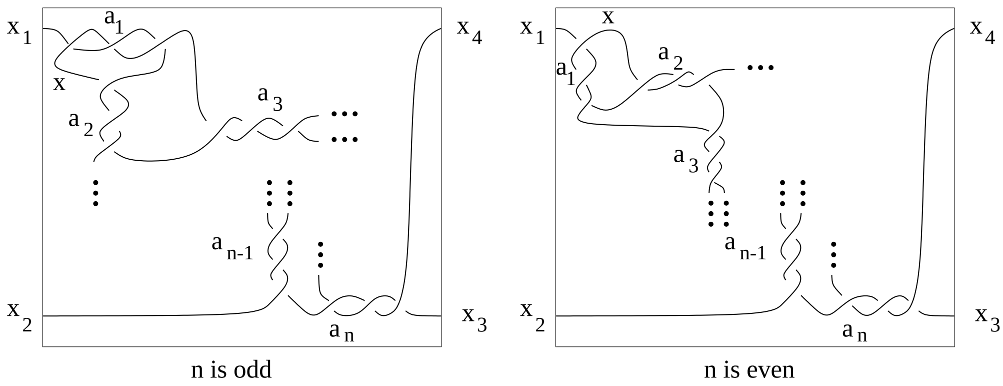


Fig. 2.5

For a Fox coloring of a rational $\frac{p}{q}$-tangle with boundary colors x_1, x_2, x_3, x_4 (Fig.2.5), one has $x_4 - x_1 = p(x - x_1)$, $x_2 - x_1 = q(x - x_1)$ and $x_3 = x_2 + x_4 - x_1$. If a coloring is nontrivial ($x_1 \neq x$) then $\frac{x_4 - x_1}{x_2 - x_1} = \frac{p}{q}$ as has been explained by Lou.

Conjecture 2.2
Let p be a fixed prime number, then

(i) Every link can be reduced to a trivial link by rational $\frac{p}{q}$-moves (q any integer).

(ii) There is a function $f(n,p)$ such that any n-tangle can be reduced to one of "basic" $f(n,p)$ n-tangles (allowing additional trivial components) by rational $\frac{p}{q}$-moves.

First we can observe that it suffices to consider $\frac{p}{q}$-moves with $|q| \leq \frac{p}{2}$, as other $\frac{p}{q}$-moves follow. Namely, we have $\frac{p}{p-q} = 1 + \frac{1}{-1+\frac{p}{q}}$ and $\frac{p}{-(p+q)} = -1 + \frac{1}{1+\frac{p}{q}}$. Thus $\frac{p}{q}$-moves generate $\frac{p}{-q \pm p}$-moves. Furthermore, we know that for p odd the $\frac{p}{1}$-move is a combination of $\frac{p}{2}$ and $\frac{p}{-2}$-moves (compare Fig.1.10). Thus, in fact, 3-move, $(2,2)$-move and $(2,3)$-move conjectures are special cases of Conjecture 2.2(i). If we analyze the case $p = 11$ we see that $\frac{11}{2} = 5 + \frac{1}{2}$, $\frac{11}{3} = 4 - \frac{1}{3}$, $\frac{11}{4} = 3 - \frac{1}{4}$, $\frac{11}{5} = 2 + \frac{1}{5}$. Thus:

Conjecture 2.3
Every link can be reduced to a trivial link (with the same space of 11-colorings) by $(2,5)$ and $(4,-3)$ moves, their inverses and mirror images.

What about the number $f(n,p)$? We know that because $\frac{p}{q}$-moves preserve p-colorings, therefore $f(n,p)$ is bounded from below by the number of subspaces of p-colorings of the $2n$ boundary points induced by Fox p-colorings of n-tangles (that is by the number of subspaces $\psi(Col_p(T))$ in Z_p^{2n}). I noted in [P-3] that for 2-tangles this number is equal to $p+1$ (even in this special case my argument was complicated). For $p=3, n=4$ the number of subspaces followed from the work of my student Tatsuya Tsukamoto and was equal to 40 [P-Ts]. The combined effort of Mietek Dąbkowski and Tsukamoto gave the number 1120 for subspaces $\psi(Col_3(T))$ and 4-tangles. That was my knowledge at the early spring of 2000. On May 2nd and 3rd I heard talks on Tits buildings (at the Banach Center in Warsaw) by J.Dymara and T.Januszkiewicz. I realized that the topic may have some connection to my work. I asked Tadek Januszkiewicz whether he sees relations and I gave him numbers 4, 40, 1120 for $p=3$. He immediately answered that most likely I counted the number of Lagrangians in Z_3^{2n-2} symplectic space, and that the number of Lagrangians in Z_p^{2n-2} is known to be equal to $\prod_{i=1}^{n-1}(p^i+1)$. Soon I constructed the appropriate symplectic form (as did Janek Dymara). I will spend most of this talk on the construction. I will end with discussion of classes of tangles for which it has been proven that $f(n,p)=\prod_{i=1}^{n-1}(p^i+1)$.

