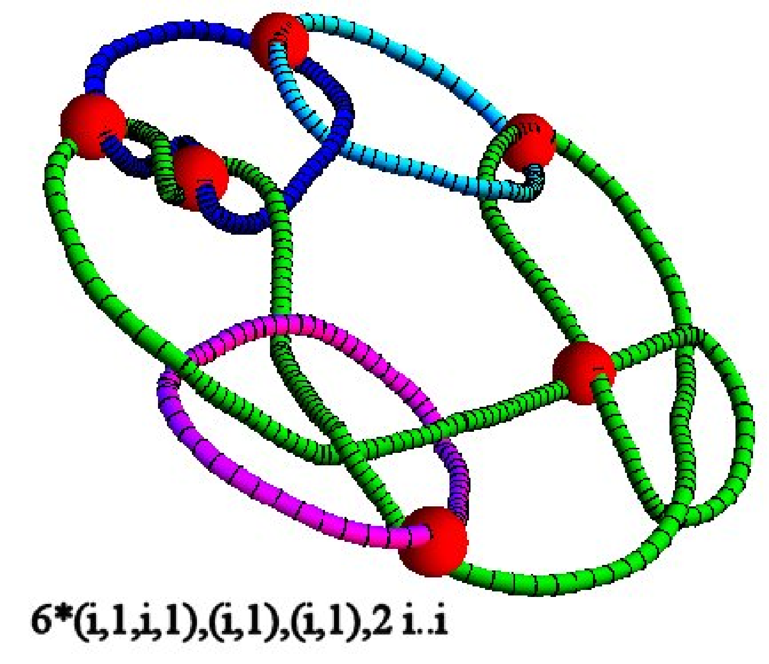
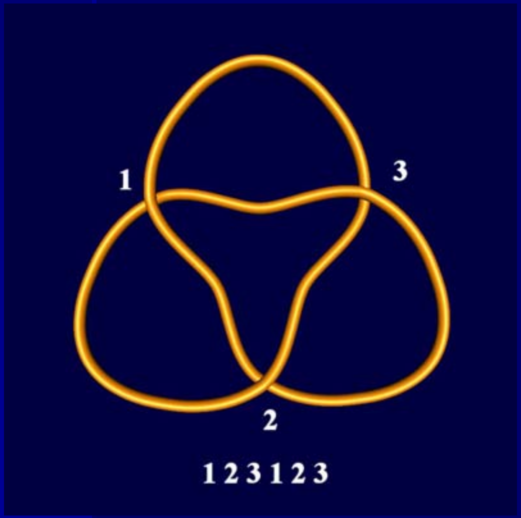
VIRTUAL KNOTS AND LINKS IN CONWAY NOTATION

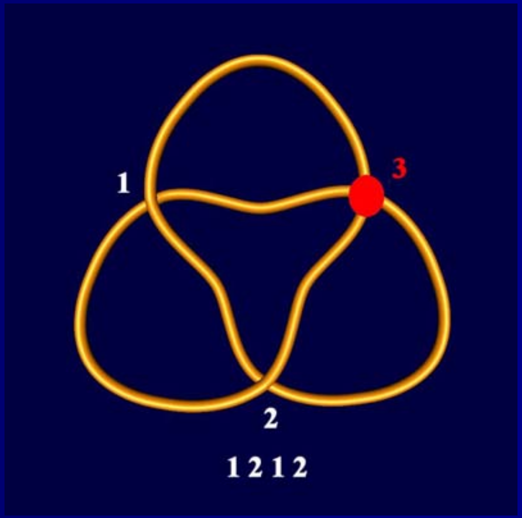


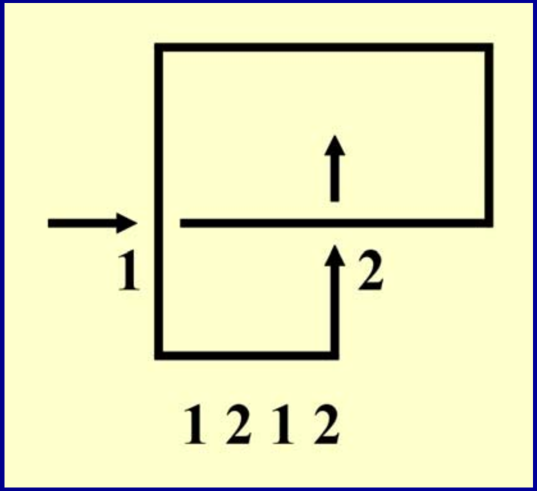
Slavik Jablan & Radmila Sazdanovic

Virtual KLs

- Interpretations
- Non-realizable Gauss codes (non-planar 4-valent graphs)
- Virtualization of crossings

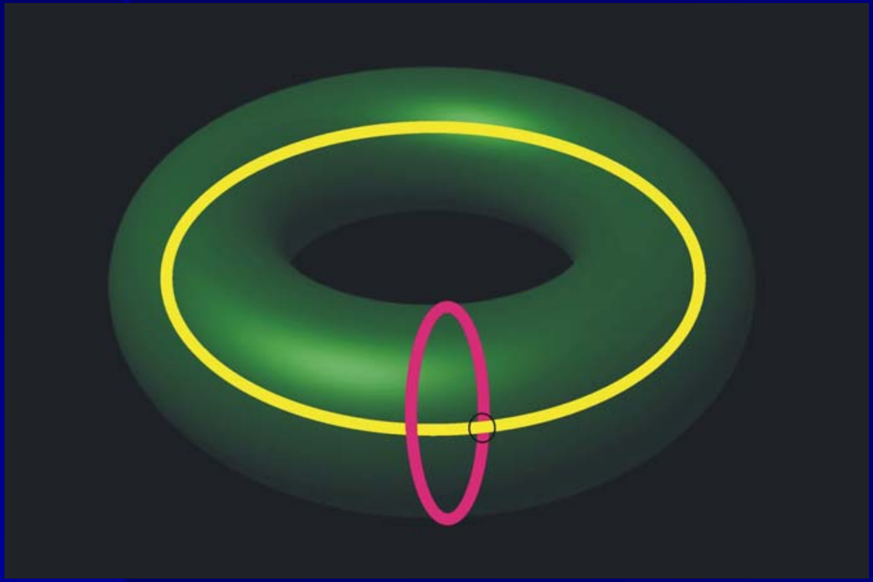




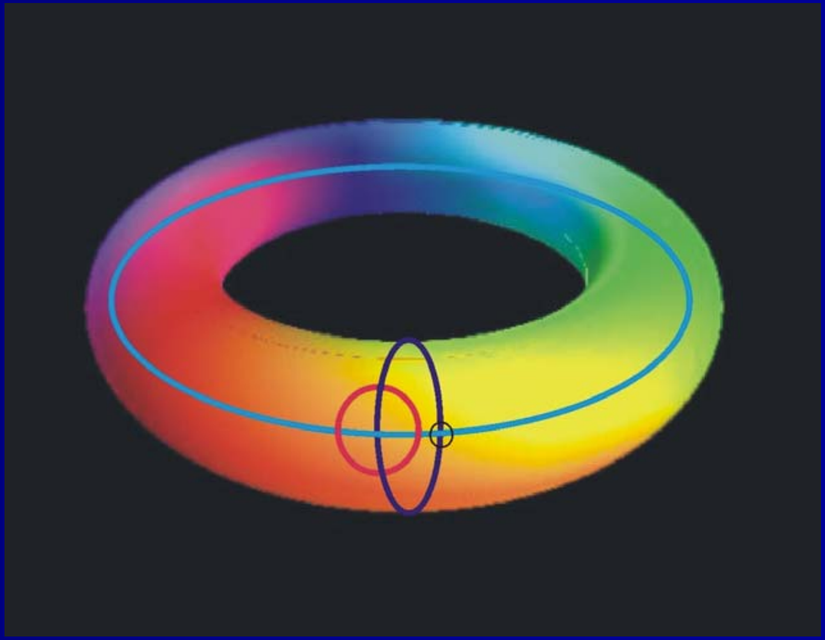


Virtual KLs (Louis Kauffman)

- KLs on different surfaces



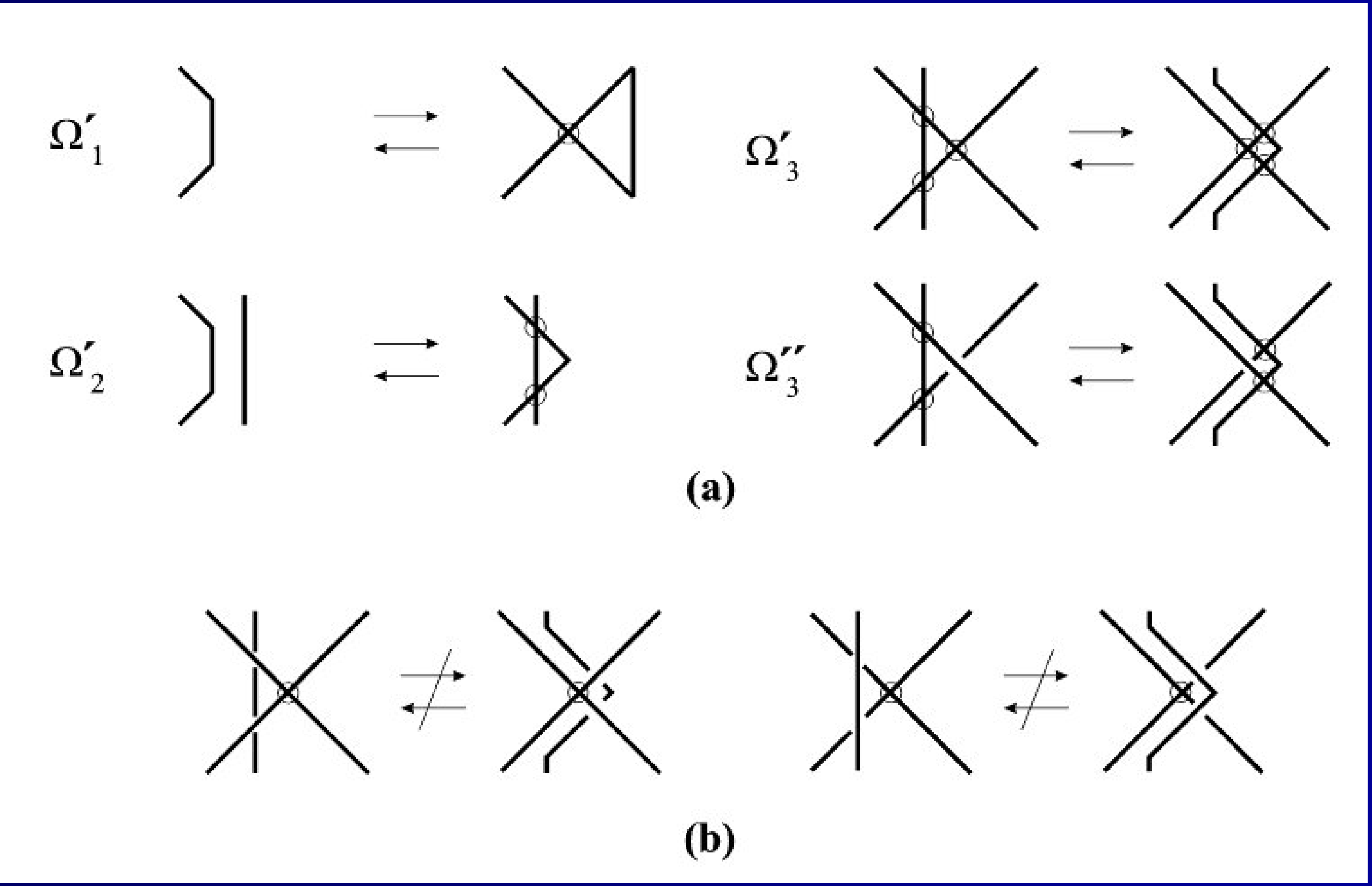
Hopf link   n=1



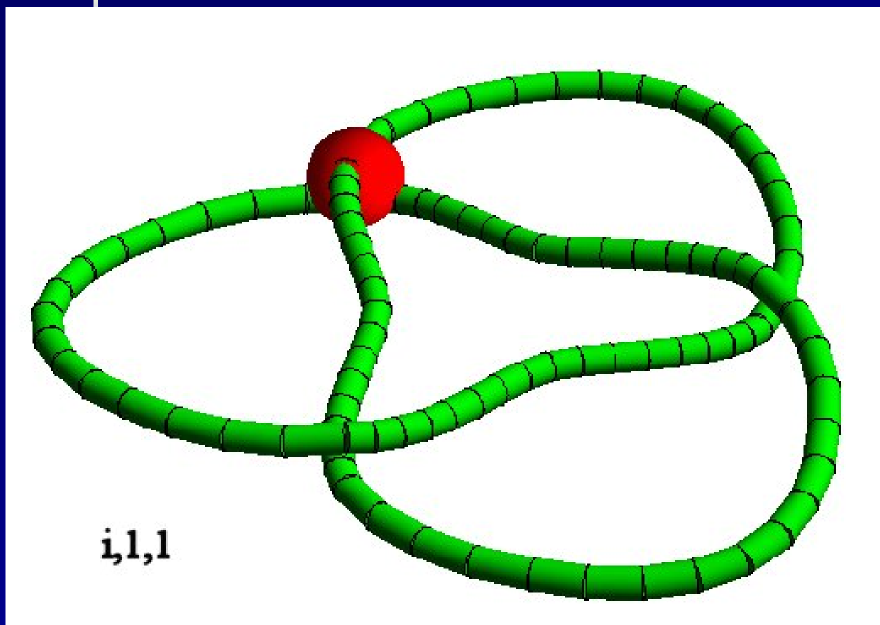
Borromean rings n=5

Virtual KLs

- Planar isotopies (Reidemeister moves)
- classical Reidemeister moves
- a) virtual Reidemeister moves
- b) forbidden move



Conway notation for virtual KLs



Gauss Code: U1+O2+O1+U2+

Kamada code: {{{1,3},{2,0}},{1,1}}

PD-notation:
PD[X[3,1,4,2],X[2,4,3,1]]
(Dror Bar Natan)

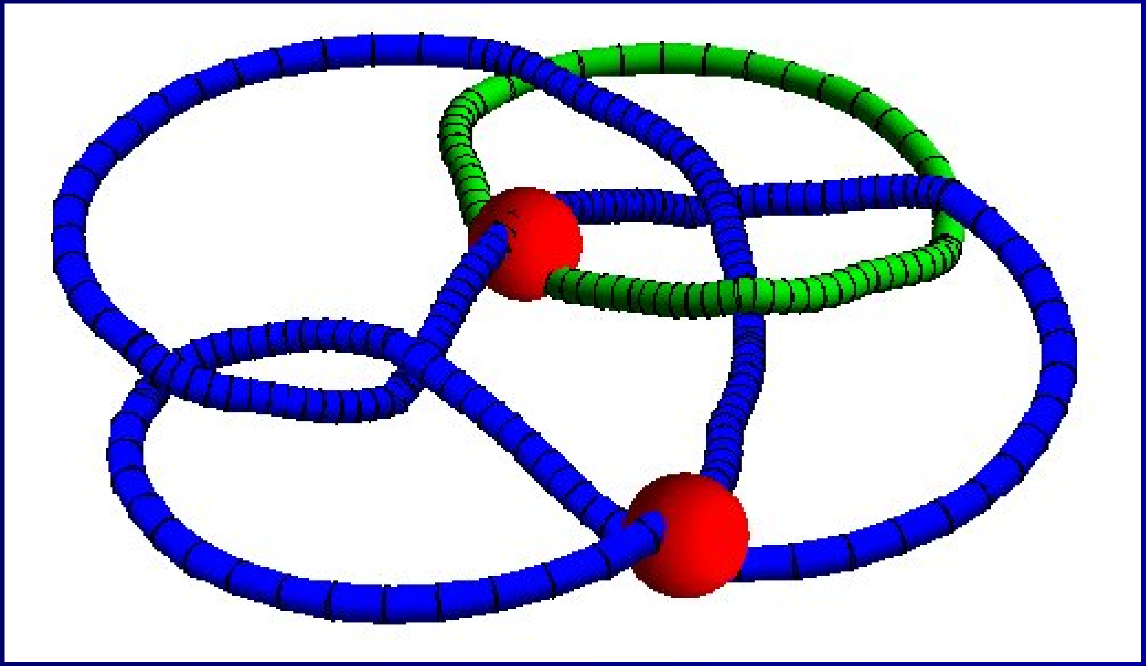
Trefoil: 3 = 1,1,1=1^3
Virtual trefoil i,1,1=i,1^2

Information about original real knot is preserved.

