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Vassiliev invariants

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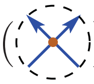
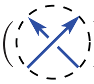
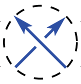
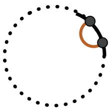
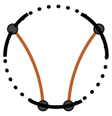
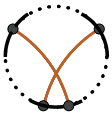
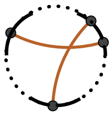
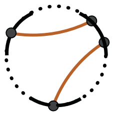
Vassiliev skein relation: $v($$) = v($$) - v($$)$

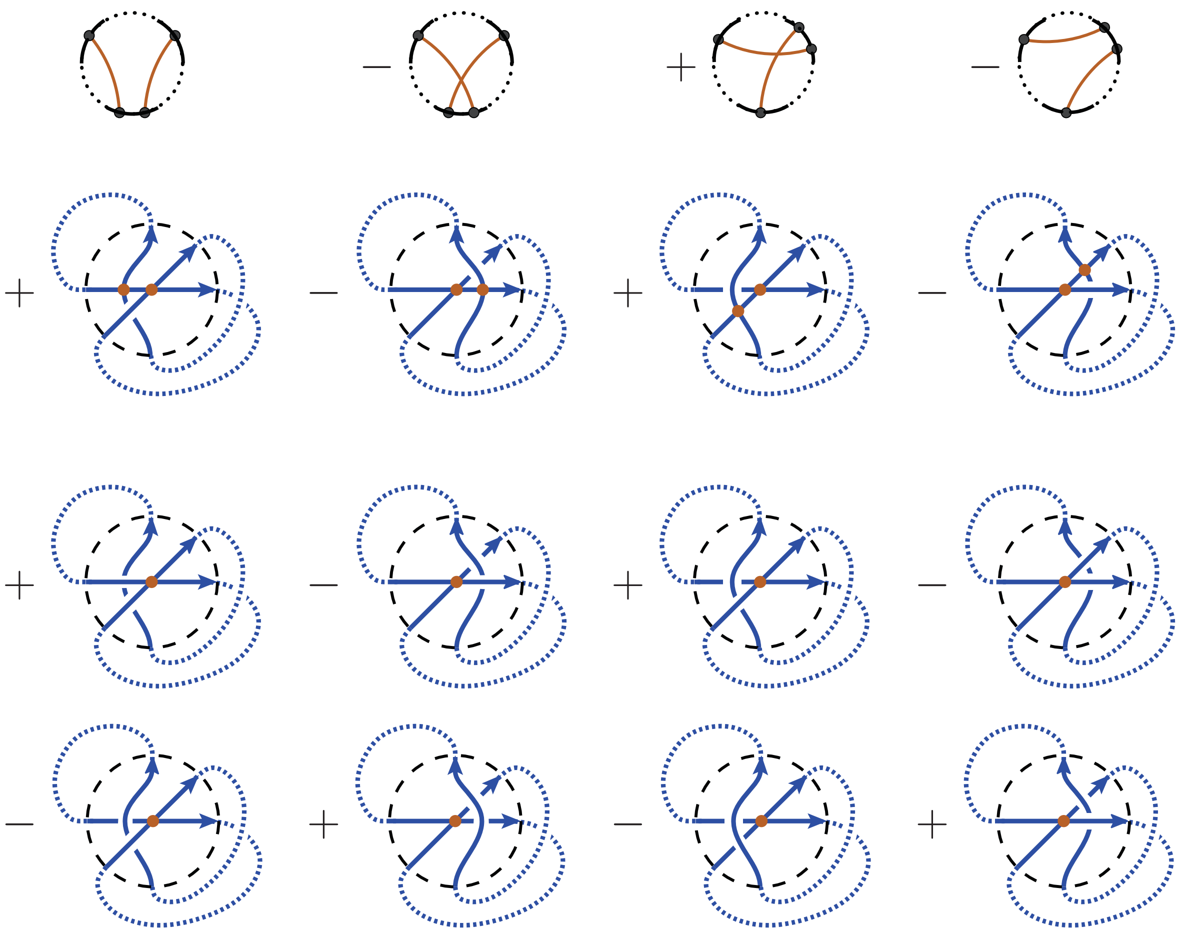
Def. *Vassiliev invariant of order* $\leqslant n$: $v|_{\mathcal{K}_{n+1}} \equiv 0$.