Consider $2n$ points on a circle (or a square) and a field $\mathbf{Z}_p$ of p-colorings of a point. The colorings of $2n$ points form $\mathbf{Z}_p^{2n}$ linear space. Let $e_1,\ldots,e_{2n}$ be its basis,

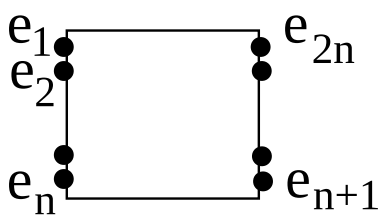


Fig. 2.6

$e_i=(0,\ldots,1,\ldots,0)$, where 1 occurs in the i-th position. Let $\mathbf{Z}_p^{2n-1}\subset \mathbf{Z}_p^{2n}$ be the subspace of vectors $\sum a_i e_i$ satisfying $\sum(-1)^i a_i=0$ (alternating condition). Consider the basis $f_1,\ldots,f_{2n-1}$ of $\mathbf{Z}_p^{2n-1}$ where $f_k=e_k+e_{k+1}$. Consider a skew-symmetric form ϕ on $\mathbf{Z}_p^{2n-1}$ of nullity 1 given by the matrix

$$\phi = \begin{pmatrix} 0 & 1 & \dots & \dots \\ -1 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & -1 & 0 \end{pmatrix}$$

that is

$$\phi(f_i, f_j) = \begin{cases} 0 & \text{if } |j-i| \neq 1 \\ \\ 1 & \text{if } \ j = i+1 \\ \\ -1 & \text{if } j = i-1. \end{cases}$$

Notice that the vector $e_1 + e_2 + \ldots + e_{2n}$ $(= f_1 + f_3 + \ldots + f_{2n-1} = f_2 + f_4 + \ldots + f_{2n})$ is ϕ-orthogonal to any other vector. If we consider $\mathbf{Z}_p^{2n-2} = \mathbf{Z}_p^{2n-1}/\mathbf{Z}_p$, where the subspace $\mathbf{Z}_p$ is generated by $e_1 + \ldots + e_{2n}$, that is, $\mathbf{Z}_p$ consists of monochromatic (i.e. trivial) colorings, then ϕ descends to the symplectic form $\hat{\phi}$ on $\mathbf{Z}_p^{2n-2}$. Now we can analyze isotropic subspaces of $(\mathbf{Z}_p^{2n-2}, \hat{\phi})$, that is subspaces on which $\hat{\phi}$ is 0 ($W \subset \mathbf{Z}_p^{2n-2}, \phi(w_1, w_2) = 0$ for $w_1, w_2 \in W$). The maximal isotropic ($(n-1)$-dimensional) subspaces of $\mathbf{Z}_p^{2n-2}$ are called Lagrangian subspaces (or maximal totally degenerated subspaces) and there are $\prod_{i=1}^{n-1}(p^i + 1)$ of them.

We have $\psi : Col_pT \to \mathbf{Z}_p^{2n}$. Our local condition on Fox colorings (Fig.2.1) guarantees that for any tangle T, $\psi(Col_pT) \subset \mathbf{Z}_p^{2n-1}$. Furthermore, the space of trivial colorings, $\mathbf{Z}_p$, always lays in Col_pT. Thus ψ descents to $\hat{\psi} : Col_pT/\mathbf{Z}_p \to \mathbf{Z}_p^{2n-2} = \mathbf{Z}_p^{2n-1}/\mathbf{Z}_p$. Now we have the fundamental question: Which subspaces of $\mathbf{Z}_p^{2n-2}$ are yielded by n-tangles? We answer this question below.

Theorem 2.4 *$\hat{\psi}(Col_pT/\mathbf{Z}_p)$ is a Lagrangian subspace of $\mathbf{Z}_p^{2n-2}$ with the symplectic form $\hat{\phi}$.*

The natural question would be whether every Lagrangian subspace can be realized by a tangle. The answer is negative for $p = 2$ and positive for $p > 2$

[D-J-P]. As a corollary we obtain a fact which was considered to be difficult before, even for 2-tangles.

Corollary 2.5 *For any p-coloring of a tangle boundary satisfying the alternating property (i.e. an element of $\mathbf{Z}_p^{2n-1}$) there is an n-tangle and its p-coloring yielding the given coloring on the boundary. In other worlds: $\mathbf{Z}_p^{2n-1} = \bigcup_T \psi_T(Col_p(T))$. Furthermore, the space $\psi_T(Col_p(T))$ is n-dimensional.*

We can say that we understand the conjectured value of the function $f(n,p)$ but when can we prove Conjecture 2.2 with $f(n,p) = \prod_{i=1}^{n-1}(p^i+1)$? In fact, we know that for $p=2$ not every Lagrangian is realized and actually $f(n,2) = \prod_{i=1}^{n-1}(2i+1)$. For rational 2-tangles Conjecture 2.2 follows almost from the definition and the generalization to 2-algebraic tangles (algebraic tangles in the Conway sense) is not difficult. In order to be able to use induction for tangles with $n>2$ we generalize the concept of the algebraic tangle:

Definition 2.6

(i) *n-algebraic tangles is the smallest family of n-tangles which satisfies:*
(0) Any n-tangle with 0 or 1 crossing is n-algebraic.
*(1) If A and B are n-algebraic tangles then $r^i(A) * r^j(B)$ is n-algebraic; r denotes here the rotation of a tangle by $\frac{2\pi}{2n}$ angle, and * denotes (horizontal) composition of tangles.*

(ii) *If in the condition (1), B is restricted to tangles with no more than k crossings, we obtain the family of (n,k)-algebraic tangles.*

(iii) *If an m-tangle, T, is obtained from an (n,k)-algebraic tangle (resp. n-algebraic tangle) by partially closing its endpoints ($2n-2m$ of them) without introducing any new crossings then T is called an (n,k)-algebraic (resp. n-algebraic) m-tangle. For $m=0$ we obtain an (n,k)-algebraic (resp. n-algebraic) link.*

Conjecture 2.2, for $p=3$, has been proven for 3-algebraic tangles [P-Ts] ($f(3,3)=40$) and $(4,5)$-algebraic tangles [Tsu] ($f(4,3)=1120$). In particular the Montesinos-Nakanishi conjecture holds for 3-algebraic and $(4,5)$-algebraic links. 40 “basic” 3-tangles are shown in Fig. 2.7.

