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Non-Gaussianity, loops and the stability of de Sitter space

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Eternal Inflation

- In Chaotic inflation, we typically have a simple EFT valid just blow the Planck scale
- Our inflating volume typically described by huge total number of e-foldings
- Thus we will typically be plagued by large IR loop contributions
- We want to understand how to deal with them!

Non-Gaussianity and loops

- The study of non-linear perturbations also enables us to study non-Gaussianity at tree-level
- Non-Gaussian signatures are one of our main windows into the underlying model of inflation
- What can we learn about the systematics of n-point correlation functions and their contribution to non-Gaussianity by studying the stability of de Sitter against loop corrections?

Outline of talk

The plan for my talk will be as follows

- Introduction
- **de Sitter space and eternity**
- **Non-linear perturbations and loops**
- **Non-Gaussianity and the stability of de Sitter space**
- Conclusion

The IR divergences in de Sitter

- The stability issue of de Sitter may teach us about the cosmological constant problem.
- The two-point function of a massless scaler field in de Sitter space is IR divergent

$$\langle \phi^2(x) \rangle = U.V. + \frac{H^2}{4\pi^2} \ln(aH/L)$$

- The IR divergency has prompted many to believe that pure de Sitter could by itself be unstable [see f.ex. Polyakov 2007].

Spectroscopy of de Sitter

Proposal: a universe with $\Lambda > 0$ is described by a finite number of *d.o.f.* $N = \ln \tilde{n}$, $\tilde{n} = e^S$ [Banks 2000]

- The entropy of pure de Sitter space

$$S = \frac{1}{4} A = \frac{3\pi}{\Lambda}$$

- which implies

$$N = \frac{3\pi}{\Lambda}$$

- If we quantize the horizon in terms of the fundamental area l_{pl}^2

$$A = 8\pi n \ , \qquad n = 0, 1, 2, \dots$$

this implies $N = \ln g(n)$, where $g(n)$ is the degeneracy factor i.e. $g(n) \sim \exp(2n)$ [Bekenstein, Mukhanov 1995].

Correspondence principle

Classical relations are valid in the limit of large quantum numbers

- Consider the quantum harmonic oscillator,

$$E = (n + 1/2)\hbar\omega \ , \qquad n = 0, 1, 2, \ldots$$

- The classical relation between energy and amplitude

$$E = \frac{m\omega^2 A^2}{2}$$

is valid in the the limit of large quantum numbers.

- From the de Sitter spectroscopy

$$\frac{1}{4}A = 2\pi n\ , \qquad n = 0, 1, 2, \ldots$$

  we see that de Sitter space with a small cosmological constant is a macroscopic system described by large quantum numbers

- The classical relation between the horizon area and the cosmological constant

$$\frac{1}{4}A = \frac{3\pi}{\Lambda}$$

  should be valid in the limit of small cosmological constant

Thus, the correspondence principle dictates that de Sitter with a small cosmological constant is (quasi) stable!

Wavefunction of the Universe

- Consider the tunneling probability from nothing to de Sitter

Wavefunction of the Universe	Hartle-Hawking	Linde	Vilenkin
Tunneling probability from nothing	$P_{HH} = e^{3\pi/G\Lambda}$	$P_L = e^{-3\pi/G\Lambda}$	$P_V = e^{-3\pi/G\Lambda}$

- It has been debated which of these wavefunctions yield the correct answer for the tunneling probability?
- de Sitter with a small cosmological constant is a macroscopic object described by large quantum numbers $\Rightarrow$ the tunneling probability into a de Sitter space with small cosmological constant must be exponentially suppressed*

*the problem of the Hartle-Hawking Wavefunction may be fixed when including matter [Firouzjahi, Sarangi, Tye, 2004]

Non-linearity parameters

Before returning to what we can we learn about non-Gaussianity from the quasi-stability of de Sitter...

