The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA International Atomic Energy Agency

2034-23

Advanced School and Conference on Knot Theory and its Applications to Physics and Biology

11 - 29 May 2009

Notes on Link Homology

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Notes on Link Homology

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April 8, 2008

Abstract

This article consists of six lectures on the categorification of the Burau representation and on link homology groups which categorify the Jones and the HOMFLY-PT polynomial. The notes are based on the lecture course at the PCMI 2006 summer school in Park City, Utah.

Contents

Introduction

These notes are based on lectures delivered by the second author at the PCMI summer school in Utah in the summer of 2006. The goal was to give an informal introduction at the graduate level to the ideas and constructions of combinatorial homology theories that categorify various quantum invariants of knots and links. We made an emphasis on the theories lifting the Jones polynomial and the HOMFLY-PT polynomial. Ideally, a link homology theory is a functor from the category of link cobordisms to some algebraic category, such as the category of abelian groups. A model example is worked out in Lectures 3-5. In Lecture 3 a categorification of the Jones polynomial to a bigraded link homology theory is sketched. In Lectures 4 and 5 we explain how to generalize this theory to tangles and tangle cobordisms. This generalization encodes a simple proof that the homology theory is functorial and extends to link cobordisms. Prior to that, in Lectures 1 and 2, we introduce a toy model of the story, a categorification of the Burau representation, which produces invariants of braids and braid cobordisms. In Lecture 6 we describe a triply-graded link homology theory categorifying the HOMFLY-PT polynomial.

In the past few years link homology has become an extensively researched area, with a significant body of literature, which we won't try to fully survey in these short introductory lectures. Although neither knot Floer homology nor contact homology is discussed here, we refer the reader to the survey papers [50], [53] on these topics and to [74] for another set of lecture notes on link homology.

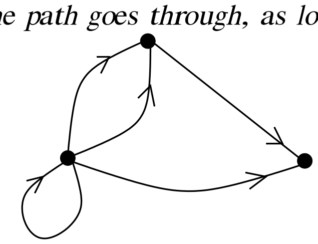
The final version of this work will appear in the AMS lecture notes from the Graduate Summer School program on Low Dimensional Topology held in Park City, Utah, on June 25 - July 15, 2006.

M.A. and M.K. would like to thank John Polking, the editor of the series, Tom Mrowka and Peter Ozsváth, the organizers of the summer school, for their encouragement and patience with the authors. The authors are indebted to Alexander Schumakovitch for providing knot homology tables for Lecture 3 and for enlightening discussions. M.A. was partially supported by the NSF grant DMS-0504199. M.K. was partially supported by the NSF grant DMS-0706924. This work was finished while the second author was at the Institute for Advanced Study, and he would like to thank the Institute and the NSF for supporting him during that time through the grant DMS-0635607.

1 A braid group action on a category of complexes

1.1 Path rings

Definition 1.1. *An oriented graph (see a picture below) consists of finitely many vertices and oriented edges. For an edge let* $\mathsf{s}(\ )$ *and* $\mathsf{t}(\ )$ *be the source and the target vertices of . A path is a concatenation of some edges* $_1, \ldots, _k$*, so that* $\mathsf{t}(\ _i) = \mathsf{s}(\ _{i+1})$ *for* $i = 1, \ldots, k-1$*. We define* $\mathsf{s}(\ )$*,* $\mathsf{t}(\ )$*, and the path length* $|\ |$ *to be* $\mathsf{s}(\ _1)$*,* $\mathsf{t}(\ _k)$*, and* k*, respectively. A path may be denoted by* $(a_1|a_2|\ldots|a_k)$ *where* a_i*'s are the vertices in the order that the path goes through, as long as there is only one such path.*



The path ring $\mathbb{Z}$ *is a free abelian group with a basis given by all the paths in , equipped with the following product: for paths and , is their concatenation if* $\mathsf{t}(\ ) = \mathsf{s}(\ )$ *and zero otherwise. We extend the product to* $\mathbb{Z}$ *by linearity; this multiplication operation is associative.*

Example 1.2. $= \overset{a}{\bullet} \xrightarrow{\alpha} \overset{b}{\bullet} \xrightarrow{\beta} \overset{c}{\bullet}$. *There are six paths in : , , , and* (a)*,* (b)*,* (c)*, where the last three are the length zero paths consisting of a vertex. Note that* $= 0$*,* $(a)(a) = (a)$*,* $(a) = = (b)$*, and so on.*

Exercise 1.3. *Check that in the above example,* $(a) + (b) + (c)$ *is the unit element of the path ring* $\mathbb{Z}$ *. For any oriented graph , the sum of the vertices (i.e. length zero paths) is the unit of* $\mathbb{Z}$ *.*

Example 1.4. $= \bullet \circlearrowright^{\alpha}$*. Then* $\mathbb{Z} = \mathbb{Z}[\ ]$ *is the polynomial ring in one variable.*

Let mod-$\mathbb{Z}$ be the category of right $\mathbb{Z}$ -modules. Each object M of mod-$\mathbb{Z}$ decomposes into the direct sum

$$M = \bigoplus_a M(a),$$

over the vertices a of the graph, as an abelian group. Multiplication by an edge with $s(\ ) = a$ and $t(\ ) = b$ is an abelian group homomorphism $M(a) \quad M(b)$. Vice versa, a right Z -module is determined by a collection of abelian groups, one for each vertex of the graph, and homomorphisms between these groups, one for each edge.

Exercise 1.5. *(1) Give a similar description of left Z -modules.*
(1) Describe Z -module homomorphisms in this language.

Converting Z into a field k, we arrive at the notion of path algebra k . These algebras have homological dimension one (any submodule of a projective module is projective), just like rings of integers in number fields and rings of functions on smooth a ne curves. Their representation theory is a spectacular story in progress; you can get a first taste of it from [15].

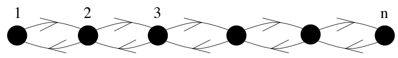
In this lecture we consider a very special quotient of a certain path ring. In general, if paths $_1, \ldots, _m$ all have the same source vertex and the same target vertex, we can quotient the path ring by the relation

$$_1 \ _1 + \ _2 \ _2 + \cdots + \ _m \ _m = 0$$

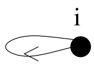
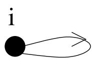
for some $_1, \ldots, _m \quad Z$.

1.2 Zigzag rings A_n

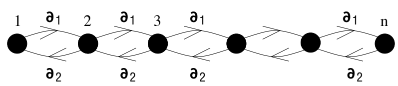
For an $n > 2$ consider the graph with vertices labelled from 1 to n and oriented edges from i to $i \pm 1$:



We define the ring A_n as the quotient of Z modulo the following relations

1. $\bullet \to \bullet \to \bullet = 0$, that is, $(i|i+1|i+2) = 0$;
2. $\bullet \leftarrow \bullet \leftarrow \bullet = 0$, that is, $(i|i-1|i-2) = 0$;
3.  = : $(i|i-1|i) = (i|i+1|i)$.

If we label all edges pointed to the right (resp. left) by ∂_1 (resp. ∂_2),

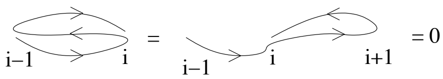


the relations become

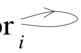
$$\partial_1^2 = 0,\ \partial_2^2 = 0,\ \partial_1\partial_2 = \partial_2\partial_1.$$

These are the relations for a bicomplex. In fact, the category of A_n-modules (either left or right) is equivalent to the category of mixed complexes of abelian groups, bounded above by n [41, 2.5.13]. The algebra A_n is isomorphic to its opposite, hence its categories of left and right modules are equivalent. If we make A_n graded, by assigning degree 1 to all arrows going to the right and degree 0 to all arrows going to the left, then the category of graded A_n-modules is isomorphic to the category of bicomplexes of abelian groups, with a suitable boundedness condition.

We will use a di erent grading on A_n, by path length. Any path of length at least three is zero in A_n. Indeed, the first two relations imply that a non-trivial path should stay within some interval $[i-1, i]$. If its length is more than two, we can flip a part of it, as illustrated below, to get zero.



It is easy to check that A_n is a free abelian group with the basis of

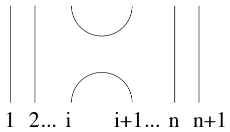
- length zero paths: $\underset{(i)}{\bullet}\ |i = 1, \ldots, n$
- length one paths: $\underset{i}{}\rightarrow\underset{i+1}{},\ \underset{i}{}\leftarrow\underset{i+1}{}\ |i = 1, \ldots, n-1$,
- length two paths: $X_i :=$  or  $|i = 1, \ldots, n$

1.3 A functor realization of the Temperley-Lieb algebra

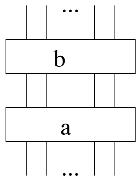
In this section we make the Temperley-Lieb algebra act by functors on the category of A_n-modules. The Temperley-Lieb algebra TL_{n+1} over the ground ring $R = \mathbb{Z}[q, q^{-1}]$ has generators $u_1, \ldots, u_n$ and relations

$$\begin{aligned} u_i^2 &= (q+q^{-1})u_i, \\ u_i u_{i\pm 1} u_i &= u_i, \\ u_i u_j &= u_j u_i,\ |i-j| > 1. \end{aligned}$$

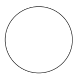
The Temperley-Lieb algebra has a graphical interpretation, via the following assignment:

$$\mathbf{u_i} =$$  ,

while the product of generators corresponds to concatenation

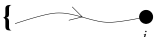
$$\mathbf{ba} =$$ 

By setting the value of the closed loop to $\mathbf{q} + \mathbf{q}^{-1}$:

 $= \mathbf{q} + \mathbf{q}^{-1}$,

and allowing arbitrary isotopies rel boundary, we obtain the relations in $\mathbf{TL_{n+1}}$ (see [26] for more).

$\mathbf{A_n}$, as a left module over itself, decomposes into the direct sum $\mathbf{A_n} = \bigoplus_{\mathbf{i}=1}^{\mathbf{n}} \mathbf{P_i}$. Here

$$\mathbf{P_i} = \mathbf{A_n}(\mathbf{i}) = \mathrm{span}_{\mathsf{Z}}\{$$  $\}.$

is a left projective $\mathbf{A_n}$-module spanned over Z by paths that end in vertex $\mathbf{i}$. As an abelian group, $\mathbf{P_i}$ is a free or rank 4 with the basis

$$\{(\mathbf{i}), (\mathbf{i} - 1|\mathbf{i}), (\mathbf{i} + 1|\mathbf{i}), \mathbf{X_i}\}$$

if $1 < \mathbf{i} < \mathbf{n}$ and free of rank 3 if $\mathbf{i} = 1, \mathbf{n}$.

Likewise, define the right projective $\mathbf{A_n}$-module

$${}_{\mathbf{i}}\mathbf{P} := (\mathbf{i})\mathbf{A_n} = \mathrm{span}_{\mathsf{Z}}\{$$  $\}.$

Exercise 1.6. *The following holds:*

$${}_{\mathbf{i}}\mathbf{P} \otimes_{\mathbf{A_n}} \mathbf{P_j} = \begin{cases} 0 \text{ if } |\mathbf{i} - \mathbf{j}| > 1, \\ \mathsf{Z}(\mathbf{i}|\mathbf{j}) \text{ if } \mathbf{j} = \mathbf{i} \pm 1, \\ \mathsf{Z}(\mathbf{i}) \oplus \mathsf{Z}\mathbf{X_i} \text{ if } \mathbf{i} = \mathbf{j} \end{cases}$$

(think of the LHS as spanned by paths that start in $\mathbf{i}$ and end in $\mathbf{j}$).

Consider A_n-bimodules

$$U_i := P_i \otimes_{\mathbb{Z}} {}_iP.$$

Claim 1.7. *There are bimodule isomorphisms*

$$\begin{aligned} U_i \otimes_{A_n} U_i &= U_i \oplus U_i, \\ U_i \otimes_{A_n} U_{i\pm 1} \otimes_{A_n} U_i &= U_i, \\ U_i \otimes_{A_n} U_j &= 0, \quad |i-j|>1. \end{aligned}$$

Proof. We prove the first equality (the rest is equally easy to check). We use Exercise 1.6:

$$\begin{aligned} U_i \otimes_{A_n} U_i &= P_i \otimes_{\mathbb{Z}} ({}_iP \otimes_{A_n} P_i) \otimes_{\mathbb{Z}} {}_iP \\ &= (P_i \otimes_{\mathbb{Z}} \mathbb{Z}(i) \otimes_{\mathbb{Z}} {}_iP) \oplus (P_i \otimes_{A_n} \mathbb{Z}X_i \otimes_{\mathbb{Z}} {}_iP) \\ &= (P_i \otimes_{\mathbb{Z}} {}_iP) \oplus (P_i \otimes_{\mathbb{Z}} {}_iP) = U_i \oplus U_i. \end{aligned}$$

□

The equalities immediately remind us of the relations in TL_{n+1} at $q=1$. The bimodule U_i plays the role of the generator u_i, tensor product of bimodules is analogous to the multiplication in TL_{n+1}, direct sum of bimodules lifts addition in the Temperley-Lieb algebra, etc. Due to the degenerate nature of our example, the tensor product $U_i \otimes_{A_n} U_j = 0$ when $|i-j|>1$ rather than just being isomorphic to the opposite tensor product. Thus, we get a bimodule realization of the quotient of TL_{n+1} at $q=1$ by the relations $u_i u_j = 0$ if $|i-j|>1$ (we will construct a non-degenerate example in Lecture 4). The unit element 1 of the Temperley-Lieb algebra corresponds to A_n, viewed as a bimodule over itself. The canonical isomorphism $A_n \otimes_{A_n} M = M$, functorial in a bimodule M, lifts the identity $1m = m$ for $m \in TL_{n+1}$.

We now bring q into the play. Recall that A_n, P_i, ${}_iP$ and U_i are graded by path length. We work with graded modules and bimodules and denote by $\{m\}$ the grading shift up by m. Redefine U_i by shifting its grading down by 1:

$$U_i = P_i \otimes_{\mathbb{Z}} {}_iP\{-1\}.$$

For instance, the element $(\mathsf{i}) \quad (\mathsf{i})$ of $\mathsf{U_i}$ now sits in degree -1. It is easy to see that there are isomorphisms of graded bimodules

$$\begin{aligned} &\mathsf{U_i} \quad {}_{\mathsf{A_n}} \mathsf{U_i} = \mathsf{U_i}\{1\} \quad \mathsf{U_i}\{-1\}, \\ &\mathsf{U_i} \quad {}_{\mathsf{A_n}} \mathsf{U_{i\pm1}} \quad {}_{\mathsf{A_n}} \mathsf{U_i} = \mathsf{U_i}, \\ &\mathsf{U_i} \quad {}_{\mathsf{A_n}} \mathsf{U_j} = 0, \ |\mathsf{i}-\mathsf{j}| > 1. \end{aligned}$$

In this way, multiplication by q becomes the grading shift $\{1\}$.

To interpret the meaning of the minus sign in our bimodule realization of the Temperley-Lieb algebra we need to work with complexes of modules and bimodules, and take a small detour in the next subsection to review their basics.

1.4 The homotopy category of complexes

Let A be an abelian category (for instance, the category of modules over some ring). Denote by Kom(A) the category with objects–complexes of objects of A and morphisms–homomorphisms of complexes. A morphism t from an object $\mathsf{M} = \{\cdots \quad \mathsf{M}^{\mathsf{i}-1} \quad \mathsf{M}^{\mathsf{i}} \quad \mathsf{M}^{\mathsf{i}+1} \quad \ldots\}$ to $\mathsf{N} = \{\cdots \quad \mathsf{N}^{\mathsf{i}-1} \quad \mathsf{N}^{\mathsf{i}} \quad \mathsf{N}^{\mathsf{i}+1} \quad \ldots\}$ is a collection of morphisms $\mathsf{t_i} : \mathsf{M}^{\mathsf{i}} \quad \mathsf{N}^{\mathsf{i}}$ that make the following diagram commute

$$\begin{tikzcd} \mathsf{M} & \cdots \arrow[r, "\mathsf{d}"] & \mathsf{M}^{\mathsf{i}} \arrow[r, "\mathsf{d}"] \arrow[d, "\mathsf{t_i}"'] & \mathsf{M}^{\mathsf{i}+1} \arrow[r, "\mathsf{d}"] \arrow[d, "\mathsf{t_{i+1}}"'] & \mathsf{M}^{\mathsf{i}+2} \arrow[r, "\mathsf{d}"] \arrow[d, "\mathsf{t_{i+2}}"'] & \cdots \\ \mathsf{N} & \cdots \arrow[r, "\mathsf{d}"] & \mathsf{N}^{\mathsf{i}} \arrow[r, "\mathsf{d}"] & \mathsf{N}^{\mathsf{i}+1} \arrow[r, "\mathsf{d}"] & \mathsf{N}^{\mathsf{i}+2} \arrow[r, "\mathsf{d}"] & \cdots \end{tikzcd}$$

Kom(A) is still an abelian category. Recall that a chain map t is null-homotopic (we write $\mathsf{t} \quad 0$) if there are maps $\mathsf{h_i} : \mathsf{M}^{\mathsf{i}} \quad \mathsf{N}^{\mathsf{i}-1}$ such that $\mathsf{t} = \mathsf{dh} + \mathsf{hd}$ (in more detail, $\mathsf{t_i} = \mathsf{d_N h_i} + \mathsf{h_{i+1} d_M}$).

$$\begin{tikzcd} \mathsf{M} & \cdots \arrow[r, "\mathsf{d}"] & \mathsf{M}^{\mathsf{i}} \arrow[r, "\mathsf{d}"] \arrow[d, "\mathsf{t}"'] \arrow[dl, phantom, "\mathsf{h}"] & \mathsf{M}^{\mathsf{i}+1} \arrow[r, "\mathsf{d}"] \arrow[d, "\mathsf{t}"'] \arrow[dl, phantom, "\mathsf{h}"] & \mathsf{M}^{\mathsf{i}+2} \arrow[r, "\mathsf{d}"] \arrow[d, "\mathsf{t}"'] \arrow[dl, phantom, "\mathsf{h}"] & \cdots \\ \mathsf{N} & \cdots \arrow[r, "\mathsf{d}"] & \mathsf{N}^{\mathsf{i}} \arrow[r, "\mathsf{d}"] & \mathsf{N}^{\mathsf{i}+1} \arrow[r, "\mathsf{d}"] & \mathsf{N}^{\mathsf{i}+2} \arrow[r, "\mathsf{d}"] & \cdots \end{tikzcd}$$

We define Com(A) as the quotient category of Kom(A) by the ideal of null-homotopic morphisms.