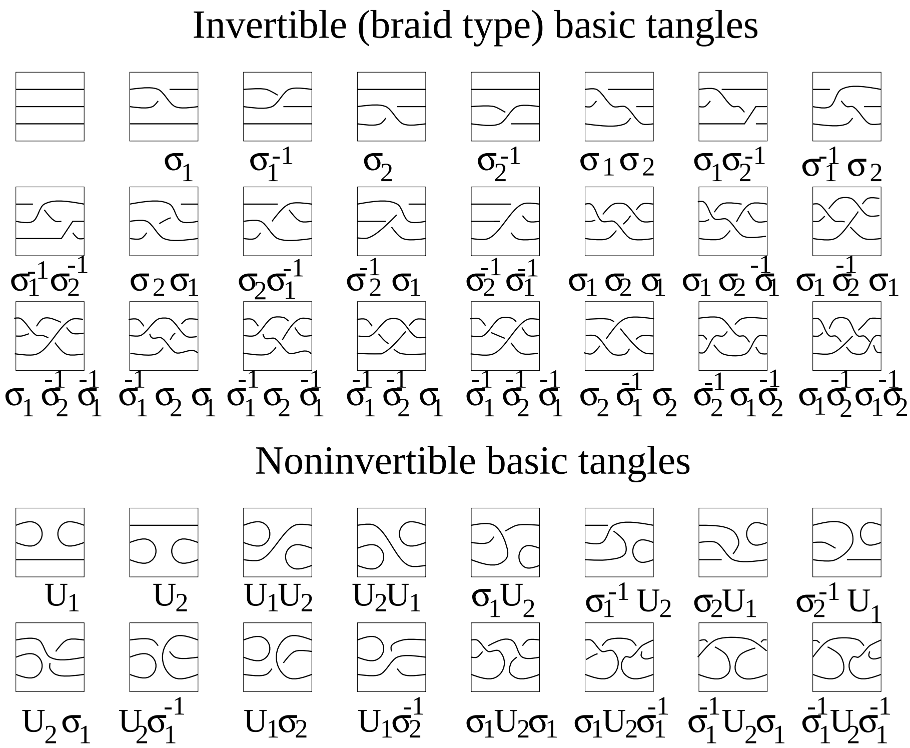


Fig. 2.7

The simplest 4-tangles which cannot be distinguished by 3-coloring for which 3-move equivalence is not yet established are illustrated in Fig.2.8. With respect to $(2,2)$ moves, the equivalence of 2-tangles in Fig.2.9 is still an open problem.

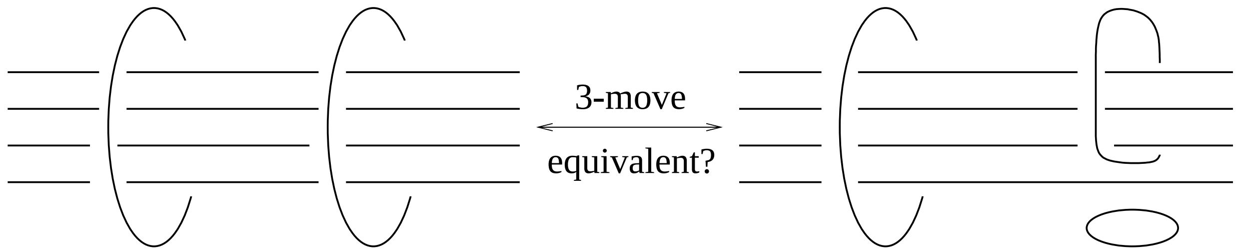


Fig. 2.8

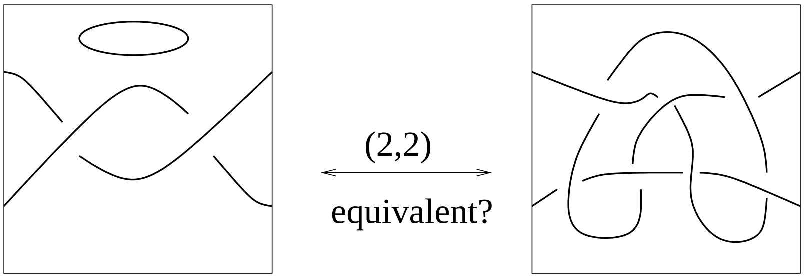


Fig. 2.9

Let me complete this talk by mentioning two generalizations of the Fox k-colorings.

In the first generalization we consider any commutative ring with the identity in place of Z_k. We construct $Col_R T$ in the same way as before with the relation at each crossing, Fig.2.1, having the form $c = 2a - b$ in R. The skew-symmetric form ϕ on R^{2n-1}, the symplectic form $\hat{\phi}$ on R^{2n-2} and the homomorphisms ψ and $\hat{\psi}$ are defined in the same manner as before. Theorem 2.4 generalizes as follows ([D-J-P]):

Theorem 2.7 *Let R be a Prinicipal Ideal Domain (PID) then, $\hat{\psi}(Col_R T/R)$ is a virtual Lagrangian submodule of R^{2n-2} with the symplectic form $\hat{\phi}$. That is $\hat{\psi}(Col_R T/R)$ is a finite index submodule of a Lagrangian in R^{2n-2}.*

The second generalization leads to racks and quandles [Joy, F-R] but we restrict our setting to the abelian case – Alexander-Burau-Fox collorings[7]. An ABF-coloring uses colors from a ring, R, with an invertible element t (e.g. $R = Z[t^{\pm 1}]$). The relation in Fig.2.1 is modified to the relation $c = (1-t)a + tb$ in R at each crossing of an oriented link diagram; see Fig. 2.10.