In all other codes virtual crossings are omitted. The information about real KL from which a virtual KL is obtained can be recovered, but in a complicated way.

Virtual KLs in Conway notation

PD[X[1,6,2,7],X[2,11,3,12],X[8,3,9,1],X[4,11,5,10],X[9,6,10,5],X[12,8,4,7]]
Conway notation: 6*i.-1.(2,i)

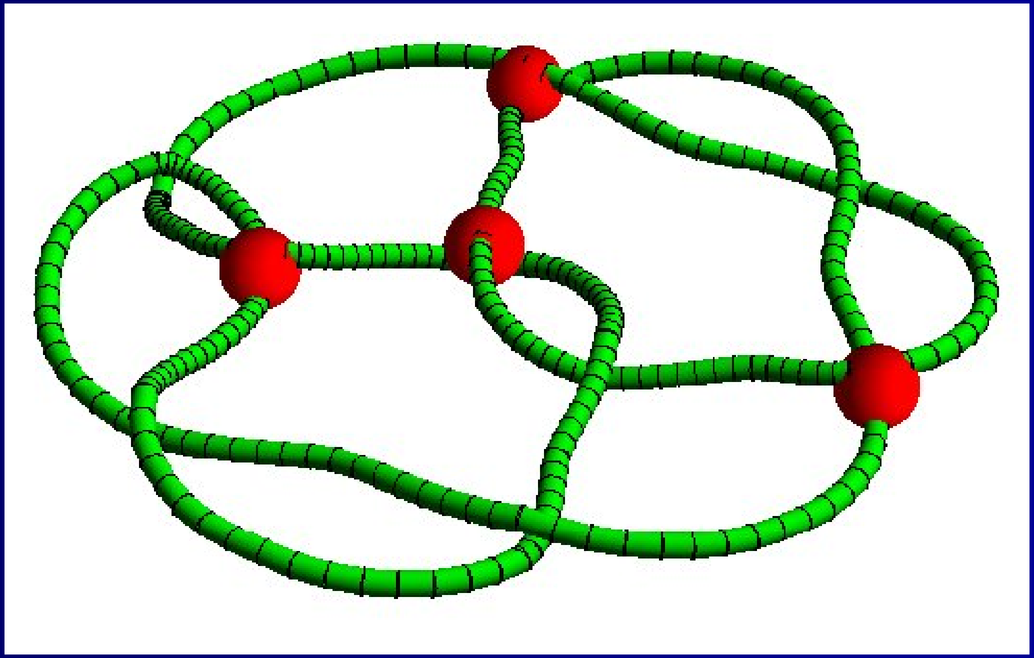


Virtual KLs in Conway notation

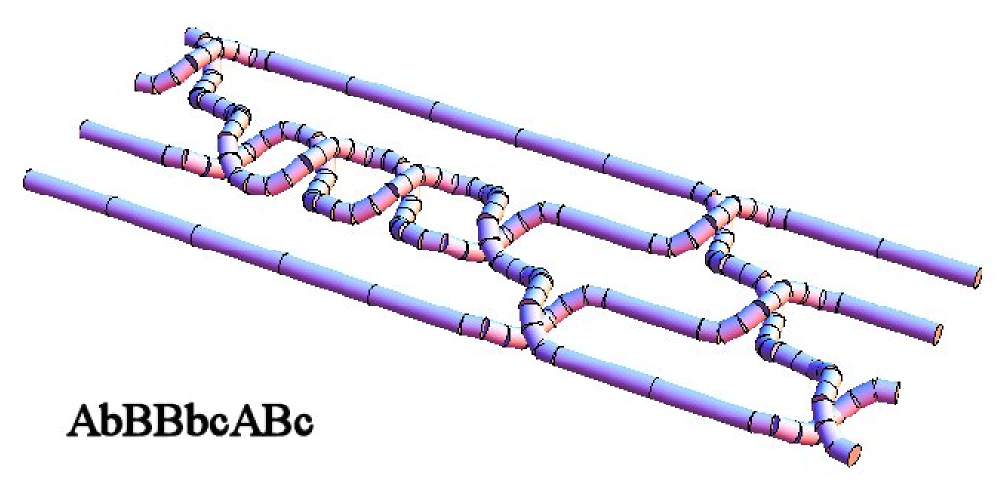
PD[X[12,6,1,7],X[10,2,11,1],X[2,10,3,9],X[11,3,12,4],X[8,4,9,5],X[5,7,6,8]] (6 crossings)

Gauss Code: O1+U2-O3-U4+U5+O6+U1+U6+O5+U3-O2-O4+ (6 crossings)

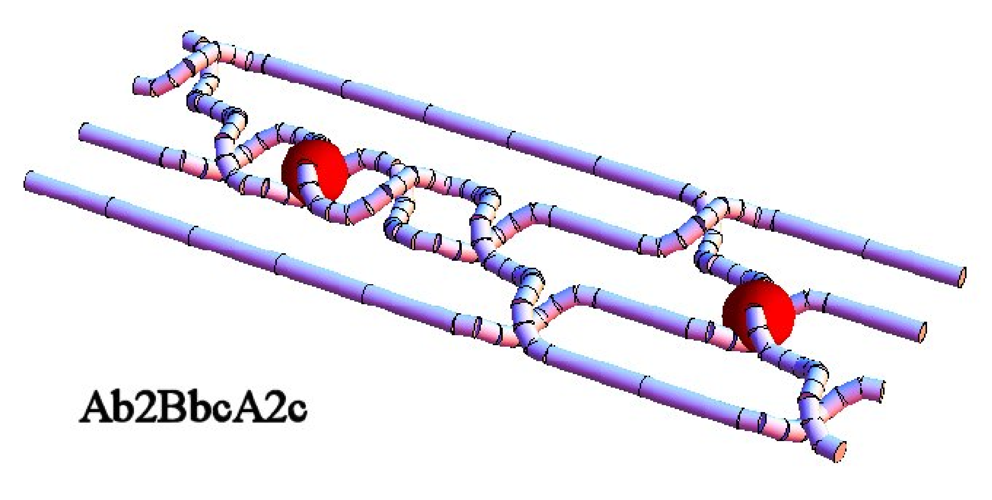
Conway notation: (1,i) 2,(i,2),(i,1,i) (10 crossings)



Virtual braids- notation

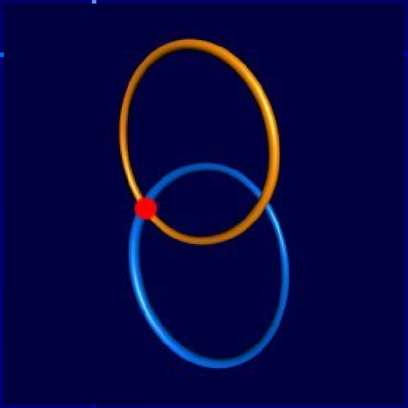


AbBBbcABc

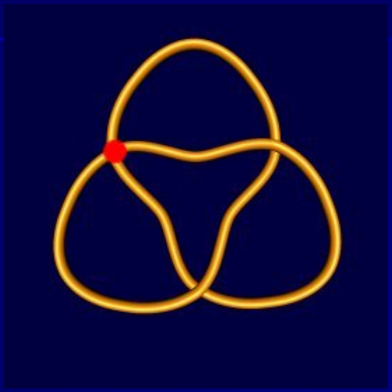


Ab2BbcA2c

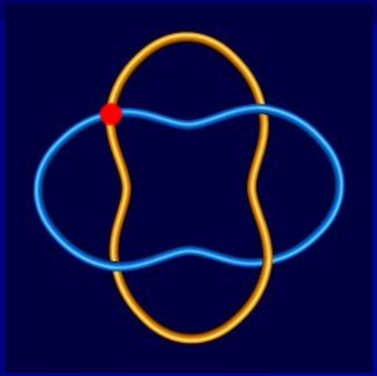
Simplest family: i,1^n (n=1,2,3,4,…)



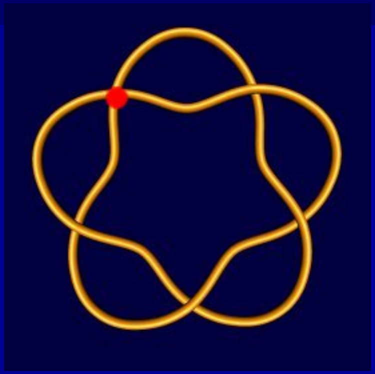
i,1^1



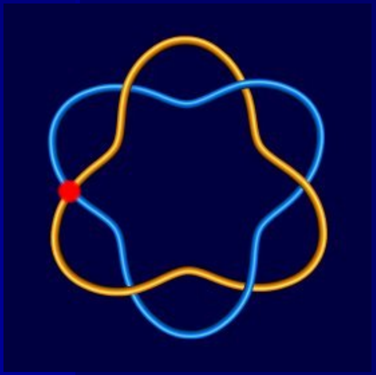
i,1,1=i,1^2



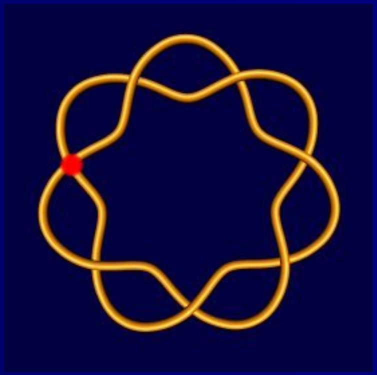
i,1,1=i,1^3



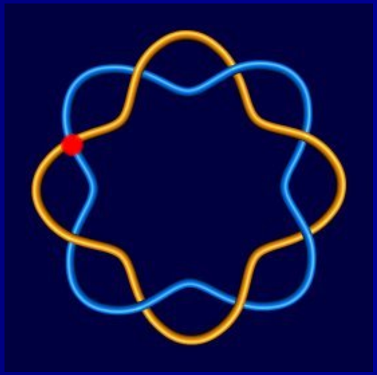
i,1,1,1,1=i,1^4



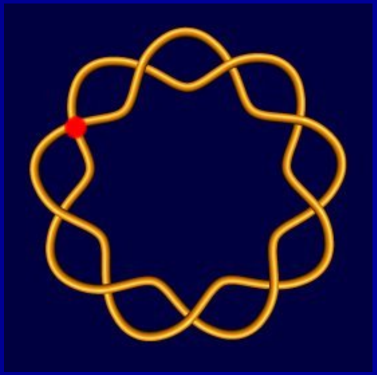
i,1^5



i,1^6



i,1^7



i,1^8

Polynomials of the family i,1,...1

Sawolek:

$i,1^1$ $1-s-t+s\,t$

$i,1^2$ $-(-1+s)\,(-1+t)\,(-1+s\,t)$

$i,1^3$ $1-s-t+s\,t$

$i,1^4$ $-(-1+s)(-1+t)(-1+st-s^2t^2+s^3t^3)$

$i,1^5$ $1-s-t+s\,t$

$i,1^6$ $-(-1+s)(-1+t)(-1+st-s^2t^2+s^3t^3-s^4t^4+s^5t^5)$

$i,1^7$ $1-s-t+s\,t$

$i,1^8$ $-(-1+s)(-1+t)(-1+st-s^2t^2+s^3t^3-s^4t^4+s^5t^5-s^6t^6+s^7t^7)$

$i,1^9$ $1-s-t+s\,t$

$i,1^{10}$ $-(-1+s)(-1+t)(-1+st-s^2t^2+s^3t^3-s^4t^4+s^5t^5-s^6t^6+s^7t^7-s^8t^8+s^9t^9)$

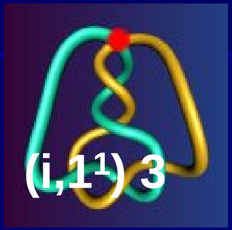
Polynomials of the family $i,1^{n}-1$

Bracket (Jones):

$i,1^1$	$1+A^2$
$i,1^2$	$1+A^2-A^6$
$i,1^3$	$1+A^2-A^6+A^{10}$
$i,1^4$	$1+A^2-A^6+A^{10}-A^{14}$
$i,1^5$	$1+A^2-A^6+A^{10}-A^{14}+A^{18}$
$i,1^6$	$1+A^2-A^6+A^{10}-A^{14}+A^{18}-A^{22}$
$i,1^7$	$1+A^2-A^6+A^{10}-A^{14}+A^{18}-A^{22}+A^{26}$
$i,1^8$	$1+A^2-A^6+A^{10}-A^{14}+A^{18}-A^{22}+A^{26}-A^{30}$
$i,1^9$	$1+A^2-A^6+A^{10}-A^{14}+A^{18}-A^{22}+A^{26}-A^{30}+A^{34}$
$i,1^{10}$	$1+A^2-A^6+A^{10}-A^{14}+A^{18}-A^{22}+A^{26}-A^{30}+A^{34}-A^{38}$

