



**The Abdus Salam  
International Centre for Theoretical Physics**



2034-25

**Advanced School and Conference on Knot Theory and its  
Applications to Physics and Biology**

*11 - 29 May 2009*

**On Free Knots and Links**

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# On Free Knots and Links

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February 2, 2009

## Abstract

Both classical and virtual knots arise as formal Gauss diagrams modulo some abstract moves corresponding to Reidemeister moves. If we forget about both over/under crossings structure and writhe numbers of knots modulo the same Reidemeister moves, we get a dramatic simplification of virtual knots, which kills all classical knots. However, many virtual knots survive after this simplification.

We construct invariants of these objects and present their applications to minimality problems of virtual knots as well as some questions related to graph-links.

One can easily generalize these results for the orientable case and apply them for solving non-invertibility problems.

The main idea behind these invariants is some geometrical construction which reduces the general equivalence to the equivalence only modulo Reidemeister - 2 move.

This paper is a sequel of the paper [5].

## 1 Introduction

In [5], for a certain class objects (a terrific simplification of virtual knots) we proved a theorem that the equivalence questions of some diagrams can often be reduced to the question of some very simple equivalence (using only Reidemeister 2 moves). To do that, we made difference between two types of crossings: the “odd” ones and the “even” ones and created a diagram-valued invariant of our objects (free knots). For some diagrams which are “irreducibly odd” the value of the invariant consists of the diagram itself, and it has been considered then only up to second Reidemeister move.

This has a lot of corollaries (already described here or still to come in forthcoming papers): for flat virtual knots (for virtual knots and their generalizations see [4]), their non-triviality, non-equivalence, non-invertibility etc.

However, the main nerve of the construction was the notion of odd crossing. Roughly speaking, we are taking a Gauss diagram and forgetting all over/under information and all writhe numbers modulo formal Reidemeister moves. Odd crossings are precisely those corresponding to *odd chords of the Gauss diagram*, i.e., those chords which intersect an odd number of other chords.

The notion of odd chord (odd crossing) is closely connected to the notion of a non-orientable atom: starting from such a four-valent (framed) graph, we may construct a checkerboard surface (see ahead) which can be either orientable or non-orientable. The presence of odd chords precisely means non-orientability of the atoms.

So, the examples we have constructed and all non-trivial results we have proved in [5] deal with only non-orientable atoms and (virtual) knots corresponding to them.

By definition, the invariant constructed in [5] “smoothes” all even crossings, and for a diagram with orientable atom we get the same value of the invariant as that of the unknot.

So, the simple non-triviality, non-invertibility, non-equivalence, and minimality results hold for a class of objects with non-orientable atoms. In particular, it has nothing to do with classical knots.

The aim of the present paper is to find another “oddness” condition for the crossings allowing to use the techniques similar to the one introduced in [5].

We give new non-trivial examples.

## 2 Basic Notions

By a *4-graph* we mean a topological space consisting of finitely many components, each of which is either a circle or a finite graph with all vertices having valency four.

A 4-graph is *framed* if for each vertex of it, the four emanating half-edges are split into two sets of edges called (*formally*) *opposite*.

A *unicursal component* of a 4-graph is either a free loop component of it or an equivalence class of edges where two edges  $a, b$  are called equivalent if there is a sequence of edges  $a = a_0, \dots, a_n = b$  and vertices  $v_1, \dots, v_n$  so that  $a_i$  and  $a_{i+1}$  are opposite at  $v_{i+1}$ .

As an example of a free graph one may take the graph of a singular link.

By a *free link* we mean an equivalence class of framed 4-valent graphs modulo the following transformations. For each transformation we assume that only one fixed fragment of the graph is being operated on (this fragment is to be depicted) or some corresponding fragments of the chord diagram. The remaining part of the graph or chord diagram are not shown in the picture; the pieces of the chord diagram not containing chords participating in this transformation, are depicted by punctured arcs. The parts of the graph are always shown in a way such that the formal framing (opposite edge relation) in each vertex coincides with the natural opposite edge relation taken from  $\mathbf{R}^2$ .

The first Reidemeister move is an addition/removal of a loop, see Fig.1

The second Reidemeister move adds/removes a bigon formed by a pair of edges which are adjacent in two edges, see Fig. 2.

Note that the second Reidemeister move adding two vertices does not impose any conditions on the edges it is applied to: we may take any two edges of the graph and connect them together as shown in Fig. 2 to get two new crossings.

The third Reidemeister move is shown in Fig.3.

Note that each of these three moves applied to a framed graph, preserves the number of unicursal components of the graph. Thus, applying these moves to graphs with a unique unicursal cycle, we get to graphs with a unique unicursal cycle.

A *free knot* is a free link with one unicursal component (obviously, the number of unicursal component of a framed 4-graph is preserved under Reidemeister moves).

Free links are closely connected to *flat virtual knots*, i. e., with equivalence classes of virtual knots modulo transformation changing over/undercrossing structure. The latter are equivalence classes of immersed curves in orientable 2-surfaces modulo homotopy and stabilization.

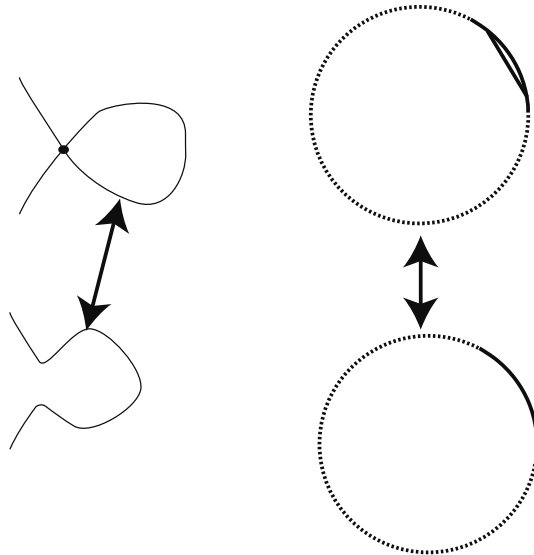


Figure 1: Addition/removal of a loop on a graph and on a chord diagram

## 2.1 Smoothings

Here we introduce the notion of smoothing, we shall often use in the sequel.