Exercise 1.8. *Check that null-homotopic morphisms constitute an ideal in* Kom(A). *First you need to define the notion of an ideal in an abelian or an additive category.*

The quotient category has the same objects as Kom(A) but fewer morphisms:

$$\mathrm{Hom}_{\mathrm{Com}(A)}(M, N) = \mathrm{Hom}_{\mathrm{Kom}(A)}(M, N)/\quad .$$

Two morphisms f, g become equal in Com(A) if their difference is null-homotopic.

Although we did not change objects when forming the quotient category, there are now more relations between them.

Exercise 1.9. *Check that for any nontrivial object K of A complexes*

$$0 \to K \xrightarrow{\mathrm{id}} K \to 0 \text{ and } 0 \to 0 \to 0 \to 0$$

are isomorphic in Com(A) *but not in* Kom(A).

The category Com(A) is no longer abelian but triangulated (see [17], [77]) and comes with the following operations.

(1) Shift. For $M \in \mathrm{Ob}\, C(A)$ define $M[j]$ to be the chain complex obtained from M by shifting it j steps to the left, $M[j]^i = M^{i+j}$, and multiplying the differential by $(-1)^j$.

(2) Cone of a morphism $f : M \to N$. The mapping cone of f is the chain complex $C(f) := M[1] \oplus N$ with the differential $D := -d_M + f + d_N$. Note that $C(f)^i = M^{i+1} \oplus N^i$.

$$\begin{tikzcd}
M^i \arrow[r, "-d_M"] \arrow[dr, "f"'] & M^{i+1} \arrow[r, "-d_M"] \arrow[dr, "f"'] & M^{i+2} \\
N^{i-1} \arrow[r, "d_N"] & N^i \arrow[r, "d_N"] & N^{i+1}
\end{tikzcd}$$

From here on we specialize to categories of modules and bimodules. For a ring A we denote by $C(A)$ the category Com($A -$ mod) of complexes of A-modules up to chain homotopies. If A is graded and we're working with graded modules, we'll use the same notation $C(A)$ for the category of complexes of graded A-modules up to chain homotopies. The differential of a complex of graded modules must preserve the grading. We can view a complex of graded A-modules as a bigraded A-module with a differential of bidegree $(1,0)$ which commutes with the action of A.

Denote by $A^e = A \otimes A^{op}$ the tensor product of A and its opposite ring. A-bimodules can also be described as left or right A^e-modules. We denote by $C(A^e)$ the category of complexes of A-bimodules up to chain homotopy (and the category of complexes of graded A-bimodules, whenever necessary).

Tensoring with a given A-bimodule is an endofunctor in the category of A-modules and an endofunctor in the category of A-bimodules. Likewise, tensoring with a complex of A-bimodules is an endofunctor in $C(A)$ and $C(A^e)$. The tensor product of $M, N \in C(A^e)$ is the complex of bimodules given by placing the bimodule $M^i \otimes_A N^j$ into the (i,j)-node of the plane and then collapsing the grading onto the principal diagonal, so that the degree k term of $M \otimes_A N$ is the direct sum

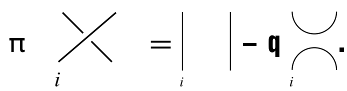
$$\bigoplus_{i \in \mathbb{Z}} M^i \otimes_A N^{k-i},$$

with the differential combining those of M and N:

$$d(m \otimes n) = d(m) \otimes n + (-1)^i m \otimes d(n), \quad m \in M^i.$$

1.5 Braid group representation

There exists a representation $\pi : Br_{n+1} \to TL_{n+1}$ of the braid group on $n+1$-strands into the group of invertible elements in the Temperley-Lieb algebra given on the standard generators of the braid group by $\pi(\sigma_i) = 1 - qu_i$. Graphically,

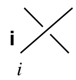


What would the meaning of $1 - qu_i$ be in our bimodule interpretation of the Temperley-Lieb algebra quotient? It should become the "difference" of graded bimodules A_n and $U_i\{1\} = P_i \otimes_{\mathbb{Z}} {}_iP$, which we interpret as the complex

$$0 \to P_i \otimes_{\mathbb{Z}} {}_iP \xrightarrow{\beta_i} A_n \to 0$$

for the bimodule homomorphism β_i which takes $x \otimes y \in P_i \otimes_{\mathbb{Z}} {}_iP$ to $xy \in A_n$. This grading-preserving map composes a path which ends in i with a path which starts in $\mathbf{i}$:

Thus,



$$(0 \quad U_i\{1\} \quad {}^{i} \quad A_n \quad 0).$$

We denote this complex of graded bimodules by R_i and normalize it so that A_n sits in cohomological degree 0.

The homomorphism π : Br_{n+1} $\quad TL_{n+1}$ takes $\;_i^{-1}$ to $1 - q^{-1}u_i$. To interpret this di erence we consider the complex R_i given by

$$0 - \quad A_n -^{i} \quad U_i\{-1\} - \quad 0,$$

with the bimodule map $\;_i$ determined by the condition

$$_i(1) = (i-1|i) \quad (i|i-1) + (i+1|i) \quad (i|i+1) + X_i \quad (i) + (i) \quad X_i$$

(for $1 < i < n$; for $i = 1, n$ omit one of the terms in the sum), and A_n placed again in cohomological degree 0.

Theorem 1.10. *There are isomorphisms in $C(A_n^e)$ of complexes of graded bimodules:*

$$R_i \quad R_{i\pm1} \quad R_i \;=\; R_{i\pm1} \quad R_i \quad R_{i\pm1}, \tag{1}$$
$$R_i \quad R_j \;=\; R_j \quad R_i, \; |i-j| > 1, \tag{2}$$
$$R_i \quad R_i \;=\; A_n = R_i \quad R_i \tag{3}$$

The last relation tell us that R_i and R_i are mutually inverse complexes of bimodules. The first two are the braid relations. The middle relation holds already in the abelian category of complexes of bimodules, before modding out by homotopies, but not the other two. The proof can be found in [35] (also see the next lecture). This action was independently discovered by R. Rouquier and A. Zimmermann [63]; its algebraic geometry counterparts are studied in [65].

The theorem implies that there is a braid group action on $C(A_n)$ and on $C(A_n^e)$ in which the generator $\;_i$ of the braid group Br_{n+1} acts on a complex M of graded A_n-modules (or bimodules) by tensoring it with R_i:

$$_i(M) := R_i \quad _{A_n} M,$$

while

$$_i^{-1}(\mathsf{M}) := \mathsf{R}_i \quad {}_{\mathsf{A}_n} \mathsf{M}.$$

More precisely, the theorem is about a group action in the *weak* sense. A *weak* action of a group G on a category C assigns an invertible functor $\mathsf{F}_g : \mathsf{C} - \quad \mathsf{C}$ to each element of G such that $\mathsf{F}_{gh} = \mathsf{F}_g \mathsf{F}_h$. For a weak action to be an action requires a specific choice of isomorphisms $\mathsf{F}_{gh} = \mathsf{F}_g \mathsf{F}_h$ for all $g, h \quad \mathsf{G}$ subject to the associativity relation that the diagram below is commutative for all $g, h, k \quad \mathsf{G}$:

$$\begin{array}{ccc} \mathsf{F}_{ghk} & \overset{=}{\longrightarrow} & \mathsf{F}_{gh}\mathsf{F}_k \\ = & & = \\ \mathsf{F}_g\mathsf{F}_{hk} & \overset{=}{\longrightarrow} & \mathsf{F}_g\mathsf{F}_h\mathsf{F}_k \end{array}$$

P. Deligne [16] gave a simple criterion for when a weak action of a braid group on a category can be upgraded to an action. His criterion holds in our case, and the weak action described above lifts to an actual action of Br_{n+1} on $\mathsf{C}(\mathsf{A}_n)$ and $\mathsf{C}(\mathsf{A}_n^e)$. Furthermore, we have:

Theorem 1.11. *The above action of the braid group* B_n *on* $\mathsf{C}(\mathsf{A}_n)$ *is faithful.*

We say that an action of the group G on a category C is *faithful* if the functors F_g are not isomorphic for di erent $g \quad \mathsf{G}$. See the next lecture for a sketch of a proof of the last theorem.

2 More on braid group actions

2.1 Invertibility of R_i

We begin the lecture with a sketch of isomorphisms:

$$\mathsf{R}_i \quad \mathsf{R}_i = \mathsf{A}_n = \mathsf{R}_i \quad \mathsf{R}_i$$

from Theorem 1.10. The double complex corresponding to $\mathsf{R}_i \quad \mathsf{R}_i$ has the form

$$\begin{array}{cccccc} & 0 & & 0 & \\ 0 & \mathsf{U}_i \quad \mathsf{U}_i & & \mathsf{A}_n \quad \mathsf{U}_i\{-1\} & 0 \\ 0 & \mathsf{U}_i\{1\} \quad \mathsf{A}_n & & \mathsf{A}_n \quad \mathsf{A}_n & 0 \\ & 0 & & 0 & \end{array}$$

Noting that $\mathsf{U_i} \quad \mathsf{U_i} = \mathsf{U_i}\{1\} \quad \mathsf{U_i}\{-1\}$ from Claim 1.7, we obtain the total compex

$$\mathsf{R_i} \quad \mathsf{R_i} = (0 \quad \mathsf{U_i}\{1\} \;^{\mathsf{d}}\; \mathsf{A_n} \quad \mathsf{U_i}\{1\} \quad \mathsf{U_i}\{-1\} \;^{\mathsf{d}}\; \mathsf{U_i}\{-1\} \quad 0).$$

Exercise 2.1. *Write an explicit formula for the di erential* d *above and check that the complex decomposes into a direct sum*

$$(0 \quad \mathsf{U_i}\{1\} \;^{1}\; \mathsf{U_i}\{1\} \quad 0) \quad (0 \quad \mathsf{A_n} \quad 0) \quad (0 \quad \mathsf{U_i}\{-1\} \;^{1}\; \mathsf{U_i}\{-1\} \quad 0).$$

The first and the last summands are null-homotopic, implying that

$$\mathsf{R_i} \quad \mathsf{R_i} = (0 \quad \mathsf{A_n} \quad 0) = \mathsf{A_n}.$$

2.2 Braid group action on complexes of projective modules $\mathsf{P_i}$ and topology of plane curves

In this section we explain how to prove that the braid group action on the homotopy category $\mathsf{C}(\mathsf{A_n})$ is faithful. For simplicity, we write (P) for the object $\mathsf{F} \; (\mathsf{P})$ given by applying the functor F to $\mathsf{P} \quad \mathsf{C}(\mathsf{A_n})$. We will give a geometric presentation of $(\mathsf{P_i})$ for all in the braid group $\mathsf{Br_{n+1}}$.

First, let's look at a couple of easy examples.

Example 2.2.

$$_{\mathsf{i}}(\mathsf{P_i}) = \mathsf{R_i} \quad \mathsf{P_i} = (0 \quad \mathsf{P_i} \quad _{\mathsf{Z}}\, _{\mathsf{i}}\mathsf{P} \quad _{\mathsf{A_n}} \mathsf{P_i} - \quad \mathsf{P_i} \quad 0)$$

The subcomplex

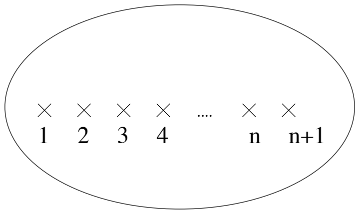
$$0 - \quad \mathsf{P_i} \quad (\mathsf{i}) \quad (\mathsf{i}) -^{1} \quad \mathsf{P_i} - \quad 0$$

is contractible and the quotient complex is $0 - \quad \mathsf{P_i} \quad \mathsf{X_i} \quad (\mathsf{i}) - \quad 0$ *with the nontrivial term in cohomological degree* -1. *The degree of* $\mathsf{X_i}$ *is two and hence* $\mathsf{P_i} \quad \mathsf{X_i} \quad (\mathsf{i}) = \mathsf{P_i}\{2\}$ *as a graded module. thus* $_{\mathsf{i}}(\mathsf{P_i}) = \mathsf{P_i}[1]\{2\}$.

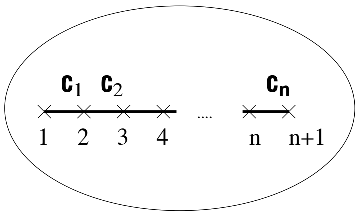
Example 2.3. *By induction on* $\mathsf{m} > 0$ *one can check that*

$$^{\mathsf{m}}_{\mathsf{i}}(\mathsf{P_{i+1}}) = (0 \quad \mathsf{P_1}\{2\mathsf{m}-1\} \xrightarrow{\mathsf{X_i}} \ldots \xrightarrow{\mathsf{X_i}} \mathsf{P_1}\{3\} \xrightarrow{\mathsf{X_i}} \mathsf{P_i}\{1\} \xrightarrow{(\mathsf{i}|\mathsf{i}+1)} \mathsf{P_{i+1}} \quad 0).$$

We set up the following ingredients. Consider a disk with $n+1$ marked points aligned on a line as below.

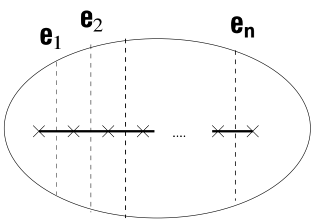


The braid group B_{n+1} is isomorphic to the the mapping class group of the disk that fixes the boundary and permutes the marked points. In particular, B_{n+1} acts on isotopy classes of simple curves in the disk which have marked points as their endpoints and don't contain marked points in their interior. We assume that the generator $\ _i$ acts on the disk by permuting the vertices i and $i+1$ counterclockwise. We fix a chain of curves $c_1, \ldots, c_n$ as follows

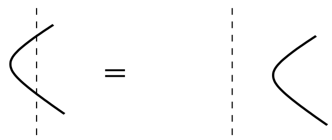


The curve c_i connects marked points i and $i+1$. The braid group action on the disc induces a braid group action on the isotopy classes of unoriented arcs that connects pairs of marked points. Any such isotopy class has the form $\ (c_i)$ for any i and some braid . Curves $c_1, \ldots, c_n$ represent some of these isotopy classes. We would like to relate the braid group action on our category of complexes with the braid group action on the isotopy classes of curves.

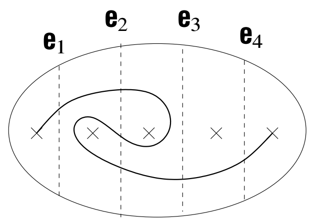
Consider vertical dotted lines $e_1, \ldots, e_n$ orthogonal to $c_1, \ldots, c_n$.



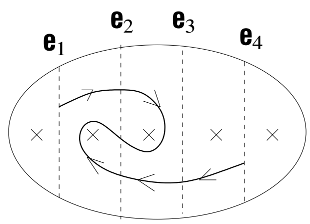
Given an isotopy class c of an arc in the disc with marked endpoints, we can choose a representative c in the minimal position relative to the system of intervals $e_1, \ldots, e_n$, in the sense that the number of intersection points of c with each of e_i is the minimal possible among curves in the isotopy class c. Such representative is unique in the appropriate sense and can be obtained from any generic diagram in c by a sequence of simplifications



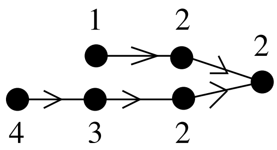
Here's an example of the minimal representative.



To a isotopy class c of arc we assign a complex $P(c)$ of projective A_n-modules as follows. Let c be the minimal representative of c. The vertical lines cut c into segments. Discard two segments containing the endpoints of c and orient each remaining segment bounded by vertical lines e_i and e_{i+1} clockwise around the marked point $i+1$, the only marked point between these vertical lines.



To each intersection point of the curve with the vertical line e_i assign number i. Now, pull the curve with these additional decorations out of the disk and draw it on the plane so that all orientations look to the right.



We next put P_i at each vertex labeled i and take the direct sum vertically. Define the differential as the sum of contributions from each arrow. To an arrow from i to $i \pm 1$ assign the module homomorphism $P_i \to P_{i\pm 1}$ which takes a to $a(i|i \pm 1)$. To an arrow from i to i assign the homomorphism $P_i \to P_i$ of multiplication by X_i. We obtain a chain complex; for the example above it has the form

$$\begin{tikzcd} & P_1 \arrow[r, "(1|2)"] & P_2 \arrow[dr, "x_2"] & \\ & & & P_2 \\ P_4 \arrow[r, "(4|3)"] & P_3 \arrow[r, "(3|2)"] & P_2 \arrow[ur, "x_2"'] & \end{tikzcd}$$

which we can also write as

$$0 \to P_4 \to P_1 \oplus P_3 \to P_2 \oplus P_2 \to P_2 \to 0.$$

In this way to an isotopy class c of arcs we assign a complex $P(c)$ of projective A_n-modules. We did not specify the overall grading shift for $P(c)$; the reader can find this and other information in [35]. It is also possible to keep track of the internal grading and view $P(c)$ as a complex of graded A_n-modules. For $c = c_i$ the complex $P(c_i) = (0 \to P_i \to 0)$.