Virtual KL family (i,1^p) q

(i,1^1) 2



(i,1^1) 3

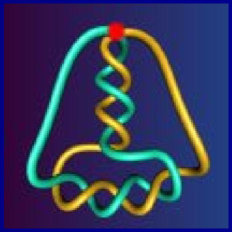
(i,1^2) 3

(i,1^1) 4

(i,1^2) 4

(i,1^1) 5

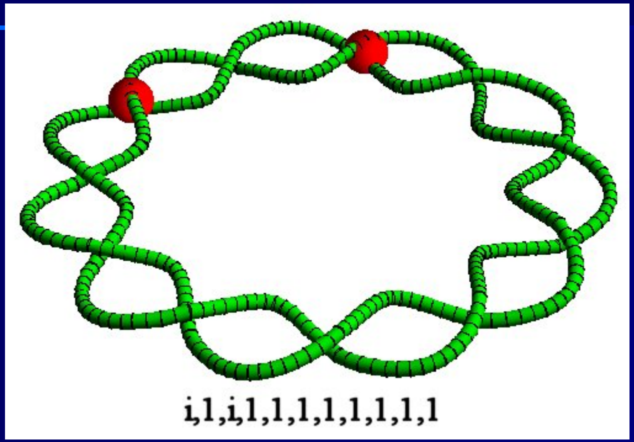
(i,1^2) 5

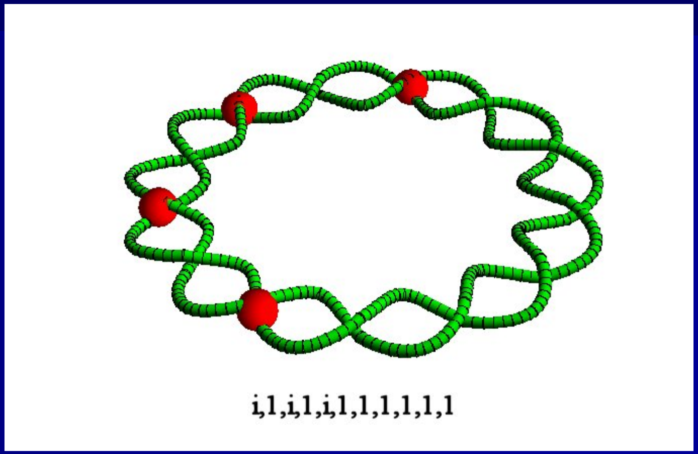


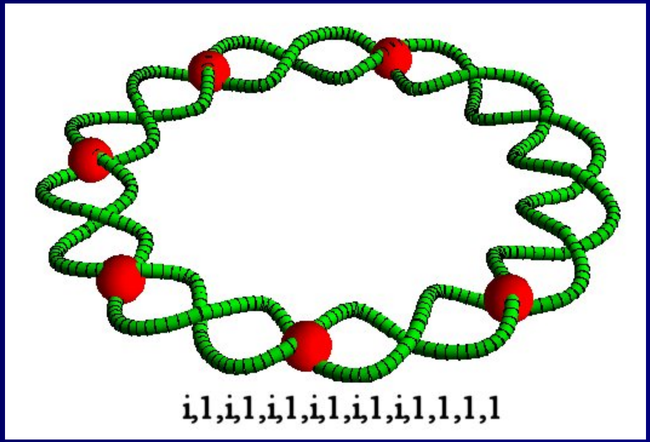
(i,1^1) 6

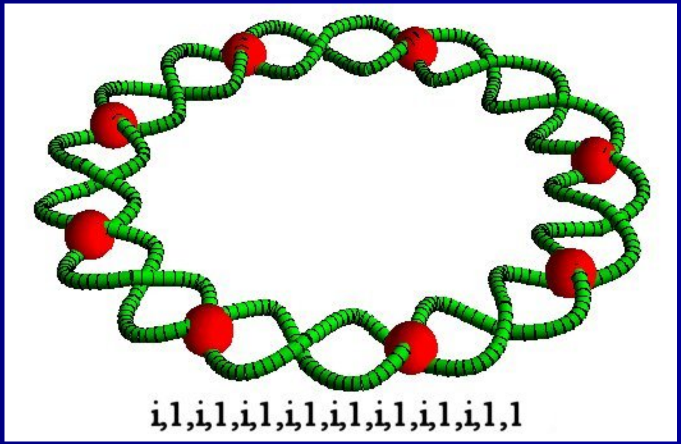
(i,1^2) 6

Bracket undistinguishable





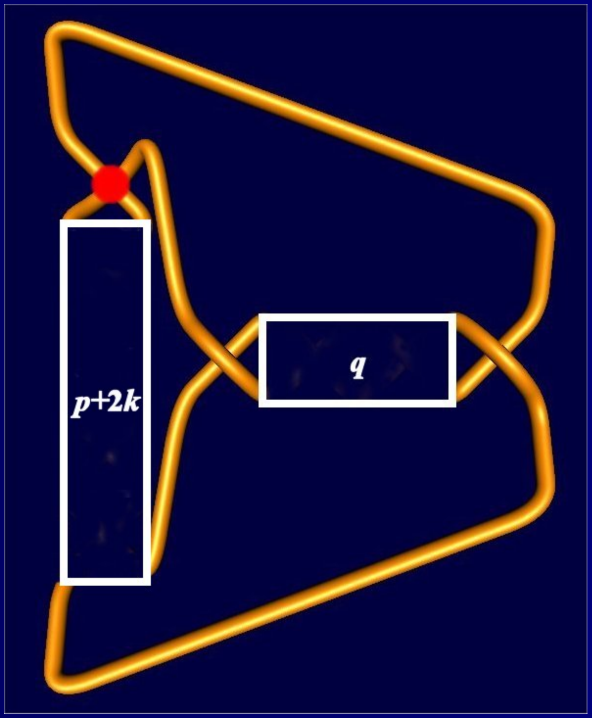


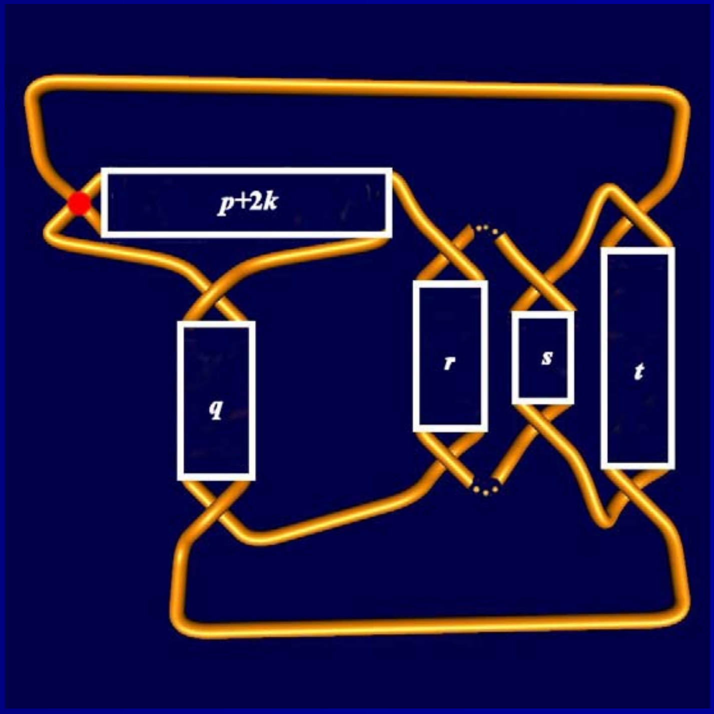


Proposition (L. Kauffman): The bracket polynomial does not detect virtual tangle that consists of a real crossing flanked by two virtual crossings.

Sawolek polynomial detects them. However:

For arbitrary k, Sawolek polynomial does not distinguish KLs $(i,1^{p})\ q$ and $(i,1^{p+2k})\ q$, KLs $(i,1^{p})\ q,r,s,\ldots t$ and $(i,1^{p+2k})\ q,r,s,\ldots,t$, $6^*(1,1^{p})\ q$ and $6^*(1,1^{p})\ (q+2k)$ *etc.* In general, on the basis of tangles $(i,1^{p+2k})\ q$ or $(i,1^{p})\ (q+2k)$ it is possible to construct infinite number of Sawolek undetectable families of *alternating prime* virtual KLs, detectable by bracket.





Definition: A virtual KL is *alternating* if it can be obtained from an alternating KL diagram by virtualization of crossings.

This definition is different from the “standard” definition of alternating virtual KL, where virtual crossings are disregarded (see, e.g., P. Zinn-Justin tables of alternating virtual links).

First fundamental difference between real and virtual KL families: all KLs belonging to the same family of real alternating KLs are detectable even by Alexander polynomial.

There exist families of virtual alternating KLs which are not detectable even by bracket polynomial.

Definition: A link is called prime if in every decomposition into a connected sum, one of the factors is unknotted. Otherwise, the link is called composite.

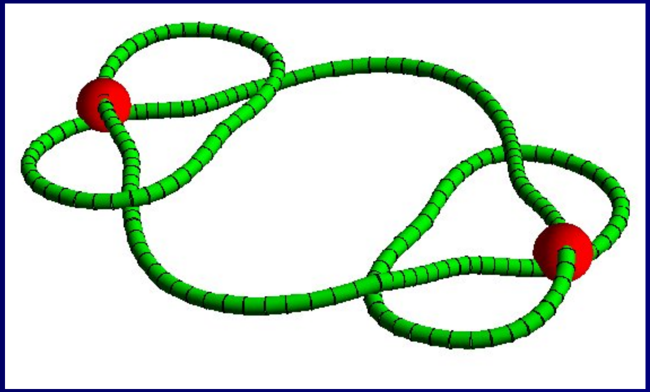
Definition: A virtual link is called prime if it can be obtained from a prime link by virtualization of crossings. Otherwise, the virtual link is called composite.

Composite virtual KLs

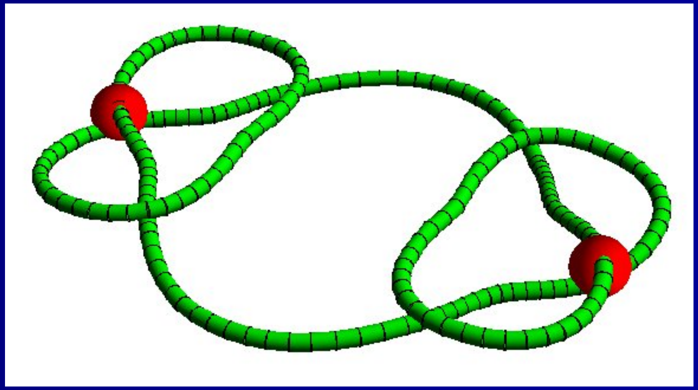
For knots, the following properties hold:

1) if K1=K2, then for any K, K1#K=K_2#K;
2) for any K1, K2, K1#K2=K2#K1 (commutativity);
3) for any K1, K2, K3, (K1#K2)#K3=K1#(K2#K3) (associativity);
4) for any K, K#1=K, where 1 is an unknot (neutral element).

For virtual KLs, composition of links (connected sum or concatenation) is not well defined, since it is positional.
Consequence: Conway notation cannot be used for composite virtual KLs!
Moreover, composition of two virtual unknots can be knotted.
Example: Kishino knot(s).



Kishino

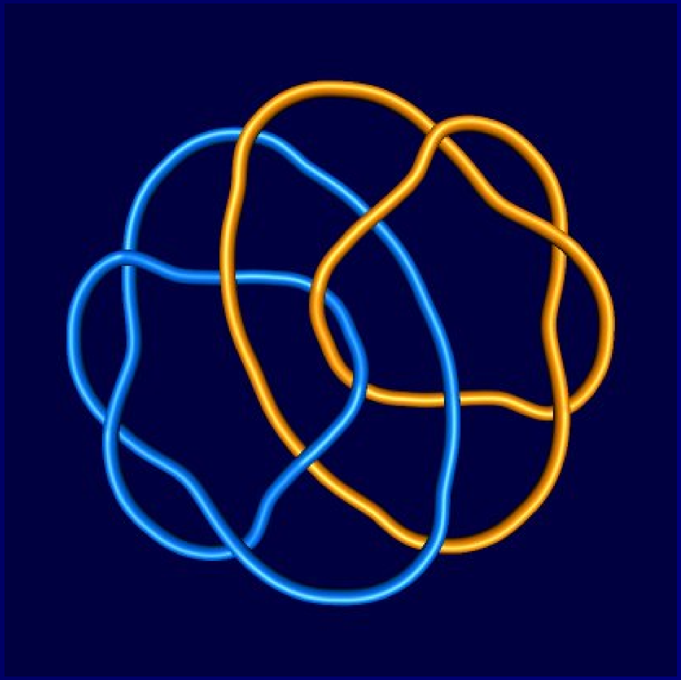


Kishino 1

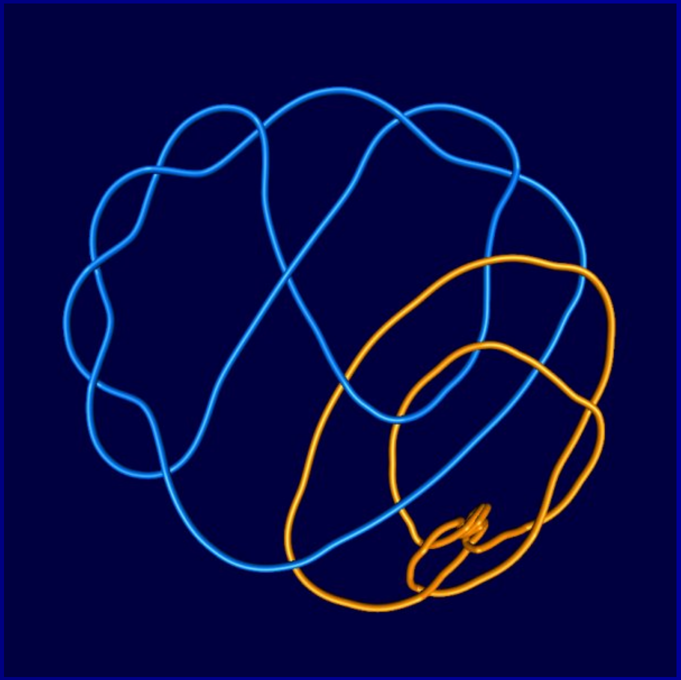
Unlink detectability

Real knots: there exist (infinite "families" of) Jones unlinks (Eliahou, Kauffman, Thistlethwaite, 2003).

OPEN QUESTION: search for Jones unknot.



9*3:-1.-1.2.-1.-1:-3

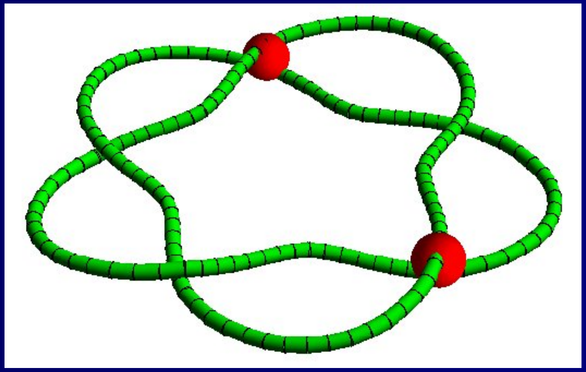


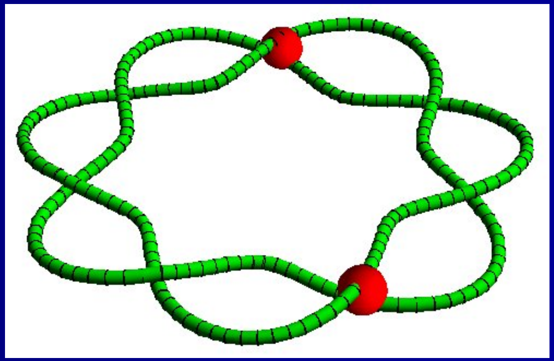
9*5 1 2:-1.-1.2.-1.-1:-5 -1 -2

Unknot detectability of virtual knots

Infinite family of prime virtual knots with trivial bracket polynomial (Jones): i,1^{k},i,-1^{k-1}, beginning from Kauffman example i,1,1,i,-1=i,1^{2},i,-1^{1}, detectable by Sawolek polynomial.