[7]The related approach was first outlined in the letter of J.W.Alexander to O.Veblen, 1919 [A-V]. Alexander was probably influenced by P.Heegaard dissertation, 1898, which he reviewed for the French translation [Heeg]. Burau was considering a braid representation but locally his relation was the same as that of Fox. According to J.Birman, Burau learned of the representation from Reidemeister or Artin [Ep], p.330.

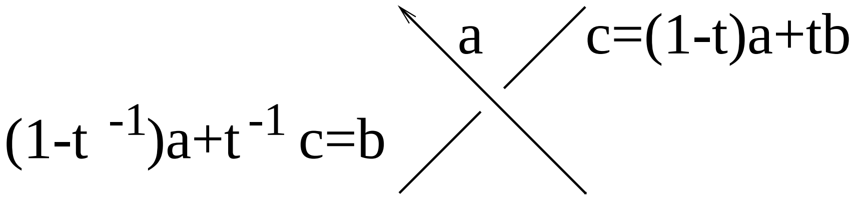


Fig. 2.10

The space R^{2n-2} has a natural Hermitian structure [Sq], but one can also find a symplectic structure and one can prove Theorem 2.7 in this setting [D-J-P].

3 Talk 3: Historical Introduction to Skein Modules.

I will discuss, in my last talk of the conference, *skein modules*, or as I prefer to say more generally, *algebraic topology based on knots*. It is my "brain child" even if the idea was also conceived by other people (most notably Vladimir Turaev), and was envisioned by John H.Conway (as "linear skein") a decade earlier. Skein modules have their origin in the observation by Alexander [Al], that his polynomials (*Alexander polynomials*) of three links, L_+, L_- and L_0 in R^3 are linearly related (Fig.3.1).

For me it started in Norman, Oklahoma in April of 1987, when I was enlightened to see that the multivariable version of the Jones-Conway (Homflypt) polynomial analyzed by Hoste and Kidwell is really a module of links in a solid torus (or more generally in the connected sum of solid tori).

I would like to discuss today, in more detail, skein modules related to the (deformations) of 3-moves and the Montesinos-Nakanishi conjecture, but first I will give the general definition and I will make a short tour of the world of skein modules.

Skein Module is an algebraic object associated to a manifold, usually constructed as a formal linear combination of embedded (or immersed) submanifolds, modulo locally defined relations. In a more restricted setting a *skein module* is a module associated to a 3-dimensional manifold, by considering linear combinations of links in the manifold, modulo properly chosen (skein) relations. It is the main object of the *algebraic topology based on knots*. When choosing relations one takes into account several factors:

(i) Is the module we obtain accessible (computable)?

(ii) How precise are our modules in distinguishing 3-manifolds and links in them?

(iii) Does the module reflect topology/geometry of a 3-manifold (e.g. surfaces in a manifold, geometric decomposition of a manifold)?.

(iv) Does the module admit some additional structure (e.g. filtration, gradation, multiplication, Hopf algebra structure)? Is it leading to a Topological Quantum Field Theory (TQFT) by taking a finite dimensional quotient?

One of the simplest skein modules is a q-deformation of the first homology group of a 3-manifold M, denoted by $\mathcal{S}_2(M;q)$. It is based on the skein relation (between unoriented framed links in M): $L_+ = qL_0$; it also satisfies the framing relation $L^{(1)} - qL$, where $L^{(1)}$ denote a link obtained from L by twisting the framing of L once in the positive direction. Already this simply defined skein module "sees" nonseparating surfaces in M. These surfaces are responsible for torsion part of our skein module [P-10].

There is more general pattern: most of analyzed skein modules reflect various surfaces in a manifold.

The best studied skein modules use skein relations which worked successfully in the classical knot theory (when defining polynomial invariants of links in R^3).

(1) The Kauffman bracket skein module, KBSM.
The skein module based on the *Kauffman bracket skein relation*, $L_+ = AL_- + A^{-1}L_\infty$, and denoted by $S_{2,\infty}(M)$, is best understood among the Jones type skein modules. It can be interpreted as a quantization of the co-ordinate ring of the character variety of $SL(2,C)$ representations of the fundamental group of the manifold M, [Bu-2, B-F-K, P-S]. For $M = F \times [0,1]$, KBSM is an algebra (usually noncommutative). It is finitely generated algebra for a compact F [Bu-1], and has no zero divisors [P-S]. The center of the algebra is generated by boundary components of F [B-P, P-S]. Incompressible tori and 2-spheres in M yield torsion in KBSM; it is a question of fundamental importance whether other surfaces can yield torsion as well.

(2) Skein modules based on the Jones-Conway (Homflypt) relation. $v^{-1}L_+ - vL_- = zL_0$, where L_+, L_-, L_0 are oriented links (Fig.3.1). These skein modules are denoted by $S_3(M)$ and generalize skein modules based on Conway relation which were hinted by Conway. For $M = F \times [0,1]$, $S_3(M)$ is a *Hopf algebra* (usually neither commutative nor co-commutative), [Tu-2, P-6]. $S_3(F \times [0,1])$ is a free module and can be interpreted as a quantization [H-K, Tu-1, P-5, Tu-2]. $S_3(M)$ is related to the algebraic set of $SL(n,C)$ representations of the fundamental group of the manifold M, [Si].