Theorem 2.4. *For any braid and any number i between 1 and n the complex $P(\,c_i)$ is homotopy equivalent to P_i.*

This theorem [35] tells us how a braid acts on projective modules P_i, and that the action can be read off the braid group action on isotopy classes of arcs.

Exercise 2.5. *Rethink Example 2.3 via this theorem.*

To prove that the braid group action on $\mathcal{C}(A_n)$ is faithful it su ces to check that for any nontrivial braid we can find some i so that P_i is not isomorphic to P_i in the homotopy category. Our description of P_i implies that if it is isomophic to P_i then $c_i = c_i$. If $c_i = c_i$ for all i then is central and is a multiple of the full twist. But it is easy to compute that the full twist takes P_i to $P_i[j]$ for some $j = 0$ (compare with Example 2.2).

Exercise 2.6. *Find this* j.

The faithfullness of the action follows, modulo Theorem 2.4, not proved in these notes.

2.3 Reduced Burau representation

A braid takes a complex of (graded) projective A_n-modules to a complex of (graded) projective A_n-modules. One can check that any finitely-generated projective A_n-module is isomorphic to a direct sum of P_i's, and the multiplicity of P_i in this decomposition is an invariant of the projective module. Likewise, any finitely-generated projective graded A_n-module is isomorphic to a direct sum of $P_i\{j\}$, and the multiplicity of $P_i\{j\}$ is an invariant of the module. For the rest of this section all modules are assumed to be left, graded and finitely generated. We introduce a formal symbol $[P]$ of each projective module P. Let $K_0(A_n)$ be the $\mathbb{Z}[q, q^{-1}]$-module generated by these symbols subject to relations

$$[P \quad Q] = [P] + [Q], \quad [P\{j\}] = q^j[P].$$

The direct sum decomposition property mentioned above implies that $K_0(A_n)$ is a free $\mathbb{Z}[q, q^{-1}]$-module generated by symbols of indecomposables $[P_1], \dots, [P_n]$. The reader familiar with K-theory will recognize $K_0(A_n)$ as the group K_0 of the category of graded finitely-generated A_n-modules (see [61] for an excellent introduction to algebraic K-theory).

Given a bounded complex P of projective A_n-modules

$$0 \quad \cdots \quad P^i \quad P^{i+1} \quad \cdots \quad 0,$$

we define its Euler characteristic as

$$(P) = \sum_i (-1)^i [P_i] \quad K_0(A_n).$$

If two complexes are chain homotopy equivalent, they have equal Euler characteristic; shifting the complex by 1 adds a minus sign to the Euler characteristic, $(P[1]) = - (P)$.

The action of the braid group on the category of complexes of projective A_n-modules (a subcategory of $C(A_n)$) descends to a $\mathbb{Z}[q, q^{-1}]$-linear action of the braid group on $K_0(A_n)$. To determine this action, we write how $_i$ acts on P_j:

$$
\begin{aligned}
{}_i(P_i) &= P_i[1]\{2\} \\
{}_i(P_{i\pm1}) &= 0 \quad P_i\{1\} \quad P_{i\pm1} \quad 0 \\
{}_i(P_j) &= P_j \text{ if } |i-j| > 1,
\end{aligned}
$$

and pass to the Euler characteristic ($[P] = [\ (P)]$ by definition):

$$
\begin{aligned}
{}_i[P_i] &= -q^2[P_i] \\
{}_i[P_{i\pm1}] &= [P_{i\pm1}] - q[P_i] \\
{}_i[P_j] &= [P_j] \text{ if } |i-j| > 1.
\end{aligned}
$$

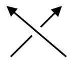
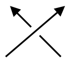
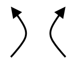
The resulting action is isomorphic to the reduced Burau representation of the braid group. Hence, we can view the braid group action on the category of complexes of projective A_n-modules as a categorification of the Burau representation.

To elements of the braid group we assign functors and it turns out that this assignment can be extended to braid cobordisms. A braid cobordism is a surface in $\mathbb{R}^4$ that goes from one braid to the other. The condition that braids have no critical points when projected onto the z-axis extends to the condition that a braid cobordism is a simple branch covering when projected onto the (z, w)-plane in $\mathbb{R}^4$. To a braid cobordism between braids and it is possible to assign a natural tranformation between functors F and F (modulo the issue of the overall sign) in a consistent way which respects compositions of braid cobordisms, see [36]. This example is the simplest way to get an algebraic invariant of a toy sector of four-dimensional topology (braid cobordisms) from homological algebra (of complexes of A_n-modules).

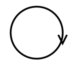
3 A Categorification of the Jones polynomial

3.1 The Jones polynomial

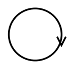
The Jones polynomial [22] is an isotopy invariant of oriented links in $\mathbb{R}^3$ that takes values in $\mathbb{Z}[q, q^{-1}]$ and satisfies the following skein relation for link diagrams:

$q^2 J$  $- q^{-2} J$  $= (q - q^{-1}) J$  . (4)

This equation implies

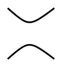
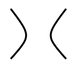
$J \;\; L$  $= (q + q^{-1}) J(L),$ (5)

that is, the Jones polynomial of the disjoint union of a link L and the unknot is the Jones polynomial of L times $q + q^{-1}$. Adding the unknot to a link multiplies the Jones polynomial by $q + q^{-1}$. Hence, it is convenient to normalize the invariant to take this value on the unknot:

J  $= q + q^{-1}.$ (6)

We also extend the invariant to the empty link, by setting $J(\;) = 1$.

Inductive simplification via the skein relation (4) implies that the Jones polynomial is uniquely determined by this relation and its value on the unknot. A simple way to prove existence was found by Louis Kau man [25]. Define the Kau man bracket polynomial D of an *unoriented* link projection D by expanding every crossing

 $=$  $- q^{-1}$  (7)

and requiring that $D = (q + q^{-1})^k$ if D is a crossingless diagram with k circles (the skein relation above di ers slightly from Kau man's original one, which has more symmetry). The Kau man bracket of a planar diagram is invariant under the Reidemeister moves up to rescaling by plus or minus a power of q. To get rid of these scaling factors, define the Kau man bracket of an *oriented* planar diagram by

$$K(D) := (-1)^{x(D)} q^{2x(D) - y(D)} \; D \; , \qquad (8)$$

where $x(D)$, respectively $y(D)$, is the number of negative  , respectively positive , crossings of D, and D is viewed as an unoriented diagram in the rightmost term of (8).

Exercise 3.1. *Show that $K(D) = J(L)$ for any planar diagram D of L.*

Thus, the Kauffman bracket is a link invariant, equal to the Jones polynomial. The two modifications of a crossing that appear on the right hand side of (7) will be called *resolutions*. We call modification $\to$ the *0-resolution*, and modification $\to$ the *1-resolution*.

3.2 Categorification and a bigraded link homology theory

In one of its manifestations, *categorification*, a term introduced by Louis Crane and Igor Frenkel [14], lifts natural numbers to vector spaces or free abelian group. Going in the opposite direction (decategorifying), to a finite-dimensional vector space V we assign its dimension $\dim(V)$ and to a finitely-generated free abelian group V its rank $\mathrm{rk}(V)$. Operations on vector spaces or free abelian groups mirror those on natural numbers. Direct sum of vector spaces corresponds to the sum of numbers, tensor product to multiplication:

$$\dim(V \oplus W) = \dim(V) + \dim(W), \quad \dim(V \otimes W) = \dim(V)\dim(W).$$

Thus, we have an informal correspondence

$$\begin{array}{rcl} n & \longleftrightarrow & \mathbb{Z}^n \text{ or } k^n, \text{ for a field } k, \\ n+m & \longleftrightarrow & V \oplus W, \text{ where } \mathrm{rk}(V) = n, \mathrm{rk}(W) = m, \\ nm & \longleftrightarrow & V \otimes W. \end{array}$$

Lifting negative numbers and differences $n - m$ requires stepping beyond the category of vector spaces and considering the category of complexes of vector spaces or free abelian groups. The analogue of the dimension of a vector space is the Euler characteristic of a complex. In the simplest instance, if positive integers n and m have become vector spaces V and W of dimension n and m, then the difference $n-m$ is the Euler characteristic of the complex $0 \longrightarrow V \xrightarrow{d} W \longrightarrow 0$ for any linear map d, with V sitting in even cohomological degree. More generally, if we have already lifted integers n and m to complexes V and W, then $n - m$ can be interpreted as the Euler characteristic of the complex $\mathrm{Cone}(f)$ for some map $f : W \longrightarrow V$ of complexes (alternatively, as the Euler characteristic of $\mathrm{Cone}(g)[\pm 1]$ for a map $g : V \longrightarrow W$).

The standard example of categorification is passing from the Euler characteristic $\chi(X)$ of a topological space X to its homology groups $H_*(X, \mathbb{Z})$. We recover the Euler characteristic by taking the alternating sum of ranks

$$\chi(X) = \sum_i (-1)^i \mathrm{rk}\, H_i(X, \mathbb{Z}). \qquad (9)$$

Homology can be built from the Euler characteristic by a lifting as above, starting with a CW-decomposition X' of X, taking the formula for $\chi(X)$ as the alternating sum of the number of i-dimensional cells, lifting each ± 1 term in the sum to the complex $0 \longrightarrow \mathbb{Z} \longrightarrow 0$ in the corresponding degree, and forming the complex $C(X')$ with the homology $H_*(X, \mathbb{Z})$. Notice the multitude of benefits that the homology of X provides compared to the Euler characteristic of X:

- The invariant is not just an integer but a graded abelian group, encoding more information about X.
- Homology extends to functor from the category of topological spaces and continuous maps modulo homotopies to the category of graded abelian groups and grading-preserving homomorphisms. Thus, it provides information about continuous maps as well, associating to $f : X \longrightarrow Y$ the homomorphism

  $$H_*(f) : H_*(X, \mathbb{Z}) \longrightarrow H_*(Y, \mathbb{Z}).$$

  The Euler characteristic does not give any information about continuous maps.
- Homology groups (singular homology) are defined for any topological space. The Euler characteristic, in its naive version, is only defined for topological spaces admitting finite CW-decomposition. Once homology becomes available the Euler characteristic can be defined for a wider range of spaces via equation (9), assuming that $H_*(X, \mathbb{Z})$ has finite rank. Still, for many spaces ($\mathbb{CP}^\infty$ or a discrete infinite space are the simplest examples) the formula (9) does not help due to $H_*(X, \mathbb{Z})$ having infinite rank. For such X the Euler characteristic cannot be defined but the homology still makes sense.
- Relatives of homology groups such as the cohomology groups of X and the K-theory of X provide even more information via the multiplication

in cohomology and in K-theory, cohomological operations, etc. We get a highly sophisticated theory called algebraic topology.

Now take a math book, find there a structure which is manifestly integral, with all the structure constants, coefficients, etc. being integers, and try to categorify it. This means consistently lifting integers to vector spaces or complexes of vector spaces. A book on combinatorics is a good place to start (you're likely to have less luck with a book on analysis since integral structures are not common there). The end result of your categorification efforts must be a richer and more beautiful structure then the one you started with, living one level above the original. Normally, in many cases your attempts will fall apart or the result will look artificial or shallow, but eventually you might be onto something.

Today we will look at a successful case of categorification–a categorificaton of the Jones polynomial. The Jones polynomial $J(L)$ of a link L takes values in $\mathbb{Z}[q, q^{-1}]$, so its coefficients are integers:

$$J(L) = \sum_j a_j(L) q^j, \quad a_j(L) \in \mathbb{Z}.$$

We will realize each coefficient as the Euler characteristic of a $\mathbb{Z}$-graded link homology theory. Taking all cofficients together, we'll get bigraded homology groups associated to a link

$$H(L) = \bigoplus_{i,j} H^{i,j}(L)$$

so that

$$a_j(L) = \sum_i (-1)^i \mathrm{rk} H^{i,j}(L), \quad j \in \mathbb{Z}, \quad J(L) = \sum_{i,j} (-1)^i q^j \mathrm{rk} H^{i,j}(L).$$

The homology will be constructed by lifting the Kauffman bracket formula for the Jones polynomial to complexes. To a diagram D we will assign a complex $C(D)$ of graded free abelian groups

$$\cdots \longrightarrow C^{i-1}(D) \xrightarrow{d} C^i(D) \xrightarrow{d} C^{i+1}(D) \xrightarrow{d} \cdots$$

with a grading-preserving differential; $C^i(D) = \bigoplus_j C^{i,j}(D)$. For a given degree j the complex restricts to a complex of free abelian groups

$$\cdots \longrightarrow C^{i-1,j}(D) \xrightarrow{d} C^{i,j}(D) \xrightarrow{d} C^{i+1,j}(D) \xrightarrow{d} \cdots,$$

with $a_j(L)$ being its Euler characteristic.

It is convenient to imagine groups $C^{i,j}(D)$ as sitting in the (i,j)-square on the plane and differential going one step to the right. The grading shift $\{1\}$ moves everything one step up, and the shift $[1]$ moves the diagram one step to the left. We refer to the i-degree as horizontal/cohomological degree and the j-degree as vertical/internal degree and also as q-degree.

First, we directly categorify the inductive formula for $\langle D \rangle$ for an unoriented diagram D and turn $\langle D \rangle$ into the Euler characteristic of a complex $\overline{C}(D)$. Next, orienting D, we'll define

$$C(D) := \overline{C}(D)[x(D)]\{2x(D) - y(D)\}, \tag{10}$$

mirroring the formula (8).

We start with the simplest diagrams. For the empty diagram $\langle \emptyset \rangle = 1$, and we define $\overline{C}(\emptyset) = \mathbb{Z}$ in bidegree $(0,0)$. For a single circle diagram $\langle \bigcirc \rangle = q + q^{-1}$. Let $A = \mathbb{Z}1 \oplus \mathbb{Z}X$ be a free graded abelian group with the basis $\{1, X\}$ such that $\deg(1) = -1$ and $\deg(X) = 1$ (the reason for notation 1 will soon become clear). The graded rank of A is $q + q^{-1}$ and we declare that

$$\overline{C}(\bigcirc) = A,$$

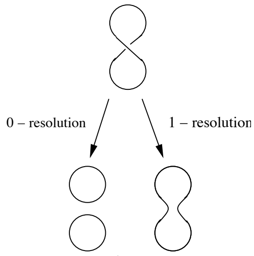
viewed as a complex of graded abelian groups $0 \longrightarrow A \longrightarrow 0$ sitting in cohomological degree 0 (necessarily with the trivial differential). In general, consider an arbitrary plane diagram D without crossings. Such diagram D consists of k disjoint circles embedded into the plane, possibly in a nested way. To such D we assign the complex $\overline{C}(D) := A^{\otimes k}$ with the trivial differential and $A^{\otimes k}$ sitting in the cohomological degree 0. The graded rank of $A^{\otimes k}$ is $(q + q^{-1})^k = \langle D \rangle$.

Next, we need to tackle diagrams with crossings and interpret the relation (7) in our framework. Assuming that complexes for the two diagrams on the right hand side of (7) have already been defined, we could look for a homomorphism of complexes

$$f : \overline{C}(\asymp) \longrightarrow \overline{C}()\,()\{1\} \tag{11}$$

and define $\overline{C}(\times)$ as the cone of f shifted one degree to the right (compare with []). The relation (7) would then hold for the Euler characteristics of these three complexes. Here's how it works in the simplest cases.

Example 1: a kinked diagram $D =$  of the trivial knot. Resolutions of the crossing produce two circles, respectively one circle:



$A^{\otimes 2}$ $\quad$ $A\{-1\}$

To the 0-resolution we assign $A^{\otimes 2}$, to the 1-resolution we assign $A\{-1\}$. The shift $\{-1\}$ mirrors multiplication by q^{-1} in the formula (7). The minus sign indicates that these two terms should live in cohomological degrees of different parity, and the simplest guess gives us the complex

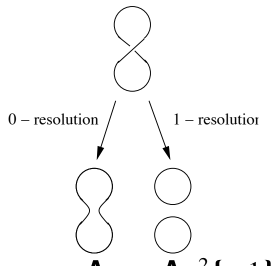
$$0 \longrightarrow A^{\otimes 2} \xrightarrow{m} A\{-1\} \longrightarrow 0, \tag{12}$$

where we placed the first term in cohomological degree 0. We denoted the differential in the complex by m since it looks like a multiplication map. This map must preserve internal grading and the cohomology of the complex should be $\mathbb{Z} \oplus \mathbb{Z}$, those of the unknot, since we want our theory to give an invariant of links and not just their diagrams. With these restrictions, there is very little choice available to us. We define

$$m(1 \otimes a) = m(a \otimes 1) = a, \text{ for } a \in A, \quad m(X \otimes X) = 0.$$

This makes A into an associative commutative unital algebra. If we shift the grading of A up by 1, the multiplication becomes grading-preserving and we can identify A with the integral cohomology ring of the 2-sphere. With this choice of m the cohomology of the complex (12) is the subgroup of $A^{\otimes 2}$ spanned by $X \otimes 1 - 1 \otimes X$ and $X \otimes X$. Thus, up to overall grading shift (which we'll take care via equation (10)), the cohomology is isomorphic to A.