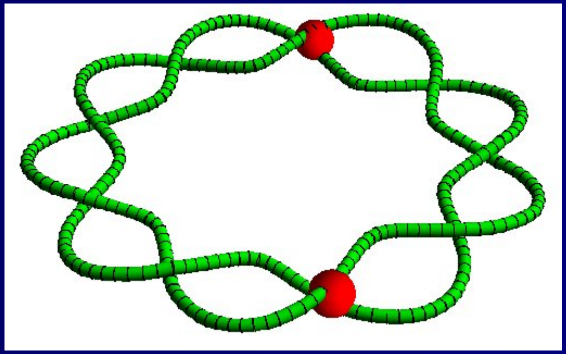
i,1^{2},i,-1

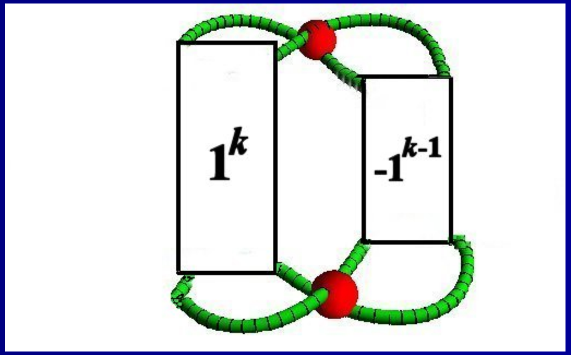




i,1^{3},i,-1^{2}

i,1^{4},i,-1^{3}

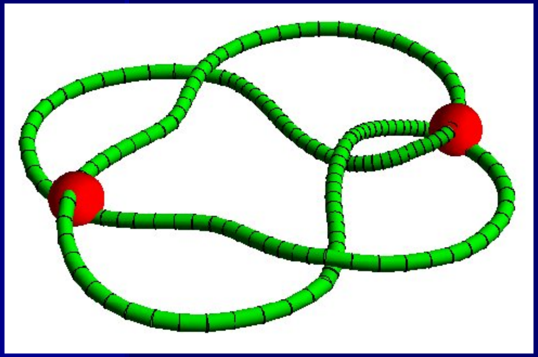




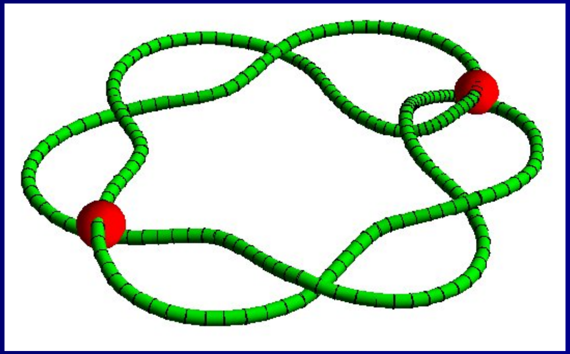
i,1^{k},i,-1^{k-1}

Family of virtual knots with trivial bracket

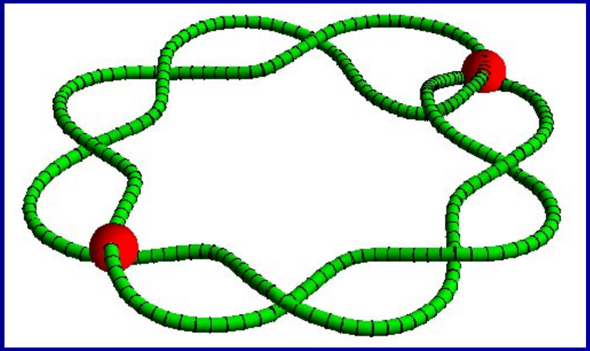
Sawolek polynomial detects them as well.



(1,i,-1) (i,1)



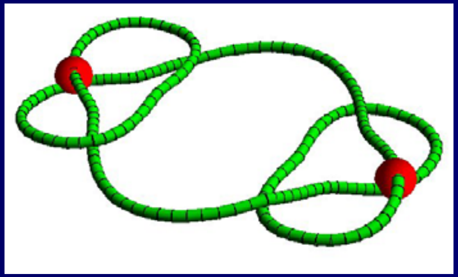
(1^2,i,-1^2) (i,1)



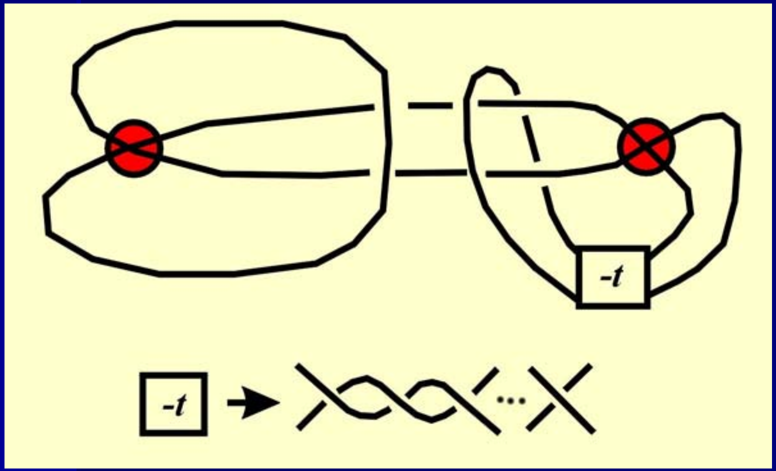
(1^3,i,-1^3) (i,1)

Composite unknot-undetectable virtual knots

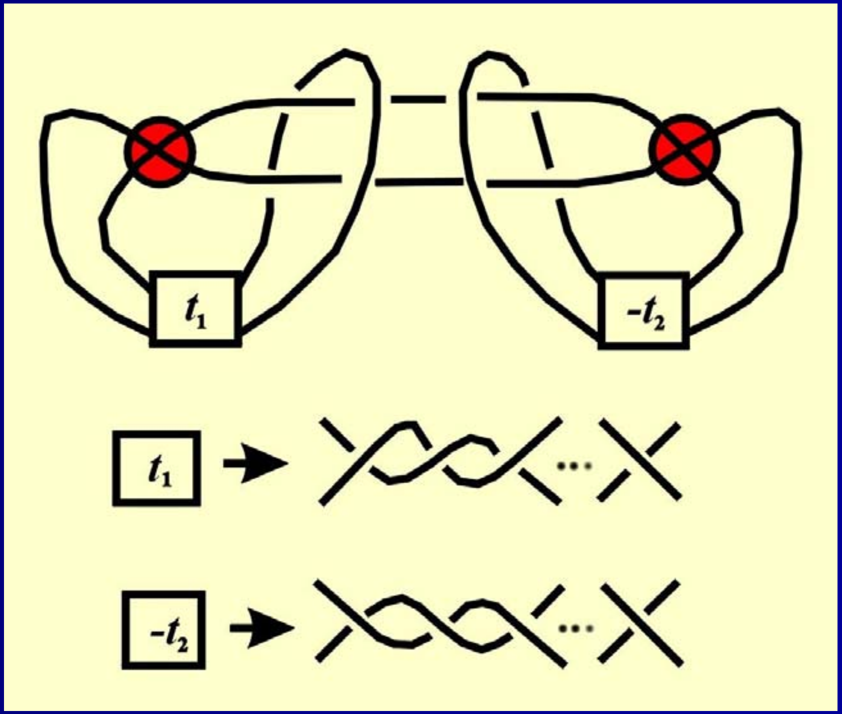
Families of composite virtual knots, beginning from Kishino knot (H. Dye), all (?) detectable by Miyazawa polynomial, 3-cabled Jones and Kauffman arrow polynomial



Kishino



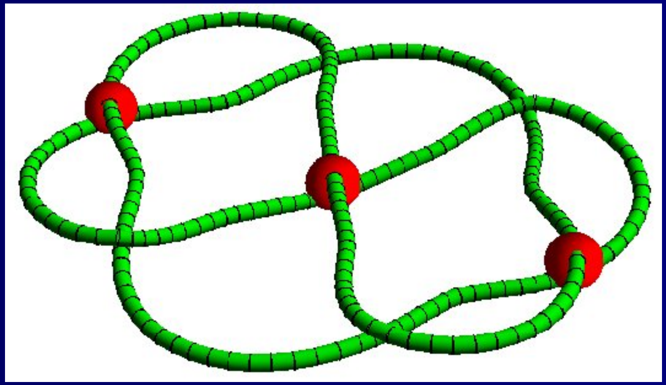
Dye B



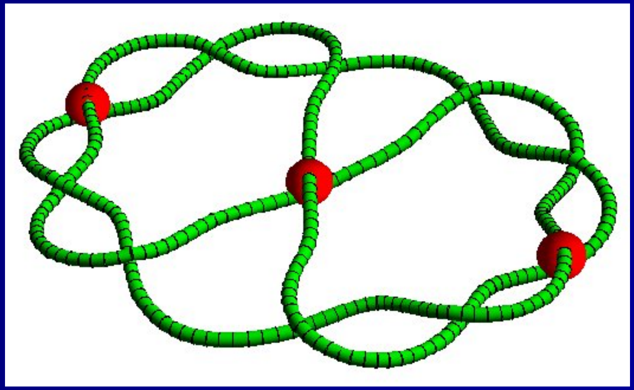
Dye A: generalization t_1, t_2

New unknot-undetectable virtual knot family

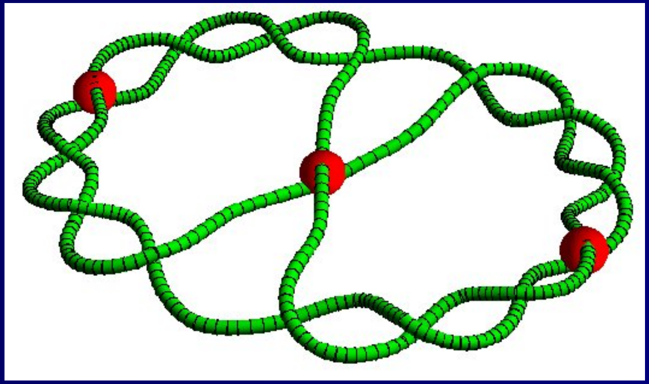
Family of prime virtual knots $(1^k,i,-1^k)$ i $(1^k,i,-1^k)$ beginning from knot (1,i,-1) i (1,i,-1), undetectable by all polynomials except 2-colored Jones.



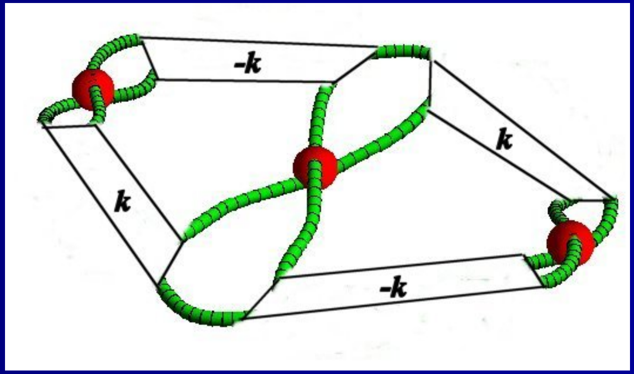
(1,i,-1) i (1,i,-1)



$(1^2,i,-1^2)$ i $(1^2,i,-1^2)$



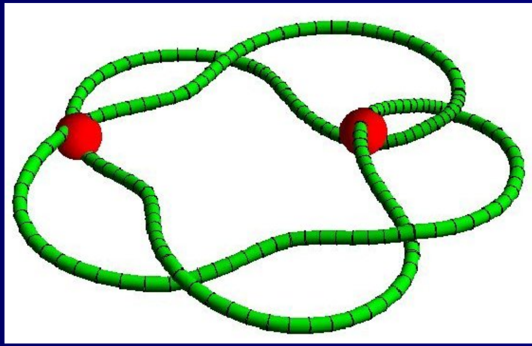
$(1^3,i,-1^3)$ i $(1^3,i,-1^3)$



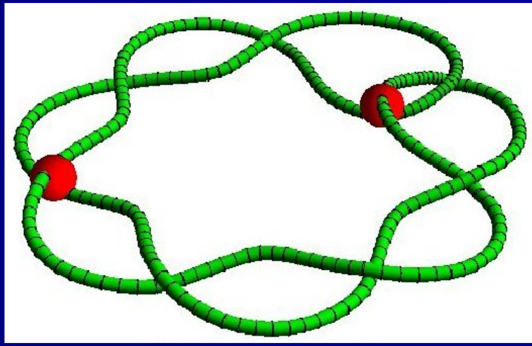
$(1^k,i,-1^k)$ i $(1^k,i,-1^k)$

Another undetectable family

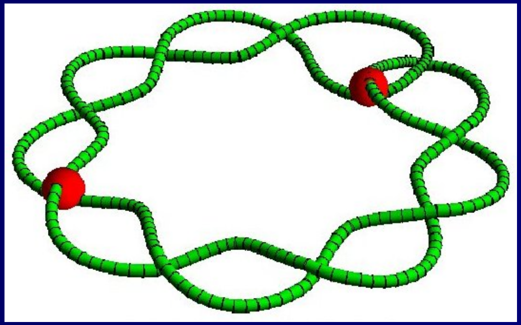
Family of prime virtual knots (1^{k+1},i,-1^{k}) i (1,i) beginning from knot (1,1,i,-1) (1,i), undetectable by all polynomials except 2-colored Jones.



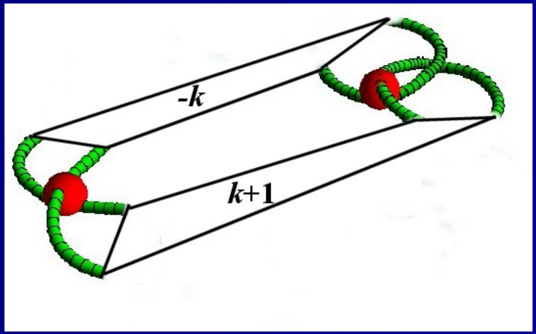
(1,1,i,-1) (1,i)



(1^3,i,-1^2) (1,i)

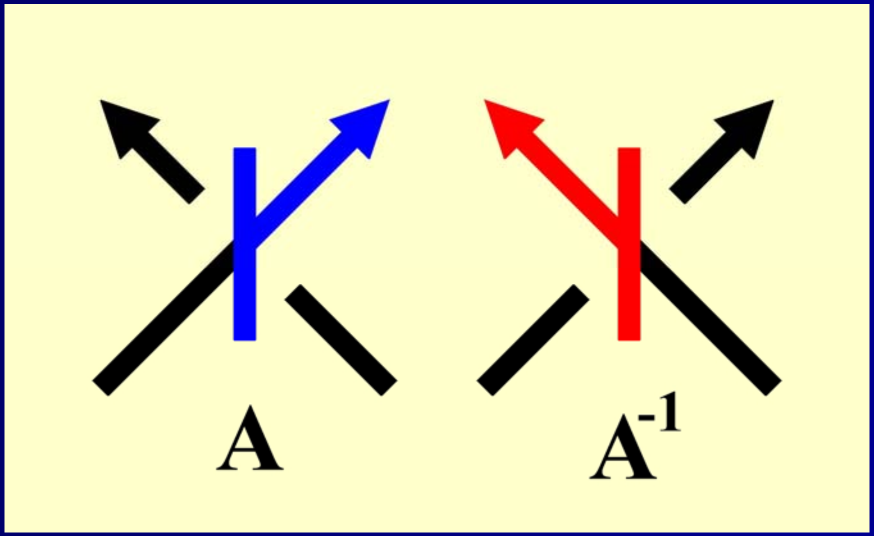


(1^4,i,-1^3) (1,i)



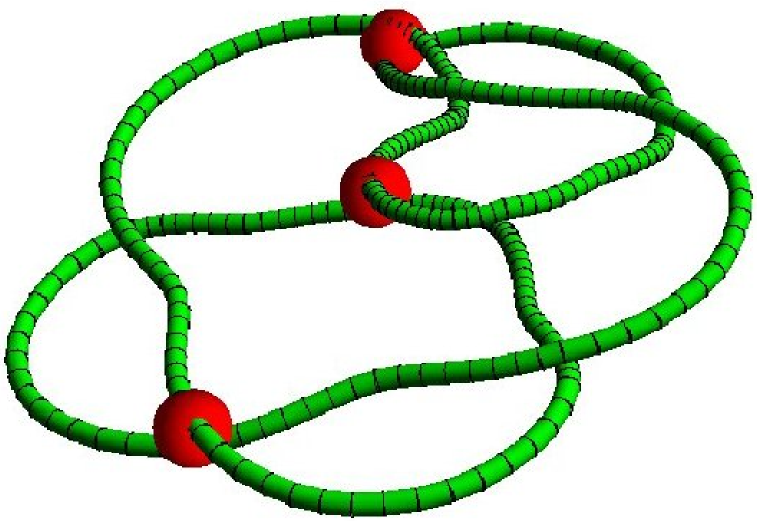
(1^{k+1},i,-1^{k}) (1,i)