$$\mathcal{V}_n := \{\text{Vassiliev invariants of order} \leqslant n\}. \ \text{symb}(v) := v|_{\mathcal{K}_n}$$

$$\mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \cdots \subseteq \mathcal{V}_n \subseteq \cdots \subseteq \cdots$$

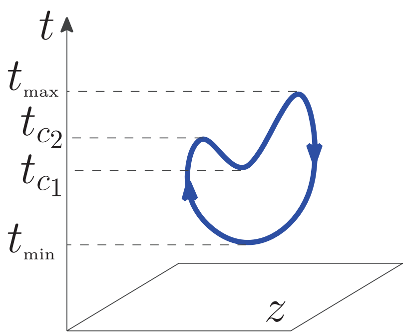
1T:  $= 0$; **4T:**  $-$  $+$  $-$  $= 0$

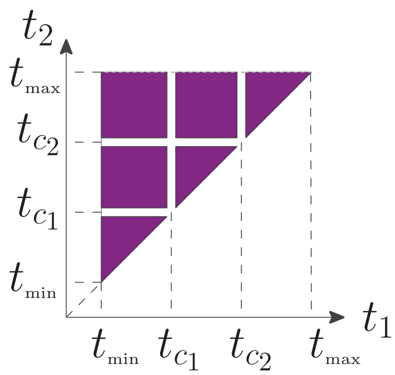
$$\mathcal{A}_n := \text{span}(\text{chord diagrams with } n \text{ chords})/(1\text{T},4\text{T})$$

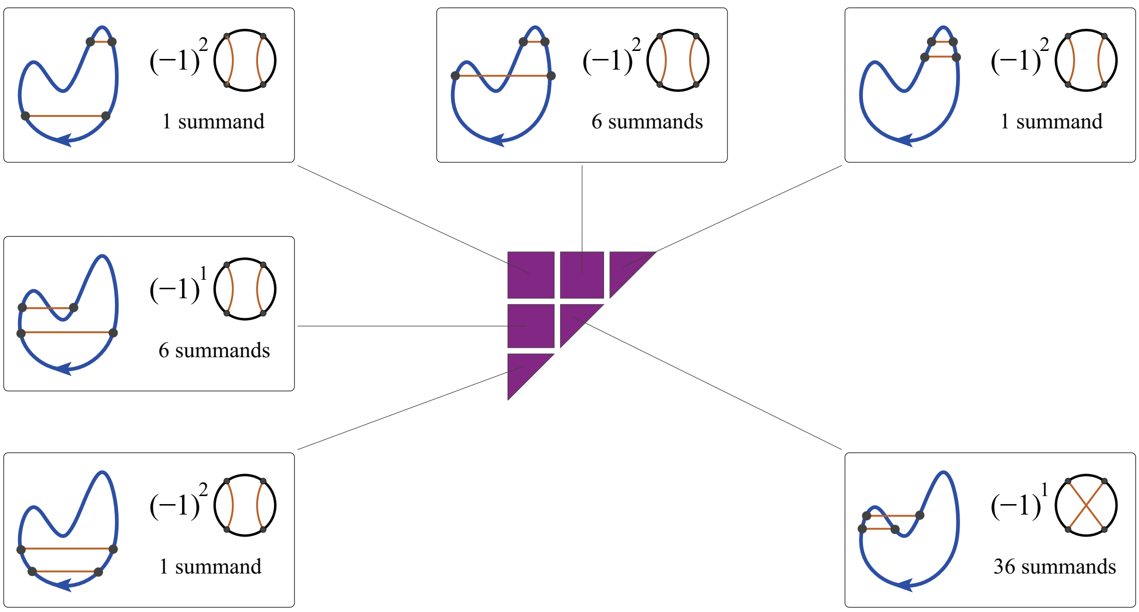


The Kontsevich integral.

