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On Free Knots

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1 Statement of the Problem

The aim of the present paper is to study *free knots*, a dramatic simplification of the notion of virtual knots also connected to finite type invariants, curves on surfaces and other objects of low-dimensional topology. It turns out that the objects of such sort (a factorization of arbitrary graphs modulo three formal Reidemeister moves) can be studied by using some very much simpler objects: in the paper we construct an invariant valued in the set of equivalent classes of the same graphs modulo only *the second Reidemeister move*. By using this invariant, one can easily prove the non-triviality of some free knots; previously, the *existence* of non-trivial free knots was not evident.

During the whole article by *graph* we mean a finite graph, possibly, having loops and/or multiple edges, not necessarily connected. Herewith, by a connected component of such graph we also admit (besides graphs in the usual sense) free loops without vertices. The number of connected components of a graph is thought to be finite.

A 4-valent graph G (possibly, with free loops) is called *framed* if for every vertex V of G the four emanating half-edges are split into two pairs of (formally) opposite edges.

Denote by G_0 the one-component free loop.

By a *unicursal component* of a framed graph we mean the equivalence class of its edges generated by the following elementary equivalence relation: two edges are elementary equivalent if they possess half-edges which attach a vertex from opposite sides.

Given a chord diagram [1] C. Then the corresponding framed graph is constructed as follows. Edges of the graph $G(C)$ are assoicated with arcs of the chord diagrams, and vertices are associated to chords of the chord diagram. Each vertex is incident to four half-edges corresponding to arcs attaching two ends of a chord. Arcs which are incident to the same chord end will correspond to the (formally) opposite half-edges. The opposite procedure is evident: having a framed graph with one unicursal component, one can generate it by a certain chord diagram, to be denoted by $C(G)$.

Thus, there is a one-to-one correspondence between framed 4-graphs with one unicursal component and chord diagrams.

For a given four-valent framed graph G with one unicursal compnent with every two vertices v_1, v_2 we associate an element $\langle v_1, v_2 \rangle$ from $\mathbf{Z}_2 = \{0, 1\}$ which is equal to one if the corresponding chords of

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[1]One may consider oriented or non-oriented graphs (singular knots) with corresponding chord diagrams being oriented or non-oriented. In this paper we restrict ourselves for the non-oriented case, though the orientation can be taken into account straightforwardly.

$C(G)$ are unlinked (recall that two chords a, b of a chord diagram C are *linked* if after removeing the two ends of a from the circle of the chord diagram, the ends of the chord b lie in different connected components.

Later we assume that all framed graphs have one unicursal component, unless otherwise specified.

A simplest example of a framed graph is a knot projection. This gives a *planar framed graph*. More generally, one can consider a virtual knot diagram where classical crossings play the role of vertices, and virtual crossings are just intersection points of images of different edges.

It is known that classical and virtual knots are encoded by *Gauss diagrams*, see [3]. The notion of framed graph is a serious simplification of the notion of Gauss diagram. First, at classical vertices we don't indicated which branch forms an overpass and which one forms an underpass; besides, we do not indicate the writhe number of the vertex; thus, we forget both arrows and signs of the Gauss diagram.

Therefore, a framed graph is a very strong simplification of a knot diagram.

Our goal is to consider framed 4-graphs modulo some moves analogous to Reidemeister moves and study the obtained graph equivalence classes.

By a *free knot* we mean an equievalence class of framed 4-valent graphs with one unicursal component modulo the following transformations. For each transformation we assume that only a fixed fragment of the graph is being operated on (this fragment is to be depicted) or some corresponding fragments of the chord diagram. The remaining part of the graph or chord diagram are not shown in the picture; the pieces of the chord diagram not containing chords participating in this transformation, are depicted by punctured arcs. The parts of the graph are always shown in a way such that the formal framing (opposite edge relation) in each vertex coincides with the natural opposite edge relation taken from $\mathbf{R}^2$.

The first Reidemeister move is an addition/removal of a loop, see Fig.1

The second Reidemeister move adds/removes a bigon formed by a pair of edges which are adjacent in two edges, see Fig. 2.

Note that the second Reidemeister move adding two vertices does not impose any conditions on the edges it is applied to.

The third Reidemeister move is shown in Fig.3.