Example 2: the opposite kink $D =$ . Similarly to example 1 take two resolutions:



$\mathcal{A} \qquad \mathcal{A}^{\otimes 2}\{-1\}$

and form the chain complex

$$0 \to \mathcal{A} \xrightarrow{\Delta} \mathcal{A}^{\otimes 2}\{-1\} \to 0.$$

It is suggestive to call the differential Δ, which is the usual symbol for comultiplication. The bases in each chain group, sorted by the internal degree j, are

j	$\mathcal{A}$	$\mathcal{A}^{\otimes 2}\{-1\}$
1	X	$X \otimes X$
-1	1	$X \otimes 1, 1 \otimes X$
-3	0	$X \otimes 1$

and we choose Δ to be

$$\Delta(1) = 1 \otimes X + X \otimes 1, \quad \Delta(X) = X \otimes X.$$

The cohomology of the resulting complex is isomorphic to $\mathcal{A}$, as a bigraded group, up to overall grading shift.

We can now guess the definition of $\overline{C}(D)$ for an arbitrary D with m crossings. Each crossing has two resolutions, and the number of complete resolutions of D is 2^m. Each complete resolutions is a crossingless diagram and has $\mathcal{A}^{\otimes k}$ assigned to it, where k is the number of circles. If r_0, r_1 are two complete resolutions that differ only in one place (near one crossing), with r_0, resp. r_1 being the 0-resolution, resp. 1-resolution there, then two things can happen. Either two circles of r_0 become one circle in r_1 or vice versa. If the first case we have a natural map $\overline{C}(r_0) \to \overline{C}(r_1)$ which is $m : \mathcal{A}^{\otimes 2} \to \mathcal{A}$ on the two $\mathcal{A}$'s corresponding to these two circles times the identity map $\mathrm{Id} : \mathcal{A}^{\otimes (k-1)} \to \mathcal{A}^{\otimes (k-1)}$ on the tensor product of the copies of $\mathcal{A}$ corresponding to circles that don't change as we go from r_0 to r_1. Thus, the map is the composition

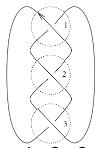
$$\overline{C}(r_0) = \mathcal{A}^{\otimes (k+1)} \xrightarrow{m \otimes \mathrm{Id}} \mathcal{A}^{\otimes k} = \overline{C}(r_1).$$

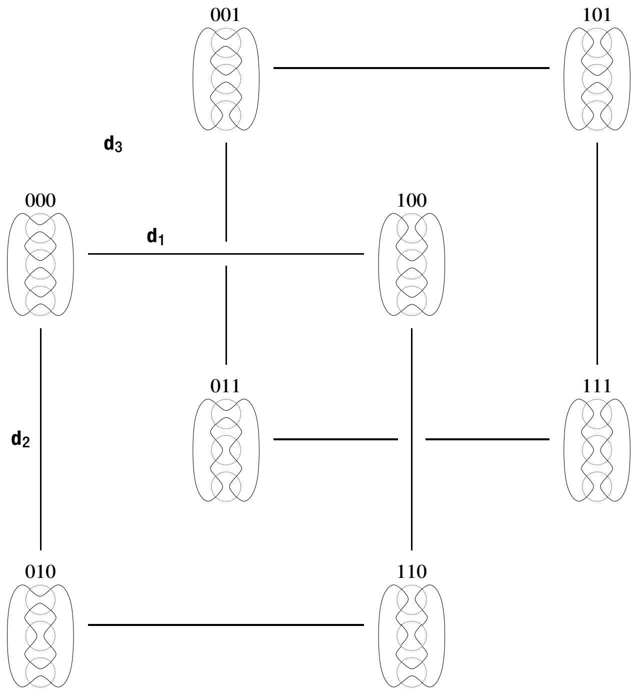
In the second case, when r_1 has more circles than r_0, we have a similar map $\overline{C}(r_0) \longrightarrow \overline{C}(r_1)$ using Δ in place of m.

Given an unoriented diagram D with m crossings we associate to it an m-dimensional cube with graded abelian groups $A^{\otimes k}$ (plus a grading shift) written in its vertices and maps $m \otimes \mathrm{Id}$, $\Delta \otimes \mathrm{Id}$ assigned to its edges.

Exercise 3.2. *Check that every square facet of this cube is a commutative diagram.*

We add signs to some of the edge maps so that each facet anticommutes and collapse the m-dimensional cube into a complex of graded abelian groups. The terms in the complex are given by direct sums of graded abelian groups sitting in vertices contained in a given hyperplane perpendicular to the main diagonal. We place the first term in cohomological degree 0 and the last in cohomological degree m. The result is a complex of graded abelian groups, denoted $\overline{C}(D)$, with a grading-preserving differential. Let us look at an example.

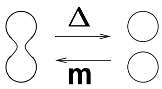
Example 3 A 3-crossing diagram $D =$  of a trefoil. We ignore the orientation of D and label the crossings by $1, 2, 3$; the resolutions are as follows. Arrows parallel the arrow labelled d_i correspond to modifications at the i-th crossing. The sequence 010 written above the bottom left diagram indicates that the first and the third crossings are modified via the 0-resolution and the second crossing–via the 1-resolution, etc.



The corresponding groups and maps are (we write $\mathbf{m}$ instead of $\mathbf{m}$ Id on top left arrows)

$$
\begin{tikzcd}
 & \mathbf{A}^{2}\{-1\} \arrow[rr, "\mathbf{m}"] \arrow[dd, "\Delta"] & & \mathbf{A}\{-2\} \arrow[dd, "\Delta"] \\
\mathbf{A}^{3} \arrow[ur, "\mathbf{m}"] \arrow[rr, "\mathbf{m}"] \arrow[dd, "\mathbf{m}"] & & \mathbf{A}^{2}\{-1\} \arrow[ur, "\mathbf{m}"] \arrow[dd, "\mathbf{m}"] & \\
 & \mathbf{A}\{-2\} \arrow[rr, "\mathbf{m}"] & & \mathbf{A}^{2}\{-3\} \\
\mathbf{A}^{2}\{-1\} \arrow[ur, "\mathbf{m}"] \arrow[rr, "\mathbf{m}"] & & \mathbf{A}\{-2\}, \arrow[ur, "\Delta"] &
\end{tikzcd}
$$

where each $\mathbf{m}$, Δ is applied according to the topology change:



$A \qquad A^{\otimes 2}$

For tensor factors for the circles that do not change, we apply Id. We add minus signs to make each square anticommutative and pass to the total complex of the cube. Due to anti-commutativity of each facet, $\mathbf{d}^2 = 0$ holds. The total complex has the form

$$\overline{\mathbf{C}}(\mathbf{D}) = \quad 0 \longrightarrow A^{\otimes 3} \xrightarrow{\mathbf{d}} \begin{array}{c} A^{\otimes 2}\{-1\} \\ \oplus \\ A^{\otimes 2}\{-1\} \\ \oplus \\ A^{\otimes 2}\{-1\} \end{array} \xrightarrow{\mathbf{d}} \begin{array}{c} A\{-2\} \\ \oplus \\ A\{-2\} \\ \oplus \\ A\{-2\} \end{array} \longrightarrow A^{\otimes 3}\{-3\} \xrightarrow{\mathbf{d}} 0 \quad .$$

For an arbitrary oriented diagram $\mathbf{D}$ we define the complex $\mathbf{C}(\mathbf{D})$ by shifting the complex $\overline{\mathbf{C}}(\mathbf{D})$ as in the formula (10). An even more elementary definition, avoiding explicit use of tensor powers, can be found in Viro [75], together with other interesting observations. The complex $\mathbf{C}(\mathbf{D})$ starts in homological degree $-\mathbf{x}(\mathbf{D})$, where $\mathbf{x}(\mathbf{D})$ is the number of negative crossings and ends in homological degree $\mathbf{y}(\mathbf{D})$, the number of positive crossings of $\mathbf{D}$. For $\mathbf{D}$ in example 3, $\mathbf{x}(\mathbf{D}) = 3$ and $\mathbf{y}(\mathbf{D}) = 0$.

Finally, define $\mathbf{H}(\mathbf{D})$ as the cohomology of the complex $\mathbf{C}(\mathbf{D})$. Note that $\mathbf{H}(\mathbf{D})$ is bigraded and, from the construction, the Euler characteristic of $\mathbf{H}(\mathbf{D})$ is the Kauffman bracket (the Jones polynomial) of $\mathbf{L}$.

Exercise 3.3. *Compute $\mathbf{H}(\mathbf{D})$ for the above diagram of the trefoil.*

The following holds [27]:

Theorem 3.4. *If two diagrams $\mathbf{D}_1$ and $\mathbf{D}_2$ are related by a chain of Reidemeister moves, the complexes of graded abelian groups $\mathbf{C}(\mathbf{D}_1)$ and $\mathbf{C}(\mathbf{D}_2)$ are homotopy equivalent and homology groups $\mathbf{H}(\mathbf{D}_1)$ and $\mathbf{H}(\mathbf{D}_2)$ are isomorphic.*

Define the link homology $\mathbf{H}(\mathbf{L}) := \mathbf{H}(\mathbf{D})$ for a diagram $\mathbf{D}$ of $\mathbf{L}$. Homology groups $\mathbf{H}(\mathbf{L})$ are known as Khovanov homology. We have

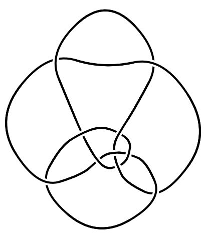
$$\mathbf{J}(\mathbf{L}) = \chi(\mathbf{H}(\mathbf{L})) = \sum_{\mathbf{i},\mathbf{j} \in \mathbb{Z}} (-1)^{\mathbf{i}} \mathbf{q}^{\mathbf{j}} \mathrm{rk} \mathbf{H}^{\mathbf{i},\mathbf{j}}(\mathbf{L}).$$

Notice that the above theorem only says that the isomorphism class of $H(L)$ as a bigraded group is an invariant of L. We'll discuss the issue of functoriality under link diffeomorphisms and, more generally, link cobordisms, in Lecture 5.

Exercise 3.5. *Let D_1, D_2 be diagrams of links L_1, L_2. Check that $C(D_1 \sqcup D_2) = C(D_1) \otimes C(D_2)$, and derive the Künneth formula for the homology of the disjoint union $L_1 \sqcup L_2$. This formula categorifies the multiplicativity property of the Jones polynomial, $J(L_1 \sqcup L_2) = J(L_1)J(L_2)$, in our normalization.*

3.3 Properties and examples

At least three programs, by D. Bar-Natan [3], A. Shumakovitch [66], D. Bar-Natan and J. Green [5] are available for computation of $H(L)$. The following tables, provided to us by A. Shumakovitch, show Khovanov homology of several knots. Given a diagram D of a knot, the complex $C(D)$ (and, hence, its homology) is nontrivial in odd internal degrees only. Thus, even q-degrees are not shown in the tables.



	-3	-2	-1	0	1	2	3	4	5	6	7
5	1										
3		4, 1_2									
1		1	6, 4_2								
-1			4	9, 6_2							
-3				7	10, 8_2						
-5					8	10, 10_2					
-7						10	8, 10_2				
-9							10	6, 8_2			
-11								8	3, 6_2		
-13									6	1, 3_2	
-15										3	1_2
-17											1

Homology of the knot 10_{121} (in Rolfsen notation)

The first table displays the homology of an alternating 10-crossing knot 10_{121}. A single integer entry a says that the homology group in that bidegree is free of rank a. Two integers a, b separated by a comma indicate that the homology group is the direct sum of Z^a and $(Z/2)^b$. For instance, $H^{5,-13} = Z^6$ and $H^{1,-3} = Z^{10} \oplus (Z/2)^8$. Calculations done by Shumakovitch [67] show that the only torsion in the homology of knots with 14 or fewer crossings is $Z/2$-torsion. Shumakovitch found several knots with 15 and 16 crossings whose homology contains a copy of $Z/4$. One of these knots is the $(4,5)$-torus knot. For more information and results on torsion see [67], [2].

Looking at the table, we notice that the homology spans horizontal degrees from -3 to 7, and the knot has homological width 10, equal to the number of crossings of the diagram. By the *homological width* of L we mean the di erence between the largest i such that $H^{i,j}(L) = 0$ for some j and the minimal i with the same property. The homological width of L gives a lower bound on the crossing number of L (the smallest number of crossings in a planar diagram of L).

Exercise 3.6. *Determine the geometric condition on D which ensures that $C(D)$ has nontrivial homology in the leftmost and in the rightmost degrees (such diagrams D are called* adequate*).*

Thus, if L has an n-crossing adequate diagram, the crossing number of L is n. This was originally proved by Thistlethwaite [73] using the 2-variable Kau man polynomial (not to be confused with the Kau man bracket, which is a one-variable polynomial). One of the Tait conjectures about the crossing number of alternating links (originally proved with the help of the Jones polynomial [25], [49]) follows from this result. This alternative approach to the Thistlethwaite theorem does not use the internal grading on the homology, only the horizontal grading, almost invisible on the level of Euler characteristic.

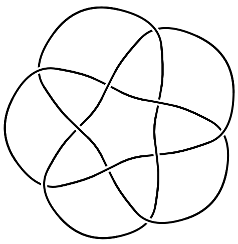
Apparently, the only known explicit relation between the 2-variable Kau -man polynomial and Khovanov homology is that they both specialize to the Jones polynomial. Yet, both can be used to prove the Thistlethwaite theorem and to give upper bounds on the Thurston-Bennequin number of Legendrian links, see L. Ng [51] and references therein.

Diagonal width of the homology gives a lower bound on the Turaev genus of a knot [45]. For relations between homology and contact topology see [54] and references therein.

The total rank of the complex $C(D)$ grows very fast as a function of the size of D. Thistlethwaite's spanning tree model for the Jones polynomial admits a categorification [78], [11], giving a complex of much smaller rank which also computes $H(D)$, but no combinatorial formula for the di erential of the resulting complex is known, except in very special cases.

The homology of 10_{121} occupies two adjacent diagonals. It is easy to see that the homology cannot lie on just one diagonal, so two is the minimum. E. S. Lee [39] proved that the homology of any alternating knot $\mathbf{L}$ lies on two adjacent diagonals consisting of $(\mathbf{i}, \mathbf{j})$ with $2\mathbf{i} - \mathbf{j} = \mathbf{c} \pm 1$ where $\mathbf{c}$ is the signature of $\mathbf{L}$. Computer calculations show this to be true for most non-alternating knots with 11 or fewer crossings as well [3], [66]. A partial explanation of this phenomenon was provided by C. Manolescu and P. Ozsváth [46] by extending Lee's result to quasi-alternating knots.

The next table shows the homology of the alternating knot 10_{123}. This knot is amphicheiral, that is, isomorphic to its mirror image.



	-5	-4	-3	-2	-1	0	1	2	3	4	5
11	1										
9		4, 1_2									
7		1	6, 4_2								
5			4	9, 6_2							
3				6	10, 9_2						
1					9	11, 10_2					
-1						11	9, 10_2				
-3							10	6, 9_2			
-5								9	4, 6_2		
-7									6	1, 4_2	
-9										4	1_2
-11											1

Homology of the knot 10_{123} (in Rolfsen notation)

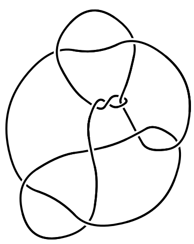
For a link $\mathbf{L}$ the mirror image $\mathbf{L}^!$ is given by reversing the orientation of the ambient $\mathbf{R}^3$ in which the 1-manifold is embedded. By reversing all crossings of a diagram $\mathbf{D}$ of $\mathbf{L}$ we obtain the diagram $\mathbf{D}^!$ of $\mathbf{L}^!$.

Exercise 3.7. *Construct an isomorphism of complexes*

$$\mathbf{C}(\mathbf{D}^!) = \mathrm{Hom}_Z(\mathbf{C}(\mathbf{D}), Z).$$

Thus, the dual $\mathbf{C}(\mathbf{D})$ of the of complex $\mathbf{C}(\mathbf{D})$ is naturally isomorphic to $\mathbf{C}(\mathbf{D}^!)$. This duality takes $\mathbf{C}^{\mathbf{i},\mathbf{j}}(\mathbf{D})$ to the dual of the free abelian group $\mathbf{C}^{-\mathbf{i},-\mathbf{j}}(\mathbf{D})$. Passing to homology, we see that the free part of $\mathbf{H}^{\mathbf{i},\mathbf{j}}(\mathbf{L})$ becomes the dual of the free part of $\mathbf{H}^{-\mathbf{i},-\mathbf{j}}(\mathbf{L}^!)$. By the free part of a finitely-generated abelian group G we mean the quotient $G/\mathrm{Tor}(G)$ of G modulo torsion. In particular, abelian groups $H^{i,j}(L)$ and $H^{-i,-j}(L^!)$ have the same rank. The torsion of $H^{i,j}(L)$ is the dual of the torsion of $H^{-i+1,-j}(L^!)$ (notice the grading shift), in particular, the two torsion groups have the same rank. The reader can see this duality in the above homology table of $10_{123} = 10^!_{123}$. The first integer in the (i,j)-entry equals the first integer in the $(-i,-j)$-entry. The torsion, only present on the upper diagonal, stays on the upper diagonal after dualization, due to the shift by 1. For instance, $H^{5,-9} = \mathbb{Z}/2$ becomes, after dualization, the torsion subgroup $\mathbb{Z}/2$ of $H^{-4,9}$.

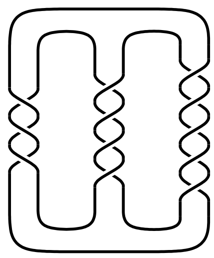
Recall that, for the singular chain complex $C(X)$ of a topological space X which computes the homology groups of X, the dual complex $C(X)$ computes the cohomology groups of X. In the link homology framework, the duality works in a different way–the underlying link is converted to its mirror image. This manifests our terminological imperfection in calling groups $H(L)$ the *homology* groups of L. We would be equally justified in calling them *cohomology* groups. In the next two lectures we'll discuss functoriality of H. To a link cobordism S from L_1 to L_2 we assign a homomorphism $H(S) : H(L_1) \longrightarrow H(L_2)$, which, over all S, gives a covariant functor. However, there is an equally natural construction that assigns to S the homomorphism going in the opposite direction, producing a contravariant functor. We see that $H(L)$ exhibits both covariant and contravariant behaviour and, in a flexible terminological environment, we are free to call $H(L)$ either homology of cohomology. Another solution is to call $H(L)$ bivariant (co)homology groups.