Kauffman arrow polynomial



Markers in a crossing

Kauffman arrow polynomial



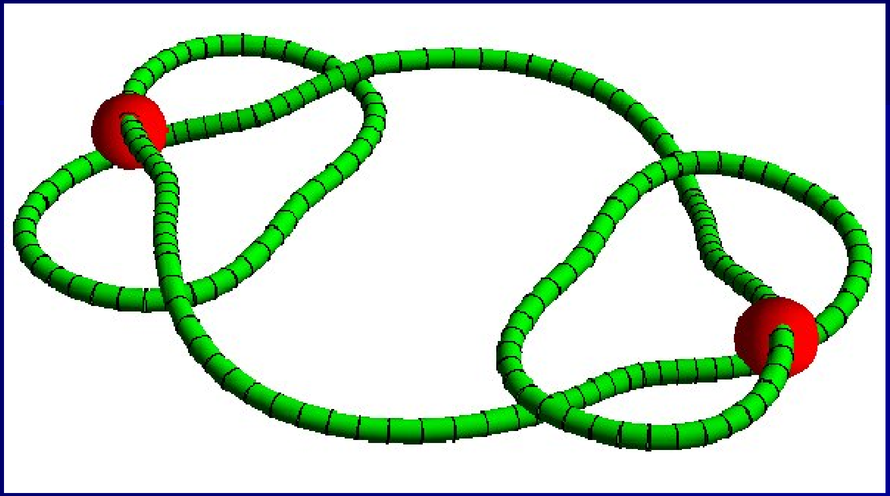
(1,i) -1,(-1,i,1),(i,-1)

```
In[25]:= K = "(1,i) -1,(-1,i,1),(i,-1)";
         fAlexVirt[K]
         fJonesVirt[K]
         fSawollek[K]
         fMiyazawaPoly[K]
         fKauffmanExtendedBracket[K]

Out[26]= 0

Out[27]= 1

Out[28]= 0

Out[29]= {1, 0}
```

Out[30]= $\left\{A - \frac{2\,K1^2}{A^3} - 2\,A\,K1^2 + \frac{K2}{A^3} + A\,K2,\ 1 - 4\,K1^2 + 2\,K2\right\}$

Virtual knot undetectable by all polynomials except 2-cabled Jones, but detectable by Kauffman arrow polynomial

Kishino knot



(-1,i,1)#(1,i,-1)

```
In[21]:= K = "(-1,i,1)#(1,i,-1)";
         fKauffmanExtendedBracket[K]
```

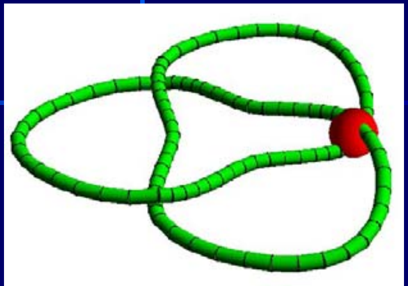
Out[22]= $\left\{1 + \frac{1}{A^4} + A^4 - 2\,K1^2 - \frac{K1^2}{A^4} - A^4\,K1^2 + 2\,K2,\ 3 - 4\,K1^2 + 2\,K2\right\}$

Virtual KLs

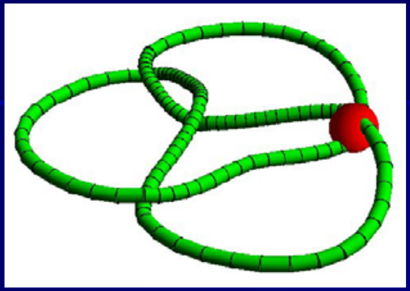
- Tables of virtual knots:
- T. Kishino
- Jeremy Green (PD), tables of virtual knots with at most n=4 crossings
- Naoko Kamada (Kamada codes ~ Gauss codes), tables of virtual knots with at most n=4 crossings
- P.Zinn-Justin: tables of alternating virtual links with at most n=8 crossings (without notation)
- Jablan-Sazdanovic: tables of prime virtual

  knots and links in Conway notation derived from knots and links with at most n=8 crossings = 2607 prime virtual knots and 3687 virtual links

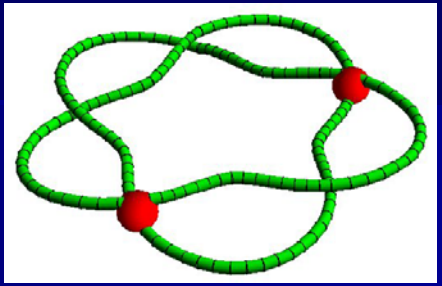
Tables of virtual knots in Conway notation



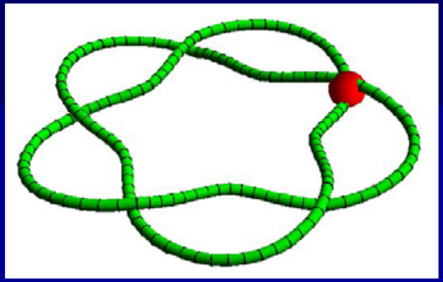
1,1,i



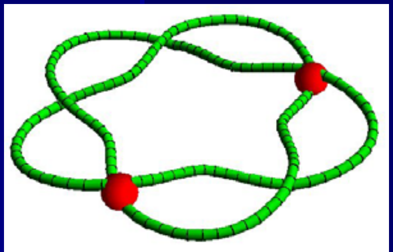
(1,1) (1,i)



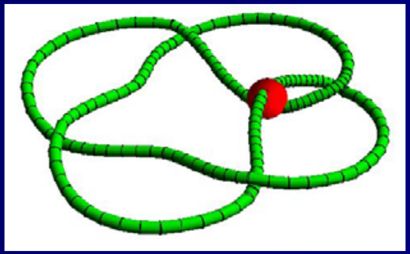
1,1,i,-1,i



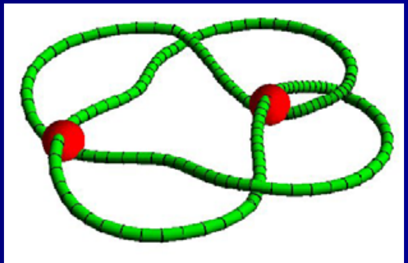
1,1,1,1,i



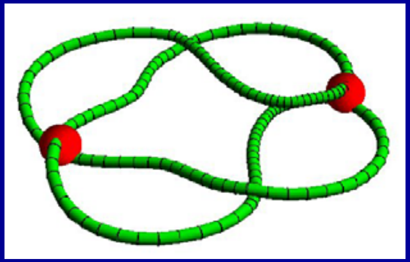
(1,1,i,1,i)



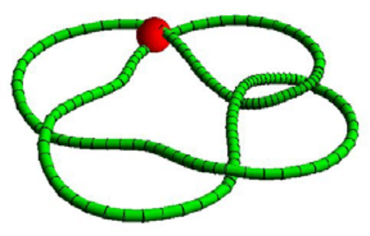
(1,1,1) (1,i)



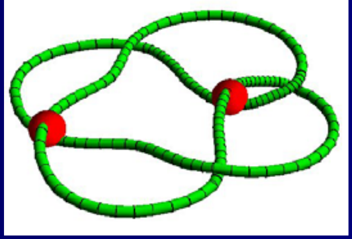
(1,i,1) (1,i)



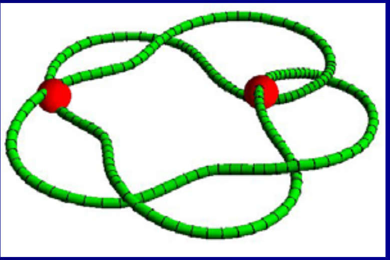
(1,i,1) (i,-1)



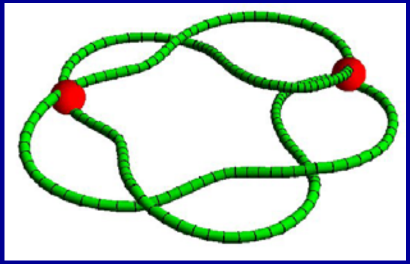
(1,1,i) (1,1)



(1,i,-1) (1,i)

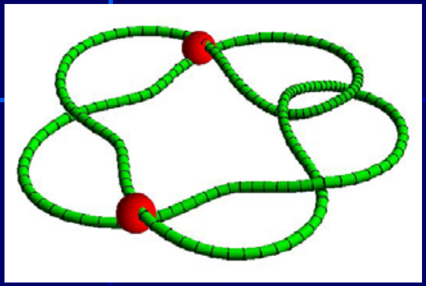


(1,1,i,-1) (1,i)

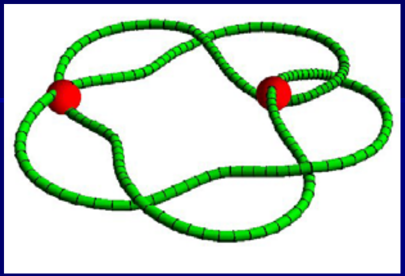


(1,1,i,-1) (i,-1)

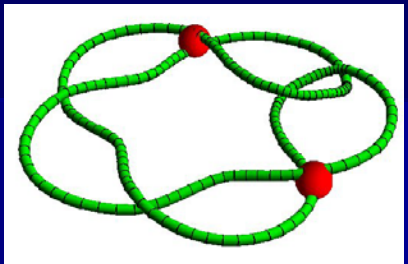
Tables of virtual knots in Conway notation



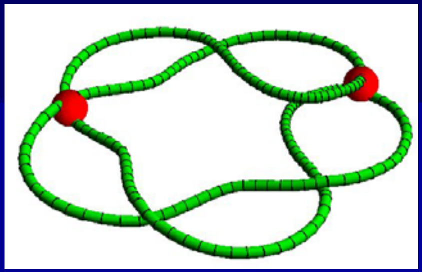
(1,i,-1,i) (1,1)



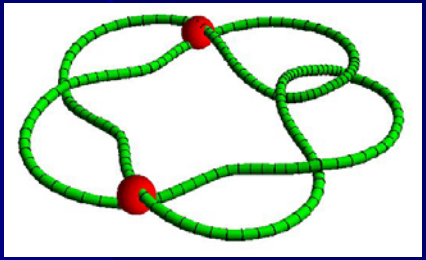
(1,1,i,1) (1,i)



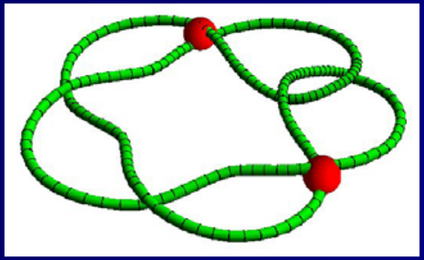
(i,1,1,i) (-1,-1)



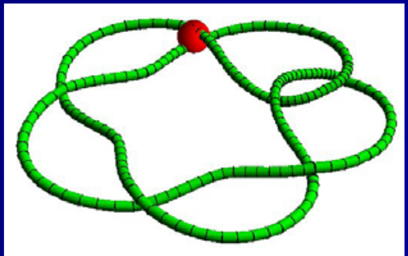
(1,1,i,1) (i,-1)



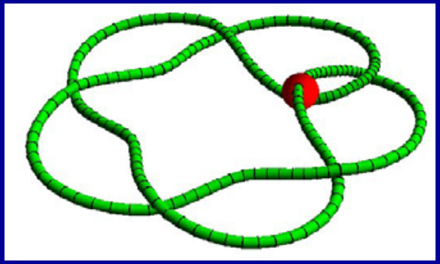
(1,i,1,i) (1,1)



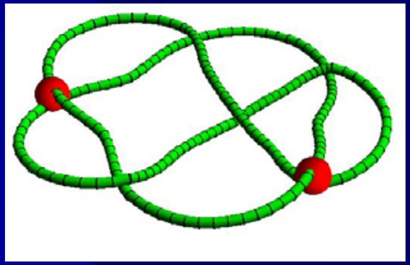
(i,1,1,i) (1,1)



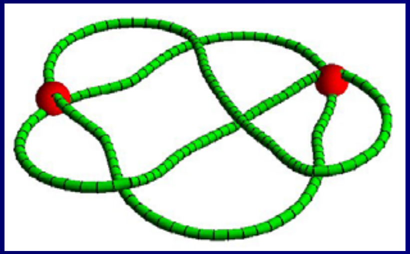
(1,1,1,i) (1,1)



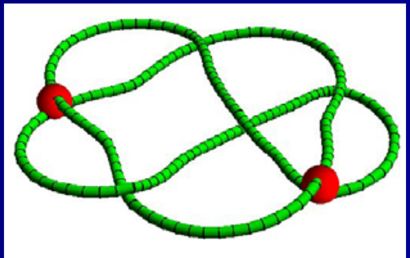
(1,1,1,1) (1,i)



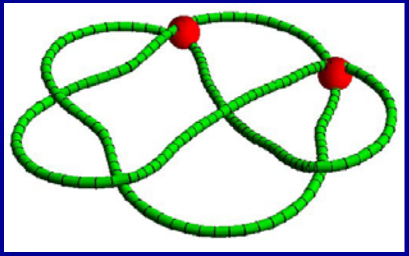
(1,i,-1) 1 (1,i)



(1,i,-1) 1 (i,-1)

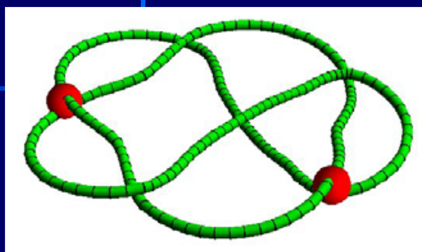


(1,i,-1) 1 (-1,i)

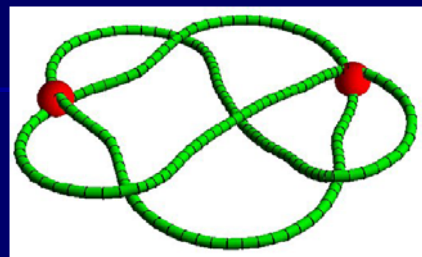


(1,1,i) -1 (i,-1)

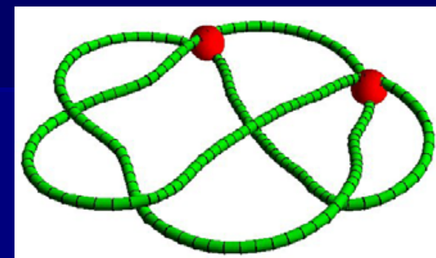
Tables of virtual knots in Conway notation



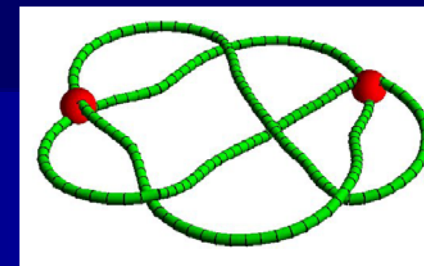
(1,i,1) -1 (1,i)



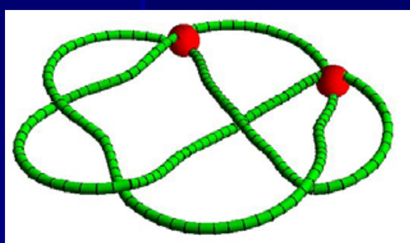
(1,i,1) -1 (i,-1)