Note that each of these three moves applied to a framed graph, preserves the number of unicursal components of the graph. Thus, applying these moves to graphs with a unique unicursal cycle, we get to graphs with a unique unicursal cycle.

A *free knot* is an equivalence class of framed graphs (with a unique unicursal cycle) modulo the transformations listed above.

What are free knots? If we consider a planar framed graph (which originates from a classical knot) then this planar graph can be easily reduced to G_0. It can be easily shown that a one-component framed graph embeddable in torus (with framing preserved) is also reducible to G_0.

Free knots are closely connected to *flat virtual knots*, i. e., with equivalence classes of virtual knots modulo transformation changing over/undercrossing structure. The latter are equivalence classes of immersed curves in orientable 2-surfaces modulo homotopy and stabilization.

Nevertheless, the equivalence of free knots is even stronger than the equivalence of flat virtual
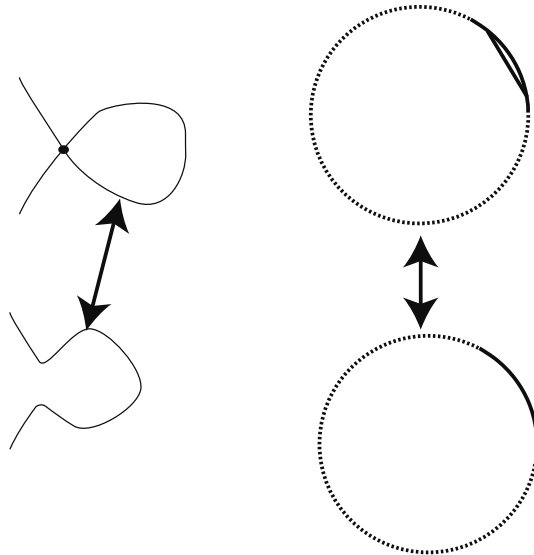
Figure 1: Addition/removal of a loop on a graph and on a chord diagram

knots: our free knots do not require any surface. Every time one applies a Reidemeister move to a regular 4-graph, one embeds this graph into a 2-surface arbitrarily (with framing preserved), apply this Reidemeister move inside the surface and then forget the surface again.

Example 1. *Consider the flat virtual Kishino knot, see Fig. 4*

It is known that this not is not trivial as a flat virtual knot: the corresponding representative lies on the sphere with two handles and it is minimal (this representative splits the sphere into a quadrilateral and an octagon).

Nevertheless, the corresponding framed graph considered by itself has a bigon formed by two edges which are adjacent in two vertices. Thus, the free knot represented by the flat Kishino knot is trivial.

The exact statement connecting virtual knots and free knots sounds as follows:

Lemma 1. *A free knot is an equivalence class of virtual knots modulo two transformations: crossing switches and virtualizations.*

A virtualization is a local transformation shown in Fig. 5.

One may think of a virtualization as way of changing the immersion of a 4-valent framed graph in plane.

Note that in the case of *free links*, i. e., framed 4-graphs with more than one unicursal components, it is much easier to find a non-trivial example.

Consider the framed graph G_2 with one vertex x and two edges a, b, each connecting x to x in such a way that the edge a is opposite to a, and the edge b is opposite to b.