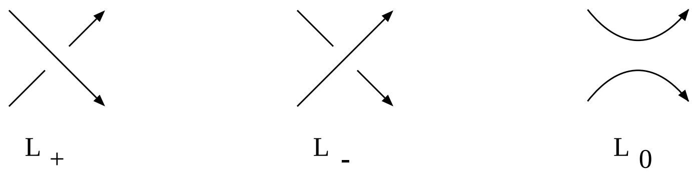


Fig. 3.1

(3) Skein modules based on the *Kauffman polynomial* relation $L_{+1} + L_{-1} = x(L_0 + L_\infty)$ (see Fig.3.2) and the framing relation $L^{(1)} - aL$. It is denoted by $S_{3,\infty}$ and is known to be free for $M = F \times [0,1]$.

(4) Homotopy skein modules. In these skein modules, $L_+ = L_-$ for self-crossings. The best studied example is the q-homotopy skein module with the skein relation $q^{-1}L_+ - qL_- = zL_0$ for mixed crossings. For $M = F \times [0,1]$ it is a quantization, [H-P-1, Tu-2, P-11], and as noted by Kaiser they can be almost completely understood using singular tori technique of Lin.

(5) Skein modules based on Vassiliev-Gusarov filtration. We extend the family of knots, $\mathcal{K}$, by singular knots, and resolve a singular crossing by $K_{cr} = K_+ - K_-$. These allows us to define the Vassiliev-Gusarov filtration: $... \subset C_3 \subset C_2 \subset C_1 \subset C_0 = R\mathcal{K}$, where C_k is generated by knots with k singular points. The k'th Vassiliev-Gusarov skein module is defined to be a quotient: $W_k(M) = R\mathcal{K}/C_{k+1}$. The completion of the space of knots with respect to the Vassiliev-Gusarov filtration, $\hat{R\mathcal{K}}$, is a *Hopf algebra* (for $M = S^3$). Functions dual to Vassiliev-Gusarov skein modules are called *finite type* or *Vassiliev invariants* of knots; [P-7].

(6) Skein modules based on relations deforming n-moves.
$\mathcal{S}_n(M) = R\mathcal{L}/(b_0L_0 + b_1L_1 + b_2L_2 + ... + b_{n-1}L_{n-1})$. In the unoriented case, we can add to the relation the term $b_\infty L_\infty$, to get $\mathcal{S}_{n,\infty}(M)$, and also, possibly, a framing relation. The case $n = 4$, on which I am working with my students, will be described, in greater detail, in a moment.

Examples (1)-(5) gave a short description of skein modules studied extensively until now. I will now spent more time on two other examples which only recently has been considered in more detail. The first example is based on a deformation of the 3-move and the second on the deformation of the $(2,2)$-move. The first one has been studied by my students Tsukamoto and Veve. I denote the skein module described in this example by $\mathcal{S}_{4,\infty}$ since it involves (in the skein relation), 4 horizontal positions and the vertical (∞) smoothing.

Definition 3.1 *Let M be an oriented 3-manifold, $\mathcal{L}_{fr}$ the set of unoriented framed links in M (including the empty link, $\emptyset$) and R any commutative ring with identity. Then we define the $(4,\infty)$ skein module as: $\mathcal{S}_{4,\infty}(M;R) = R\mathcal{L}_{fr}/I_{(4,\infty)}$, where $I_{(4,\infty)}$ is the submodule of $R\mathcal{L}_{fr}$ generated by the skein relation:*
$b_0L_0 + b_1L_1 + b_2L_2 + b_3L_3 + b_\infty L_\infty = 0$ *and the framing relation:*
$L^{(1)} = aL$ *where a, b_0, b_3 are invertible elements in R and b_1, b_2, b_∞ are any fixed elements of R (see Fig.3.2).*

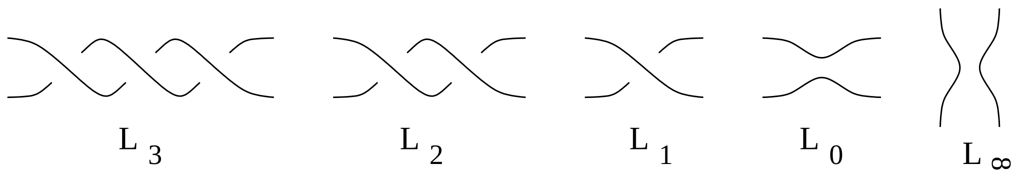


Fig. 3.2

The generalization of the Montesinos-Nakanishi conjecture says that $\mathcal{S}_{4,\infty}(S^3, R)$ is generated by trivial links and that for n-tangles our skein module is generated by $f(n,3)$ basic tangles (with possible trivial components). This would give a generating set for our skein module of S^3 or D^3 with $2n$ boundary points (an n-tangle). In [P-Ts] we analyzed extensively the possibility that trivial links, T_n, are linearly independent. This may happen if $b_\infty = 0$ and $b_0b_1 = b_2b_3$. These lead to the following conjecture:

Conjecture 3.2 *(1) There is a polynomial invariant of unoriented links, $P_1(L) \in Z[x,t]$, which satisfies:*

(i) Initial conditions: $P_1(T_n) = t^n$, where T_n is a trivial link of n components.

(ii) Skein relation $P_1(L_0) + xP_1(L_1) - xP_1(L_2) - P_1(L_3) = 0$ where L_0, L_1, L_2, L_3 is a standard, unoriented skein quadruple (L_{i+1} is obtained from L_i by a right-handed half twist on two arcs involved in L_i; compare Fig.3.2).