Let  $G$  be a framed graph, let  $v$  be a vertex of  $G$  with four incident half-edges  $a, b, c, d$ , s.t.  $a$  is opposite to  $c$  and  $b$  is opposite to  $d$  at  $v$ .

By *smoothing* of  $G$  at  $v$  we mean any of the two framed 4-graphs obtained by removing  $v$  and repasting the edges as  $(a, b), (c, d)$  or as  $(a, d), (b, c)$ , see Fig. 4.

Herewith, the rest of the graph (together with all framings at vertices except  $v$ ) remains unchanged.

We may then consider further smoothings of  $G$  at *several* vertices.

## 3 Atoms and orientability.

### The source-sink condition

Our further strategy is as follows: in many situations, it is easier to find *links* rather than *knots* with desired non-triviality properties. So, we shall first define a map from free 1-component links to  $\mathbf{Z}_2$ -linear combinations of 2-component links, and then we shall study the latter by an invariant similar to that constructed in [5].

Ideologically, the first map is a simplified version of Turaev's cobracket [8] which establishes a structure of Lie coalgebra on the set of curves immersed in 2-surfaces (up to some equivalence, the Lie *algebra* structure was introduced by Goldman in a similar way).

We shall need Turaev's construction (Turaev's  $\Delta$ ) to get a 2-component free link from a 1-component one.

The second map takes a certain state sum for a 2-component free link, where we distinguish between two types of crossings, and smooth only crossings of the first type. What should these