	-2	-1	0	1	2	3	4	5	6	7	8	9
1	**1**											
-1		$\mathbf{1}_2$										
-3		**1**	**2**									
-5			**1**	$\mathbf{1}, \mathbf{1}_2$	**1**							
-7				**1**	$\mathbf{1}, \mathbf{1}_2$	$\mathbf{1}, \mathbf{1}_2$						
-9					**1**	$\mathbf{2}, \mathbf{1}_2$	$\mathbf{1}, \mathbf{1}_2$					
-11						**1**	$\mathbf{2}, \mathbf{1}_2$	$\mathbf{2}, \mathbf{1}_2$				
-13							**1**	$\mathbf{1}, \mathbf{1}_2$	$\mathbf{1}, \mathbf{2}_2$			
-15								**1**	**2**	$\mathbf{1}, \mathbf{1}_2$		
-17										**1**	$\mathbf{1}, \mathbf{1}_2$	
-19											**1**	$\mathbf{1}_2$
-21												**1**

Homology of the knot 11^n_{31} (in Knotscape notation)

In the third table we see an 11-crossing non-alternating knot whose homology occupies 3 adjacent diagonals. This knot is adequate, and the width of the homology is 11. This diagram D has 2 negative and 9 positive crossings, and the groups are bounded by homological degrees -2 and 9. Unlike the previous two examples, each homology group has small rank (at most two).



	-7	-6	-5	-4	-3	-2	-1	0
13	**1**							
11		1_2						
9		**1**	**1**					
7				**1**, 1_2				
5				**1**	1_2			
3					**1**	**1**		
1							1_2	**1**
-1							**1**	**1**

Homology of the (-3,4,5)-pretzel knot

The fourth table shows the (-3,4,5)-pretzel knot and its homology. This time the rank of each group is at most one. Homological width of this knot equals 7, much less then 12, the crossing number of the knot.

The simplest knot known to have odd torsion in its homology is the (5,6)-torus knot, see the table below. It has a $\mathbf{Z}/3$-summand in bidegree (14,-43) and $\mathbf{Z}/5$-summands in bidegrees (11,-35) and (12,-49).

This is a positive knot (all the crossings look like in the planar diagram D given by the closure of the braid $(\ _1\ _2\ _3\ _4)^6$), so it has homology groups in nonnegative homological degrees only. You can check by hand the correctness of the table in the homological degree 0 by computing the kernel of the leftmost differential

$$0 \longrightarrow C^0(D) \xrightarrow{d} C^1(D) \longrightarrow \dots$$

in the diagram D. For various results on homology of positive and torus knots and links see [68], [69].

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
-19	1														
-21	1														
-23			1												
-25				1_2	1										
-27				1	1		1								
-29						1	1	1_2	1						
-31						1		$1, 1_2$	2						
-33								1		$1, 1_2$	1				
-35										2	1_2	$1_2\boxed{1_5}$	1		
-37											1_2	2	1		
-39													$1_2\boxed{1_5}$	1	
-41													1	1	
-43															$\boxed{1_3}$

Homology of the (5,6)-torus knot

4 Flat tangles and bimodules

4.1 Two-dimensional TQFTs and Frobenius algebras

Definition 4.1. *A 2-dimensional TQFT (topological quantum field theory) is a tensor functor from the category of two-dimensional oriented cobordisms between oriented closed one-manifolds to an additive tensor category.*

Such functor F satisfies $F(X \quad Y) = F(X) \quad F(Y)$ for 1-manifolds X, Y and $F(f \quad g) = F(f) \quad F(g)$ for cobordisms f, g. We won't define here additive tensor categories, but rather provide examples:

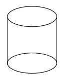
- the category of vector spaces over a field,
- the category of graded vector spaces over a field,
- the category of free modules over a commutative ring R,
- the category of complexes of free modules over a commutative ring R modulo chain homotopies.

In these examples the tensor structure is the obvious one (the tensor product is taken over R in the last two). We restricted to free R-modules since, in homological algebra, for general R-modules the tensor product $M \quad {}_R N$ must be redefined via a free or projective resolution of M or N.

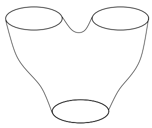
A 2-dimensional TQFT F with values in the category of free R-modules must assign R to the empty 1-manifold, $F(\ ) = R$, and some free R-module A to the circle, $F(\bigcirc) = A$. Then

$$F(\underbrace{\bigcirc \cdots \bigcirc}_{j}) = A^{\ j}$$

since the functor F is tensor. To the identity cobordism F assigns the identity map

F  $= \mathrm{id}_A.$

To the *inverted pants* cobordism

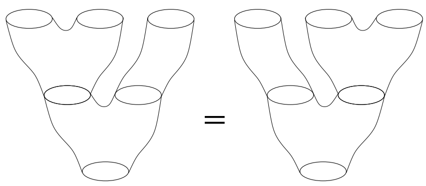


$A^{\ 2}$

m

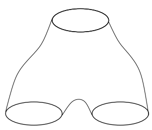
A

F assigns a homomorphism $m : A^{\ 2} - \ A$, associative due to the equality of cobordisms

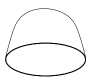


The two upper legs of the multiplication cobordisms may be permuted without changing the di eomorphism type of the cobordism. Thus m is commutative as well.

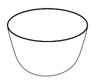
The following three cobordisms induce three more maps between tensor power of A, denoted $\Delta,\ $ and $\ $, respectively, which make A into a commutative Frobenius algebra over R.



A $\quad\Delta\quad$ $A^{\ 2}$



k $\quad$ A



A $\quad$ k

It is well known [37], [6, Section 4.3], [23] that two-dimensional TQFTs with the target category being the category of free modules over a commutative ring R are classified by commutative Frobenius R-algebras A. The Frobenius property means that

$$A \cong A^* := \mathrm{Hom}_R(A, R)$$

as an A-module. The isomorphism takes $1 \in A$ to an R-linear trace map $\varepsilon : A \to R$ which is non-degenerate, meaning that the above map $a \to \varepsilon(a\,\cdot)$ from A to A^* is an isomorphism. When R is a field, ε is nondegenerate iff $\forall a \in A \setminus \{0\}\ \exists b$ such that $\varepsilon(ab) \neq 0$. Given ε as above, we can reconstruct Δ as the dual of m:

$$\Delta : A \cong A^* \xrightarrow{m^*} (A \otimes A)^* \cong A \otimes A.$$

Example 4.2. *The direct sum of even-dimensional cohomology groups* $H^{\mathrm{even}}(M, R)$ *of a closed oriented* $2n$*-dimensional manifold* M *is a commutative Frobenius* R*-algebra, with the trace map given by the integration over the fundamental* $2n$*-cycle.*

Example 4.3. *Let* R *be a field and* $f \in \mathbb{C}[x_1, \dots, x_m]$ *a polynomial. If the quotient algebra* A *of* $\mathbb{C}[x_1, \dots, x_m]$ *by the ideal generated by all partial derivatives* $\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_m}$ *is finite-dimensional then* A *is Frobenius. This example comes up in singularity theory, see [1].*

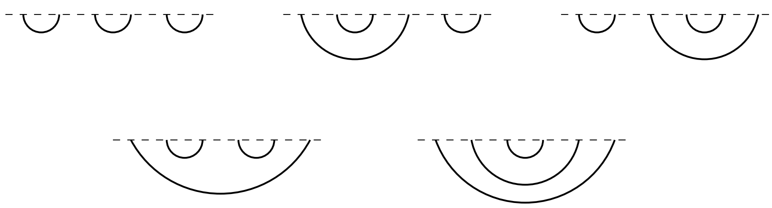
The above two types of Frobenius algebras have a nonempty intersection. For instance, if $f = x^{n+1} \in \mathbb{Q}[x]$ in the second example then $A = \mathbb{Q}[x]/(x^n)$, isomorphic to the cohomology ring of the complex projective space $\mathbb{CP}^{n-1}$. Notice that we only get a countable number of commutative Frobenius algebras (up to isomorphism) from Example 4.2, but an uncountable number from Example 4.3. Both examples are important sources of 2-dimensional TQFT's.

4.2 Algebras H^n

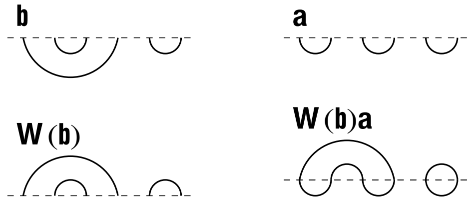
Our goal in this lecture and the next one is to extend link homology to tangles and tangle cobordisms. We start with an arbitrary Frobenius algebra A over R and construct an invariant of flat (or crossingless) tangles.

Consider $2n$ points on a horizontal line and denote by B^n the set of crossingless matchings of these points by n arcs lying in the lower half-plane. The cardinality of B^n is the n-th Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

Example 4.4. *The set* B^3 *has 5 elements*



Let $W(b)$ be the reflection of a matching b along the horizontal line. For $a, b \quad B^n$, the composition $W(b)a$ makes sense and can be viewed as a closed 1-manifold.



Applying the functor F to it, we get $F(W(b)a)$, which is a tensor power of A.

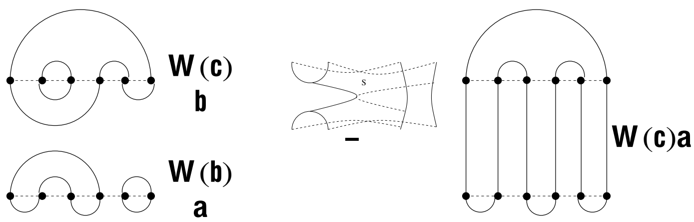
For each $n \geq 0$ we define the ring H^n by

$$H^n := {}_{a,b \quad B^n} F(W(b)a).$$

The multiplication in H^n is built out of compositions

$$F(W(c)b) \quad F(W(b)a) - \quad F(W(c)a)$$

induced by cobordisms from $W(c)bW(b)a$ to $W(c)a$ which contract b with $W(b)$:



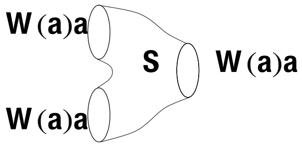
$$F$$

$$F(W(c)b) \quad F(W(b)a) \qquad - \qquad F(W(c)a)$$

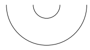
$$H^n \quad H^n \qquad -^m \qquad H^n$$

For $\mathbf{x}$ $\mathbf{F}(\mathbf{W}(\mathbf{d})\mathbf{c})$ and $\mathbf{y}$ $\mathbf{F}(\mathbf{W}(\mathbf{b})\mathbf{a})$ the product $\mathbf{xy} = 0$ if $\mathbf{c} = \mathbf{b}$. For $\mathbf{n} = 0$ the set $\mathbf{B}^0$ contains only the empty diagram, and $\mathbf{H}^0 = \mathbf{R}$, the ground ring.

Example 4.5. *The set* $\mathbf{B}^1$ *consists of the single diagram* $\{\ \smile\ \}$ *which we denote* $\mathbf{a}$. $\mathbf{H}^1 = \mathbf{F}(\mathbf{W}(\mathbf{a})\mathbf{a}) = \mathbf{A}$. *The product is given by* $\mathbf{F}(\mathbf{S}) : \mathbf{A}^{2} \quad \mathbf{A}$, *where*



Thus, the product in $\mathbf{H}^1$ *is the multiplication in* $\mathbf{A}$ *and* $\mathbf{H}^1 = \mathbf{A}$ *as an associative* $\mathbf{R}$*-algebra.*

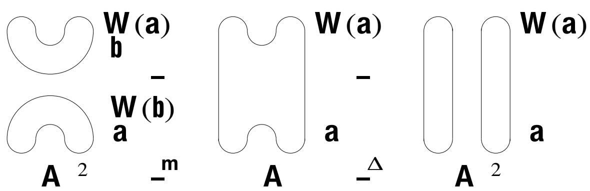
Example 4.6. *For* $\mathbf{n} = 2$ *the set* $\mathbf{B}^2$ *has two elements* $\mathbf{a} = \smile \quad \smile$ *and* $\mathbf{b} =$ .

$$\mathbf{H}^2 = \mathbf{F}(\mathbf{W}(\mathbf{a})\mathbf{a}) \quad \mathbf{F}(\mathbf{W}(\mathbf{b})\mathbf{a}) \quad \mathbf{F}(\mathbf{W}(\mathbf{a})\mathbf{b}) \quad \mathbf{F}(\mathbf{W}(\mathbf{b})\mathbf{b})$$

An example of multiplication

$$\mathbf{F}(\mathbf{W}(\mathbf{a})\mathbf{b}) \quad \mathbf{F}(\mathbf{W}(\mathbf{b})\mathbf{a}) - \quad \mathbf{F}(\mathbf{W}(\mathbf{a})\mathbf{a})$$

is the composition of morphisms:



Exercise 4.7. *Functor* $\mathbf{F}$ *is defined on oriented cobordisms only. Find a consistent way to equip 1-manifolds* $\mathbf{W}(\mathbf{b})\mathbf{a}$ *and the multiplication cobordisms with orientations to make legitimate the above application of functor* $\mathbf{F}$.

Exercise 4.8. *Use the functoriality of* $\mathbf{F}$ *to show that the multiplication in* $\mathbf{H}^{\mathbf{n}}$ *is associative.*

Exercise 4.9. *Define* $1_a \in H^n$ *as* $1^{\otimes n} \in A^{\otimes n} = F(W(a)a)$. *Show that* $x1_a = x$ *for any* $x \in F(W(b)a)$ *and* $1_a y = y$ *for any* $y \in F(W(a)b)$. *Check that* $\{1_a\}_{a \in B^n}$ *are mutually orthogonal idempotents and*

$$1 = \sum_{a \in B^n} 1_a$$

is the unit element of H^n.

Remark 4.10. *Observe the similarity with the setting of the ring* A_n *from Lecture 1, with idempotents* 1_a *of* H^n *analogous to idempotents* (i) *of* A_n. *The latter were used to define projective* A_n*-modules* $P_i = A_n(i)$. *Likewise, any* $a \in B^n$ *produces a left projective* H^n*-module*

$$P_a := H^n 1_a = \bigoplus_{b \in B^n} F(W(b)a)$$

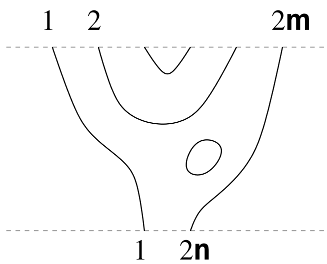
and a right projective H^n*-module*

$$_aP := 1_a H^n = \bigoplus_{b \in B^n} F(W(a)b).$$

To summarize, H^n is a unital associative R-algebra built out of a commutative Frobenius R-algebra A. For $n > 1$ the algebra H^n is noncommutative.

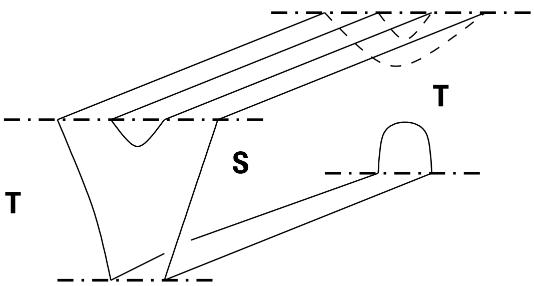
4.3 Flat tangles and their cobordisms

By a *flat* or *crossingles* tangle we mean a finite collection of arc and circles properly embedded in $R \times [0, 1]$.



We require that the number of top endpoints be even, which implies that the number of bottom endpoints is even as well, and call a flat tangle with $2m$ top and $2n$ bottom endpoints a flat (m, n)-tangle. We also fix once and for all the position of $2n$ points on R, to make flat tangles easy to compose. The composition of a flat (k, m)-tangle and a flat (m, n)-tangle is a flat (k, n)-tangle.

By a cobordism S between flat (m, n)-tangles T, T we mean a surface properly embedded in $R \times [0, 1] \times [0, 1]$ with the boundary comprised of T, T and the product 1-manifold $\partial T \times [0, 1] = \partial T \times [0, 1]$. We think of $\partial T \times \{0\}$ as the boundary of T and $\partial T \times \{1\}$ as the boundary of T.



The upper part of T is shown by dashed lines; the corner R's of the 3-manifold $R \times [0, 1] \times [0, 1]$ are indicated by dashed-dotted lines. Surface S has $4n + 4m$ corner points.

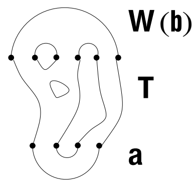
There are two ways to compose these cobordisms. If S_1 is a cobordism from T to T and S_2 a cobordism from T to T, we can glue them along T to produce the cobordism $S_2 \ S_1$ from T to T. If S_1 is a cobordism between flat (m, n)-tangles T_1 and T_1 and S_2 a cobordism between flat (k, m)-tangles T_2 and T_2, we can compose S_2 and S_1 along the one-manifold $\{2m \text{ points}\} \times [0, 1]$ to get the cobordism $S_2 S_1$ from $T_2 T_1$ to $T_2 T_1$.

This structure can be encoded into the 2-category of flat tangle cobordisms. The objects of this 2-category are nonnegative integers n, one-morphisms from n to m are flat (m, n)-tangles T, two-morphisms from T to T are isotopy classes rel boundary of flat tangle cobordisms S.