$$Z(K) := \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int_{\substack{t_{\min}<t_1<\cdots<t_m<t_{\max} \\ t_j \text{ are noncritical}}} \sum_{P=\{(z_j,z_j')\}} (-1)^{\downarrow} D_P \bigwedge_{j=1}^{m} \frac{dz_j - dz_j'}{z_j - z_j'}$$

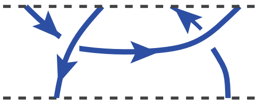
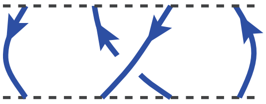
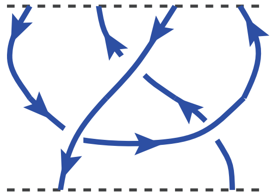




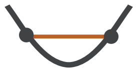


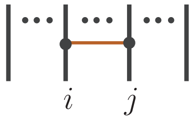
Horizontal invariance

Tangle multiplication

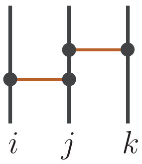
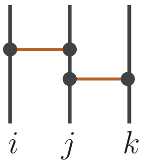
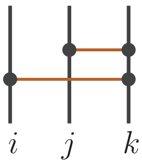
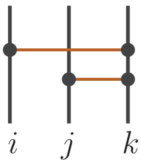
$T_1 =$  ; $T_2 =$  ; $T_1{\cdot}T_2 =$ 

Tangled chord diagrams.

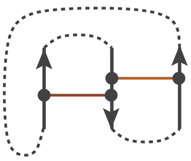
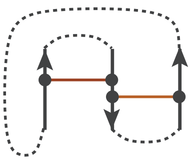
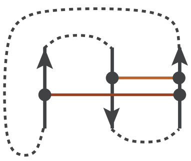
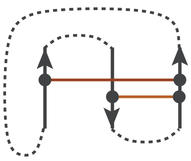
Tangled 1T relation:  $= 0$ or  $= 0\,.$

Tangled 4T relation: $t_{ij} := (-1)^{\downarrow}$  ,

$$[t_{ij} + t_{ik}, t_{jk}] = 0$$

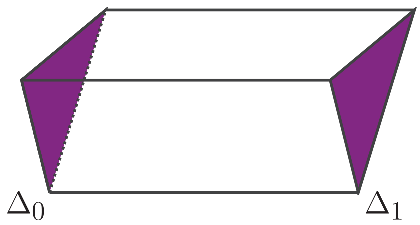
$(-1)^{\downarrow}$  $- (-1)^{\downarrow}$  $+ (-1)^{\downarrow}$  $- (-1)^{\downarrow}$  $= 0\,.$

Example.

 $-$  $-$  $+$  $= 0$

$$\omega_{ij} := \frac{dz_i - dz_j}{z_i - z_j}\,, \qquad \Omega_{ij} := t_{ij}\frac{dz_i - dz_j}{z_i - z_j}\,.$$

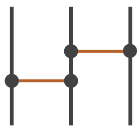
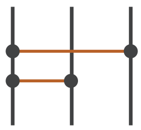
Horizontal deformation: T_λ.

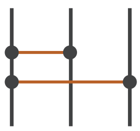
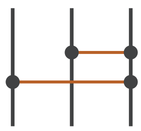
$\Delta = \Delta_0 \times [0,1] =$  .

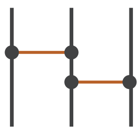
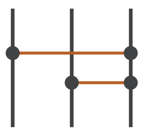
Stokes' theorem: $\int_{\partial\Delta} \Omega = \int_{\Delta} d\Omega = 0$, since $d\Omega = 0$.

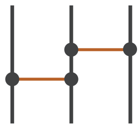
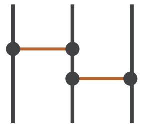
$\partial\Delta = \Delta_0 - \Delta_1 + \sum \{faces\}$. We prove that $\Omega\Big|_{\{face\}} = 0$.