(2) There is a polynomial invariant of unoriented framed links, $P_2(L) \in Z[A^{\pm 1}, t]$ which satisfies:

(i) Initial conditions: $P_2(T_n) = t^n$,

(ii) Framing relation: $P_2(L^{(1)}) = -A^3 P_2(L)$ where $L^{(1)}$ is obtained from a framed link L by a positive half twist on its framing.

(iii) Skein relation: $P_2(L_0) + A(A^2 + A^{-2})P_2(L_1) + (A^2 + A^{-2})P_2(L_2) + AP_2(L_3) = 0$.

The above conjectures assume that $b_\infty = 0$ in our skein relation. Let us consider, for a moment, the possibility that b_∞ is invertible in R. Using the "denominator" of our skein relation (Fig.3.3) we get the relation which allows us to compute the effect of adding a trivial component to a link L (we write t^n for the trivial link T_n):

$$(*) \quad (a^{-3}b_3 + a^{-2}b_2 + a^{-1}b_1 + b_0 + b_\infty t)L = 0$$

When considering the "numerator" of the relation and its mirror image (Fig.3.3) we obtain formulas for Hopf link summands, and because the unoriented Hopf link is amphicheiral thus we can eliminate it from our equations to get the formula (**):

$$b_3(L\#H) + (ab_2 + b_1 t + a^{-1}b_0 + ab_\infty)L = 0.$$

$$b_0(L\#H) + (a^{-1}b_1 + b_2 t + ab_3 + a^2 b_\infty)L = 0.$$

$$(**) \quad ((b_0b_1 - b_2b_3)t + (a^{-1}b_0^2 - ab_3^2) + (ab_0b_2 - a^{-1}b_1b_3) + b_\infty(ab_0 - a^2b_3))L = 0.$$

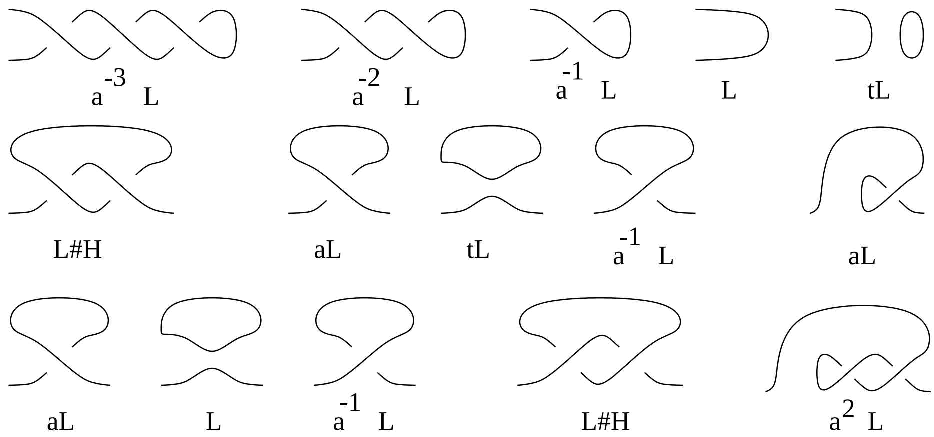


Fig. 3.3

It is possible that (*) and (**) are the only relations in the module. Precisely, we ask whether $\mathcal{S}_{4,\infty}(S^3;R)$ is the quotient ring $R[t]/(\mathcal{I})$ where t^i represents the trivial link of i components and $\mathcal{I}$ is the ideal generated by (*) and (**) for $L = t$. The pleasant substitution which satisfies the relations is: $b_0 = b_3 = a = 1$, $b_1 = b_2 = x$, $b_\infty = y$. This may lead to the polynomial invariant of unoriented links in S^3 with value in $Z[x,y]$ and the skein relation $L_3 + xL_2 + xL_1 + L_0 + yL_\infty = 0$.

What about the relations to Fox colorings? One such a relation, already mentioned, is the use of 3-colorings to estimate the number of basic n-tangles (by $\prod_{i=1}^{n-1}(3^i+1)$) for the skein module $\mathcal{S}_{4,\infty}$. I am also convinced that $\mathcal{S}_{4,\infty}(S^3;R)$ contains full information on the space of Fox 7-colorings. It would be a generalization of the fact that the Kauffman bracket polynomial contains information about 3-colorings and the Kauffman polynomial contains information about 5-colorings. In fact, François Jaeger told me that he knew how to get the space of p-colorings from a short skein relation (of the type of $(\frac{p+1}{2}, \infty)$). Unfortunately François died prematurely in 1997 and I do not know how to prove his statement[8].

[8] If $|Col_p(L)|$ denotes the order of the space of Fox p-colorings of the link L then among $p+1$ links $L_0, L_1, ..., L_{p-1}$, and L_∞, p of them has the same order of $|Col_p(L)|$ and one has its order p times larger [P-3]. This leads to the relation of type (p,∞). The relation between Jones polynomial (or the Kauffman bracket) and $Col_3(L)$ has the form: $Col_3(L) = 3|V(e^{\pi i/3}|^2$ and the formula relating the Kauffman polynomial and $Col_5(L)$ has the form: $Col_5(L) = 5|F(1, e^{2\pi i/5} + e^{-2\pi i/5})|^2$. This seems to suggest that the formula which Jaeger discovered involved Gaussian sums.

Finally let me describe shortly the skein module related to the $(2,2)$-move conjecture. Because a $(2,2)$-move is equivalent to the rational $\frac{5}{2}$-move, I will denote the skein module by $\mathcal{S}_{\frac{5}{2}}(M;R)$.