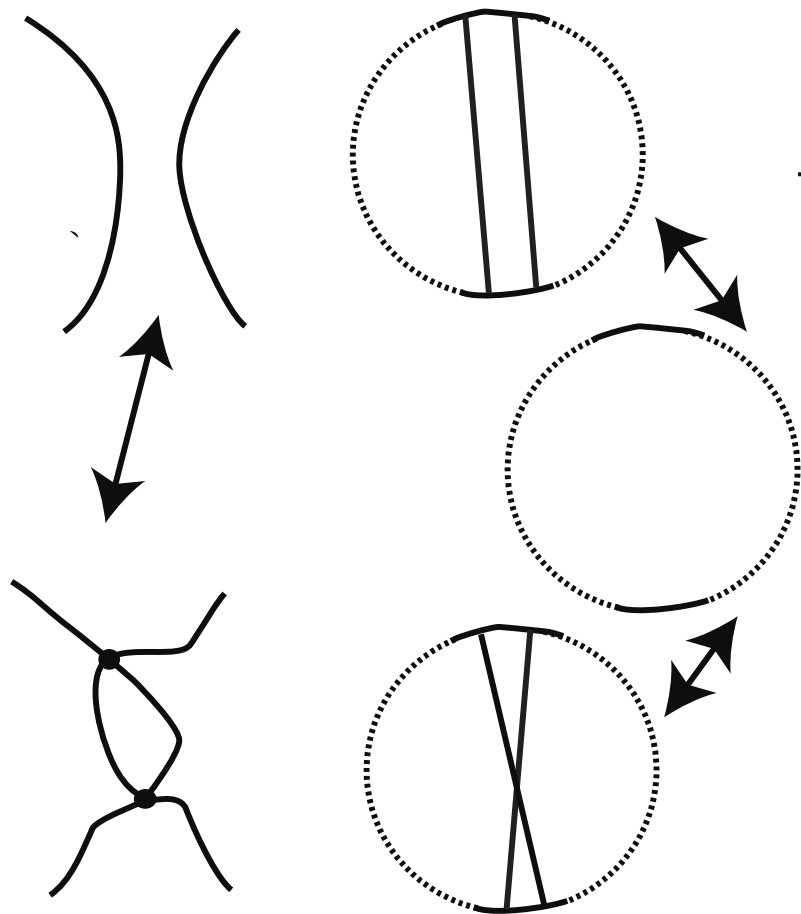


Figure 2: The second Reidemeister move and two chord diagram versions of it

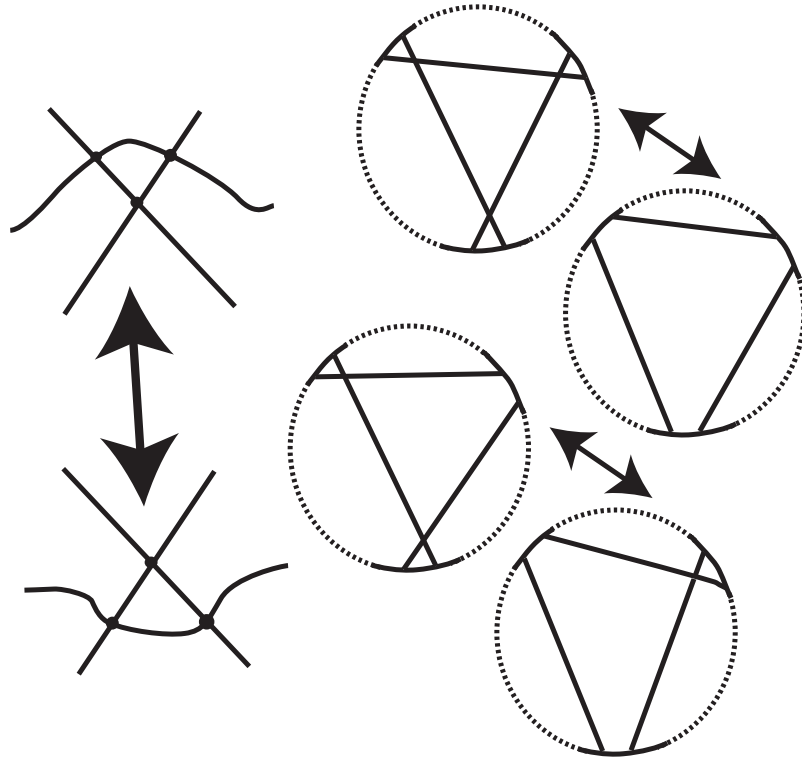


Figure 3: The third Reidemeister move and its chord diagram versions

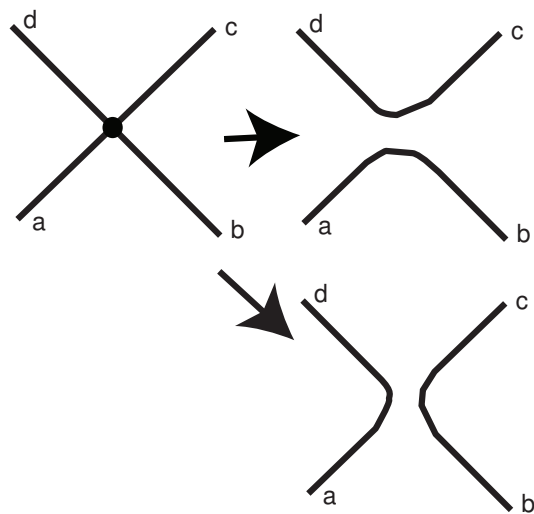


Figure 4: Two smoothings of a vertex of for a framed graph

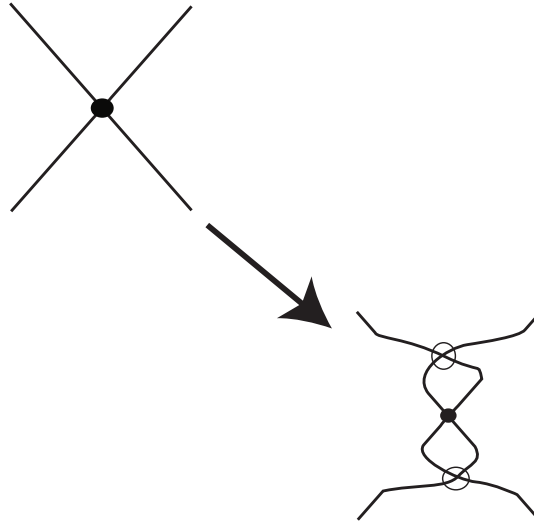


Figure 5: Virtualisation

“two types” mean, will be discussed later.

In some sense, the invariant  $[\cdot]$  of free knots constructed in [5] is a diagrammatic extension of a terrifically simplified Alexander polynomial (we forget about the variable and signs taking  $\mathbf{Z}_2$ -coefficients). The invariant  $\{\cdot\}$  suggested in the present paper is in the same sense an extension of the terrifically simplified Kauffman bracket, but again we use diagrams as coefficients.

Altogether, these two constructions provide an example of non-trivial and minimal diagrams of free knots with orientable atoms.

**Definition 1.** An atom (originally introduced by Fomenko, [1]) is a pair  $(M, \Gamma)$  consisting of a 2-manifold  $M$  and a graph  $\Gamma$  embedded in  $M$  together with a colouring of  $M \setminus \Gamma$  in a checkerboard manner. An atom is called orientable if the surface  $M$  is orientable. Here  $\Gamma$  is called the frame of the atom, whence by genus (atoms and their genera were also studied by Turaev [9], and atom genus is also called the Turaev genus [9]) (Euler characteristic, orientation) of the atom we mean that of the surface  $M$ .

Having an atom  $V$ , one can construct a virtual link diagram out of it as follows. Take a generic immersion of atom’s frame into  $\mathbb{R}^2$ , for which the formally opposite structure of edges coincides with the opposite structure induced from the plane.

Put virtual crossings at the intersection points of images of different edges and restore classical crossings at images of vertices ‘as above’. Obviously, since we disregard virtual crossings, the most we can expect is the well-definiteness up to detours. However, this allows us to get different virtual link types from the same atom, since for every vertex  $V$  of the atom with four emanating half-edges  $a, b, c, d$  (ordered cyclically on the atom) we may get two different clockwise-orderings on the plane of embedding,  $(a, b, c, d)$  and  $(a, d, c, b)$ . This leads to a move called *virtualisation*.

**Definition 2.** By a virtualisation of a classical crossing of a virtual diagram we mean a local transformation shown in Fig. 5.

The above statements summarise as

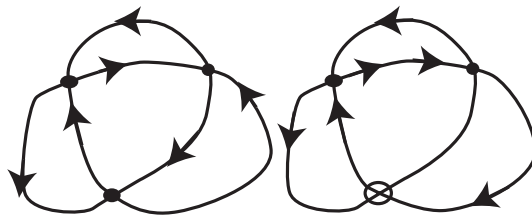


Figure 6: The source-target condition

**Proposition 1.** (see, e.g., [6]). *Let  $L_1$  and  $L_2$  be two virtual links obtained from the same atom by using different immersions of its frame. Then  $L_1$  differs from  $L_2$  by a sequence of detours and virtualisations.*

At the level of Gauss diagrams, virtualisation is the move that does not change the writhe numbers of crossings, but inverts the arrow directions. So, atoms just keep the information about signs of Gauss diagrams, but not of their arrows.

A further simplification comes when we want to forget about the signs and pass to flat virtual links (see also [10]): in this case we don't want to know which branch forms an overpass at a classical crossing, and which one forms an underpass. So, the only thing we should remember is its frame with opposite edge structure of vertices (the *A-structure*). Having that, we take any atom with this frame and restore a virtual knot up to virtualisation and crossing change.

The *A-structure* of an atom's frame is exactly a 4-valent framed graph.

This perfectly agrees with the fact that **free links are virtual links modulo virtualization and crossing changes.**

Having a framed 4-graph, one can consider *all atoms* which can be obtained from it by attaching black and white cells to it. In fact, it turns out that for a given framed 4-graph either all such surfaces are orientable or they are all non-orientable.

To see this, one should introduce the *source-sink* orientation. By a *source-sink* orientation of a 4-valent framed graph we mean an orientation of all edges of this graph in such a way that for each vertex some two opposite edges are outgoing, whence the remaining two edges are incoming.