To a given flat (m, n)-tangle T we assign the R-module

$$F(T) := \bigoplus_{a \ B^n, b \ B^m} F(W(b) T a).$$

In other words, we consider all possible ways to close up T by crossingless matchings a and b at the bottom and the top, respectively, to produce a closed 1-manifold $W(b)Ta$, and apply the functor F to each closure.



$F(T)$ is actually an (H^m, H^n)-bimodule, that is, it has a right H^m-action and a commuting left H^n-action. In the rest of the notes, we call an (H^m, H^n)-bimodule simply an (m, n)-bimodule, and assume that in these bimodules the left and the right action of R are equal. The action of H^m on $F(T)$ comes from maps

$$\begin{array}{ccccc} F(W(c)b) & \times & F(W(b)Ta) & - & F(W(c)Ta) \\ & & & & \\ H^m & \times & F(T) & - & F(T) \end{array}$$

Example 4.11. *F applied to the identity flat (n, n)-tangle produces H^n viewed as an H^n–bimodule:*

$$F \;\; \Big| \; \Big| \; \Big| \; \Big| \; \Big| \; \Big| \;\; = H^n.$$

Example 4.12. *A crossingless matching $a \quad B^m$ is a flat $(m, 0)$-tangle and the bimodule $F(a)$ is simply the left H^m-module P_a, see Remark 4.10 (notice that $(m, 0)$-bimodules are just left H^m-modules, since $H^0 = R$, the ground ring). Likewise, $F(W(a)) = {}_aP$ is a right projective H^m-module.*

Example 4.13. *A flat $(0, 0)$-tangle T is a closed 1-manifold embedded in the plane, $(0, 0)$-bimodules are just R-modules, and $F(T) = A^{\;\;r}$, where r is the number of components of T.*

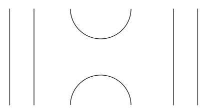
The composition of flat tangles T_2T_1 corresponds to the tensor product of bimodules

$$F(T_2T_1) = F(T_2) \quad {}_{H_m} F(T_1)$$

for a flat (k, m)-tangle T_2 and a flat (m, n)-tangle T_1, see [28, Theorem 1].

If we fix n and consider only flat (n, n)-tangles T and H^n-bimodules $F(T)$ we get a functor realization of the Temperley-Lieb algebra TL_{2n}. In Lecture

1 we already constructed a realization of TL_{n+1} by A_n-bimodules, with the generators

$u_i =$ 

of the Temperley-Lieb algebra represented by A_n-bimodules U_i. Recall that $U_i \otimes U_j = 0$ if $|i-j|>1$, while $u_i u_j = 0$ in TL algebra, so our bimodule realization was degenerate. On the other hand, $F(T)$ is a non-trivial bimodule for any flat (n,n)-tangle T. Also, in the current setting a closed loop evaluates to the R-module A. For instance, $F(u_i u_i) = F(u_i) \otimes_R A.$ When we pass to ranks, the value of the closed loop becomes the rank of A as a free R-module, a positive integer.

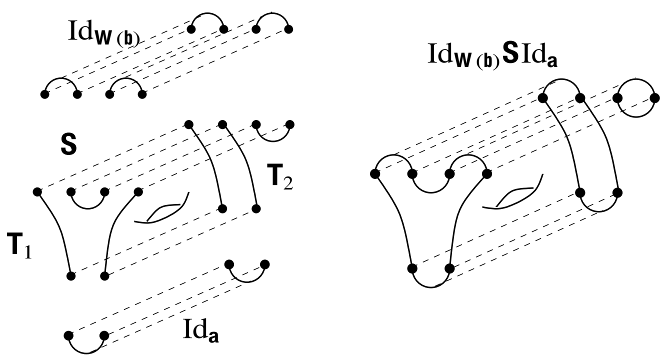
To get more general values for the closed loop we extend the framework of commutative Frobenius R-algebras A and rings H^n to the graded case, by requiring that A be a graded R-algebra and the morphism $F(S)$ associated with a 2-dimensional cobordism S be homogeneous of degree proportional to the Euler characteristic of S. Under these assumptions, the rings H^n and bimodules $F(T)$ become graded [28]. A closed loop still corresponds to A. Upon decategorification, closed loop evaluates to the graded rank of A as R-module and takes value in $\mathbb{N}[q,q^{-1}]$. In the simplest nontrivial case of the graded pair (R,A) described in Lecture 3 the value of the closed loop is $q+q^{-1}$, the standard value of the loop in the Temperley-Lieb algebra.

Let S be a cobordism in $\mathbb{R} \times [0,1] \times [0,1]$ between flat (m,n)-tangles T_1 and T_2. To S we assign a homomorphism

$$F(S) \ : \ F(T_1) \longrightarrow F(T_2)$$

between (m,n)-bimodules. For $a \in B^n$ and $b \in B^m$ we can compose S with the identity cobordism Id_a from a to a and the identity cobordism $\mathrm{Id}_{W(b)}$ from $W(b)$ to $W(b)$ to get the cobordism $\mathrm{Id}_{W(b)} S \mathrm{Id}_a$ from the closed 1-manifold

$\mathbf{W}(\mathbf{b})\mathbf{T}_1\mathbf{a}$ to $\mathbf{W}(\mathbf{b})\mathbf{T}_2\mathbf{a}$.



This cobordism induces a homomorphism $\mathbf{F}(\mathbf{W}(\mathbf{b})\mathbf{T}_1\mathbf{a}) \longrightarrow \mathbf{F}(\mathbf{W}(\mathbf{b})\mathbf{T}_2\mathbf{a})$. Summing over all $\mathbf{a}$ and $\mathbf{b}$ we get a homomorphism of $(\mathbf{m},\mathbf{n})$-bimodules $\mathbf{F}(\mathbf{S}) : \mathbf{F}(\mathbf{T}_1) \longrightarrow \mathbf{F}(\mathbf{T}_2)$. It is straightforward to check that $\mathbf{F}(\mathbf{S})$ is natural with respect to both types of compositions of cobordisms. When all the properties are written down, we discover that $\mathbf{F}$ becomes a 2-functor from the 2-category of flat tangle cobordisms to the 2-category of $(\mathbf{m},\mathbf{n})$-bimodule homomorphisms. Objects of the latter 2-category are nonnegative integers $\mathbf{n}$, 1-morphisms from $\mathbf{n}$ to $\mathbf{m}$ are $(\mathbf{m},\mathbf{n})$-bimodules and 2-morphisms are bimodule homomorphisms. Composition of bimodules is given by the tensor product. The 2-functor $\mathbf{F}$ is the identity on objects, $\mathbf{n} \longmapsto \mathbf{n}$, takes a flat tangle $\mathbf{T}$ to the bimodule $\mathbf{F}(\mathbf{T})$, and flat tangle cobordism $\mathbf{S}$ to the bimodule homomorphism $\mathbf{F}(\mathbf{S})$. This 2-functor converts topological information about 1-manifolds embedded in the plane and 2-manifolds embedded in $\mathbb{R}^3$ into the algebraic information provided by bimodules and bimodule homomorphisms.

The following table summarizes the construction of the 2-functor $\mathbf{F}$.

	2-category of flat tangle cobordisms	$\xrightarrow{\mathbf{F}}$	2-category of bimodule homomorphisms
objects	$\mathbf{n} = 0, 1, 2, \dots$	$\longrightarrow$	$\mathbf{n} = 0, 1, 2, \dots$
1-morphisms	flat $(\mathbf{m},\mathbf{n})$-tangles $\mathbf{T}$	$\longrightarrow$	$(\mathbf{m},\mathbf{n})$-bimodules $\mathbf{F}(\mathbf{T})$
2-morphisms	flat tangle cobordisms $\mathbf{S}$	$\longrightarrow$	bimodule homomorphisms $\mathbf{F}(\mathbf{S})$

5 A homological invariant of tangles and tangle cobordisms

5.1 An invariant of tangles

In the previous lecture we described a 2-functor from the 2-category of cobordisms between flat tangles to the 2-category of bimodule maps. Such functor exists for any Frobenius R-algebra A. In this lecture, for a specific (R, A), we extend the construction to a 2-functor from the 2-category of tangle cobordisms to the 2-category of homomorphisms between complexes of bimodules (up to chain homotopy). Going one dimension up, from flat tangles, which are 2-dimensional objects, to tangles, which are 3-dimensional, exactly corresponds to passing from the abelian category of bimodules to the triangulated category of complexes of bimodules. Ditto for cobordisms, which are 3-dimensional between flat tangles and 4-dimensional between tangles. Thus, in the framework described here, the key transformation in algebra

abelian categories $=$ triangulated categories

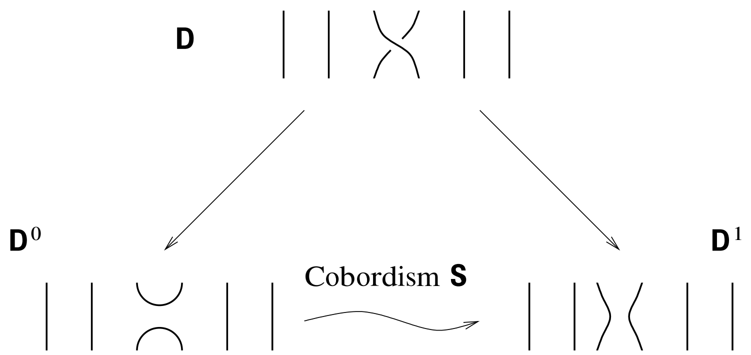
is mirrored in low-dimensional topology by the transformation

(2+1)-dimensional structures $=$ (3+1)-dimensional structures.

We specialize the construction of the previous lecture to $R = Z$ and $A = Z[X]/(X^2)$ with the trace $(X) = 1,\ (1) = 0$. The ring H^n is made graded by defining $H^n = \ _{a,b\ \ B^n} F(W(b)a)\{n\}$. The multiplication in H^n is grading-preserving and $\deg(1_a) = 0$. For a flat (m, n)-tangle T the (m, n)-bimodule $F(T)$ is graded.

Start with an oriented (m, n)-tangle T. Just as in the flat case, we assume that $T \quad R^2 \times [0, 1]$ has $2n$ bottom endpoints placed in the standard position on the plane $R^2 \times \{0\}$ and $2m$ top endpoints in standard position on $R^2 \times \{1\}$. Oriented tangles can be composed in the same way as flat tangles, assuming that the orientations at the endpoints match. A tangle cobordism S between (m, n)-tangles T_1, T_2 is an oriented smooth surface in $R^2 \times [0, 1]^2$ subject to the boundary conditions mirroring those for flat tangles. In particular, the boundary of S consists of four pieces, two of which are T_1, T_2 and the other two are product 1-manifolds. Let $CobT$ be the 2-category of tangle cobordisms. Its objects are finite sequences of pluses and minuses, its 1-morphisms are tangles with prescribed orientations at the endpoints, and 2-morphisms are tangle cobordisms up to rel boundary isotopies.

We choose a diagram D of a tangle T, a generic projection of T onto the plane, with the endpoints projecting in the standard way. Assume that D has a single crossing. Let D^0, D^1 be unoriented flat tangles obtained by resolving the crossing of D.



Let S be the simplest cobordism between D^0 and D^1; it has one saddle point for the projection $S \subset \mathbb{R} \times [0,1]^2 \rightarrow [0,1]$. Cobordism S induces a degree 1 morphism $F(S) : F(D^0) \rightarrow F(D^1)$ between graded (m,n)-bimodules. We define $F(D)$ as the chain complex

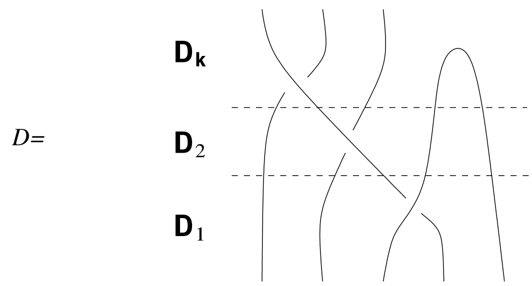
$$F(T) := (0 \rightarrow F(D^0) \xrightarrow{F(S)} F(D^1)\{-1\} \rightarrow 0)$$

of graded bimodules with a grading-preserving differential, with $F(D^0)$ in homological degree 0. Recalling that D is a diagram of an oriented tangle T, we set

$$F(D) := F(D)[x(D)]\{2x(D) - y(D)\},$$

where, as before, $x(D)$ and $y(D)$ is the number of negative and positive crossings of D.

If D is an arbitrary tangle diagram, decompose D as the composition of diagrams with at most one crossing each, $D = D_k \cdots D_2 D_1$



and define

$$\mathbf{F}(\mathbf{D}) := \mathbf{F}(\mathbf{D}_k) \quad \cdots \quad \mathbf{F}(\mathbf{D}_2) \quad \mathbf{F}(\mathbf{D}_1),$$

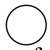
with the tensor product over rings $\mathbf{H}^k$, for suitable $\mathbf{k}$'s equal to half the number of top/bottom endpoints for intermediate diagrams $\mathbf{D}_i$. This is a complex of graded $(\mathbf{m}, \mathbf{n})$-bimodules. Let $\mathcal{C}_{m,n}$ be the category of complexes of graded $(\mathbf{m}, \mathbf{n})$-bimodules up to chain homotopies.

Theorem 5.1. *If diagrams $\mathbf{D}_1, \mathbf{D}_2$ of an $(\mathbf{m}, \mathbf{n})$-tangle $\mathbf{T}$ are related by a chain of Reidemeister moves, then $\mathbf{F}(\mathbf{D}_1) = \mathbf{F}(\mathbf{D}_2)$ in $\mathcal{C}_{m,n}$.*

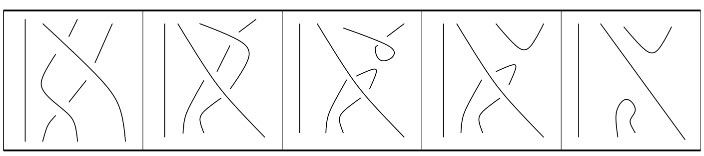
The proof consists of constructing an explicit homotopy equivalence $\mathbf{F}(\mathbf{D}_1) = \mathbf{F}(\mathbf{D}_2)$ for two diagrams related by a Reidemeister move [28].

Example 5.2. $\mathbf{n} = \mathbf{m} = 0$. *In this case the tangle $\mathbf{T}$ is a link, and the ring $\mathbf{H}^0 = \mathbb{Z}$. $\mathbf{F}(\mathbf{D}) = \mathbf{C}(\mathbf{D})$ is a complex of graded abelian groups, and its homology groups $\mathbf{H}(\mathbf{F}(\mathbf{D}))$ coincide with the link homology $\mathbf{H}(\mathbf{T})$.*

5.2 Tangle cobordisms

Let $\mathbf{S}$ be a tangle cobordism from tangle $\mathbf{T}_0$ to tangle $\mathbf{T}_1$. The two tangles can be represented by their planar diagrams $\mathbf{D}_0, \mathbf{D}_1$. Likewise, we can represent $\mathbf{S}$ by a sequence of planar diagrams of intersections of $\mathbf{S}$ and $\mathbb{R}^2 \times [0,1] \times \{\mathbf{t}\}$ for various $\mathbf{t}$ $[0,1]$. Such representation is called a *movie* of $\mathbf{S}$. Consecutive diagrams $\mathbf{D}_0 = \mathbf{D}^0, \mathbf{D}^1, \ldots, \mathbf{D}^m = \mathbf{D}_1$ in a movie differ by either a Reidemeister move or a move that corresponds to going through a critical point of the projection $\mathbf{S} \to [0,1]$. There are 3 types of critical point moves. The saddle move  $)\,($ correspond to passing through an index 1 critical point. Creation and annihilation moves  correspond to critical points of index 0 and 2, respectively. An example of a movie is as follows.

$\mathbf{S} =$



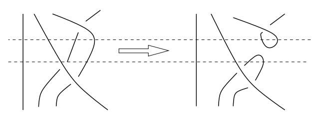
$\mathbf{D}_0 = \mathbf{D}^0 \qquad \mathbf{D}^1 \qquad \mathbf{D}^2 \qquad \mathbf{D}^3 \qquad \mathbf{D}^4 = \mathbf{D}_1$

We often denote a movie representing a cobordism $\mathbf{S}$ by $\mathbf{S}$ as well. To a Reidemeister move between $\mathbf{D}^i$ and $\mathbf{D}^{i+1}$ we assign the isomorphism $\mathbf{F}(\mathbf{D}^i) \xrightarrow{=}$

$F(D^{i+1})$ mentioned earlier. To a degree 0 critical point move from D^i to D^{i+1} we assign the map

$$F(D^i) = F(D^i) \otimes \mathbb{Z} \xrightarrow{\mathrm{Id}\otimes\iota} F(D^i) \otimes A = F(D^{i+1})$$

induced by the unit map $\iota : \mathbb{Z} \to A$ from $F(\emptyset)$ to $F(\text{circle})$. To a degree 2 critical point move we assign the map induced by the trace homomorphism $\epsilon : A \to \mathbb{Z}$. The map assigned to a degree 1 critical point move was essentially described earlier in the lecture, as the bimodule homomorphism induced by the standard cobordism between two resolutions of a crossing. For instance, in S shown above, diagrams D^1 and D^2 are related by such a move (saddle move). We can decompose each D^1 and D^2 as the composition of 3 diagrams, with only the middle diagrams being different



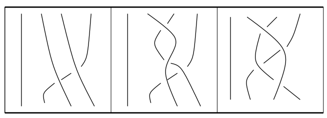
D^1 D^2

and define $F(D^1) \to F(D^2)$ as the composition of the identity map on the first and the third terms and the homomorphism of the middle terms induced by the saddle point cobordism between two crossingless tangles.