Definition 3.3 *Let M be an oriented 3-manifold, $\mathcal{L}_{fr}$ the set of unoriented framed links in M (including the empty link, $\emptyset$) and R any commutative ring with identity. Then we define the $\frac{5}{2}$-skein module as: $\mathcal{S}_{\frac{5}{2}}(M;R) = R\mathcal{L}_{fr}/(I_{\frac{5}{2}})$ where $I_{\frac{5}{2}}$ is the submodule of $R\mathcal{L}_{fr}$ generated by the skein relation:*
(i) $b_2L_2 + b_1L_1 + b_0L_0 + b_\infty L_\infty + b_{-1}L_{-1} + b_{-\frac{1}{2}}L_{-\frac{1}{2}} = 0$,
its mirror image:
$(\bar{i})$ $b'_2L_2 + b'_1L_1 + b'_0L_0 + b'_\infty L_\infty + b'_{-1}L_{-1} + b'_{-\frac{1}{2}}L_{-\frac{1}{2}} = 0$
and the framing relation:
$L^{(1)} = aL$, where $a, b_2, b'_2, b_{-\frac{1}{2}}, b'_{-\frac{1}{2}}$ are invertible elements in R and b_1, b'_1, b_0, b'_0, $b_{-1}, b'_{-1}, b_\infty$, and b'_∞ are any fixed elements of R. The links $L_2, L_1, L_0, L_\infty, L_{-1}$, $L_{\frac{1}{2}}$ and $L_{-\frac{1}{2}}$ are illustrated in Fig.3.4.[9]

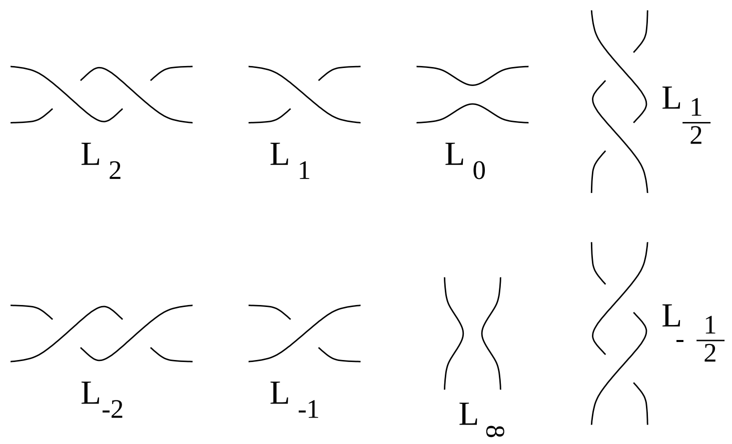


Fig. 3.4

If we rotate the figure from the relation (i) we obtain:
(i') $b_{-\frac{1}{2}}L_2 + b_{-1}L_1 + b_\infty L_0 + b_0L_\infty + b_1L_{-1} + b_2L_{-\frac{1}{2}} = 0$
One can use (i) and (i') to eliminate $L_{-\frac{1}{2}}$ and to get the relation:
$(b_2^2 - b^2_{-\frac{1}{2}})L_2 + (b_1b_2 - b_{-1}b_{-\frac{1}{2}})L_1 + ((b_0b_2 - b_\infty b_{-\frac{1}{2}})L_0 + (b_{-1}b_2 - b_1b_{-\frac{1}{2}})L_{-1} +$
$(b_\infty b_2 - b_0b_{-\frac{1}{2}})L_\infty = 0$.

[9]Our notation is based on Conway's notation for rational tangles. However it differs from it by a sign. The reason is that the Conway convention for a positive crossing is generally not used in the setting of skein relations.

Thus either we deal with the shorter relation (essentially that of the fourth skein module described before) or all coefficients are equal to 0 and therefore (assuming that there are no zero divisors in R) $b_2 = \varepsilon b_{-\frac{1}{2}}$, $b_1 = \varepsilon b_{-1}$, and $b_0 = \varepsilon b_\infty$. Similarly we would get: $b'_2 = \varepsilon b'_{-\frac{1}{2}}$, $b'_1 = \varepsilon b'_{-1}$, and $b'_0 = \varepsilon b'_\infty$. Here $\varepsilon = \pm 1$. Assume for simplicity that $\varepsilon = 1$. Further relations among coefficients follows from the computation of the Hopf link component using the amphicheirality of the unoriented Hopf link. Namely, by comparing diagrams in Figure 3.5 and its mirror image we get:

$$L\#H = -b_2^{-1}(b_1(a + a^{-1}) + a^{-2}b_2 + b_0(1 + T_1))L$$

$$L\#H = -{b'_2}^{-1}(b'_1(a + a^{-1}) + a^2 b'_2 + b'_0(1 + T_1))L.$$

Possibly, the above equalities give the only other relations among coefficients (in the case of S^3), but I would present below the simpler question (assuming $a = 1, b_x = b'_x$ and writting t^n for T_n).