This *free link* is not equivalent to the trivial one because of the following simple *invariant* of two-component free links. Consider a 2-component free link and calculate the *parity* of the number of crossings formed by two components. This parity is obviously invariant under Reidemeister moves.
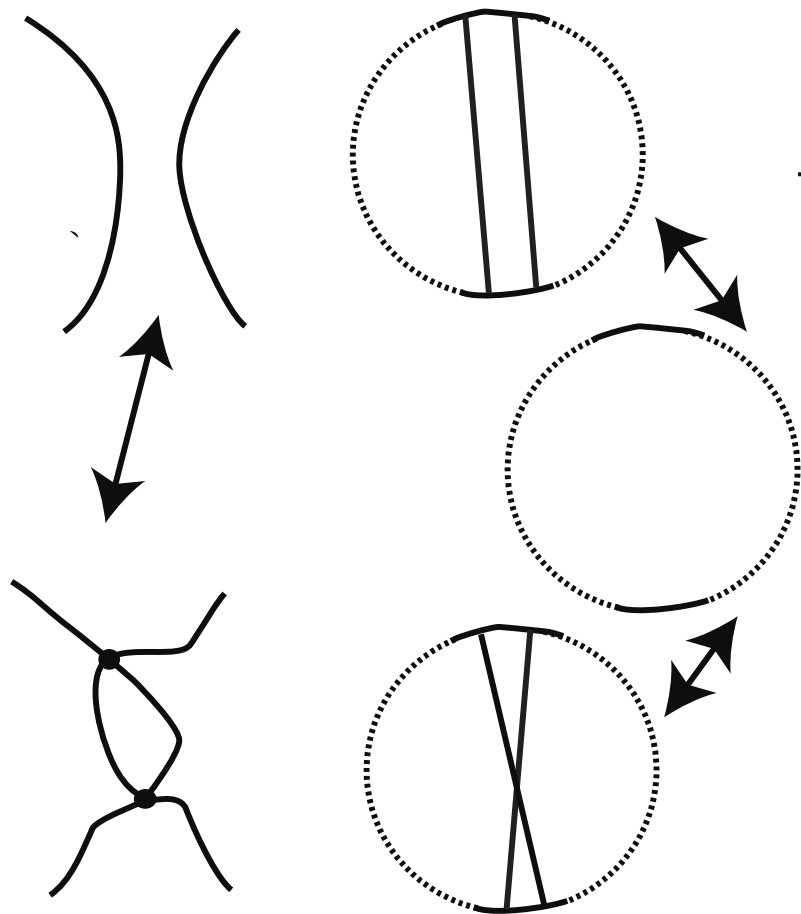
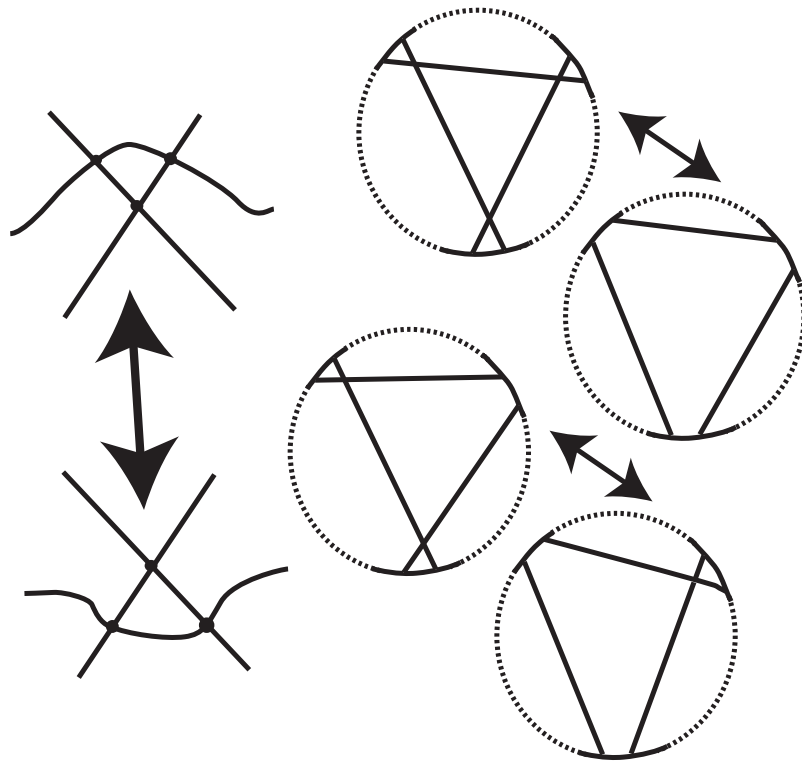Figure 2: The second Reidemeister move and two chord diagram versions of it