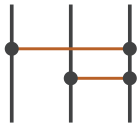
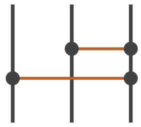
Restriction to the face $\{t_k = t_{k+1}\}$:

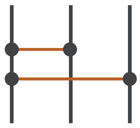
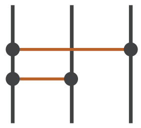
$(-1)^{\downarrow}$  $\omega_{12} \wedge \omega_{23}$ + $(-1)^{\downarrow}$  $\omega_{12} \wedge \omega_{13}$

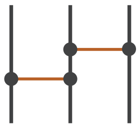
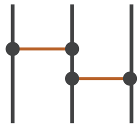
$+(-1)^{\downarrow}$  $\omega_{13} \wedge \omega_{12}$ + $(-1)^{\downarrow}$  $\omega_{13} \wedge \omega_{23}$

$+(-1)^{\downarrow}$  $\omega_{23} \wedge \omega_{12}$ + $(-1)^{\downarrow}$  $\omega_{23} \wedge \omega_{13}$

$= \Bigg((-1)^{\downarrow}$  $- \; (-1)^{\downarrow}$  $\Bigg) \, \omega_{12} \wedge \omega_{23}$

$+ \Bigg((-1)^{\downarrow}$  $- \; (-1)^{\downarrow}$  $\Bigg) \, \omega_{23} \wedge \omega_{31}$

$+ \Bigg((-1)^{\downarrow}$  $- \; (-1)^{\downarrow}$  $\Bigg) \, \omega_{31} \wedge \omega_{12}$

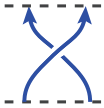
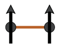
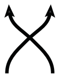
$= \Bigg((-1)^{\downarrow}$  $- \; (-1)^{\downarrow}$  $\Bigg) (\omega_{12} \wedge \omega_{23} + \omega_{23} \wedge \omega_{31} + \omega_{31} \wedge \omega_{12}) = 0,$

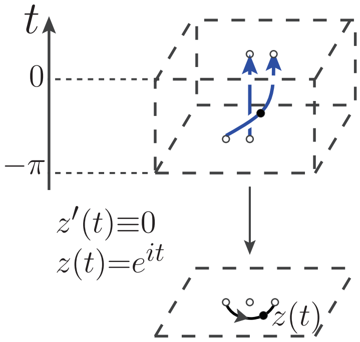
because of the *Arnold identity*:

$$f + g + h = 0 \implies \frac{df}{f} \wedge \frac{dg}{g} + \frac{dg}{g} \wedge \frac{dh}{h} + \frac{dh}{h} \wedge \frac{df}{f} = 0$$

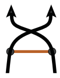
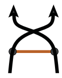
(in our case $f = z_1 - z_2$, $g = z_2 - z_3$, $h = z_3 - z_1$)

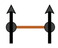
$\square$

Example. $R := Z\Big($  $\Big) = \exp\Big($  $/2\Big) \cdot$ 

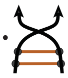
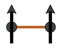
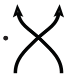


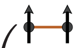
Indeed, for one chord:

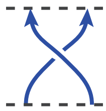
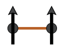
$\dfrac{1}{2\pi i}\displaystyle\int_{-\pi}^{0}$  $\cdot \dfrac{dz - dz'}{z - z'} = \Big(\dfrac{1}{2\pi i}\displaystyle\int_{-\pi}^{0}\dfrac{de^{it}}{e^{it}}\Big)\cdot$ 

$= \frac{1}{2}\cdot$  $=\Big($  $/2\Big)\cdot$  .

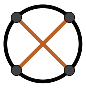
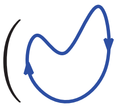
For two chords:

$\dfrac{1}{4\pi^2}\displaystyle\int_{-\pi}^{0}\int_{t_1}^{0}$  $\cdot\, dt_2\, dt_1 = \Big(\dfrac{1}{4\pi^2}\displaystyle\int_{-\pi}^{0} -t_1\, dt_1\Big)\cdot$  $= \dfrac{1}{2}\Big($  $/2\Big)^2\cdot$  .