We define $F(S)$ as the composition of homomorphisms $F(D^i) \to F(D^{i+1})$ associated to frame changes $D^i \to D^{i+1}$. Notice that homomorphisms associated to Reidemeister moves are invertible in $\mathcal{C}_{m,n}$, the category of complexes of graded (m,n)-bimodules modulo chain homotopies. For instance, for the movie S drawn above, the homomorphism $F(D^1) \to F(D^2)$ is the only non-invertible one.

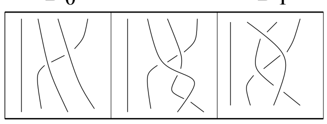
It is a theorem of S. Carter and M. Saito [7] that two movies represent the same tangle cobordism S iff they can related by a sequence of *movie moves*. A movie move converts a certain sequence of frames to another sequence of frames representing the same cobordism. Here's an example:

S_0 =



D_0 D_1

S_1 =



D_0 D_1

We refer the reader to [7] or [29] for a complete list of movie moves.

Theorem 5.3. *If movies* $\mathbf{S}_0$ *and* $\mathbf{S}_1$ *represent isotopic tangle cobordisms,* $\mathbf{F}(\mathbf{S}_0) = \pm\mathbf{F}(\mathbf{S}_1)$ *in* $\mathbf{C}_{\mathbf{m},\mathbf{n}}$.

Proving this statement amount to checking the invariance of $\mathbf{F}$ up to a sign for each movie move. We only explain it for the above example of a movie move. We have two morphisms: $\mathbf{F}(\mathbf{S}_0), \mathbf{F}(\mathbf{S}_1) : \mathbf{F}(\mathbf{D}_0) \quad \mathbf{F}(\mathbf{D}_1)$. They are isomorphisms of complexes of bimodules in the homotopy category $\mathbf{C}_{\mathbf{n},\mathbf{n}}$, since both movie moves $\mathbf{S}_0, \mathbf{S}_1$ are compositions of Reidemeister moves. Then $\mathbf{f} = \mathbf{F}(\mathbf{S}_1)^{-1}\mathbf{F}(\mathbf{S}_0)$ is an isomorphism of $\mathbf{F}(\mathbf{D}_0)$. Note that $\mathbf{F}(\mathbf{D}_0)$ is an invertible complex of bimodules, since it is represented by a braid. Tensoring it with the complex associated with the inverse braid will give us the identity bimodule $\mathbf{H}^{\mathbf{n}}$. Therefore, the group of automorphisms of $\mathbf{F}(\mathbf{D}_0)$ in $\mathbf{C}_{\mathbf{n},\mathbf{n}}$ is isomorphic to the group of automorphisms of $\mathbf{H}^{\mathbf{n}}$. Automorphisms of the $\mathbf{H}^{\mathbf{n}}$-bimodule $\mathbf{H}^{\mathbf{n}}$ are multiplications by invertible central elements of $\mathbf{H}^{\mathbf{n}}$. Moreover, automorphisms in the category $\mathbf{C}_{\mathbf{n},\mathbf{n}}$ should preserve the internal grading of $\mathbf{H}^{\mathbf{n}}$. It is a simple exercise to check that there are only two such automorphisms, $\pm\mathrm{Id}$. Thus, $\mathbf{f} = \pm\mathrm{Id}$ and $\mathbf{F}(\mathbf{S}_1) = \pm\mathbf{F}(\mathbf{S}_0)$.

This argument works for any movie move where each movie is a sequence of Reidemeister moves. Taking care of other movie moves is only slightly more complicated [29].

Putting everything together, we get a (projective) 2-functor from the 2-category of oriented tangle cobordisms $\mathbf{TC}$ to the 2-category $\mathbf{C}$. The objects of the latter are nonnegative integers, 1-morphisms are complexes of graded $(\mathbf{m},\mathbf{n})$-bimodules, and 2-morphisms are homogeneous homomorphisms of complexes modulo chain homotopies. The word *projective* refers to the sign indeterminancy in Theorem 5.3. Objects of $\mathbf{TC}$ are even length sequences of pluses and minuses. One-morphisms are tangles with prescribed orientations at the top and bottom endpoints; 2-morphisms are isotopy classes of tangle cobordisms. The 2-functor $\mathbf{F}$ assigns $\mathbf{n}$ to a signed sequence of length $2\mathbf{n}$, complex of graded bimodules $\mathbf{F}(\mathbf{T})$ to a tangle $\mathbf{T}$, and homomorphism $\pm\mathbf{F}(\mathbf{S})$ to a cobordism $\mathbf{S}$.

Specializing from tangles to links, we get a (projective) functor from the category of link cobordisms to the category of bigraded abelian groups [21], [29]. To a link $\mathbf{L}$ it assigns Khovanov homology $\mathbf{H}(\mathbf{L})$, to a link cobordism $\mathbf{S}$ between $\mathbf{L}_0$ and $\mathbf{L}_1$ it assigns a homomorphism $\pm\mathbf{H}(\mathbf{S}) : \mathbf{H}(\mathbf{L}_0) - \quad \mathbf{H}(\mathbf{L}_1)$ of bidegree $(0, - \ (\mathbf{S}))$. The sign indeterminacy in Theorem 5.3 has been eliminated by D. Clark, S. Morrison and K. Walker [13] and C. Caprau [10] at the cost of certain decorations of tangles and cobordisms.

For more information on rings H^n and related topological invariants see [12], [70], [71] and references therein.

5.3 Equivariant versions and applications

The ring H^n and complexes $C(D)$ can be defined for any commutative Frobenius R-algebra A. Requiring that $C(D)$ be invariant under the first Reidemeister move implies the condition that A is a rank two free R-module [31]. It turns out that there are many examples of rank two Frobenius pairs (R, A) giving rise to link homology, tangle and tangle cobordism invariants. They were originally described by D. Bar-Natan [4] in a more categorical language, avoiding the use of H^n.

One of these rank two Frobenius pairs is given by $R = \mathbb{Q}[t]$ and $A = \mathbb{Q}[X]$, with the condition that $X^2 = t$, making A an R-algebra. The trace map is $(X) = 1,$ $(1) = 0$. It is natural to think of R as the $SU(2)$-equivariant cohomology of a point and A as the $SU(2)$-equivariant cohomology of the 2-sphere, with $SU(2)$ acting via the surjective homomorphism onto $SO(3)$:

$$
\begin{aligned}
R &= H_{SU(2)}(pt, \mathbb{Q}) = H\ (BSU(2), \mathbb{Q}) = H\ (HP^{\infty}, \mathbb{Q}) = \mathbb{Q}[t], \\
A &= H_{SU(2)}(S^2, \mathbb{Q}) = H_{SO(2)}(pt, \mathbb{Q}) = \mathbb{Q}[X].
\end{aligned}
$$

Construction of Lecture 3, done for this (R, A), produces a functorial link homology theory, which we denote H_t. It extends to tangles and tangle cobordisms via the framework described in Lecture 4 and the first two sections of Lecture 5. A cobordism S between links L_0, L_1 induces a homomorphism $H_t(S) : H_t(L_0) - \ \ H_t(L_1)$, well-defined up to overall minus sign. Groups $H_t(L)$ are bigraded finitely-generated $\mathbb{Q}[t]$-modules, with the multiplication by t shifting the bigrading by $(0, 4)$. Let $Tor(L) \ \ H_t(L)$ be the torsion submodule; it consists of elements of $H_t(L)$ annihilated by some power of t. Homomorphisms $H_t(S)$ take $Tor(L_1)$ into $Tor(L_2)$, thus Tor is a functorial subtheory of H_t. Let $H\ (L) = H_t(L)/Tor(L)$. The quotient theory H is functorial with respect to link cobordisms, and each $H\ (L)$ is a free bigraded $\mathbb{Q}[t]$-module. It follows from [40] that $H\ (L)$ has rank 2^m, where m is the number of components of L. Moreover, when L is a knot, $H\ (L)$ lives in cohomological degree 0 and

$$H\ (L) = \mathbb{Q}[X]\{-s(L) - 1\}$$

for some even integer $s(L)$ called the Rasmussen invariant of L. The Rasmussen invariant tells us where the internal grading of $H(L)$ starts: in q-degree $-s(L)-1$. By writing $\mathbb{Q}[X] = \mathbb{Q}[t]\cdot 1 \oplus \mathbb{Q}[t]\cdot X$, we see two copies of $\mathbb{Q}[t]$ with the relative grading shift by 2.

Rasmussen [58] showed that, given a connected cobordism S between knots L_0, L_1, the induced homomorphism of $\mathbb{Q}[X]$-modules $H(S) : H(L_0) \to H(L_1)$ is nontrivial. Moreover, this homomorphism has degree $-\chi(S)$. Since

$$H(L_0) = \mathbb{Q}[X]\{-s(L_0)-1\}, \quad H(L_1) = \mathbb{Q}[X]\{-s(L_1)-1\},$$

nontriviality of the homomorphism implies that the absolute value of the difference $s(L_0) - s(L_1)$ is bounded by twice the genus of S,

$$|s(L_0) - s(L_1)| \leq 2g(S) = -\chi(S).$$

In particular, $L \longmapsto s(L)$ descends to a homomorphism from the knot concordance group to $2\mathbb{Z}$. Specializing to cobordisms from the trivial knot to L, one gets a lower bound on the slice genus of L:

$$|s(L)| \leq 2g_4(L).$$

The slice genus $g_4(L)$ of a knot L is the minimum genus of a smooth oriented surface in the four-ball D^4 that bounds $L \subset S^3 = \partial D^4$. It is also the minimum genus of a cobordism between the trivial knot and L.

Generally, both the slice genus $g_4(L)$ and the Rasmussen invariant $s(L)$ are very difficult to compute. For positive knots, however, the computation of $s(L)$ is straightforward. If L is a positive knot with a positive diagram D, the complex $C_t(D)$ starts in cohomological degree 0, and $H^0_t(D)$ is the kernel of the differential $C^0_t(D) \longrightarrow C^1_t(D)$. This allows us to determine $H^0_t(D)$, its quotient $H(D)$, and find the Rasmussen invariant $s(L)$. It is equal to $n+1-c$, where n is the number of crossings of D, and c is the number of Seifert circles. At the same time, Seifert's algorithm gives a Seifert surface for L of genus $\frac{n+1-c}{2}$, so the ordinary genus $g(L) \leq \frac{n+1-c}{2}$. The chain of inequalities

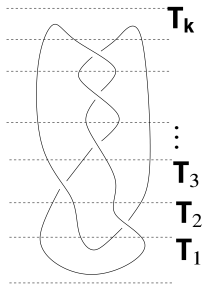
$$g(L) \geq g_4(L) \geq \frac{|s(L)|}{2} = \frac{n+1-c}{2} \geq g(L)$$

implies that all of them are equalities, and the slice genus of L is $\frac{n+1-c}{2}$. As a special case, this argument proves the Milnor conjecture, also known as Kronheimer-Mrowka theorem, that the slice genus of the torus knot $T_{p,q}$ is

$\frac{(p-1)(q-1)}{2}$. The first proof of the Milnor conjecture was given by P. Kronheimer and T. Mrowka [38] via Donaldson theory. The above much more recent proof, due to Rasmussen, is algebraic. The sketch, presented here, uses the graded theory H_t instead of its filtered version, utilized by Rasmussen [58].

Two 1's in the zero column of the homology table for $T_{5,6}$, depicted at the end of Lecture 3, are all that's left of $H\ (T_{5,6})$ in the homology groups $H(T_{5,6})$, where $t = 0$ and $Q[X]$ becomes A. The Rasmussen invariant of this torus knot equals 20.

Extension of the link homology to tangles, in addition to giving an easy proof of functoriality, also helps with computing link homology, as demonstrated by D. Bar-Natan and J. Green, who produced a fast program for computing Khovanov homology [5]. One puts a link L in "thin" position, namely a position minimizing the number of intersection points of horizontal planes with L, as illustrated below.

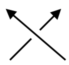
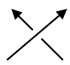
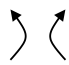


Write L as the product of tangles with at most one crossing each, $L = T_k \dots T_2 T_1$. To compute $H(L) = F(L)$, one starts with $F(T_1)$, then computes $F(T_2T_1)$, $F(T_3T_2T_1)$, etc. Each $F(T_i \dots T_2T_1)$ is a complex of projective H^m-modules, where $2m$ is the number of top endpoints of T_i. At each step, one simplifies $F(T_i \dots T_2T_1)$ as much as possible by removing null-homotopic components, isomorphic to $0 \quad P_a \ ^1 \ P_a \quad 0$, where P_a are projective H^m-modules described earlier and labelled by crossingless matchings a. After that, the reduced complex is tensored with $F(T_{i+1})$, and the simplification procedure is repeated. This method gives the most e cient algorithm at present for computing link homology.

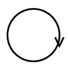
6 Categorifications of the HOMFLY-PT polynomial

6.1 The HOMFLY-PT polynomial and its generalizations

The HOMFLY-PT polynomial [20], [56] is a generalization of the Jones polynomial which is determined by the skein relation

aP  $- a^{-1}P$  $= (q - q^{-1})P$ 

and its value on the unknot

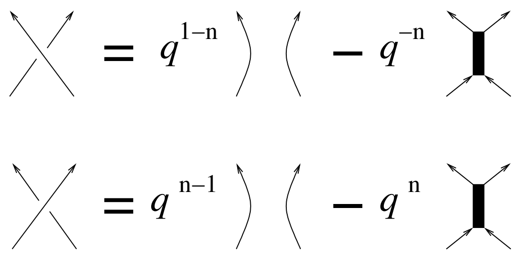
P  $= \dfrac{a - a^{-1}}{q - q^{-1}}.$

For $P(L)$ to really be a (Laurent) polynomial, one should change variables in the above formulas by introducing $b = q - q^{-1}$, for then $P(L) \quad Z[a^{\pm 1}, b^{\pm 1}]$. Variables a, q are natural from the representation-theoretical viewpoint, though, since the one-variable specialization $P_n(L) := P_{a=q^n}(L)$ of $P(L)$ for $n > 0$ can be extended [60] to an invariant of tangles via representation theory of $U_q(sl(n))$, a Hopf algebra deformation of the universal enveloping algebra $U(sl(n))$. For the first few values of n, the polynomial $P_n(L)$ is as follows:

- $P_0(L)$ is the Alexander polynomial of L.
- $P_1(L) = 1$ for all L is a trivial invariant.
- $P_2(L) = J(L)$ is the Jones polynomial of L.

We already discussed a categorification of $P_2(L)$. A categorification of $P_0(L)$ (the Alexander polynomial) has been constructed by P. Ozsváth, Z. Szabó [52] and, independently, J. Rasmussen [57]. It is a bigraded homology theory, known as the knot Floer homology, which comes in several versions and has found a multitude of applications in low-dimensional topology, see [53] and references therein. The invariant $P_1(L)$ is trivial, so there's nothing to categorify. Polynomial $P_n(L)$ was categorified in [30] for $n = 3$ (see [42], [48] for extension to tangles and for equivariant versions) and in [33] for all $n > 1$.

Both constructions employ a generalization of the Kauffman's bracket decomposition of the crossing, which for $n > 2$ takes the form



Each crossing of a diagram is resolved in two possible ways, and a complete resolution of a diagram produces a planar graph of a particular type. The invariant $P_n(L)$ extends to these graphs and has a positive evaluation on each of them: $P_n(\Gamma) \in \mathbb{N}[q, q^{-1}]$ for a planar graph Γ. To categorify $P_n(L)$ one first categorifies $P_n(\Gamma)$, which become graded dimensions of graded $\mathbb{Q}$-vector spaces $H_n(\Gamma)$ (that's the trickiest part of the construction). These vector spaces are then put into the vertices of an m-dimensional cube, where m is the number of crossings of the diagram, and for each edge of the cube one constructs a linear map between the spaces. Homology of D is defined as the homology of the total complex of the cube and checked to be invariant under the Reidemeister moves. The result [33] is a family of bigraded link homology theories

$$H_n(L) = \bigoplus_{i,j \in \mathbb{Z}} H_n^{i,j}(L), \quad n > 1,$$

with each group $H_n^{i,j}(L)$ a finite-dimensional $\mathbb{Q}$-vector space, and the Euler characteristic

$$P_n(L) = \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim H_n^{i,j}(L)$$

(when $n = 2$, we recover Khovanov homology, tensored with $\mathbb{Q}$.) These invariants extend to tangles and tangle cobordisms in a way conceptually similar to the one described in Lectures 4 and 5 for $H(L)$.

There are now several strikingly different constructions of graded [44] and bigraded [72], [9], [43], [47] homology theories of links which exhibit behavior similar to H_n and might all be isomorphic to H_n or mild modifications of the latter (see also [64], [8], [70] for prior constructions in the $n = 2$ case).

In the rest of the lecture we discuss a triply-graded homology theory [34], [32] which categorifies the 2-variable HOMFLY-PT polynomial $P(L)$.

6.2 Hochschild homology

Let R be a ring, and M (resp. N) be a right (resp. left) R-module. The tensor product $M \otimes_R N$ is an abelian group. In homological algebra various functors need to be redefined to make them exact in a suitable category. For general M, tensor product with M is only a right exact functor on the category of left R-modules. To convert it into an exact functor (on a bigger category, say the derived category of the original category), we consider a projective resolution of M, that is, a chain complex of projective R-modules (P_i, ∂_i) so that the vertical chain map in the following diagram is a quasi-isomorphism (i.e. isomorphism on homology).