Figure 3: The third Reidemeister move and its chord diagram versions
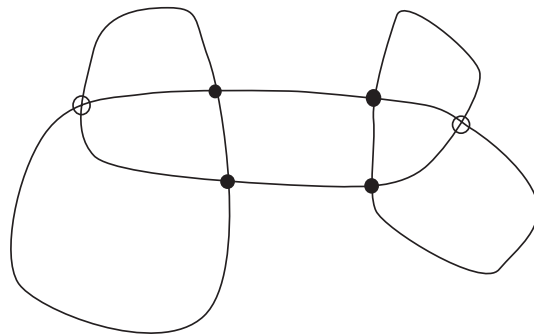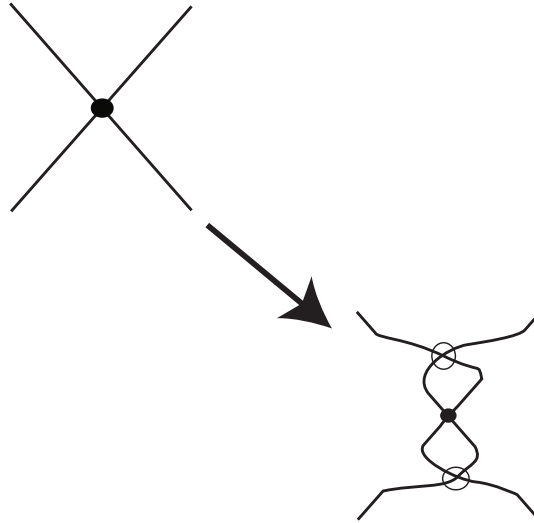

Figure 4: The Flat Kishino knot

Figure 5: Virtualization

Thus, G_2 is not equivalent to the unlink (formed by two disjoint free loops).

In the case of knots we shall also seek some *oddness* which can not be got rid of by using Reidemeister moves.

Given a framed 4-graph G (with one unicursal component), and let $V = \{v_1, \ldots, v_n\}$ be the set of vertices of G. Consider the linear $\mathbb{Z}_2$-space $L(G)$ generated by vectors $v_1, \ldots, v_n$. We shall identify subsets of the set of vertices of G with vectors from $L(G)$ and we shall consider sums of elements from $L(G)$ as boolean sums in 2^V.

A vertex of G is called *even*, if the number of vertices incident to it is even, and *odd* otherwise.

For a vertex a, we denote by E_a the sum of vertices incident to a (by definition, we assume that the vertex is not incident to itself).

We emphasize some evident properties which hold under Reidemeister moves.

The first Reidemeister move corresponds to an addition/removal of an even vertex, and the pairwise incidence relation of the remaining vertices does not change.

The second Reidemeister move adds (removes) two vertices a, b of the same parity, herewith $E_a + E_b$ is either 0 or coincides with $a + b$; after applying the Reidemeister move the parity of the remaining vertices does not change. Neither does the pairwise incidence of the remaining vertices.

Note that for every chord diagram, the chords are naturally split into *equivalence classes*: $a \sim b$ if and only if $E_a + E_b = 0$ or $E_a + E_b = a + b$. We call a set of equivalence classes of chords a *bunch*; in every bunch, all chords are either pairwise linked or pairwise unlinked. Every second Reidemeister move adds/deletes two chords from the same bunch. Note that this does not affect the bunches (equivalences of the remaining chords).

Note also that the intersection index of two chords from different bunches α, β does not depend on the particular choice of these chords, thus one can write $\langle \alpha, \beta \rangle$.

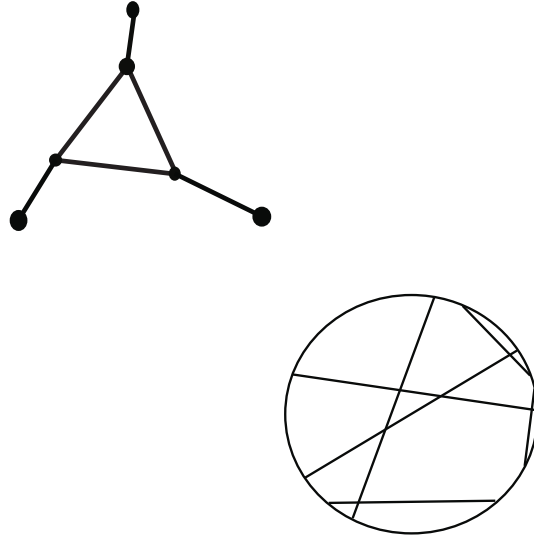When performing the third Reidemeister move, we have three vertices (chords) a, b, c, for which

Figure 6: An irreducibly odd graph and its chord diagram

$E_a + E_b + E_c \subset \{a, b, c\}, |E_a + E_b + E_c| = 0$ or 2. After performing the move, we get instead of a, b, c three vertices a', b', c' with pairwise switched incidences (w.r.t. a, b, c) the remaining incidences are unchanged: for $f, g \notin \{a, b, c\} : \langle f, g \rangle$ remains unchanged an $\langle f, a \rangle = \langle f, a' \rangle; \langle f, b \rangle = \langle f, b' \rangle; \langle f, c \rangle = \langle f, c' \rangle$. Therefore it follows that, e. g., the number of odd vertices amongst a, b, c is **even** (is equal to zero or two).