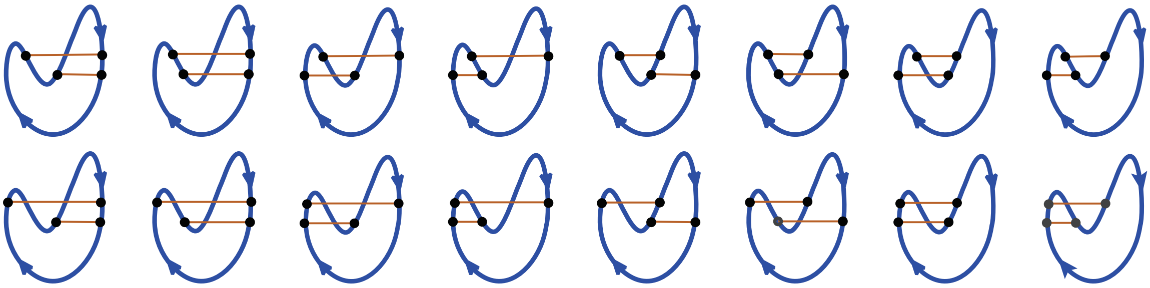
For n chords we will have: $\dfrac{1}{n!}\Big($  $/2\Big)^n\cdot$  .

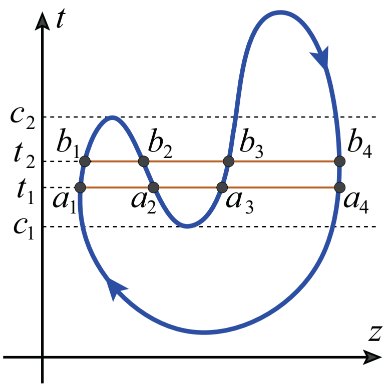
Exercise. $R^{-1} := Z\Big($  $\Big) = \exp\Big(-$  $/2\Big)\cdot$ 

Example.

The coefficient of the chord diagram  in $Z\Big($$\Big)$.

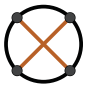
Out of the total number of 51 pairings the following 16 contribute to the coefficient:





$a_{jk} := a_k - a_j$;
$(jk) \in A := \{(12), (13), (24), (34)\}$;

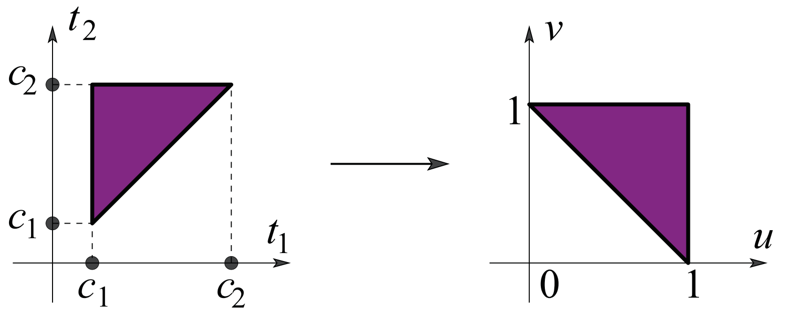
$b_{lm} := b_m - b_l$;
$(lm) \in B := \{(13), (23), (14), (24)\}$.

The coefficient of  is equal to

$$\frac{1}{(2\pi i)^2} \int_{\Delta} \sum_{(jk)\in A} \sum_{(lm)\in B} (-1)^{j+k+l+m} d\ln a_{jk} \wedge d\ln b_{lm}$$

$$= -\frac{1}{4\pi^2} \int_{\Delta} \sum_{(jk)\in A} (-1)^{j+k+1} d\ln a_{jk} \wedge \sum_{(lm)\in B} (-1)^{l+m-1} d\ln b_{lm}$$

$$= -\frac{1}{4\pi^2} \int_{\Delta} d\ln \frac{a_{12}a_{34}}{a_{13}a_{24}} \wedge d\ln \frac{b_{14}b_{23}}{b_{13}b_{24}} =$$