The following statement is left to the reader as an exercise

**Example 1.** *Let  $G$  be a 4-valent framed graph. Then the following conditions are equivalent:*

1.  $G$  admits a source-sink orientation
2. At least one atom obtained from  $G$  by attaching black and white cells is orientable.
3. All atoms obtained from  $G$  by attaching black and white cells are orientable.

*Moreover, if  $G$  has one unicursal component, then each of the above conditions is equivalent to the following:*

*Every chord of the corresponding Gauss diagram  $C(D)$  is odd.*

We give two examples: for a planar 4-valent framed graph we present a source-sink orientation (left picture, Fig. 6), and for a non-orientable 4-valent framed graph (right picture, Fig.6, the artefact of immersion is depicted by a virtual crossing) we see that the source-sink orientation taken from the left crossing leads to a contradiction for the right crossing.



### 3.1 The sets $\mathbb{Z}_2\mathfrak{G}$ and $\tilde{\mathbb{Z}}_2\mathfrak{G}$

Let  $\mathfrak{G}$  be the set of all equivalence classes of framed graphs with one unicursal component modulo second Reidemeister moves. Consider the linear space  $\mathbb{Z}_2\mathfrak{G}$ . By  $\tilde{\mathbb{Z}}_2\mathfrak{G}$  we denote the space of all equivalence classes of framed graphs (with arbitrarily many components) by the second Reidemeister move with all graphs with free loops taken to be zero.

Having a framed 4-graph, one can consider it as an element of  $\mathbb{Z}_2\mathfrak{G}$  or of  $\tilde{\mathbb{Z}}_2\mathfrak{G}$ . It is natural to try simplifying it: we call a graph in  $\mathbb{Z}_2\mathfrak{G}$  *irreducible* if no decreasing second Reidemeister move can be applied to it. We call a graph in  $\mathbb{Z}_2\mathfrak{G}$  *irreducible* if it has no free loops and no decreasing second Reidemeister move can be applied to it.

The following theorem is trivial

**Theorem 1.** *Every 4-valent framed graph  $G$  with one unicursal component considered as an element of  $\mathbb{Z}_2\mathfrak{G}$  has a unique irreducible representative, which can be obtained from  $G$  by consecutive application of second decreasing Reidemeister moves.*

*Every 4-valent framed graph  $G$  considered as an element of  $\tilde{\mathbb{Z}}_2\mathfrak{G}$  is either equal to 0 or has a unique irreducible representative. In both cases, the reduction can be held by monotonous decreasing of the diagram by using second Reidemeister move and, if at some point one gets a free loop, the diagram is equal to zero.*

This allows to recognize elements  $\mathbb{Z}_2\mathfrak{G}$  and  $\tilde{\mathbb{Z}}_2\mathfrak{G}$  easily, which makes the invariants constructed in the previous subsection digestable.

In particular, the minimality of a framed 4-graph in  $\mathbb{Z}_2\mathfrak{G}$  or  $\tilde{\mathbb{Z}}_2\mathfrak{G}$  is easily detectable: one should just check all pairs of vertices and see whether any of them can be cancelled by a second Reidemeister move (or in  $\mathbb{Z}_2\mathfrak{G}$  one should also look for free loops).

Denote by  $\tilde{\mathbb{Z}}_2\mathfrak{G}_k$  the subspace of  $\tilde{\mathbb{Z}}_2\mathfrak{G}$  generated by  $k$ -component free links.

## 4 The Turaev cobracket

There is a simple and fertile idea due to Goldman [2] and Turaev [8] of transforming two-component curves into one-component curves and vice versa.

Here we simplify Turaev's idea for our purposes and call it "Turaev's  $\Delta$ ".

We shall construct a map from  $\mathbb{Z}_2\mathfrak{G}$  to  $\tilde{\mathbb{Z}}_2\mathfrak{G}_2$  as follows.

In fact, to define the map  $\Delta$ , one may require for a free knot to be oriented. However, we can do without.

Given a framed 4-graph  $G$ . We shall construct an element  $\Delta(G)$  from  $\tilde{\mathbb{Z}}_2\mathfrak{G}_2$  as follows. For each crossing  $c$  of  $G$ , there are two ways of smoothing it. One way gives a knot, and the other smoothing gives a 2-component link  $G_c$ . We take the one giving a 2-component link and write

$$\Delta(G) = \sum_c G_c \in \tilde{\mathbb{Z}}_2\mathfrak{G}_2 \quad (1)$$

**Theorem 2.**  *$\Delta(G)$  is a well defined mapping from  $\mathbb{Z}_2\mathfrak{G}$  to  $\tilde{\mathbb{Z}}_2\mathfrak{G}_2$*

The proof is standard and follows Turaev's original idea. One should consider the three Reidemeister move. The first move adds a new summand which has a free loop (the latter assumed to be trivial in  $\tilde{\mathbb{Z}}_2\mathfrak{G}_2$ ); for the second Reidemeister move we get two new identical summands, which cancel each other because we are dealing with  $\mathbb{Z}_2$  coefficients. For the third Reidemeister

moves the LHS and the RHS will lead to the summands identical up to second Reidemeister moves.

*We call the conditions described above the parity conditions.*

## 5 Two Types of Crossings: Reducing all Reidemeister Moves to the Second Reidemeister move

Assume we have a certain class of knot-like objects which are equivalence classes of **diagrams** modulo *three Reidemeister moves*. Assume for this class of diagrams (e.g. 4-valent framed graphs) there is a fixed rule of distinguishing between two types of crossings (called even and odd) such that:

- 1) Each crossing taking part in the first Reidemeister move is even, and after adding/deleting this crossing the parity of the remaining crossings remains the same.
- 2) Each two crossings taking part in the second Reidemeister move are either both odd or both even, and after performing these moves, the parity of the remaining crossings remains the same.
- 3) For the third Reidemeister move, the parities of the crossings which do not take part in the move remain the same.

Moreover, the parities of the three pairs of crossings are the same in the following sense: there is a natural one-to-one correspondence between pairs of crossings  $A - A', B - B', C - C'$  taking part in the third Reidemeister move, see Fig. 5.