- The size of non-Gaussianity from the bispectrum is parameterized in terms of f_{NL} is defined by

$$\langle \zeta_{k_1}\zeta_{k_2}\zeta_{k_3}\rangle = (2\pi)^3\delta(\sum_i \vec{k_i})\frac{6}{5}f_{NL}\left[P_\zeta(k_1)P_\zeta(k_2) + 2\ perm.\right]$$

- Similarly for the trispectrum, one can define a τ_{NL}

$$\langle \zeta_{k_1}\zeta_{k_2}\zeta_{k_3}\zeta_{k_4}\rangle = (2\pi)^3\delta(\sum_i \vec{k_i})\frac{1}{2}\tau_{NL}\left[P_\zeta(k_1)P_\zeta(k_2)P_\zeta(k_{14}) + 11\ perm.\right]$$

- So, an order of magnitude estimate gives

$$f_{NL} \approx \frac{\langle\zeta^3\rangle}{P_\zeta^2}\ , \qquad \tau_{NL} \approx \frac{\langle\zeta^4\rangle}{P_\zeta^3}\ , \qquad f_{NL}^{(n)} \sim \frac{\langle\zeta^n\rangle}{P_\zeta^{(n-1)}}$$

Second order action of perturbations

The bispectrum of non-Gaussianity can be obtained from the third order action

- Second order action in uniform curvature gauge [Mukhanov]

$$S_2 = \frac{1}{2}\int a^3\left[\delta\dot{\phi}^2 - \partial^i\delta\phi\partial_i\delta\phi - V''\delta\phi^2 + \frac{\dot{\phi}_c^2}{H^2}V\delta\phi^2 + 2\frac{\dot{\phi}_c^2}{H^2}\left(\frac{\ddot{\phi}_c^2}{H\dot{\phi}_c} + \frac{\dot{\phi}_c^2}{2H^2}\right)\delta\phi^2\right]$$

Third order action of perturbations

- Third order action [Maldacena, 2002]

$$
\begin{aligned}
S_3 \quad = \quad & \int a^3 \left[-\frac{1}{4}\frac{\dot{\phi}_c}{H}\dot{\delta\phi}^2\delta\phi - \frac{1}{4}\frac{\dot{\phi}_c}{H}\delta\phi(\partial\delta\phi)^2 - \dot{\delta\phi}\partial^i\chi_1\partial_i\delta\phi \right. \\
& + \frac{3}{8}\frac{\dot{\phi}_c^3}{H}\delta\phi^3 - \frac{1}{4}\frac{\dot{\phi}_c}{H}V_{,\phi\phi}\delta\phi^3 - \frac{1}{6}V_{,\phi\phi\phi}\delta\phi^3 + \frac{1}{4}\frac{\dot{\phi}_c^3}{H^2}\delta\phi^2\dot{\delta\phi} + \frac{1}{4}\frac{\dot{\phi}_c^2}{H}\delta\phi^2\partial^2\chi_1 \\
& \left. + \frac{1}{4}\frac{\dot{\phi}_c}{H}\left(-\delta\phi\partial^i\partial^j\chi_1\partial_i\partial_j\chi_1 + \delta\phi\partial^2\chi_1\partial^2\chi_1\right) \right]
\end{aligned}
$$

$$
\alpha_1 = \frac{1}{2}\frac{\dot{\phi}_c}{H}\delta\phi\,, \qquad \partial^2\chi_1 = -\frac{1}{2}\frac{\dot{\phi}_c}{H}\dot{\delta\phi} - \frac{1}{2}\dot{\phi}_c\frac{\dot{H}}{H^2}\delta\phi + \frac{1}{2}\frac{\ddot{\phi}}{H}\delta\phi
$$