Change of variables (reversing orientation):

$$u = \frac{a_{12}a_{34}}{a_{13}a_{24}},$$

$$v = \frac{b_{14}b_{23}}{b_{13}b_{24}}.$$

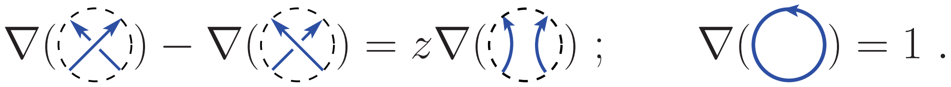
$$= \frac{1}{4\pi^2}\int_{\Delta'} d\ln u \wedge d\ln v \;=\; \frac{1}{4\pi^2}\int_0^1 \left(\int_{1-u}^1 d\ln v\right)\frac{du}{u}$$

$$= -\frac{1}{4\pi^2}\int_0^1 \ln(1-u)\frac{du}{u} \;=\; \frac{1}{4\pi^2}\sum_{k=1}^{\infty}\int_0^1 \frac{u^k}{k}\frac{du}{u}$$

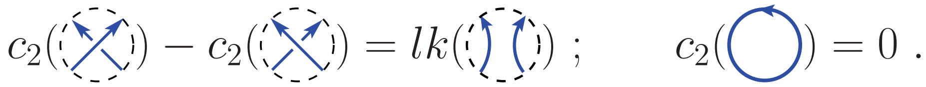
$$= \frac{1}{4\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2} \;=\; \frac{\zeta(2)}{4\pi^2} = \frac{1}{24}\;.$$

Goussarov Theorem

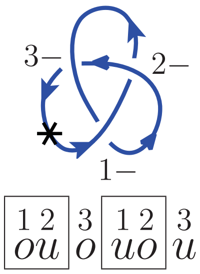
The Conway polynomial



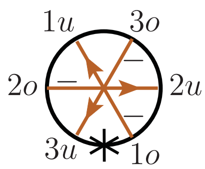
$$\nabla(K) = 1 + c_2(K)z^2 + c_4(K)z^4 + \dots$$

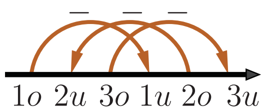


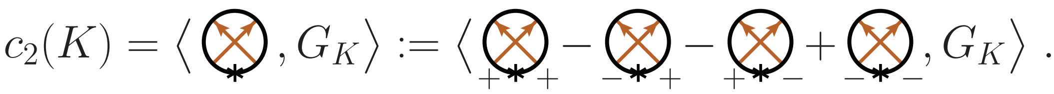
Polyak–Viro firmula: $$c_2(K) = \sum_{\substack{i\,j\,i\,j \\ ouuo}} \varepsilon_i \varepsilon_j \ .$$



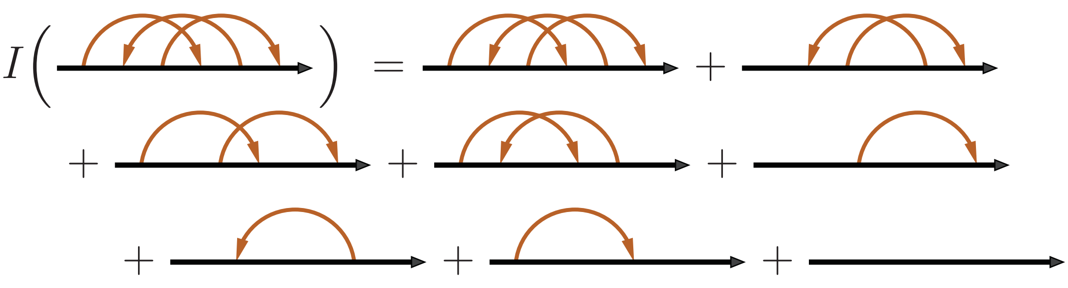
Gauss diagrams of 3_1:







Map $I : \mathbb{Z}[\mathrm{GD}] \to \mathbb{Z}[\mathrm{GD}]$, $\qquad I(D) := \sum_{D' \subseteq D} D'$