We require that the *parity* of  $A$  coincides with that of  $A'$ , the *parity* of  $B$  coincides with that of  $B'$  and the parity of  $C$  coincides with that of  $C'$ .

We also require that the number of odd crossings among the three crossings in question ( $A, B, C$ ) is even (that is, is equal to 2 or 0).

Having such objects with a prescribed rule satisfying the above properties, one can define an invariant polynomials of our knot-like objects (to be more precise, this leads to an invariant *mapping* from knot-like objects to a terrifically simpler class of objects).

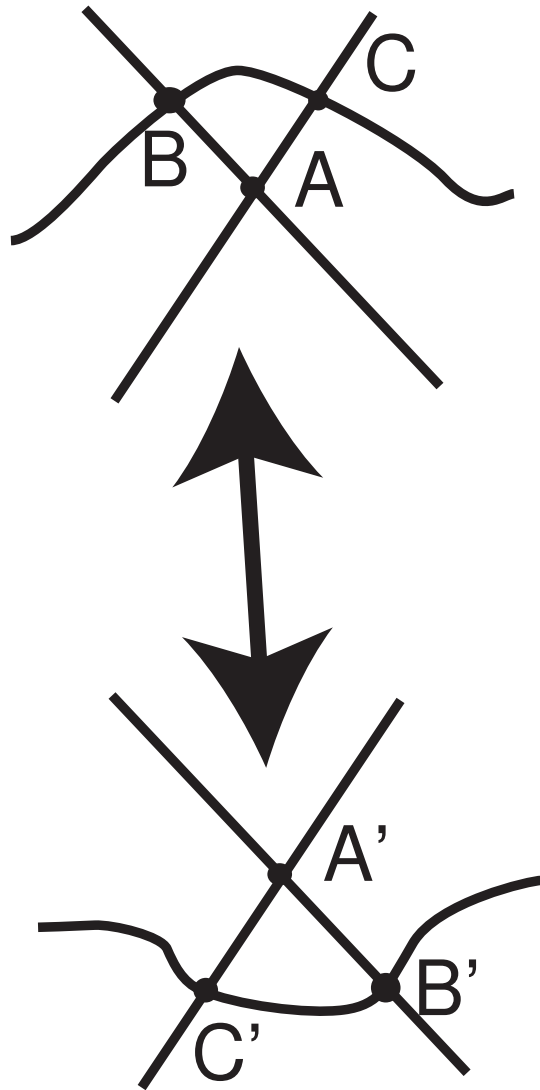
In particular, this will lead us to one invariant of graph-knots and one invariant of graph-links. The first invariant is introduced in [5].

It counts all *rotating circuits*, i.e. circuits going along all the edges of the graph once and switching from each edge to a non-opposite edge. Let us be more specific.

If one takes these circuits for a classical knot with an appropriate signs and weights (powers of  $t$ ), one would get the Alexander polynomial.

Here, we introduce two ingredients: instead of circuits rotating at all crossings, we let circles rotate only at “even” crossings, and leave odd crossings as they are. Besides, instead of signs and powers of  $t$ , we add some framed graphs modulo relations as coefficients.

The scheme for the Kauffman bracket polynomial is the same with the only difference that we count *all states* with weights being either polynomials or chord diagrams: by a state we mean a way of smoothing of all vertices (or, resp., all odd vertices). The only difference between a smoothing and a rotating circuit is that for a smoothing the number of circles may be arbitrary.



## 5.1 The “Alexander-like” bracket

Let us be more specific. We concentrate on free knots and call a vertex of a 4-valent graph corresponding to a free knot *odd* if and only if the corresponding chord of the chord diagram corresponding to the framed graph is odd.

It is left for the reader as an exercise to check the **parity conditions**. We shall construct a map from free knots to  $\mathbb{Z}_2\mathfrak{G}$ .

Consider the following sum

$$[G] = \sum_{s \text{ even}, 1 \text{ comp}} G_s, \quad (2)$$

which is taken over all smoothings in all *even* vertices, and only those summands are taken into account where  $G_s$  has one unicursal component.

Thus, if  $G$  has  $k$  even vertices, then  $[G]$  will consist of at most  $2^k$  summands, and if all vertices of  $G$  are odd, then we shall have exactly one summand, the graph  $G$  itself.

The “Alexander-like” *bracket* (to be denoted by  $[\cdot]$ ) is defined as follows:

$$[G] = \sum_{s \text{ even}} G_s, \quad (3)$$

where  $G_s$  is considered as an element in  $\mathbb{Z}_2\mathfrak{G}$ .

**Theorem 3.** ([5]) *The mapping  $G \mapsto [G]$  is well defined, i. e.,  $[G]$  does not depend on the representative of the free knot corresponding to  $G$ .*

This theorem is proved in [5]. The idea behind the proof relies on the comparison of diagrams obtained from each other by Reidemeister moves with parity conditions taken into account.

We call a four-valent framed graph having one unicursal component *odd* if all vertices of this graph are odd. We call an odd graph *irreducibly odd* if for every two distinct vertices  $a, b$  there exists a vertex  $c \notin \{a, b\}$  such that  $\langle a, c \rangle \neq \langle b, c \rangle$ .

Theorem 3 yields the following

**Corollary 1.** *Let  $G$  be an irreducibly odd framed 4-graph with one unicursal component. Then any representative  $G'$  of the free knot  $K_G$ , generated by  $G$ , has a smoothing  $\tilde{G}$  having the same number of vertices as  $G$ . In particular,  $G$  is a minimal representative of the free knot  $K_G$  with respect to the number of vertices.*

The simplest example of an irreducibly odd graph is depicted in Fig. 7.

## 6 The Kauffman-like bracket

Thus, we have proved that the free knot  $K$  having Gauss diagram shown in Fig.7 is minimal: every diagram of this knot has at least 6 odd vertices.

The reason is that  $[K]$  consists of one diagram representing  $K$  itself, and  $K$  is not simplifiable in the category  $\mathbb{Z}_2\mathfrak{G}$ .