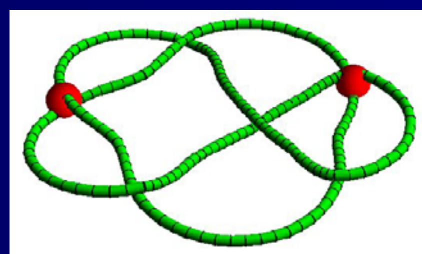
(1,1,i) -1 (i,1)



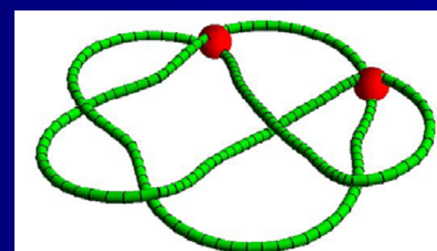
(1,i,-1) 1 (i,1)



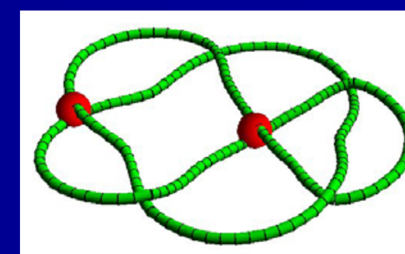
(1,1,i) 1 (i,1)



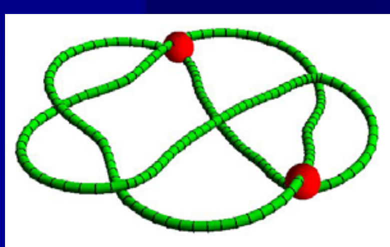
(1,i,1) 1 (i,-1)



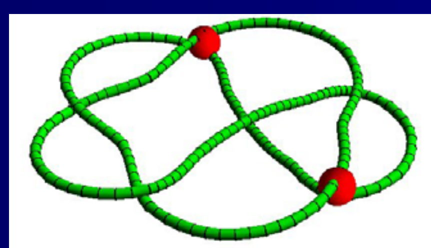
(1,1,i) 1 (i,-1)



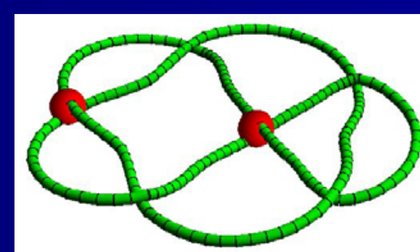
(1,i,-1) i (1,1)



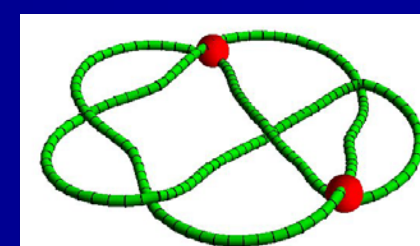
(1,1,i) -1 (1,i)



(1,1,i) -1 (-1,i)

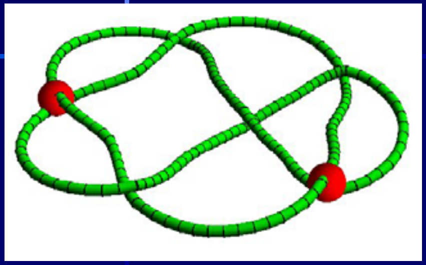


(1,i,1) i (1,1)

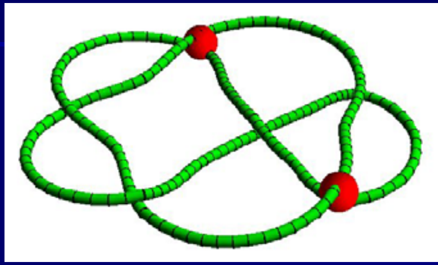


(1,1,i) 1 (1,i)

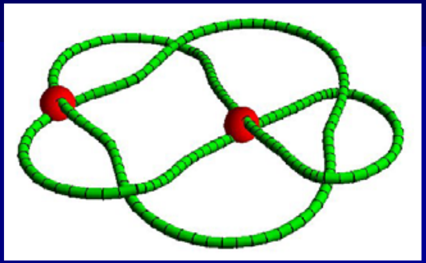
Tables of virtual knots in Conway notation



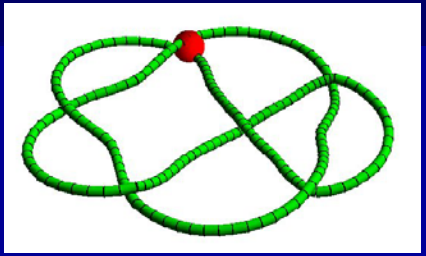
(1,i,1) 1 (1,i)



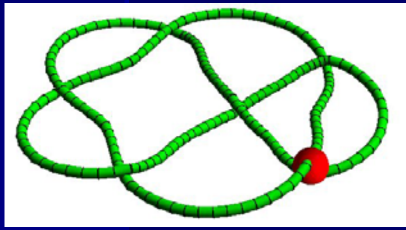
(1,1,i) 1 (-1,i)



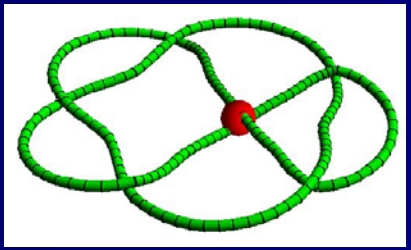
(1,i,1) i (-1,-1)



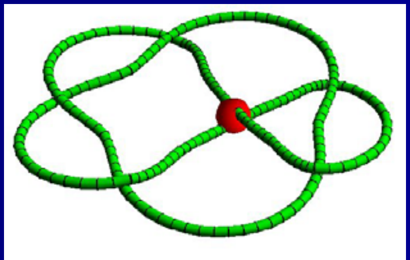
(1,1,i) 1 (1,1)



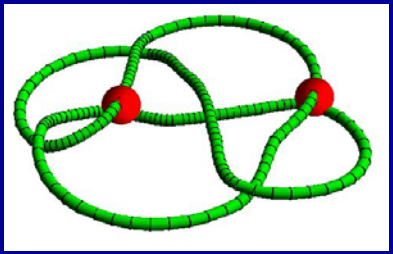
(1,1,1) 1 (1,i)



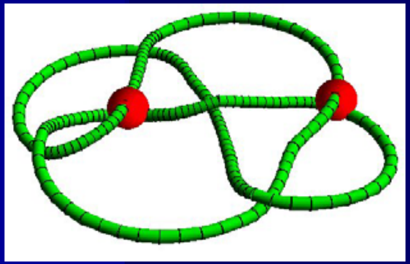
(1,1,1) i (1,1)



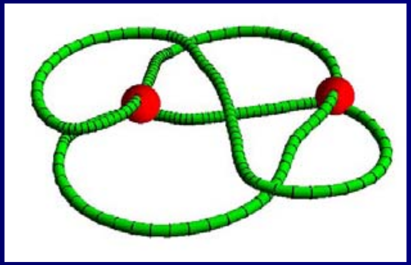
(1,1,1) i (-1,-1)



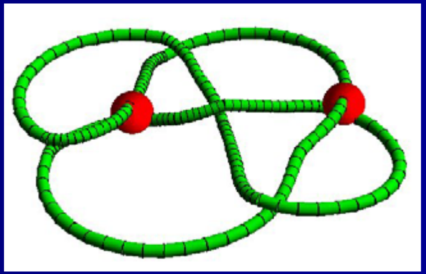
(1,i) -1 -1 (i,1)



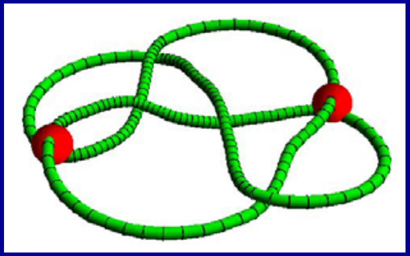
(1,i) 1 -1 (i,1)



(1,i) -1 1 (i,-1)

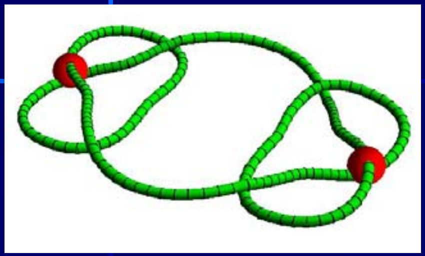


(1,i) 1 1 (i,-1)

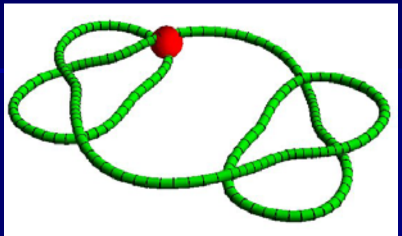


(1,i) -1 -1 (-1,i)

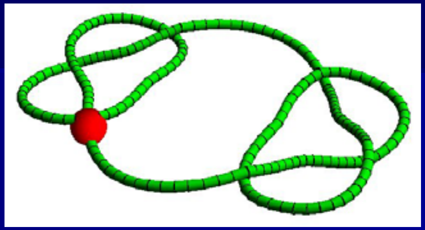
Tables of virtual knots in Conway notation



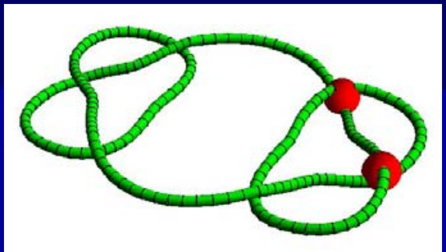
(1,i,1)#(1,i,-1)



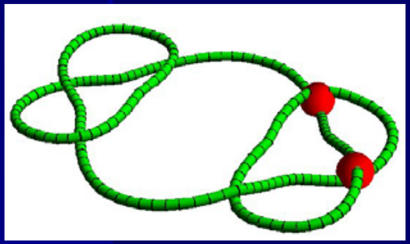
(1,1,1)#(i,-1,-1)



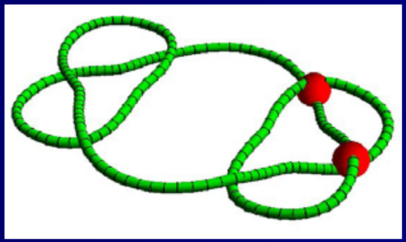
(1,1,1)#(1,1,i)



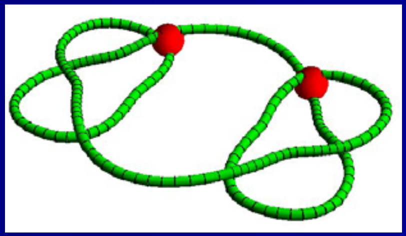
(1,1,i)#(1,i,-1)



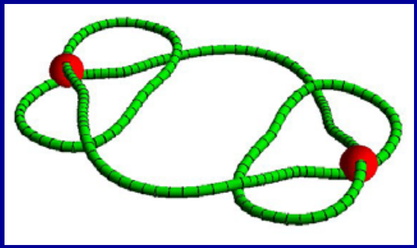
(1,1,i)#(-1,i,1)



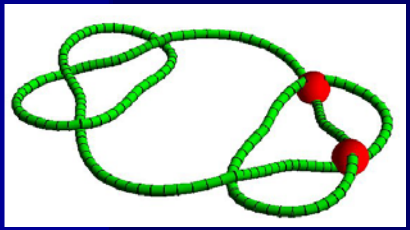
(1,1,i)#(-1,i,-1)



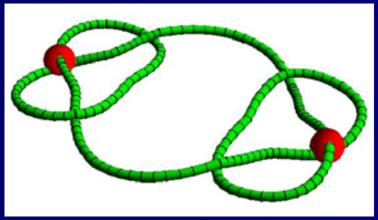
(1,1,i)#(i,-1,-1)



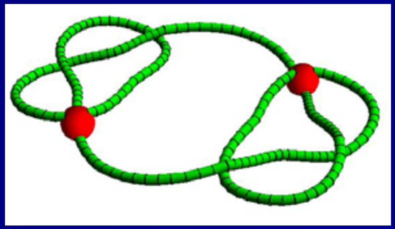
(1,i,1)#(-1,i,-1)



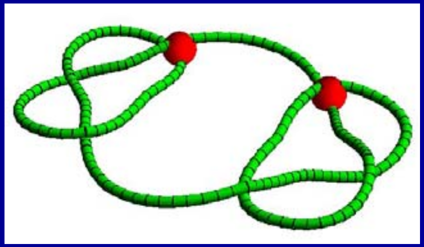
(1,1,i)#(1,i,1)



(1,1,i)#(1,i,1)

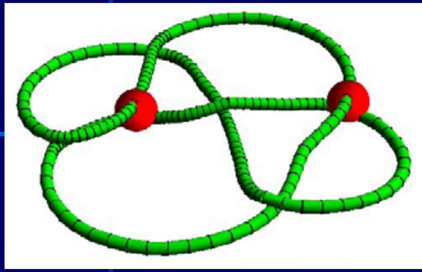


(1,1,i)#(1,1,i)

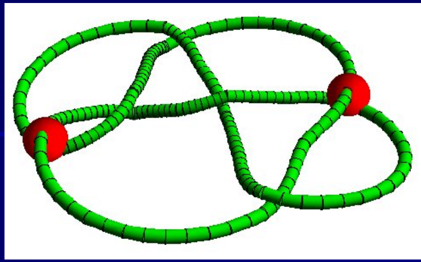


(1,1,i)#(i,1,1)

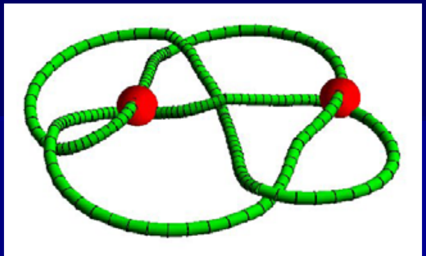
Tables of virtual knots in Conway notation



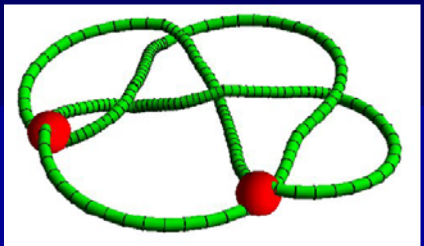
(1,i) 1 -1 (i,-1)



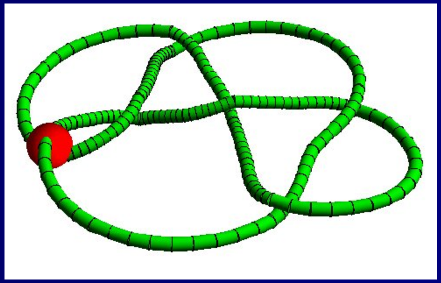
(1,i) 1 1 (1,i)



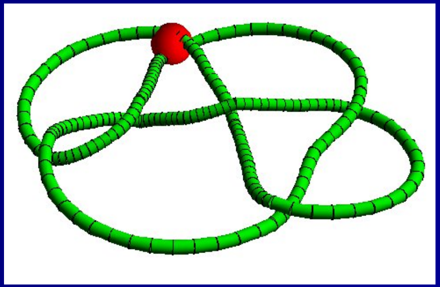
(1,i) 1 1 (i,1)



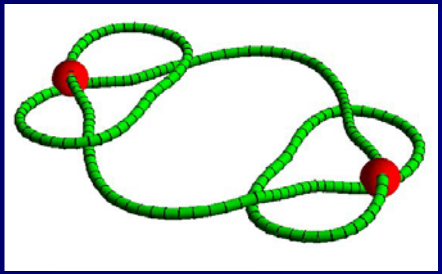
(i,1) 1 1 (1,i)