Herewith the parity of a coincides with that of a', the parity of b coincides with that of b', and the parity of c coincides with that of c'

Thus, to decrease the number of vertices of the diagram by using Reidemeister move, it is necessary to have either an even vertex (to be able to apply the first Reidemeister move) or a couple of vertices having the same incidence with any of the remaining vertices. The third Reidemeister move can be applied if for some vertices a, b, c we have $E_a + E_b + E_c \subset \{a, b, c\}$.

Thus, a natural class of graphs arises, where neither a simplification nor a third Reidemeister move can be applied in turn.

For example, so are the graphs with all vertices odd and such that for every two distinct vertices a, b there exists a vertex $c \notin \{a, b\}$ such that $\langle a, c \rangle \neq \langle b, c \rangle$.

We call such graphs *irreducibly odd*.

The simplest example of an irreducibly odd graph is depicted in Fig. 6.

Assume an irreducibly odd graph G generates a free knot K. By definition it is impossible to simplify G in one turn. It turns out that the representative G of the knot K is indeed *minimal*: any other representative of K has the number of vertices at least as many as those of G.

The main idea behind the proof is that in certain cases the equivalence recognition problem with respect to all three Reidemeister moves can be reduced to the equivalence recognition problem with respect to only Reidemeister 2 move, which is significantly easier.
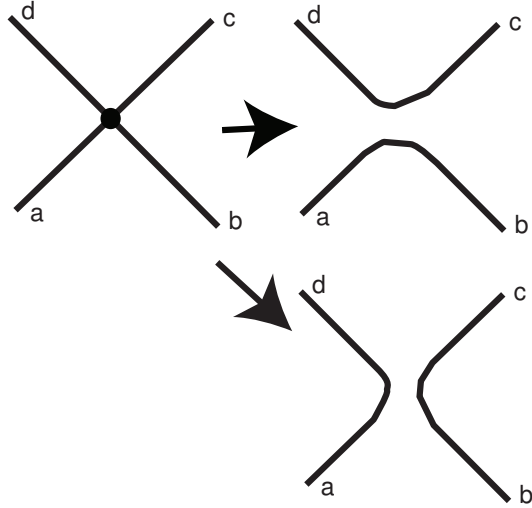
Figure 7: Two smoothings of a vertex of for a framed graph

Now, we are going to construct an invariant map of free knots valued in some objects considered modulo only the second Reidemeister moves.

2   The Main Theorem

Let $\mathfrak{G}$ be the set of all equivalence classes of framed graphs with one unicursal component modulo second Reidemeister moves. Consider the linear space $\mathbb{Z}_2\mathfrak{G}$.

Let G be a framed graph, let v be a vertex of G with four incident half-edges a, b, c, d, s.t. a is opposite to c and b is opposite to d at v.

By *smoothing* of G at v we mean any of the two framed 4-graphs obtained by removing v and repasting the edges as $a - b$, $c - d$ or as $a - d, b - c$, see Fig. 7.

Herewith, the rest of the graph (together with all framings at vertices except v) remains unchanged.

We may then consider further smoothings of G at *several* vertices.

Consider the following sum

$$[G] = \sum_{s\ even.,1\ comp} G_s, \tag{1}$$

which is taken over all smoothings in all *even* vertices, and only those summands are taken into account where G_s has one unicursal component.

Thus, if G has k even vertices, then $[G]$ will contain at most 2^k summands, and if all vertices of G are odd, then we shall have exactly one summand, the graph G itself.

Consider $[G]$ as an element of $\mathbb{Z}_2\mathfrak{G}$. In this case it is evident, for instance, that if all vertices of G are even then $[G] = [G_0]$: by construction, all summands in the definition of $[G]$ are equal to $[G_0]$, it can be easily checked that the number of such summands is odd.

Now, we are ready to formulate the main theorem:

Theorem 1. *If G and G' represent the same free knot then in $\mathbb{Z}_2\mathfrak{G}$ the following equality holds:* $[G] = [G']$.