Fourth order action of perturbations

- The trispectrum is obtainable from the fourth order action [MSS, 2006], [Seery, Lidsey, MSS, 2006]

$$S_4 = \int a^3 \Big[-\frac{1}{24} V_{,\phi\phi\phi\phi}\delta\phi^4 + \frac{1}{2}\partial_j\chi_1\partial^j\delta\phi\partial_m\chi_1\partial^m\delta\phi - \delta\dot{\phi}\partial_j\chi_2\partial^j\delta\phi$$
$$+(\alpha_1^2\alpha_2 - \frac{1}{2}\alpha_2^2)(-6H^2+\dot{\phi}^2) + \frac{\alpha_1}{2}\Big\{ -\frac{1}{3}V_{,\phi\phi\phi}\delta\phi^3 - 2\alpha_1^2 V_{,\phi}\delta\phi$$
$$+\alpha_1\left(-\partial^i\delta\phi\partial_i\delta\phi - V_{,\phi\phi}\delta\phi^2\right) - 2\partial_i\partial_j\chi_2\partial^i\partial^j\chi_1 + 2\partial^2\chi_2\partial^2\chi_1$$
$$+2\dot{\phi}\partial_j\chi_2\partial^j\delta\phi + 2\delta\dot{\phi}\partial_j\chi_1\partial^j\delta\phi\Big\}\Big]$$

$$\alpha_2 = \frac{\dot{\phi}_c^2}{8H^2}\delta\phi^2 + F(\delta\phi,\dot{\phi}) \; ,$$

$$\partial^2\chi_2 = \frac{3}{8}\frac{\dot{\phi}_c^2}{H}\delta\phi^2 + \frac{3}{4}\frac{\ddot{\phi}_c}{\dot{\phi}_c}\delta\phi^2 - \frac{a^2}{4H}(\partial\delta\phi)^2 - \frac{1}{4H}\delta\dot{\phi}^2 + \frac{\dot{\phi}_c}{2H}\partial_i\chi_1\partial_i\delta\phi$$
$$+\frac{1}{4H}\left((\partial^2\chi_1)^2 - (\partial_i\partial_j\chi_1)^2\right) - \frac{V}{H}F(\delta\phi,\delta\dot{\phi})$$

$$F(\delta\phi,\delta\dot{\phi}) = \frac{1}{2H}\partial^{-2}\left[\partial^2\alpha_1\partial^2\chi_1 - \partial_i\partial_j\alpha_1\partial_i\partial_j\chi_1 + \partial_i\delta\dot{\phi}\partial_i\delta\phi + \delta\dot{\phi}\partial^2\delta\phi\right]$$

Tree-level stuff

From these results the Bispectrum and Trispectrum has been calculated in single field slow-roll inflation

- $S_2 \sim \mathcal{O}(1) \quad \Rightarrow \quad \langle \delta\phi^2 \rangle \propto H^2$ and
  $\zeta = \epsilon^{-1/2}\delta\phi/M_p \quad \Rightarrow \quad \langle \zeta^2 \rangle \propto \epsilon^{-1}(H/M_p)^2$
- $S_3 \sim \mathcal{O}(\epsilon^{1/2}) \quad \Rightarrow \quad \langle \delta\phi^3 \rangle \propto \epsilon^{1/2} H^4/M_p \quad \Rightarrow$
  $\langle \zeta^3 \rangle \propto \epsilon^{-1}(H/M_p)^4 \propto \epsilon \mathcal{P}_\zeta^2 \quad \Rightarrow \quad f_{NL} \propto \epsilon$ [Maldacena, 2002]
- $S_4 \sim \mathcal{O}(1) \quad \Rightarrow \quad \langle \delta\phi^4 \rangle \propto H^6/M_p^2 \quad \Rightarrow$
  $\langle \zeta^4 \rangle \propto \epsilon^{-2}(H/M_p)^6 \propto \epsilon \mathcal{P}_\zeta^3 \quad \Rightarrow \quad \tau_{NL} \propto \epsilon$

  [Seery, Lidsey, MSS, 2006], [Seery, MSS, Vernizzi, 2008]

Perhaps slightly surprisingly

$$f_{NL} \sim \tau_{NL}$$

Loops and IR divergences

The third and fourth order actions of perturbations also enables us to calculate the loop corrections to the power-spectrum