Theorem (Goussarov). *For any $v \in \mathcal{V}_n$ there is a function $c : \mathbb{Z}[GD] \to \mathbb{Z}$ such that $v = c \circ I$ and $c(D) = 0$ for $|D| > n$.*

Inverse map: $\quad I^{-1}(D) := \sum_{D' \subseteq D} (-1)^{|D-D'|} D'$

$$\boxed{c = v \circ I^{-1}}$$

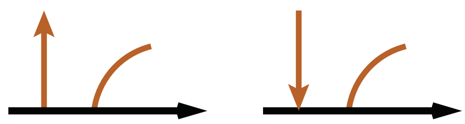
However one should extend v to non-realizable Gauss diagrams.

Mixed Gauss diagrams with chords representing singular knots.

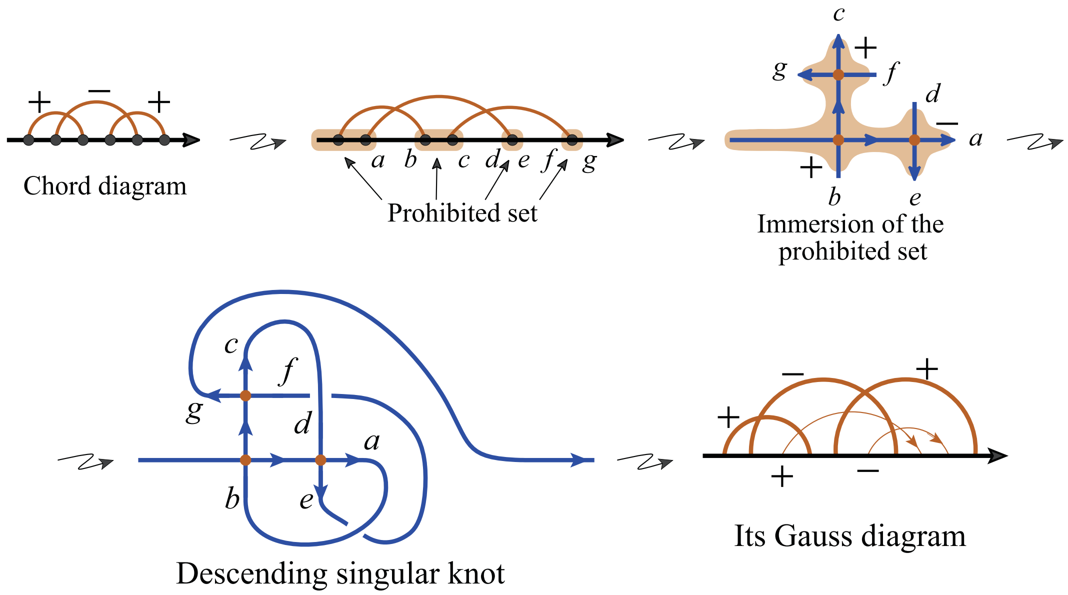
A Gauss diagram is *descending* if

(1) *all the arrows are directed to the right*, and

(2) *no endpoint of an arrow can be followed by the left endpoint of a chord.*

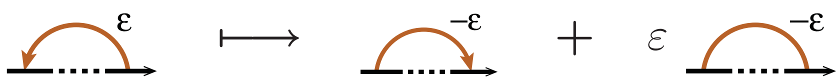
Forbidden situations:  .

Lemma. *Each long chord diagram with signed chords underlies a unique (up to isotopy) singular classical long knot that has a descending Gauss diagram.*



A map P making a diagram "more descending":

(1) Replace all the left-pointing arrows of D by the right-pointing according to the Vassiliev skein relation:



(2) Remove "prohibited pairs":

For a (non-realizable) Gauss diagram D there is a number m such that $P^m(D)$ is a linear combination of descending diagrams, modulo the diagrams with more than n chords.

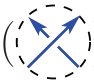
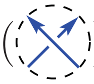
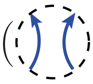
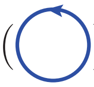
Extend of v to non-realizable Gauss diagrams.