Theorem 1 yields the following

Corollary 1. *Let G be an irreducibly odd framed 4-graph with one unicursal component. Then any representative G' of the free knot K_G, generated by G, has a smoothing $\tilde{G}$ having the same number of vertices as G. In particular, G is a minimal representative of the free knot K_G with respect to the number of vertices.*

2.1 The set $\mathbb{Z}_2\mathfrak{G}$

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Having a framed 4-graph, one can consider it as an element of $\mathbb{Z}_2\mathfrak{G}$. It is natural to try simplifying it: we call a graph in $\mathbb{Z}_2\mathfrak{G}$ *irreducible* if no decreasing second Reidemeister move can be applied to it. We call a graph in $\mathbb{Z}_2\mathfrak{G}$ *irreducible* if it has no free loops and no decreasing second Reidemeister move can be applied to it.

The following theorem is trivial

Theorem 2. *Every 4-valent framed graph G with one unicursal component considered as an element of $\mathbb{Z}_2\mathfrak{G}$ has a unique irreducible representative, which can be obtained from G by consequtive application of second decreasing Reidemeister moves.*

This allows to recognize elements $\mathbb{Z}_2\mathfrak{G}$ easily, which makes the invariants constructed in the previous subsection digestable.

In particular, the minimality of a framed 4-graph in $\mathbb{Z}_2\mathfrak{G}$ is easily detectable: one should just check all pairs of vertices and see whether any of them can be cancelled by a second Reidemeister move (or in $\mathbb{Z}_2\mathfrak{G}$ one should also look for free loops) .

Proof of the Corollary. By definition of $[G]$ we have $[G] = G$. Thus if G' generates the same free knot as G we have $[G'] = G$ in $\mathbb{Z}_2\mathfrak{G}$.

Consequently, the sum representing $[G']$ in $\mathfrak{G}$ contains at least one summand which is a-equivalent to G. Thus G' has at least as many vertices as G does.

Moreover, the corresponding smoothing of G' is a diagram, which is a-equivalent to G. One can show that under some (quite natural) "rigidity" condition this will yield that one of smoothings of G' coincides with G. ◻

Proof of the Theorem. Let us check the invariance $[G] \in \mathbb{Z}_2\mathfrak{G}$ under the three Reidemeister moves.

Let G' differ from G by a first Reidemeister move, so that G' has one vertex more than G. By definition this vertex is even, and when calculating $[G']$ this vertex has to be smoothed in order to get one unicursal curve in total.

Thus, we have to take only one of two smoothings of the given vertex, see Fig. 8

Thus there is a natural equivalence between smoothings of G having one unicursal component, and smoothings of G' with one unicursal component. Moreover, this equivalence yields a termwise identity between $[G]$ and $[G']$.
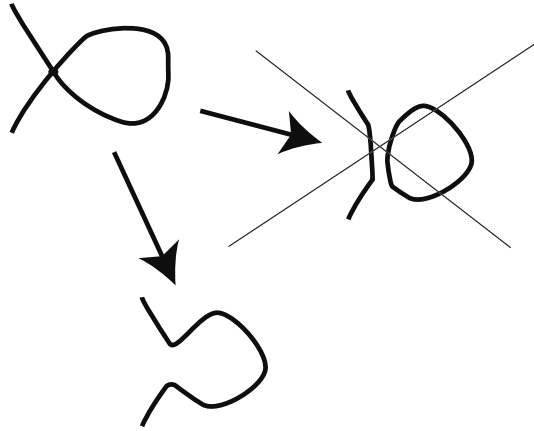
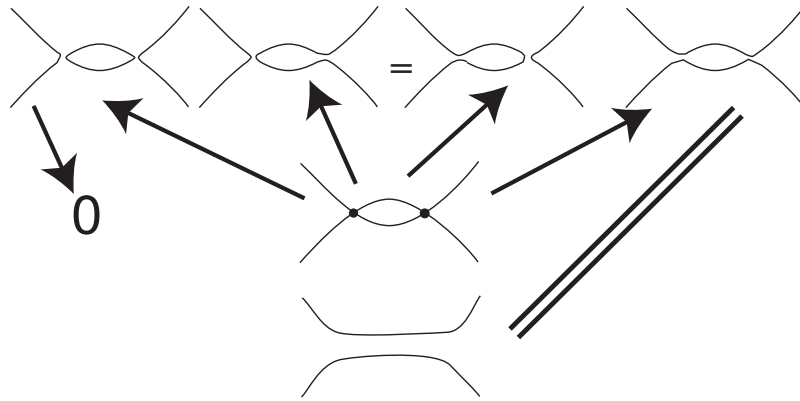Figure 8: The two smoothings — good and bad — of a loop

Figure 9: Smoothings of two even vertices

Now, let G' be obtained from G by a second Reidemeister move adding two vertices.