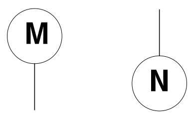
$$\begin{tikzcd} \cdots \arrow[r] & P_2 \arrow[r, "\partial_2"] \arrow[d] & P_1 \arrow[r, "\partial_1"] \arrow[d] & P_0 \arrow[r] \arrow[d] & 0 \\ & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

This simply means that the chain complex is exact everywhere except the last term, where $M = P_0/\mathrm{Im}\,\partial_1$.

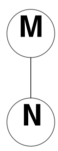
Consider the chain complex $M \otimes^L N := (P_i, \partial_i) \otimes_R N = (P_i \otimes_R N, \partial_i \otimes \mathrm{id})$. We call the i-th homology of this complex *the i-th derived tensor product* of M and N. It is known that derived tensor products do not depend on the choice of projective resolution, and that if we projectivize N instead of M, we get the same answer as well.

Exercise 6.1. *Determine the derived tensor product of Z-modules Z_n and Z_m.*

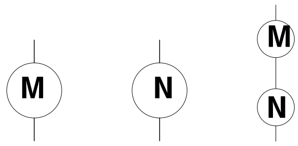
We represent a right R-module M and a left R-module N graphically as



R-action is depicted by wires, and "left" is seen as "up", "right" is "down". Turn your head 90 degrees clockwise or, alternatively, turn the paper 90 degrees counterclockwise to see the match. We represent $M \otimes^L N$ by the following picture:



If $\mathbf{M}$ and $\mathbf{N}$ are $\mathbf{R}$-bimodules, we depict them and their derived tensor product $\mathbf{M} \overset{\mathbf{L}}{\otimes} \mathbf{N}$ as follows:



(compare with Lecture 4). The top and the bottom wires of the diagram for a bimodule $\mathbf{M}$ indicate left and right actions of $\mathbf{R}$. A single vertical undecorated wire denotes $\mathbf{R}$ viewed as an $\mathbf{R}$-bimodule.

The space of $\mathbf{R}$-coinvariants of an $\mathbf{R}$-bimodule $\mathbf{M}$ is

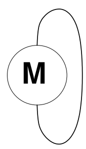
$$\mathbf{M_R} := \mathbf{M}/[\mathbf{R}, \mathbf{M}],$$

the quotient of $\mathbf{M}$ by the abelian subgroup generated by expressions $\mathbf{rm} - \mathbf{mr}$ over all $\mathbf{r} \in \mathbf{R}, \mathbf{m} \in \mathbf{M}$. The functor $\mathbf{M} \mapsto \mathbf{M_R}$ is right exact.

Remark 6.2. *"Quotient object" functors (such as the $\mathbf{R}$-coinvariants functor) between abelian categories are often right exact. The "subobject" functors tend to be left exact. An example of a "subobject" functor is $\mathbf{M} \longmapsto \mathbf{M^R}$, which to a bimodule $\mathbf{M}$ assigns its $\mathbf{R}$-invariants*

$$\mathbf{M^R} := \{\mathbf{m} \in \mathbf{M} | \mathbf{rm} = \mathbf{mr} \text{ for all } \mathbf{r} \in \mathbf{R}\}.$$

Graphically, passing from $\mathbf{M}$ to its $\mathbf{R}$-coinvariants should correspond to joining the two wires of $\mathbf{M}$. If we imagine elements of $\mathbf{R}$ moving along the wires, the equations $\mathbf{rm} = \mathbf{mr}$ that hold in $\mathbf{M_R}$ for all $\mathbf{r}$ and $\mathbf{m}$ mean that $\mathbf{r}$ can jump from the top to the bottom wire and back without changing the value of the diagram. The easiest way to achieve this geometrically is by closing off the two ends of the diagram:



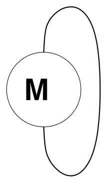
$$= \mathbf{M}/[\mathbf{R}, \mathbf{M}] = \mathbf{M_R}. \tag{13}$$

This is only an approximation, though, since $\mathbf{M} \longmapsto \mathbf{M_R}$ is not left exact and we need to form its derived functor, known as Hochschild homology. Notice that $\mathbf{R}$-bimodules are the same as left (or right) modules over the ring

$\mathbf{R^e} := \mathbf{R} \otimes \mathbf{R^{op}}$. If we are viewing $\mathbf{R}$ as a $\mathbf{k}$-algebra, for some commutative ring $\mathbf{k}$, often a field, the tensor product in the definition of $\mathbf{R^e}$ should be taken over $\mathbf{k}$. The group $\mathbf{M_R}$ equals the tensor product $\mathbf{M} \otimes_{\mathbf{R^e}} \mathbf{R}$ of a right $\mathbf{R^e}$-module $\mathbf{M}$ and a left $\mathbf{R^e}$-module $\mathbf{R}$ (right and left here can be transposed). We define the $\mathbf{i}$-th Hochschild homology of $\mathbf{M}$ as the $\mathbf{i}$-th derived functor of the tensor product:

$$\begin{aligned} \mathrm{HH}_{\mathbf{i}}(\mathbf{R}, \mathbf{M}) &:= \mathrm{H}_{\mathbf{i}}(\mathbf{M} \otimes^{\mathbf{L}}_{\mathbf{R^e}} \mathbf{R}), \\ \mathrm{HH}_\bullet(\mathbf{R}, \mathbf{M}) &:= \bigoplus_{\mathbf{i}\geq 0} \mathrm{HH}_{\mathbf{i}}(\mathbf{R}, \mathbf{M}). \end{aligned}$$

Going back to diagrammatics, we should interpret the closure of a bimodule diagram as taking the entire Hochschild homology of $\mathbf{M}$ rather than just $\mathbf{M_R}$, its degree 0 part:

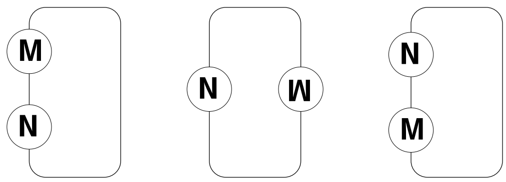


$$= \mathrm{HH}_\bullet(\mathbf{R}, \mathbf{M}).$$

The Hochschild homology exhibits "tracial" behaviour, since there are (functorial in $\mathbf{M}$ and $\mathbf{N}$) isomorphisms

$$\mathrm{HH}_\bullet(\mathbf{R}, \mathbf{M} \otimes^{\mathbf{L}} \mathbf{N}) = \mathrm{HH}_\bullet(\mathbf{R}, \mathbf{N} \otimes^{\mathbf{L}} \mathbf{M}).$$

These isomorphisms acquire topological interpretation



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that is, bimodule boxes can be dragged along the wires. Hochschild homology of a bimodule can be viewed as a categorification of the trace of a linear operator.

To compute the Hochshild homology of $\mathbf{M}$, it suffices to construct a projective resolution of $\mathbf{R}$ as an $\mathbf{R}$-bimodule, tensor with $\mathbf{M}$ over $\mathbf{R^e}$ and take homology.

Example 6.3. *Let* $R = \mathbb{Q}[x]$, *viewed as a* $\mathbb{Q}$*-algebra. Note that* $R^{\mathrm{op}} = R$*. The following is a free resolution of* R *as* $R \otimes R$*-module called* Koszul resolution
$$0 \to \mathbb{Q}[x] \otimes \mathbb{Q}[x] \xrightarrow{\delta} \mathbb{Q}[x] \otimes \mathbb{Q}[x] \to 0,$$
where δ *is the* $R \otimes R$*-module map determined by* $\delta(1 \otimes 1) = x \otimes 1 - 1 \otimes x$. *Tensoring with* M*, we get*

$$(M \otimes_{R^e} (0 \to R^e \to R^e \to 0)) = (0 \to M \xrightarrow{\delta} M \to 0),$$

where $\delta(m) = xm - mx$ *(notice that* $M \otimes_{R^e} R^e = M$*). Therefore,*

$$\mathrm{HH}_0(R, M) = M_R, \quad \mathrm{HH}_1(R, M) = M^R,$$

and all higher Hochschild homology groups vanish.

Example 6.4. $R = \mathbb{Q}[x_1, \ldots, x_n]$. *We have a resolution of* R *by free* R^e*-modules*

$$\bigotimes_{i=1}^{n} (0 \to \mathbb{Q}[x_i] \otimes \mathbb{Q}[x_i] \xrightarrow{\delta_i} \mathbb{Q}[x_i] \otimes \mathbb{Q}[x_i] \to 0)$$
$$= (0 \to R \otimes R \to \cdots (R \otimes R)^{\oplus \binom{n}{k}} \to \cdots \to R \otimes R \to 0),$$

where $\delta_i(1 \otimes 1) = x_i \otimes 1 - 1 \otimes x_i$. *One may consider the* k*-th term as*

$$(R \otimes R)^{\oplus \binom{n}{k}} = R \otimes R \otimes \Lambda^k V,$$

where $V := \mathrm{span}_{\mathbb{Q}}\{y_1, \ldots, y_n\}$, *and the differential is given by*

$$d(z_1 \otimes z_2 \otimes y_{r_1} \wedge \cdots \wedge y_{r_k}) =$$
$$\sum_{j=1}^{k} (-1)^j (x_{r_j} z_1 \otimes z_2 - z_1 \otimes z_2 x_{r_j}) y_{r_1} \wedge \cdots \widehat{y_{r_j}} \cdots \wedge y_{r_k}.$$

Tensoring with M*, we get the complex* $0 \to M \to \cdots \to M^{\oplus \binom{n}{k}} \to \cdots \to M \to 0$ *which computes Hochschild homology of* M*. For the boundary terms, we get*
$$\mathrm{HH}_0(R, M) = M_R, \quad \mathrm{HH}_n(R, M) = M^R.$$

6.3 A categorification of the HOMFLY-PT polynomial

We use $\mathbf{R} = \mathbb{Q}[x_1, \ldots, x_n]$ as in Example 6.4. For the transposition $s_i = (i, i+1)$ in the symmetric group $\mathbf{S_n}$ let

$$\mathbf{R_i} := \mathbf{R}^{s_i} = \mathbb{Q}[x_1, \ldots, x_{i-1}, x_i + x_{i+1}, x_i x_{i+1}, x_{i+2}, \ldots, x_n] \quad \mathbf{R},$$

be the space of s_i-invariants under the permutation action of $\mathbf{S_n}$ on $\mathbf{R}$. As an $\mathbf{R_i}$-module, $\mathbf{R}$ is free of rank 2 and can be written as $\mathbf{R} = \mathbf{R_i} \cdot 1 \quad \mathbf{R_i} \cdot x_i$. We set the degree of x_i to 2; this makes $\mathbf{R}$ and $\mathbf{R_i}$ into graded rings. Then $\mathbf{B_i} := \mathbf{R} \quad_{\mathbf{R_i}} \mathbf{R}\{-1\}$ is a graded $\mathbf{R}$-bimodule, and we have

$$\mathbf{B_i} \quad_{\mathbf{R}} \mathbf{B_i} = \mathbf{R} \quad_{\mathbf{R_i}} (\mathbf{R_i} \cdot 1 \quad \mathbf{R_i} \cdot x_i) \quad_{\mathbf{R_i}} \mathbf{R}\{-1\} = \mathbf{B_i}\{1\} \quad \mathbf{B_i}\{-1\}.$$

Recall that bimodules $\mathbf{U_i}$ in Lecture 1 satisfy the same relation. In Lecture 1 we formed chain complexes $(0 \quad \mathbf{U_i} \quad \mathbf{A_n} \quad 0)$ and $(0 \quad \mathbf{A_n} \quad \mathbf{U_i} \quad 0)$ corresponding to the braid $\ _i$; these complexes gave rise to a braid group action in the homotopy category of complexes. An analogous theory exists for bimodules $\mathbf{B_i}$. Form bimodule complexes

$$\begin{aligned} \mathbf{C_i} &:= (0 \quad \mathbf{B_i}\{1\} = \mathbf{R} \quad_{\mathbf{R_i}} \mathbf{R} \xrightarrow{m} \mathbf{R} \quad 0), \\ \mathbf{C_i} &:= (0 \quad \mathbf{R} \quad \mathbf{B_i}\{-1\} \quad 0), \end{aligned}$$

where $\ (1) := (x_i - x_{i+1}) \quad 1 + 1 \quad (x_i - x_{i+1})$ and in both complexes $\mathbf{R}$ sits in cohomological degree 0.

Theorem 6.5. *(R.Rouquier [62]) In the category of complexes of graded* $\mathbf{R}$*-bimodules modulo homotopic to zero morphisms there are the following isomorphisms:*

$$\begin{aligned} \mathbf{C_i} \quad \mathbf{C_i} &= \mathbf{R} = \mathbf{C_i} \quad \mathbf{C_i}, \\ \mathbf{C_i} \quad \mathbf{C_{i+1}} \quad \mathbf{C_i} &= \mathbf{C_{i+1}} \quad \mathbf{C_i} \quad \mathbf{C_{i+1}}, \\ \mathbf{C_i} \quad \mathbf{C_j} &= \mathbf{C_j} \quad \mathbf{C_i}, \text{ if } |i-j| > 0. \end{aligned}$$

The theorem say that there is a weak braid group action on the homotopy category of complexes of graded $\mathbf{R}$-modules. Rouquier also showed that this weak action lifts to a genuine action. Moreover, just like in the example of Lectures 1 and 2, this braid group action extends to an action of the category of braid cobordisms [36].

This is reminiscent of our previous categorifications, via rings $\mathbf{A_n}$ and $\mathbf{H^n}$:

$$\mathbf{A_n}: \quad \text{Braid cobordisms} \quad \overset{\text{categorification}}{} \quad \text{Burau representations}$$

$$\mathbf{H^n}: \quad \text{Tangle cobordisms} \quad \overset{\text{categorification}}{} \quad \text{Jones polynomials}$$

It turns out that the braid group action via complexes of $\mathbf{R}$-bimodules leads to a categorification of the HOMFLY-PT polynomial of braid closures.

Starting with an arbitrary braid word , which is a product of $_i$'s and their inverses, form the corresponding complex of graded $\mathbf{R}$-bimodules, denoted $\mathbf{C}(\ )$, the tensor product of $\mathbf{C_i}$'s and $\mathbf{C_i}$'s:

$$\mathbf{C}(\ ): \quad \cdots \overset{\mathbf{d}}{} \mathbf{C^j}(\ ) \overset{\mathbf{d}}{} \mathbf{C^{j+1}}(\ ) \overset{\mathbf{d}}{} \cdots.$$

Now, take the Hochschild homology of each term. The di erential map $\mathbf{d}$ induces a mapping of Hochschild homology groups, and we obtain the chain complex

$$\cdots \quad \mathrm{HH}(\mathbf{R}, \mathbf{C^j}(\ )) \overset{\mathrm{HH}(\mathbf{d})}{} \mathrm{HH}(\mathbf{R}, \mathbf{C^{j+1}}(\ )) \quad \cdots.$$

Each $\mathrm{HH}(\mathbf{R}, \mathbf{C^j}(\ ))$ is a $\mathbf{Q}$-vector space with two gradings: the Hochschild grading and the internal grading (the grading of $\mathbf{R}$ by $\deg \mathbf{x_i} = 2$). Therefore, taking the homology, we get a triply-graded vector space

$$\mathrm{H}(\mathrm{HH}(\mathbf{R}, \mathbf{C}(\ )), \mathrm{HH}(\mathbf{d})) = \mathrm{HHH}(\ ).$$

This triply-graded vector space needs an overall shift, as desribed by Hao Wu [79]. With it in place, we have

Theorem 6.6. *Triply-graded homology groups* $\mathrm{HHH}(\ )$ *is an invariant of the link* $\hat{}$*, the closure of braid . The Euler characteristic of* $\mathrm{HHH}(\ )$ *is the HOMFLY-PT polynomial of the link* $\hat{}$*.*

This construction simplifies a categorification of the HOMFLY-PT polynomial in [34]. There are some problems with this homology theory: HHH is not functorial under link cobordisms,for instance due to the theory being infinite-dimensional on non-empty links. It might be possible to make it finite-dimensional by setting several $\mathbf{x_i}$'s in $\mathbf{R}$ to 0, one for each component of the link. It is not clear, though, why the finite-dimensional version should be functorial under tangle cobordisms. One would also like to define HHH in a more natural way and extend it to tangles. Overall, it is an open problem to develop the homology theory HHH and make it as aethetically pleasing as the one described in lectures 3-5.

There are ideas on how to generalize this theory to the so-called colored HOMFLY polynomials and relate it to topological strings [19], [18]. The homology was computed for a number of knots by Rasmussen [59], and earlier, via a computer program, by Ben Webster.

The ring $\mathbf{R}$ has a geometric interpretation as the $\mathbf{GL}(\mathbf{n})$-equivariant cohomology of the variety of full flags in $\mathbf{C}^{\mathbf{n}}$. This interpretation was extended to the Hochschild homology of indecomposable summands of $\mathbf{C}(\ )$ in [76].

Similarities between the Hochschild homology and link homology were originally observed in [55].

To summarize, we've seen three categorifications in these lectures.

- Rings $\mathbf{A_n}$ lead to a categorification of the reduced Burau representation of the braid group. Braids act by complexes of $\mathbf{A_n}$-bimodules. The theory can be extended to give invariants of braid cobordisms via homomorphisms of complexes of bimodules.

- Bimodules over rings $\mathbf{H^n}$ give a categorification of the Temperley-Lieb algebra. Complexes of bimodules produce an invariant of tangles; homomorphisms of complexes–invariants of tangle cobordisms. The construction specializes to a bigraded homology theory of links categorifying the Jones polynomial.

- Suitable bimodules over polynomial rings $\mathbf{R}$ give rise to a braid group action. The action extends to braid cobordisms. Taking Hochschild homology produces a triply-graded link homology theory categorifying the HOMFLY-PT polynomial.

Exercise 6.7. *Choose an integral structure that appears in combinatorics, or algebra, or topology, etc. and categorify it.*

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