However, this argument is not applicable to free knots with no odd crossings. Thus, we give two more examples using “Kauffman bracket like” techniques for other free knots.

Let  $K$  be a two-component free considered as an element from  $\tilde{\mathbb{Z}}_2\mathfrak{G}_2$ . We shall construct a map  $\{\cdot\} : K \mapsto \{K\}$  valued in  $\tilde{\mathbb{Z}}_2\mathfrak{G}$  as follows.

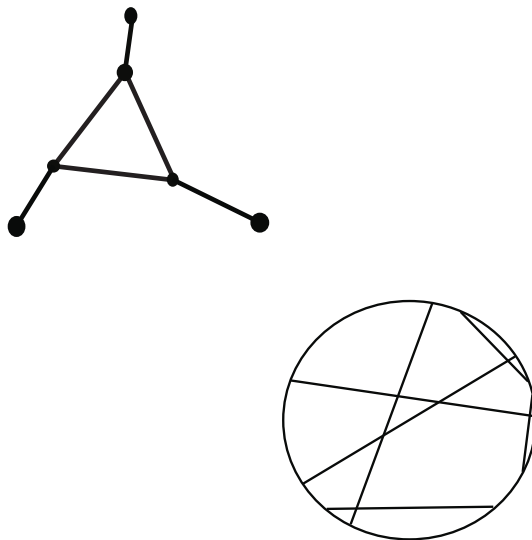


Figure 7: An irreducibly odd graph and its chord diagram

Take a framed four-valent graph  $G$  representing  $K$ . By definition, it has two components. Now, a vertex of  $G$  is called *odd* if it is formed by two different components, and *even* otherwise.

**The parity conditions can be checked straightforwardly.**

Now, we define

$$\{G\} = \sum_s G_s, \tag{4}$$

where we take the sum over all smoothings of all even vertices, and consider the smoothed diagrams  $K_s$  as elements of  $\tilde{\mathbb{Z}}_2\mathfrak{G}$ . In particular, we take all elements of  $K_s$  with free loops to be zero.

**Theorem 4.** *The bracket  $\{K\}$  is an invariant of two-component free links, that is, for two graphs  $G$  and  $G'$  representing the same two-component free link  $K$  we have  $\{G\} = \{G'\}$  in  $\tilde{\mathbb{Z}}_2\mathfrak{G}$ .*

*Proof.* The proof is very similar to that of Theorem 3. Indeed, we have to consider two diagrams that differ by a Reidemeister move and show that the corresponding brackets  $\{\cdot\}$  are equal in  $\tilde{\mathbb{Z}}_2\mathfrak{G}$ .

Let us check the invariance  $[G] \in \tilde{\mathbb{Z}}_2\mathfrak{G}$  under the three Reidemeister moves.

Let  $G'$  differ from  $G$  by a first Reidemeister move, so that  $G'$  has one vertex more than  $G$ . By definition this vertex is even (it is formed by one component), and when calculating  $[G']$  this vertex has to be smoothed in order to get one unicursal curve in total.

Thus, we have to take only one of two smoothings of the given vertex, see Fig. 8

Thus there is a natural equivalence between smoothings of  $G$  having one unicursal component, and smoothings of  $G'$  with one unicursal component. Moreover, this equivalence yields a termwise identity between  $[G]$  and  $[G']$ .

Now, let  $G'$  be obtained from  $G$  by a second Reidemeister move adding two vertices.

These two vertices are either both even or both odd (that is, two branches belong to the same component of the free link or to different components).

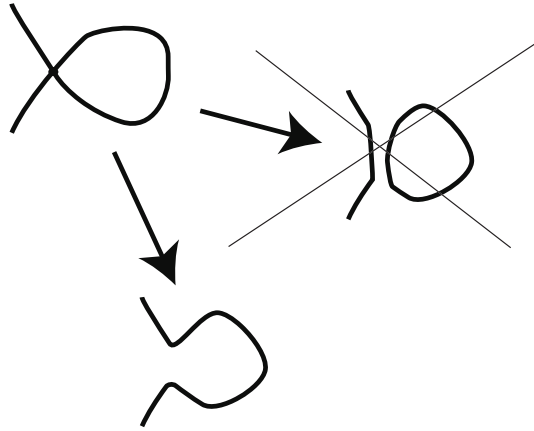


Figure 8: The two smoothings — good and bad — of a loop

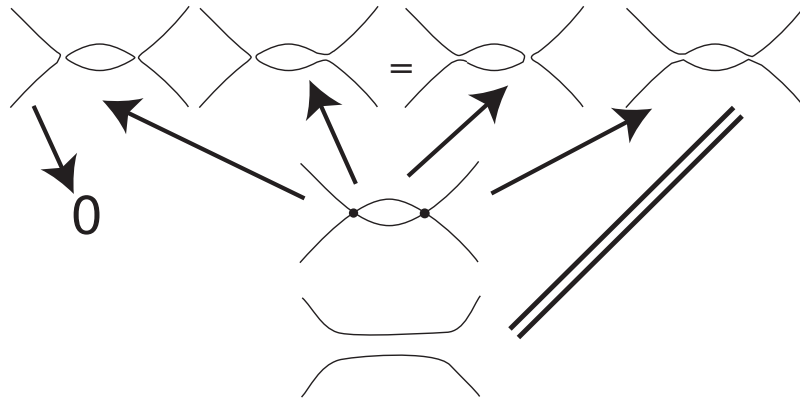


Figure 9: Smoothings of two even vertices

If both added vertices are odd, then the set of smoothings of  $G$  is in one-to-one correspondence with that of  $G'$  and the corresponding summands for  $[G]$  and for  $[G']$  differ from each other by a second Reidemeister move.

If both vertices are odd then one has to consider different smoothings of these vertices shown in. 9.

The smoothings shown in the upper-left Fig. 9, yield to free loops, so they do not count in  $[G'] \in \tilde{\mathbb{Z}}_2 \mathfrak{G}$ .