These two vertices are either both even or both odd.

If both added vertices are odd, then the set of smoothings of G is in one-to-one correspondence with that of G' and the corresponding summands for $[G]$ and for $[G']$ differ from each other by a second Reidemeister move.

If both vertices are odd then one has to consider different smoothings of these vertices shown in. 9.

The smoothings shown in the upper-left Fig. 9, yield more than one unicursal component (there is a free loop), so they do not count in $[G']$.

The second-type and third-type smoothings (the second and the third pictures in the top row of 9) give the same impact to $\mathbb{Z}_2\mathfrak{G}$, thus, they reduce in $[G']$. Finally, the smoothings corresponding to the upper-right Fig. 9 are in one-to-one correspondence with smoothings of G, thus we have a term-wise equality of terms $[G]$ and those terms of $[G']$, which are not cancelled by comparing the two middle pictures.
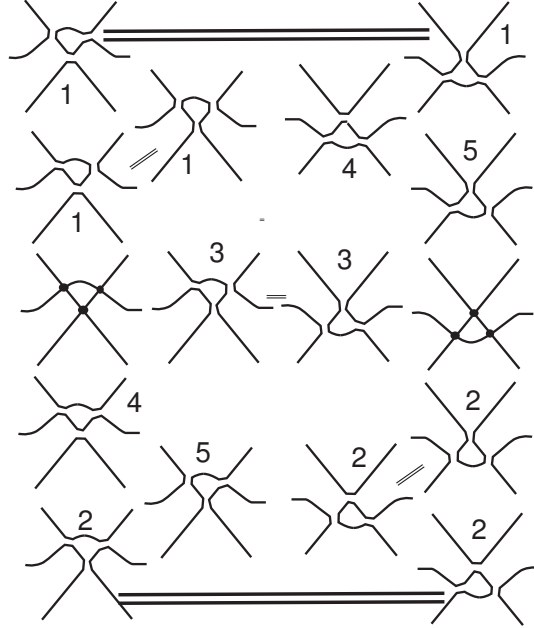
Figure 10: Correspondences of smoothings with respect to Ω_3 with three even vertices

If G and G' differ by a third Reidemeister move, then the following two cases are possible: either all vertices taking part in the third Reidemeister move are even, or two of them are odd and one is even.

If all the three vertices are even, there are seven types of smoothings corresponding to $[G]$ (and seven types of smoothings corresponding to $[G']$): in each of the three vertex we have two possible smoothings, and one case is ruled out because of a free loop (which yields at least two unicursal components, thus having no impact in $[G]$ or $[G']$). When considering G, three of these seven cases coincide (this triple is denoted by 1), so, in $\mathbb{Z}_2 \mathfrak{G}$ it remains exactly one of these two cases. Amongst the smoothings of the diagram G', the other three cases coincide (they are marked by 2). Thus, both in $[G]$ and $[G']$ there are five types of summands marked by $1, 2, 3, 4, 5$.

These five cases are in one-to-one correspondence (see Fig. 10) and they yield the equality $[G] = [G']$.

If amongst the three vertices taking part in Ω_3 we have exactly one even vertices (say $a \to a'$), we get the situation depicted in Fig. 11.