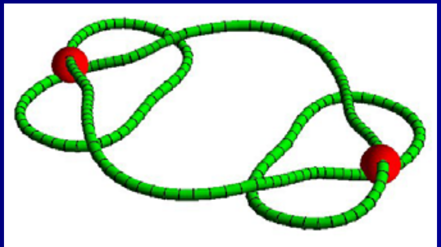
(1,1) 1 1 (1,i)



(1,1) 1 i (1,1)



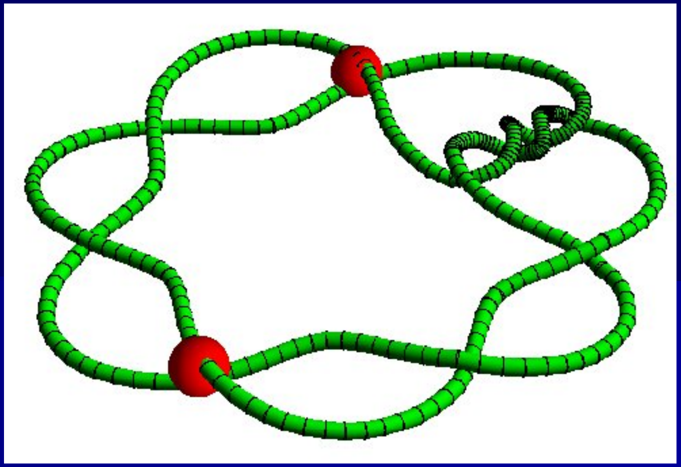
(1,i,-1)#(-1,i,1)



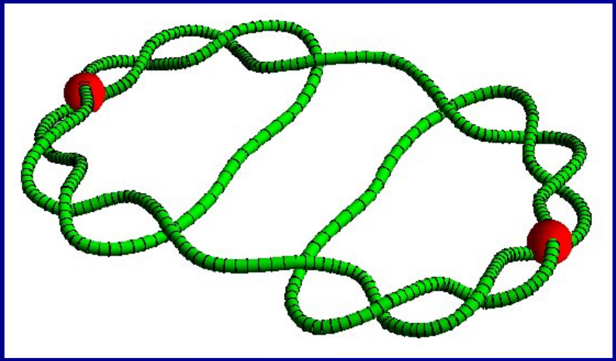
(1,i,-1)#(1,i,-1)

Families of undetectable knots

1,1,i,-1,i	1^k,i,-1^{k-1},i
(1,i,-1) (1,i)	(1^k,i,-1^k) (1,i)
(1,1,i,-1) (1,i)	(1^k,i,-1^k) (1,i)
(1,i,-1,i) (1,1)	(1^k,i,-1^k,i) (1^{2m})
(1,i,-1) 1 (1,i)	(1^k,i,-1^k) 1 (1,i)
(1,i,-1) 1 (i,1)	(1^k,i,-1^k) 1 (i,1)
(1,i,-1) i (1,1)	(1^k,i,-1^k) i (1,1)
(i,1) -1 -1 (1,i)	(-1^k,i,1^k) -1 -1 (1^m,i,-1^m)
(1,i,-1) (1,i,-1)	(1^k,i,-1^k) (1^k,i,-1^k)
(1,i,-1) (-1,i,1)	(1^k,i,-1^k) (-1^k,i,1^k)

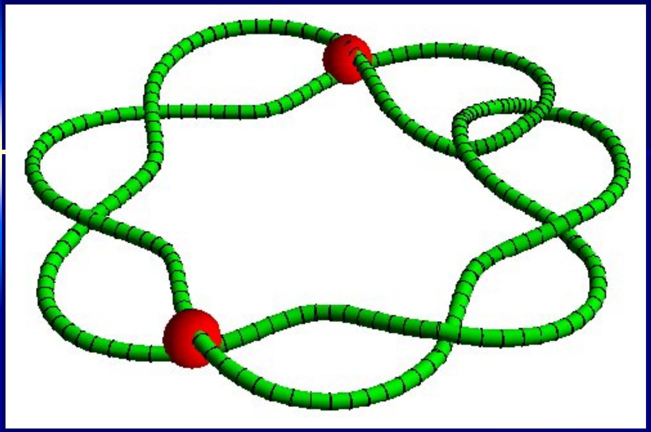


Subfamily (1^{2k},i,-1^{2k},i) (1^{2m}) with all trivial polynomials except 2-colored Jones (example k=1,m=2)



(1^k,i,-1^k) (1^k,i,-1^k)

Knot (1,1,i,-1,-1,i) (1,1) undetectable by arrow polynomial



Extended bracket and arrow bracket for this knot and all knots from the family (1^{2k},i,-1^{2k},i) (1^{2m}) reduces to classical bracket. This family can be distinguished from unknot only by 2-cabled Jones polynomial.

The same property holds for knots:

(1,1) (1,i) i i (1,1)

(1,1,i,-1,-1,i) (-1,-1)

(1,i,-1,-1,i) (1,1)

(1,1) (1^{k+1},i,-1^{k}) i i (1,1) (k=0,1,…)

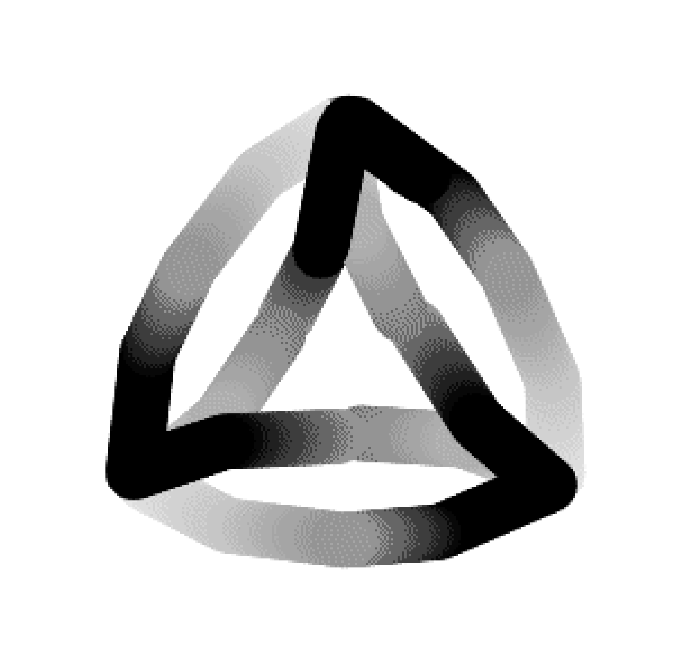
(1^{2k},i,-1^{2k},i) (-1^{2m})

(1^{2k-1},i,1^{2k+1},i) (1,1)

Virtual knots with trivial polynomials

- Among 55 prime virtual knots derived from knots with at most n=6 crossings, 8 of them have trivial bracket. Each of them is extendable to a family of undetectable virtual knots.
- From 2607 prime virtual knots derived from knots with at most n=8 crossings, 331 (about 12%) have trivial bracket. Among these 331 knots, 21 have trivial Sawollek polynomial, and 12 knots have all trivial polynomials including Miyazawa polynomial. From them, Kauffman arrow polynomial detects 5, and remaining 7 knots are undetectable by Kauffman arrow polynomial as well. All of them have non-trivial 2-cabled Jones.
- Conjecture: 2-cabled Jones polynomial detects all undetectable prime virtual knots. 3-cabled Jones detects all undetectable composite knots (Dye).

Unlinking number and Bernhard-Jablan Conjecture



Definition:

The *unlinking number* *u* (*L*) of a link *L* is the minimal number of crossing changes required to obtain an unlink from the link *L*. The minimum is taken over all projections of *L*.

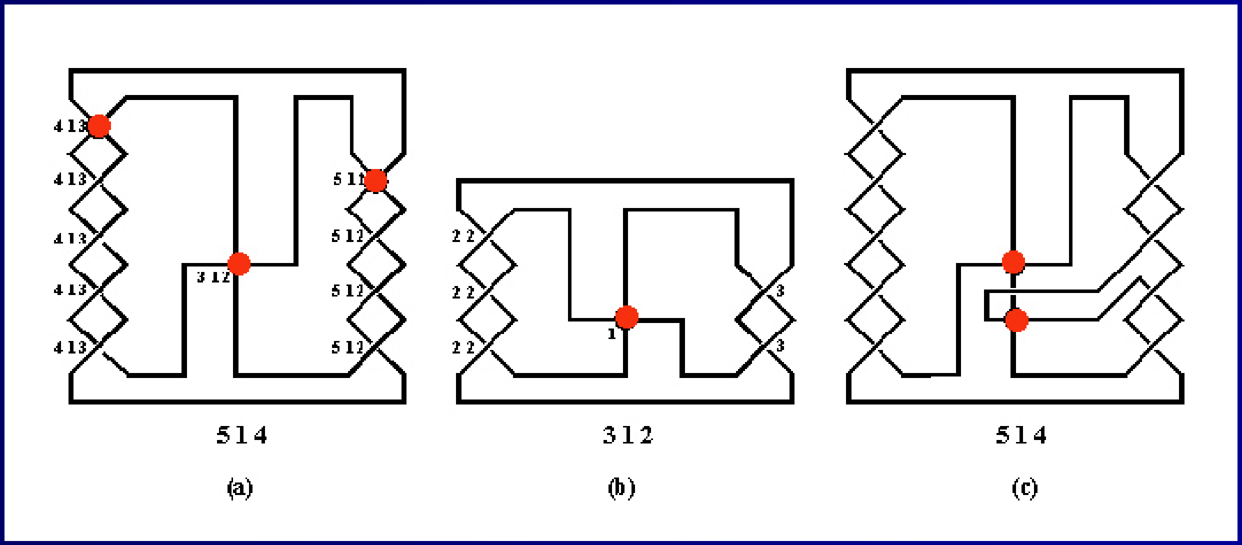
CLASSICAL DEFINITION

it is allowed to make an ambient isotopy after each crossing change and then continue with the unlinking process with the newly obtained projection

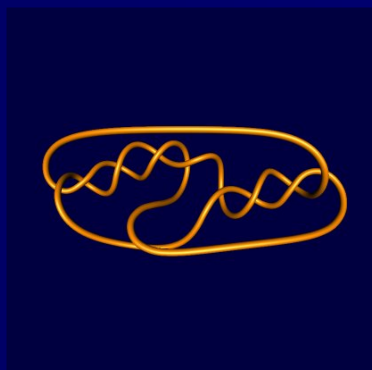
STANDARD DEFINITION

all crossing changes must be done simultaneously in a fixed projection

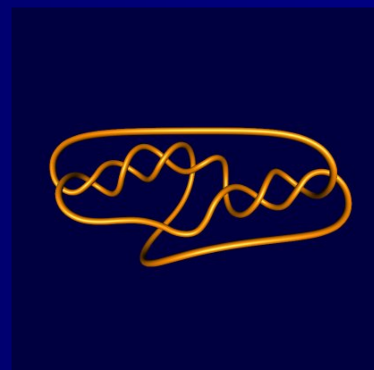
If in the standard definition we restrict “all projections” to “all minimal projections”, we cannot always obtain correct unlinking number (Nakanishi-Bleiler example).



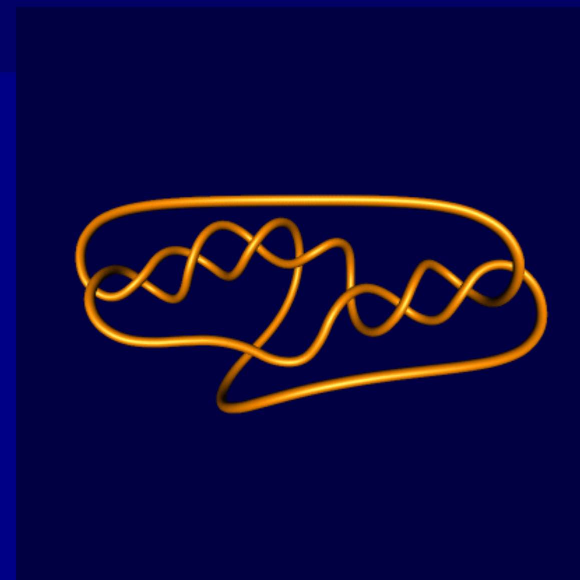
Unknotting 5 1 4 (10_8)

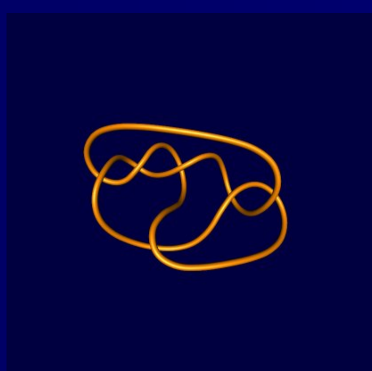


5 1 4



5 -1 4





3 1 2

3 -1 2

BERNHARD- JABLAN CONJECTURE

1. $u(L) = 0$, where L is the unknot (unlink)
2. $u(L) = \min u(L^-)+1$, where the minimum is taken over all minimal projections of links L^-, obtained from a minimal projection of L by one crossing change.

Algorithm:
Start with a minimal projection of KL
Make a crossing change
Minimize KL obtained
Repeat steps 2&3 for all vertices of a starting KL
Repeat steps 1-4 for all newly obtained KLs until unknot is obtained.

BJ-unlinking number is the number of steps in this recursive unlinking process.

BJ-conjecture: BJ-unlinkig number of every link L is equal to its unlinking number.

BJ-unlinking numbers for virtual links

All unlinking numbers are computed from minimal diagrams, where after every crossing change *KL* diagram is minimized.

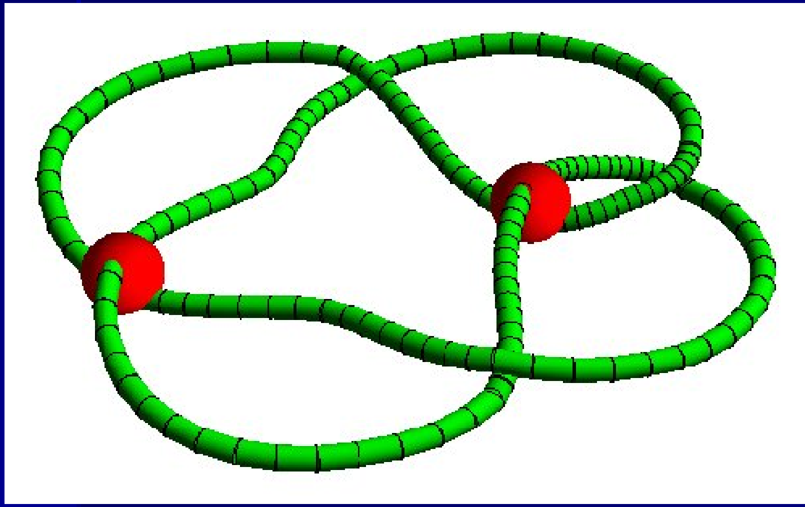
Crossing changes:
1) virtualization of a crossing;
2) real crossing change.

Definition:
Virtual BJ-unlinking number vu_{BJ} is the minimal number of vertex virtualizations (crossing changes 1) necessary to obtain unlink.
Mixed BJ-unlinking number mu_{BJ} is the minimal number of virtualizations and real crossing changes (crossing changes 1 and 2) necessary to obtain unlink.
Real BJ-unlinking number ru_{BJ} is the minimal number of real crossing changes (crossing changes 2) necessary to obtain unlink.