- From S_4 we can calculate the one-loop corrected power spectrum of inflaton perturbations in the "in-in" formalism

$$\langle \zeta^4(t) \rangle = \left\langle U_{int}^{-1} \zeta^4(t) U_{int}(t, t_0) \right\rangle \ , \qquad U_{int} = T e^{-i \int_{t_0}^{t} H_{int}(t') dt'}$$

which gives [MSS, 2006]

$$\mathcal{P}(\eta_0, k) \quad \approx \quad \frac{H^2}{4\pi^2} \left[1 - \left(\frac{1}{16}\epsilon + \frac{1}{2}(2\epsilon - \eta) - \frac{3}{8}(2\epsilon - \eta)\mathrm{Ci}(-2k\eta_0) \right) \langle \delta\phi^2 \rangle \right]$$

- With $V(\phi) = \lambda M_p^{4-\alpha} \phi^\alpha$ and assuming chaotic inflation

$$\Delta\phi = \dot{\phi}\Delta t < \delta\phi \ \Rightarrow \phi_i < \lambda^{-1/6} M_p \ , \qquad \Delta t = 1/H$$

$$\langle \delta\phi \rangle = \int_{a_i H_i}^{a\Lambda} \frac{dk}{k} \mathcal{P}_{\delta\phi}(k) \propto N^{(4+\alpha)/2} \ , \qquad N \approx \frac{1}{2\alpha M_p^2} \phi_i^2$$

Size of one-loop corrections

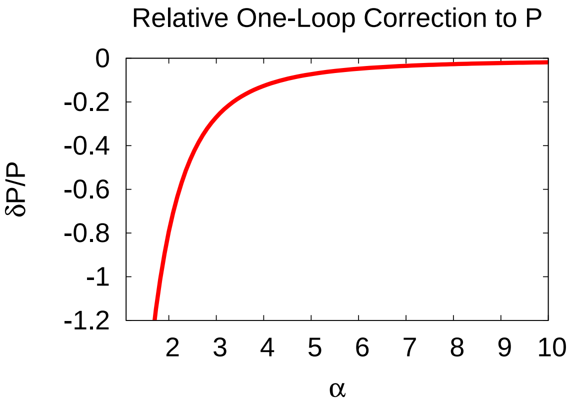

Figure: The relative one-loop correction $\delta\mathcal{P}/\mathcal{P}$ to the power spectrum

Resumming the loops

Large one-loop corrections requires resummation of the loops to all orders in perturbation theory [Riotto, MSS, 2008]

- Consider an $O(N)$ invariant toy model in the large N limit

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi_i\partial^\mu\phi_i - m^2\phi_i\phi_i - \frac{\lambda}{4N}(\phi_i\phi_i)^2\right)$$

- From the 2PI effective action we can derive a *gap* equation for the two-point function $\langle\phi(x)\phi(x')\rangle$ valid to all orders
- The gap equation can be solved self-consistently for the variance $\langle\phi^2(x)\rangle$ yielding

$$\langle\phi^2(x)\rangle = H^2\,\frac{\tanh(\frac{\sqrt{\lambda}}{4\pi^2}\ln a)}{\sqrt{\lambda}}$$

Resumming the loops

- To lowest order in the expansion tanh we obtain

$$\langle \phi^2 \rangle = \frac{H^2}{4\pi^2} \ln a$$

- The asymptotic solution for $\ln a \to \infty$ is [Starobinsky,Yokoyama, 1994], [Tsamis, Woodard, 2005]

$$\langle \phi^2 \rangle = \frac{H^2}{\sqrt{\lambda}}$$

- It acts like a regulating non-perturbative mass for the two-point function

$$m_{np}^2 = \frac{\sqrt{3\lambda}}{4\pi} H^2$$

Non-perturbative enhancement of non-Gaussianities

Example:

- At tree-level the four point function of the $O(N)$ field is
  [Bernardeau, Uzan, 2003]

$$\langle \psi_{k_1} \dots \psi_{k_4} \rangle \simeq \frac{3\lambda}{N} \delta^{(3)}(\vec{k}_1 + \cdots + \vec{k}_4) H^4 \frac{k_1^3 + k_2^3 + k_3^3 + k_4^3}{24 k_1^3 k_2^3 k_3^3 k_4^3} (\gamma + \log(-k\eta))$$