From this figure we see that those smoothings where a (resp., a') is smoothed *vertically*, give identical summands in $[G]$ and in $[G']$, and those smoothings where a and a' are smoothed *horizontally*, are in one-to-one correspondence for G and G', and the corresponding summands are obtained by applying two second Reidemeister moves. This proves that $[G] = [G']$ in $\mathbb{Z}_2 \mathfrak{G}$. $\qquad\square$
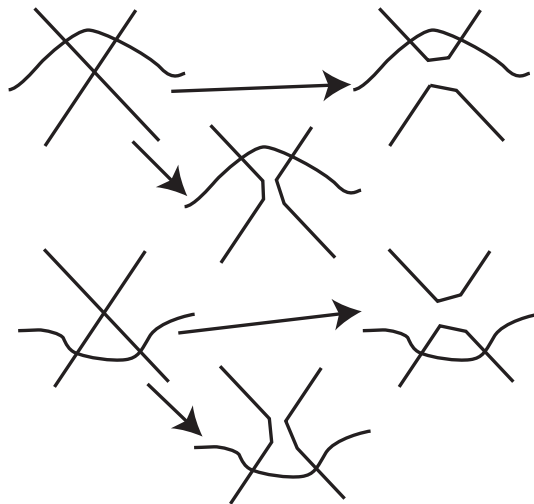
Figure 11: Correspondence between smoothings for Ω_3 with one even vertex

3 A functorial mapping

It turns out that the odd axioms listed above lead to a simple and powerful map on the set of free knots and, more generally, on the set of virtual knots.

Let K be a virtual knot diagram. Let f be a diagram obtained from K by making all *odd* crossings virtual. In other words, we remove all odd chords.

The following theorem follows from definitions.

Theorem 3. *f is a well-defined map on the set of all virtual knots. For a virtual knot diagram K, $f(K) = K$ iff the atom corresponding to K is orientable. Otherwise, $f(K)$ has strictly less classical crossings than K.*

This theorem leads to a new proof of a partial case the following

Theorem 4 (First proved by O.Ya. Viro and V.O.Manturov, 2005, first published in [4])**.** *The set of virtual links with orientable atoms is closed. In other words, if two virtual diagrams K and K' have orientable atoms and they are equivalent, then there is a sequence of diagrams $K = K_0 \to K_1 \cdots \to K_n = K'$ all having orientable atoms where K_i is obtained from K_i by a Reidemeister move.*

The case of this theorem for knots (1-component links) is then proved as follows: take any sequence $K = L_0 \to \cdots \to L_m = K'$ and apply the map f to it as many times as necessary (that is, to make all intermediate knot diagrams having orientable atom). This leads to the desired sequence.

Thus, odd chords of Gauss diagrams are holders of the "atom non-orientability" condition for virtual knots (one-component links). It would be interesting to find an appropriate "parity" for the case of arbitrary component links.

The map f is also important because it gives rise to a natural filtration on the set of virtual knots: $\mathcal{K}_1 \subset \mathcal{K}_\infty \in \mathcal{K}_\in \subset \cdots \subset \mathcal{K}_\backslash$ where $\mathcal{K}_\backslash$ denotes the set of all virtual knots K where $f^n(K)$ has a diagram with orientable atom. In particular, all classical knots are in $\mathcal{K}_1$.

4   Further applications

Theorem 1 can be treated independently from knots. It allows to establish non-equivalence of some quandratic forms over $\mathbb{Z}_2$, which are considered up to a certain quite natural and simple equivalence: (de)stabilizations, i.e., addition/removal of a vector perpendicular to the whole space, addition/removal of a pair of "similar" vectors, and a simple transformation corresponding to the Reidemeister 3 move.

We construct an invariant of such quandratic forms valued in formal $\mathbf{Z}_2$-linear combinations of much simpler equivalence classes of quadratic forms. These classes are actually classified by intersection matrices of odd bunches.

At the level of virtual knots, Corollary 1 immediately yields minimality of some virtual knot diagrams.

Actually, the only thing we need from knots and chord diagrams are intersection graphs [2]. Thus, the results given above generalize for the case of *graph-links* by Ilyutko-Manturov [4] and looped graphs by Traldi and Zulli [**?**] which are equivalence classes of some graphs modulo formal Reidemeister moves (which are defined with respect to the information coming from the intersection matrix).

The approach given above proves the *non-realizability* of some looped graphs and *graph-links*.

It suffices to take the graph P_5, representing the 1-frame of the pentagon pyramid. This graph is irreducibly odd, thus any corresponding looped graph in sense of [7] has no realizable representative: if there were any realizable representative, one would easily obtain a realizability of a chord diagram whose intersection graph contains P_5 as a subgraph.

The same argument works for graph-links [4] with *non-orientable atoms*, [6].

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