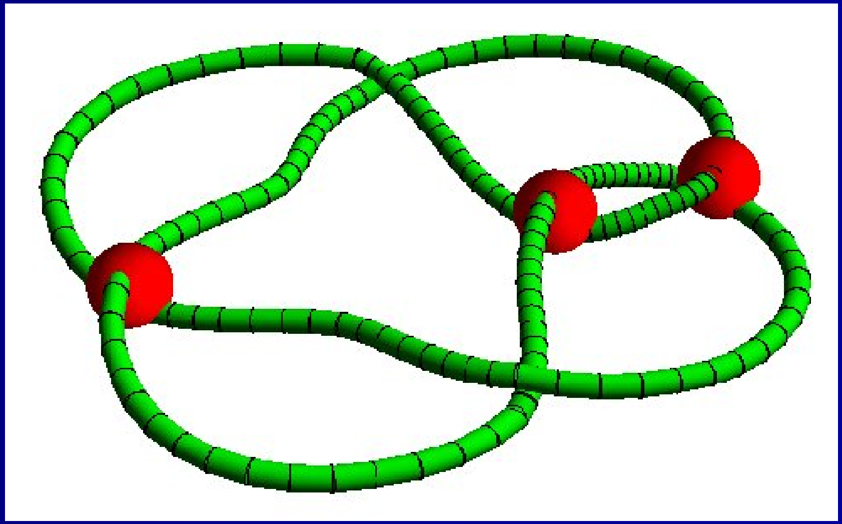
For real knots all three unlinking numbers are finite, but for virtual links the real unlinking number is not always finite, because real crossing change is not necessarily an unlinking operation for virtual links.

Unlinking numbers of virtual KLs

Example: virtual knot with vu_{BJ}=mu_{BJ}=1, and infinite ru_{BJ}.

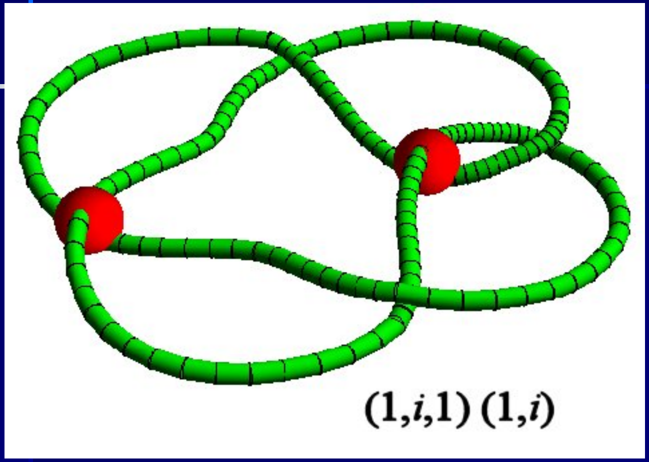


(1,i,1) (i,1)

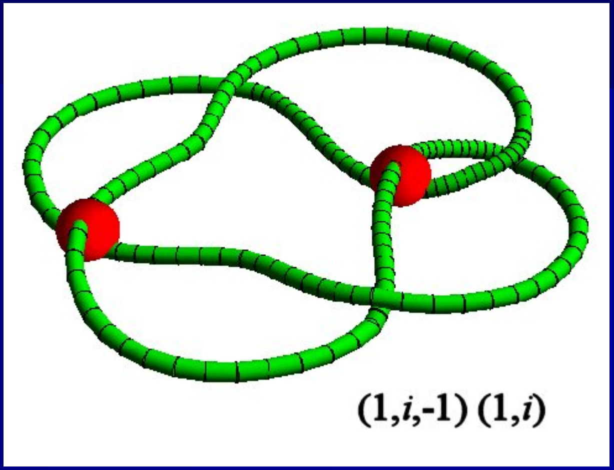


(1,i,1) (i,i) = unknot

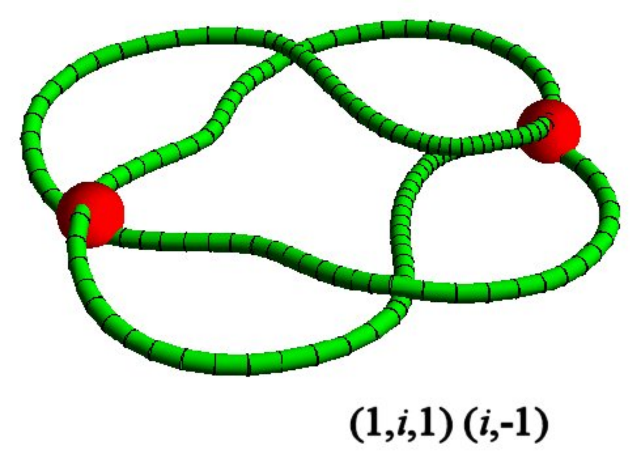
Infinite real unknotting number



(1,*i*,1) (1,*i*)

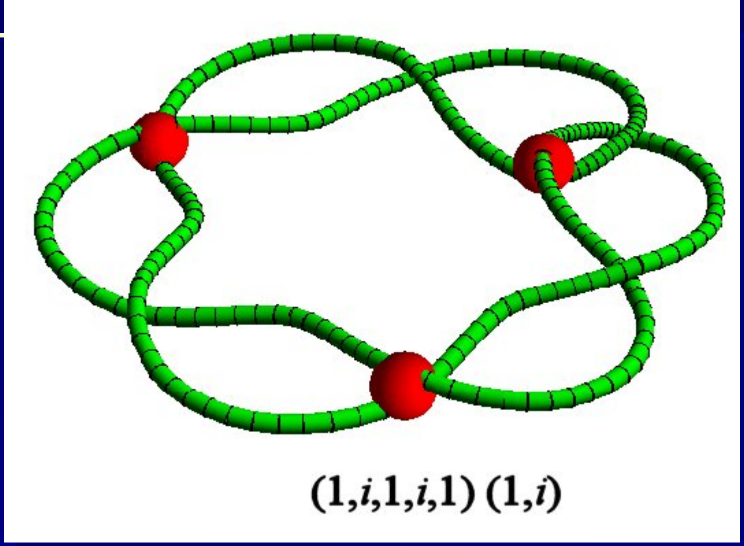


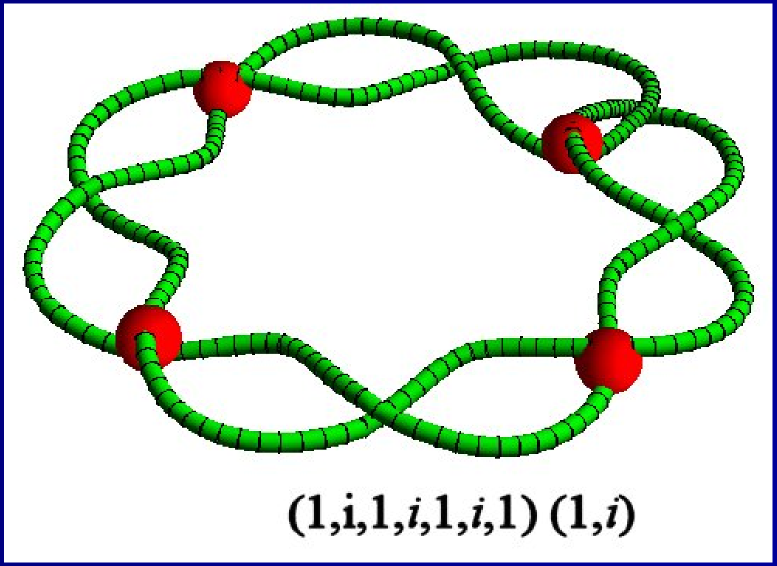
(1,*i*,-1) (1,*i*)



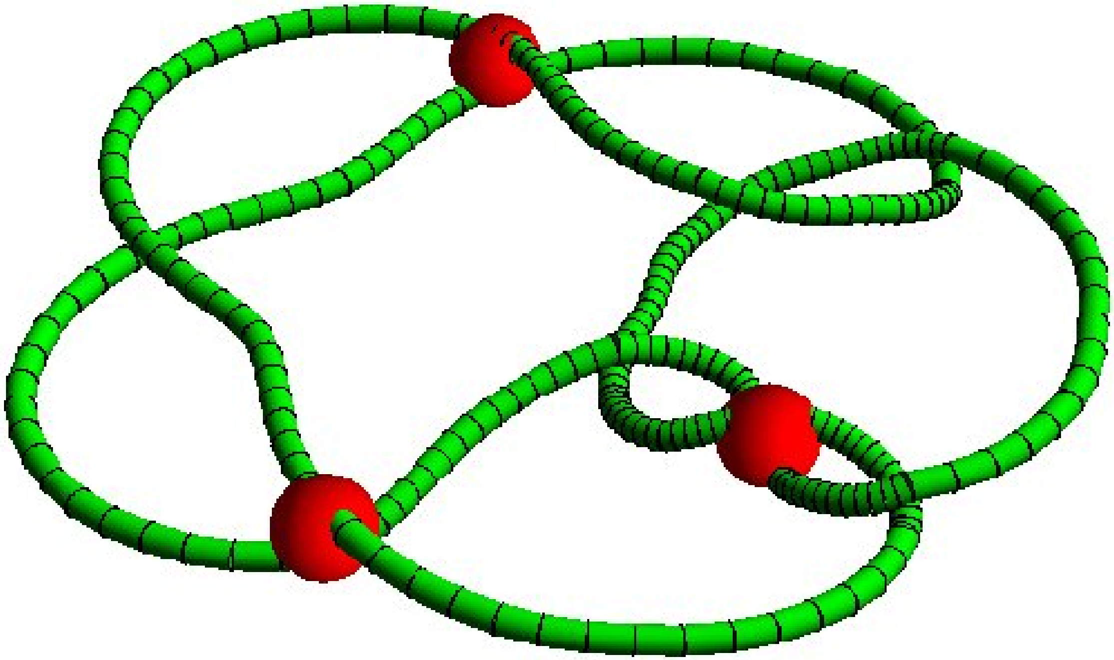
(1,*i*,1) (*i*,-1)

Family of virtual knots with infinite unknotting number



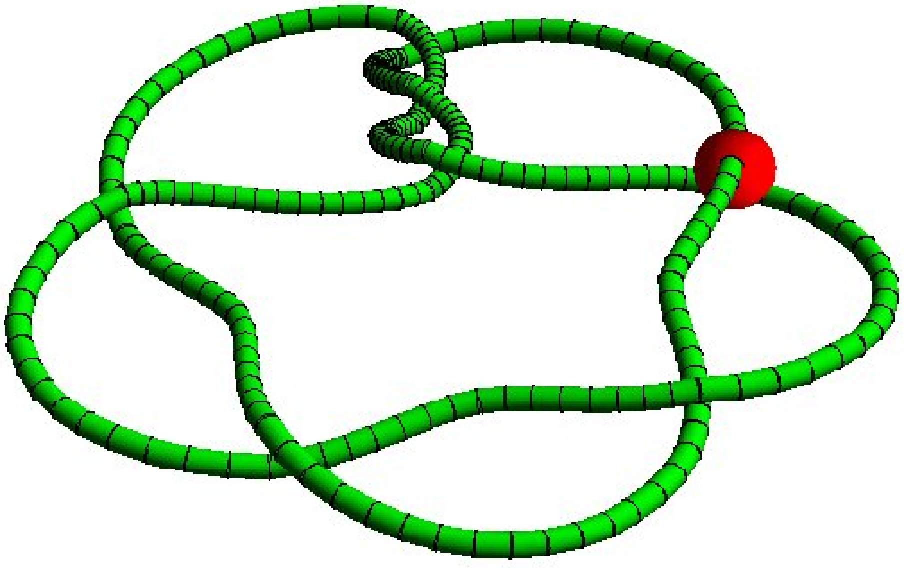


Nakanishi-Bleiler for virtual unknotting number: fixed projection can be unknotted with ate least 3 virtual crossing changes, and vu_{BJ}=2



(1,*i*,1) (*i*,-1,*i*) (-1,-1)

Virtual knot with $vu_{BJ}=ru_{BJ}=3$, and mixed unknotting number $mu_{BJ}=2$



(1,1,1,*i*) (1,1,1)

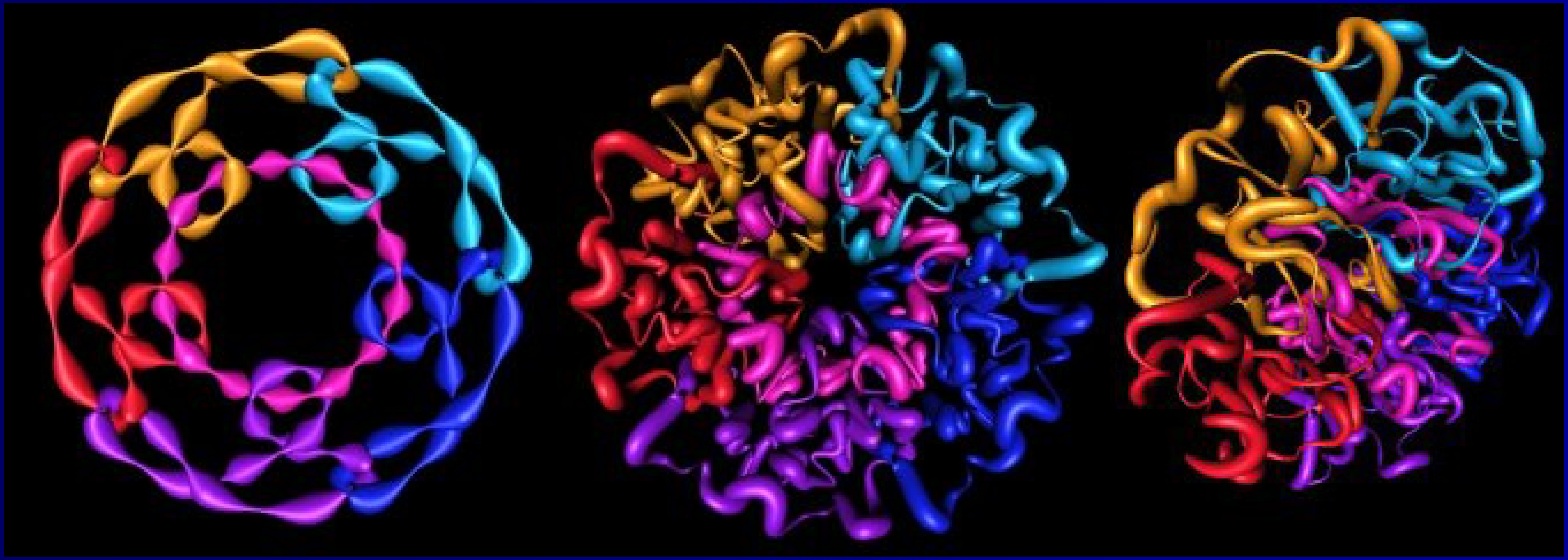
Unknotting and BJ-Conjecture for virtual KLs

- Real crossing changes: “real” ru_{BJ} (possibly infinite!). Example: (1,i,1) (i,1): non-minimal diagrams?
- Virtualization of real crossings: virtual vu_{BJ} (finite, upper limit, difficult knots)
- Mixed unknotting mu_{BJ} (“Unknott me fast as you can!”)
- Rational virtual unlinks (general form)
- Virtual links with unlinking number 1

Virtual KLs

- Open questions:
- Link tables (?)
- tangles, classification (rational, stellar, polyhedral), u_{BJ}, hard unknots, amphicheirality, undetectability, new powerful invariants (Kauffman extended bracket), connections with real knots (Jones unknot) (L.Kauffman, D. Silver and S. Williams), unlinking number...

Acknowledgements: Kauffman arrow polynomial (Louis Kauffman), cabled Jones (Jeremy Green), Miyazawa polynomials (Naoko Kamada), undetectable knots (Heather Dye)



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