- At one-loop, we expect IR divergences as usual
- But, the non-perturbative mass $m^2_{\rm np}$ regulate the log-divergent integrals

$$\langle \psi_{k_1} \dots \psi_{k_4} \rangle \approx \frac{12\pi\sqrt{3\lambda}}{N} \delta^{(3)}(\vec{k}_1 + \cdots + \vec{k}_4) H^4 \frac{k_1^3 + k_2^3 + k_3^3 + k_4^3}{24 k_1^3 k_2^3 k_3^3 k_4^3}$$

The tri-spectrum is non-perturbatively enhanced by a factor $12\pi/\sqrt{3\lambda}$, while the IR divergence has disappeared [Riotto, MSS, 2008]

de Sitter Limit

What we can we learn about non-Gaussianity from the quasi-stability of de Sitter?

We saw that $S_3 = O(\epsilon^{1/2})$ while $S_2 = O(1)$, $S_4 = O(1)$ implying

$$f_{NL} \sim \tau_{NL}$$

Naively we would perhaps have expected $\tau_{NL} << f_{NL}$

- We may wonder what requires the third order action, $S_3(\delta\phi)$, to be slow-roll suppressed, when there is nothing which forces the second and fourth order actions $S_2(\delta\phi)$, $S_4(\delta\phi)$ to be slow-roll suppressed

A toy example

- Assume that we have unsuppressed third order terms. The interaction Hamiltonian for the perturbations will have the form

$$H_I = \int d^3x a^3 \left[\delta\phi(\ddot{\phi}_c + 3H\dot{\phi}_c + V') + gO(\delta\phi^3) + \ldots\right]$$

  with the $O(\delta\phi^3)$ toy term being any operator of the type $\delta\phi^3$, $\delta\dot{\phi}^2\delta\phi$, $\delta\dot{\phi}\partial^{-2}(\partial_i\delta\dot{\phi}\partial^i\delta\phi)$
- The term $O(\delta\phi^3)$ will give a contribution to the tadpole diagram in the figure.

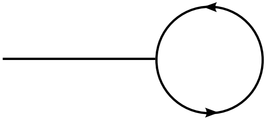

Figure: Tadpole diagram

Tadpole destabilization of de Sitter?

- The tadpole condition yields the one-loop correction to the equation of motion of the classical background field

$$0 = \langle \delta\phi \rangle = \ddot{\phi}_c + 3H\dot{\phi}_c + V' + g\Gamma_t \ ,$$

where $g\Gamma_t$ is the amputated tadpole contribution

- In the simplest case of a massless scalar field and with $O(\delta\phi^3) = H\delta\phi^3$, the tadpole contribution would become

$$g\Gamma_t = 3gH \left\langle \delta\phi^2 \right\rangle = \frac{3g}{4\pi^2} H^4 t$$

- In this case, the time-independent de Sitter solution is destabilized by the tadpole.

Tadpole destabilization of de Sitter?

- If the toy term has a more complicated form involving derivatives, the infrared divergency will generally approach a constant
- In this case the solution to with $V' = 0$ and $g\Gamma_t = const.$ at late times is

$$\phi_c(t) = -\frac{g\Gamma_t}{3H} t$$

- This is inconsistent with a time-independent de Sitter solution.

$\Rightarrow$ if unsuppressed third order terms were allowed, classical de Sitter space with a massless scalar field would be destabilized

The systematics of non-Gaussianity

Since any unsuppressed odd order terms in the action would lead to a non-vanishing tadpole contribution

$\Rightarrow$ all odd order terms in the action should be slow-roll suppressed, while even order terms are not slow-roll suppressed