The second-type and third-type smoothings (the second and the third pictures in the top row of 9) give the same impact to  $\mathbb{Z}_2 \mathfrak{G}$ , thus, they reduce in  $[G']$ . Finally, the smoothings corresponding to the upper-right Fig. 9 are in one-to-one correspondence with smoothings of  $G$ , thus we have a term-wise equality of terms  $[G]$  and those terms of  $[G']$ , which are not cancelled by comparing the two middle pictures.

If  $G$  and  $G'$  differ by a third Reidemeister move, then the following two cases are possible: either all vertices taking part in the third Reidemeister move are even, or two of them are odd and one is even.

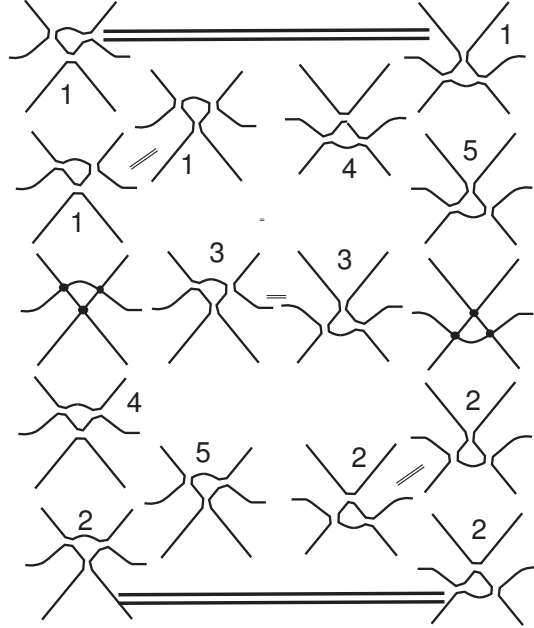


Figure 10: Correspondences of smoothings with respect to  $\Omega_3$  with three even vertices

If all the three vertices are even, there are seven types of smoothings corresponding to  $[G]$  (and seven types of smoothings corresponding to  $[G']$ ): in each of the three vertex we have two possible smoothings, and one case is ruled out because of a free loop. When considering  $G$ , three of these seven cases coincide (this triple is denoted by 1), so, in  $\mathbb{Z}_2\mathfrak{G}$  it remains exactly one of these two cases. Amongst the smoothings of the diagram  $G'$ , the other three cases coincide (they are marked by 2). Thus, both in  $[G]$  and  $[G']$  there are five types of summands marked by 1, 2, 3, 4, 5.

These five cases are in one-to-one correspondence (see Fig. 10) and they yield the equality  $[G] = [G']$ .

If amongst the three vertices taking part in  $\Omega_3$  we have exactly one even vertices (say  $a \rightarrow a'$ ), we get the situation depicted in Fig. 11.

From this figure we see that those smoothings where  $a$  (resp.,  $a'$ ) is smoothed *vertically*, give identical summands in  $[G]$  and in  $[G']$ , and those smoothings where  $a$  and  $a'$  are smoothed *horizontally*, are in one-to-one correspondence for  $G$  and  $G'$ , and the corresponding summands are obtained by applying two second Reidemeister moves. This proves that  $[G] = [G']$  in  $\mathbb{Z}_2\mathfrak{G}$ .  $\square$

We extend  $K$  to  $\tilde{\mathbb{Z}}_2\mathfrak{G}_2$  by linearity.

## 7 New Examples

**Statement 1.** *The free link  $L_1$  shown in Fig. 12 is minimal and the corresponding atom is orientable.*

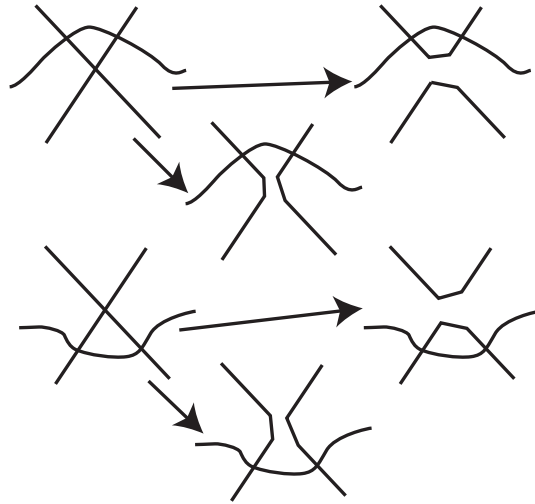


Figure 11: Correspondence between smoothings for  $\Omega_3$  with one even vertex

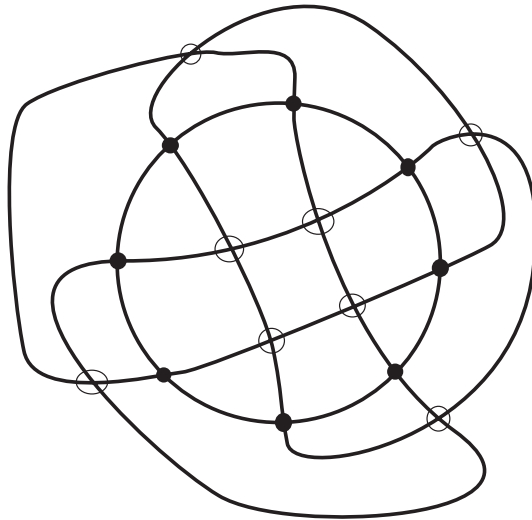


Figure 12: Minimal free two-component link



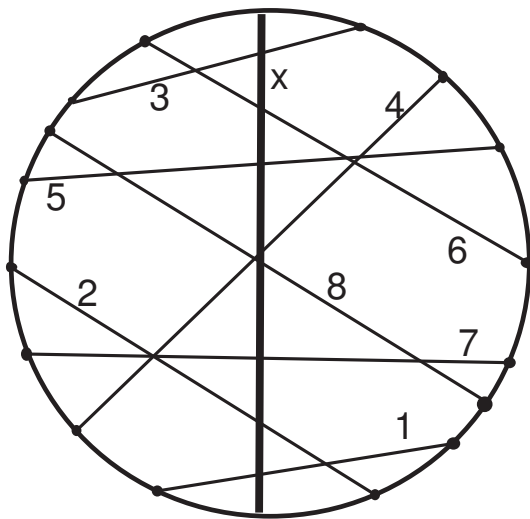


Figure 13: A Minimal Free Knot Gauss Diagram

*Proof.* The orientability of any of the corresponding atoms can be checked straightforwardly: one can easily verify the source-sink condition.