If D is a descending Gauss diagram with signed chords, there exists precisely one singular classical knot K which has a descending diagram with the same signed chords. We set

$v(D) := v(K)$. Now, if D is an arbitrary diagram, then we apply the previous algorithm to obtain a linear combination $\sum a_i D_i$ of descending diagrams. Set $v(D) := \sum a_i v(D_i)$.

$\square$

Vassiliev invariants coming from the HOMFLYPT polynomial

$aP($  $) - a^{-1}P($  $) = zP($  $)$; $P($  $) = 1$.

Make a substitution $a = e^h$ and take the Taylor expansion $P(K) = \sum_{k,l} p_{k,l}(K) h^k z^l$.

Goussarov's Lemma.

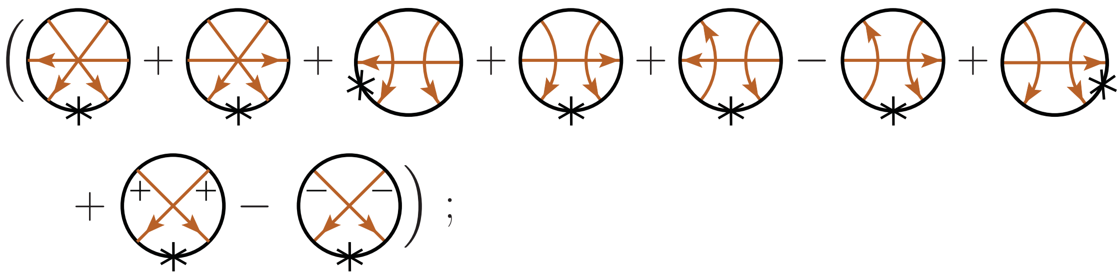
The coefficient $p_{k,l}$ is a Vassiliev invariant of order $\leqslant k + l$.

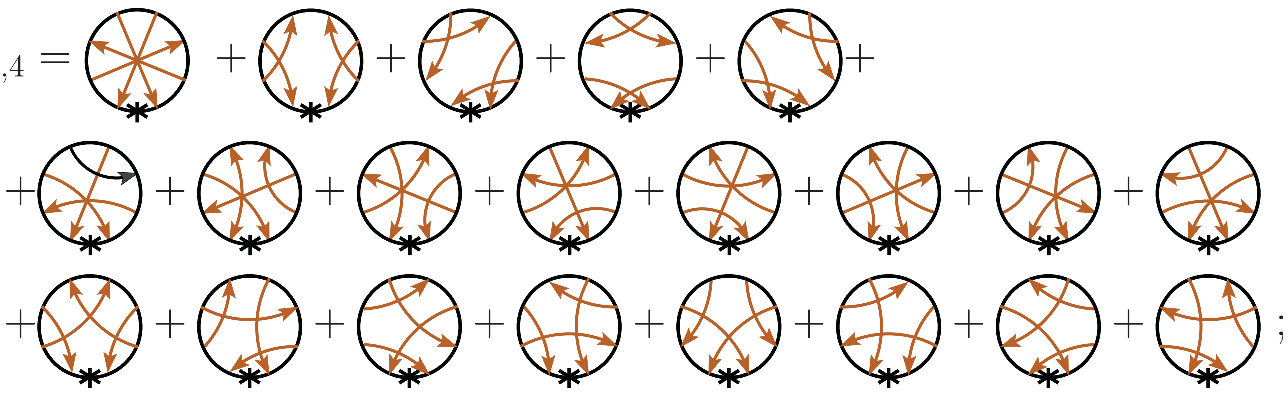
$$p_{k,l}(K) =: \langle A_{k,l}, G_K \rangle$$

Combinations $A_{k,l}$ for small k and l.

$A_{0,2} =$  ; $\qquad A_{2,0} = 0$;

$A_{0,3} = 0$; $\qquad A_{3,0} = -4A_{1,2}$;

$A_{1,2} = -2$  ;

$A_{0,4} =$  ;

$A_{2,2} =$ 78 terms.