- We can then extrapolate our results from $n \leq 4$ to any n, and conjecture the order of magnitude of any n-point correlation function in single field inflation
- We can also generalize the nonlinearity parameter f_{NL} to n'th order up to a numerical factor of order one

$$\langle \zeta^n \rangle \approx f_{NL}^{(n)} \mathcal{P}_\zeta^{n-1}$$

The systematics of non-Gaussianity

Order	$S(\delta\phi)$	$S(\zeta_n)$	$S(\zeta)$
2nd	$\mathcal{O}(1)$	$\mathcal{O}(\epsilon)$	$\mathcal{O}(\epsilon)$
3rd	$\mathcal{O}(\epsilon^{1/2})$	$\mathcal{O}(\epsilon^2)$	$\mathcal{O}(\epsilon)$
4rd	$\mathcal{O}(1)$	$\mathcal{O}(\epsilon^2)$	$\mathcal{O}(\epsilon)$
$2n$th	$\mathcal{O}(1)$	$\mathcal{O}(\epsilon^n)$	$\mathcal{O}(\epsilon)$
$(2n+1)$th	$\mathcal{O}(\epsilon^{1/2})$	$\mathcal{O}(\epsilon^{n+1})$	$\mathcal{O}(\epsilon)$

Table: Slow-roll order of the action to n'th order.

[Jarnhus, MSS, 2007]

The systematics of non-Gaussianity

p	$\langle \delta\phi^p \rangle$	$\langle \zeta^p \rangle$	$f_{NL}^{(p)} \approx \langle \zeta^p \rangle / \mathcal{P}_\zeta^{p-1}$
2	$\mathcal{O}(H^2)$	$\mathcal{O}(\epsilon^{-1} H^2)$	$\mathcal{O}(1)$
3	$\mathcal{O}(\epsilon^{1/2} H^4)$	$\mathcal{O}(\epsilon^{-1} H^4)$	$\mathcal{O}(\epsilon)$
4	$\mathcal{O}(H^6)$	$\mathcal{O}(\epsilon^{-2} H^6)$	$\mathcal{O}(\epsilon)$
$2n$	$\mathcal{O}(H^{2p-2})$	$\mathcal{O}(\epsilon^{-p/2} H^{2p-2})$	$\mathcal{O}(\epsilon^{p/2-1})$
$2n+1$	$\mathcal{O}(\epsilon^{1/2} H^{2p-2})$	$\mathcal{O}(\epsilon^{(1-p)/2} H^{2p-2}))$	$\mathcal{O}(\epsilon^{(p-1)/2})$

Table: Slow-roll order of the n-point functions and generalized nonlinearity parameter.

[Jarnhus, MSS, 2007]

The systematics of non-Gaussianity

- It is easy to verify that the action $S(\delta\phi)$ to any even order, $2n$, in perturbations will be unsuppressed in the slow-roll parameters, as it will contain contributions from ${\alpha^{(2)}}^{2n}$
- Similarly, to odd orders $\alpha^{(2)}$ will always appear in combination with some $\alpha^{(n)}$ or $\chi^{(n)}$ to odd order, which are slow-roll suppressed
- As an example we predict that the nonlinearity parameter related to the 5- and 6-point function is $f_{NL}^{(5)} = f_{NL}^{(6)} = \epsilon^2$
- Including gravitational wave modes one can have unsuppressed odd n-terms in the action with gravitational waves, γ_{ij}, of the form $\partial_i\delta\phi\partial_j\delta\phi\gamma^{ij}$
- But terms like $\delta\phi\dot{\gamma}_{ij}\dot{\gamma}^{ij}$ have to be slow-roll suppressed. This agrees with [Maldacena, 2002]

Conclusions

- The quasi-stability of de Sitter can be used to make systematic predictions for the order of magnitude of non-Gaussianity.
- On the other hand, there is nothing so far in our studies of non-linear perturbation theory that seems to vindicate the point of view that de Sitter is unstable
- The IR divergences that appears order by order in perturbation theory, which could be a problem for the stability of de Sitter, go away when they are resummed to all orders in perturbation theory
- However, the IR divergences do lead to a non-perturbative (but finite) enhancement of correlation functions

These issues are important in typical models of eternal inflation and should be fully understood!