To prove minimality, let us consider the bracket  $\{L_1\}$ . By construction,  $\{L_1\}$  consists of only one diagram, the diagram  $L_1$  itself. Since  $L_1$  is minimal in  $\mathbb{Z}_2\mathfrak{G}$ , we see that for every link  $L'_1$  equivalent to  $L_1$  there is a smoothing of  $L'_1$  at some vertices equivalent to  $L_1$ . So,  $L'_1$  has at least eight crossings.  $\square$

**Statement 2.** *The free knot  $K_1$  shown in Fig. 13 is minimal.*

*Proof.* Consider  $\Delta(K_1)$ . By construction, it consists of nine summands, each of which representing a two-component free link. These summands are constructed by smoothing exactly one crossing (one chord). If we smooth the chord  $x$ , we get the link  $L_1$  depicted above. Thus,  $\Delta(K_1) = L_1 + \sum L_i$ , where the all  $L_i$ 's are two-component links.

We claim that none of the links  $L_i$  is equivalent to  $L_1$  as a free link.

Indeed, the initial Gauss diagram of  $K_1$  has 9 chords, and there is only one chord  $x$  which is linked with any other chords. Thus, if we smooth the diagram along a chord distinct from  $x$ , we would get a 2-component link, say,  $L_i$  with at least one crossing formed by one and the same component. Thus, by definition of  $\{\cdot\}$ , we see that  $\{L_i\}$  is a sum of diagrams having strictly less than 8 crossings. Consequently,  $L_i \neq L_1$ .

Thus we have proved that for  $\{\Delta(K_1)\}$  has exactly one diagram with minimal crossing number 8. Taking into account the invariance of  $\{\cdot\}$ , we see that  $K_1$  has at least 9 crossings.

Moreover, we have indeed proved that every diagram of  $\Delta$  has at least one smoothing equivalent to  $L_1$ .  $\square$

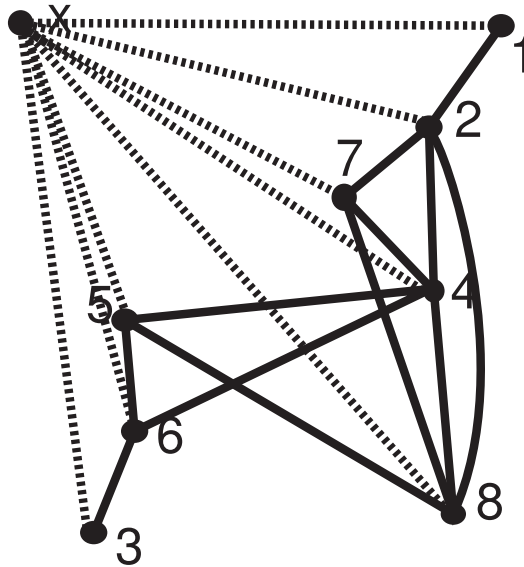


Figure 14: The intersection graph of the minimal chord diagram

## 8 Post Scriptum. Odds and Ends.

In [5] we gave an example of a looped graph (or graph-knot, for definition see [7, 3]) which has no realisable representative, i. e. which is not equivalent to any virtual knot. However, this graph was consisting of only *odd vertices*, i. e., each vertex has odd valency.

Below we present an example of a looped graph with all vertices of even valency, which has no realizable representative.

To do that, let us analyse the example shown in Fig. 13. This Gauss diagram has all even chords (this guarantees the orientability of the corresponding atom). There is exactly one chord which is linked with any of the remaining chords: the chord  $x$ . This guarantees that the smoothing at  $x$  gives a 2-component link where every crossing is formed by 2 components, whence the smoothing at any other crossing gives a 2 components gives a 2-component link having at least one crossing formed by one component. This means, that if we then apply the bracket  $\{\cdot\}$ , these remaining diagrams will lead to diagrams with strictly smaller number of crossings (than eight).

Finally, there is no room to perform the second decreasing Reidemeister move to the graph obtained by smoothing along  $x$ . This is guaranteed by the fact that in the initial chord diagram there is no pair of chords  $a, b$  both distinct from  $x$  and such that

For reader's convenience, the intersection graph of this chord diagram looks as shown in Fig. 14.

Quite analogously, one considers the looped graph in the sense [7] with “Gauss diagram intersection graph” given by the non-realizable graph shown in Fig. 15.

All vertices of this diagram have even valency, and there is exactly one vertex connected to all the remaining vertices. Thus, applying Turaev's  $\Delta$  to it, one gets 7 2-component diagrams:  $A + \sum_i B_i$ , exactly one of which has all “even crossings”, that is, only for one diagram  $A$  every crossing belongs to both components of the two component link. For any other diagram  $B_i, i =$

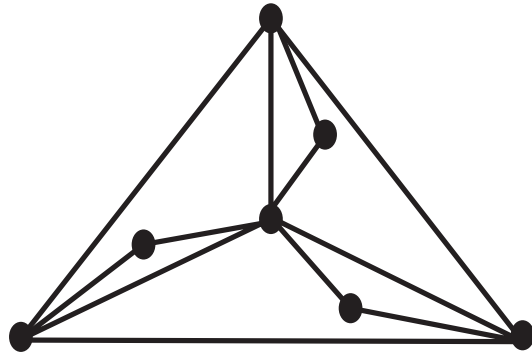


Figure 15: The intersection graph of a non-realizable knot

$1, \dots, 6$ , there is at least one crossing formed by branches of the same component.

So, the diagram  $A$  is not cancelled by any of  $B_i$ 's. One easily checks that  $A$  is non-realizable, thus, the looped graph with Gauss diagram shown in Fig. 15 has no realizable representative.

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