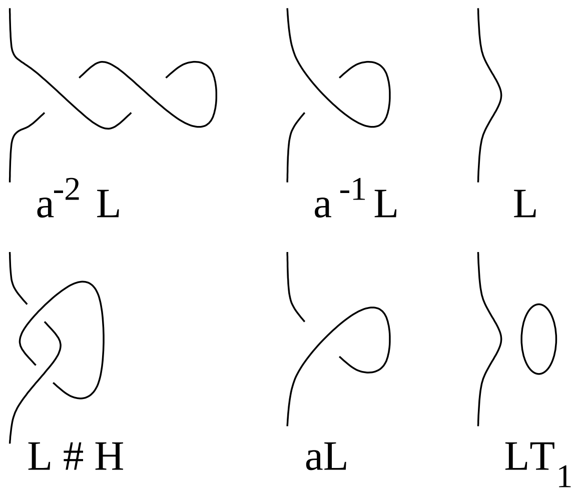


Fig. 3.5

Question 3.4 *Is there a polynomial invariant of unoriented links in* S^3, $P_{\frac{5}{2}}(L) \in Z[b_0, b_1, t]$, *which satisfies the following conditions?*

(i) *Initial conditions:* $P_{\frac{5}{2}}(T_n) = t^n$, *where* T_n *is a trivial link of* n *components.*

(ii) Skein relations

$$P_{\frac{5}{2}}(L_2) + b_1 P_{\frac{5}{2}}(L_1) + b_0 P_{\frac{5}{2}}(L_0) + b_0 P_{\frac{5}{2}}(L_\infty) + b_1 P_{\frac{5}{2}}(L_{-1}) + P_{\frac{5}{2}}(L_{-\frac{1}{2}}) = 0.$$

$$P_{\frac{5}{2}}(L_{-2}) + b_1 P_{\frac{5}{2}}(L_{-1}) + b_0 P_{\frac{5}{2}}(L_0) + b_0 P_{\frac{5}{2}}(L_\infty) + b_1 P_{\frac{5}{2}}(L_1) + P_{\frac{5}{2}}(L_{\frac{1}{2}}) = 0.$$

Notice that by taking the difference of our skein relations one get the interesting identity:

$$P_{\frac{5}{2}}(L_2) - P_{\frac{5}{2}}(L_{-2}) = P_{\frac{5}{2}}(L_{\frac{1}{2}}) - P_{\frac{5}{2}}(L_{-\frac{1}{2}}).$$

Nobody has yet seriously studied the skein module $\mathcal{S}_{\frac{5}{2}}(M; R)$ so everything you can get will be a new exploration, even a table of the polynomial $P_{\frac{5}{2}}(L)$ for small links, L.
I wish you luck!

4 Acknowledgments

I am grateful to Prof. Zbigniew Oziewicz for his generous hospitality in Cuautitlan.

5 Added in proof – the Montesinos-Nakanishi 3-move conjecture

A preliminary calculation performend by my student Mietek Dąbkowski (February 21, 2002) shows that the Montesinos-Nakanishi 3-move conjecture does not hold for the Chen link (Fig. 1.7). Below is the text of the abstract we have sent for the Knots in Montreal conference organized by Steve Boyer and Adam Sikora in April 2002.

Authors: Mieczysław Dąbkowski, Józef H. Przytycki (GWU)
Title: Obstructions to the Montesinos-Nakanishi 3-move conjecture.
Yasutaka Nakanishi asked in 1981 whether a 3-move is an unknotting operation. This question is called, in the Kirby's problem list, *the Montesinos-Nakanishi Conjecture*. Various partial results have been obtained by Q.Chen, Y.Nakanishi, J.Przytycki and T.Tsukamoto. Nakanishi and Chen presented

examples which they couldn't reduce (the Borromean rings and the closure of the square of the center of the fifth braid group, $\bar{\gamma}$, respectively). The only tool, to analyze 3-move equivalence, till 1999, was the Fox 3-coloring (the number of Fox 3-colorings is unchanged by a 3-move). It allowed to distinguish different trivial links but didn't separate Nakanishi and Chen examples from trivial links. The group of tricolorings of a link L corresponds to the first homology group with Z_3 coefficients of the double branched cover of a link L, $M_L^{(2)}$, i.e.

$$Tri(L) = H_1(M_L^{(2)}, Z_3) \oplus Z_3$$

We find more delicate invariants of 3-moves using homotopy in place homology and we consider the fundamental group of $M_L^{(2)}$.

We define an nth Burnside group of a link as the quotient of the fundamental group of the double branched cover of the link divided by all relations of the form $a^n = 1$. For $n = 2, 3, 4, 6$ the quotient group is finite.

The third Burnside group of a link is unchanged by 3-moves.

In the proof we use the "core" presentation of the group from the diagram; that is arcs are generators and each crossing gives a relation $c = ab^{-1}a$ where a corresponds to the overcrossing and b and c to undercrossings.

The Montesinos-Nakanishi 3-move conjecture does not hold for Chen's example $\hat{\gamma}$.

To show that $\hat{\gamma}$ has different third Burnside group than any trivial link it suffices to show that the following element, P, of the Burnside free group $B(4,3) = \{x, y, z, t : (a)^3\}$ is nontrivial: $P = uwtu^{-1}w^{-1}t^{-1}$ where
$u = xy^{-1}zt^{-1}$ and $w = x^{-1}yz^{-1}t$.

With the help of GAP it has been achieved!! (Feb. 21, 2002).

We have confirmed our calculation using Magnus program.

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To have a taste of Alexander letter, here is the quotation from the beginning of the interesting part: "When looking over Tait

on knots among other things, He really doesn't get very far. He merely writes down all the plane projections of knots with a limited number of crossings, tries out a few transformations that he happen to think of and assumes without proof that if he is unable to reduce one knot to another with a reasonable number of tries, the two are distinct. His invariant, the generalization of the Gaussian invariant ... for links is an invariant merely of the particular projection of the knot that you are dealing with, - the very thing I kept running up against in trying to get an integral that would apply. The same is true of his "Beknottednes".

Here is a genuine and rather jolly invariant: take a plane projection of the knot and color alternate regions light blue (or if you prefer, baby pink). Walk all the way around the knot and ..."

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