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Hamiltonian Perturbations of Hyperbolic PDEs and Applications

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HAMILTONIAN PERTURBATIONS OF HYPERBOLIC PDES AND APPLICATIONS

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Abstract. These are the lecture notes of the course given at the ICTP School "Integrable Systems and Scientific Computing", June 15 – 20, 2009.

Contents

1. Lecture 1

In order to build an efficient numerical code for solving a PDE it is often useful to establish simple qualitative properties of the equation.

Dichotomy 1: slow and fast variables. Consider a PDE

$$u_t = f(u) \tag{1.1}$$

on the space of 2π-periodic functions $u(x)$. For a constant initial data

$$u(x,t)_{t=0} \equiv u_0$$

the solution remains constant in x but not in time. Indeed, the problem reduces to solving an ODE

$$\dot u = f(u), \quad u(t=0) = u_0.$$

Moreover if the initial data is not a constant but a slow varying function of x then the solution to (1.1) will not be a slow varying function in time:

$$u_x = o(1) \quad \text{but} \quad u_t = O(1).$$

We say that the unknown function u of eq. (1.1) is a *fast variable*.

In order to formalize the definition let $\epsilon > 0$ be a small parameter. We say that $u(x)$ is a slow varying function if it changes by ϵ on the distances of order 1. Thus the higher is the order of a spatial derivatives of a slow varying function the smaller it is:

$$u_x = O(\epsilon), \quad u_{xx} = O(\epsilon^2), \quad u_{xxx} = O(\epsilon^3) \quad \text{etc.} \tag{1.2}$$

Slow varying functions of time are defined in a similar way.

More generally, consider a PDE of the form

$$u_t = f(u; u_x, u_{xx}, \dots, u^{(m)}) \tag{1.3}$$

for some $m > 0$. Assume that the r.h.s. is analytic at the point

$$(u = u_0,\ u_x = 0,\ u_{xx} = 0, \dots). \tag{1.4}$$

Then the time-dependence of the solution with a slow-varying initial data will not be slow unless the value of the r.h.s. at the point (1.4) vanishes:

$$f(u_0,\ 0,\ 0, \dots) = 0. \tag{1.5}$$

Let us now consider another example:

$$u_t = u\, u_x. \tag{1.6}$$

It is easy to see that already at the moment $t = 0$ the time derivatives of the solution with a slow varying initial data $u(x, 0) = u^0(x)$ decay with the number of the derivative like in (1.2):

$$u_t|_{t=0} = u^0(x)u^0_x(x) = O(\epsilon), \quad u_{tt}|_{t=0} = \left(u^0\right)^2 u^0_{xx} + 2u^0\left(u^0_x\right)^2 = O\left(\epsilon^2\right) \quad \text{etc.} \tag{1.7}$$

The above considerations extend without major changes to systems of PDEs

$$\mathbf{u}_t = \mathbf{f}(\mathbf{u}; \mathbf{u}_x, \mathbf{u}_{xx}, \dots, \mathbf{u}^{(m)}), \quad x \in \mathbb{R}, \quad \mathbf{u} = (u^1, \dots, u^n) \in \mathcal{B} \subset \mathbb{R}^n. \tag{1.8}$$

The r.h.s. will be assumed to be analytic in the derivatives at the points

$$\mathbf{u}_x = 0, \quad \mathbf{u}_{xx} = 0, \dots \quad \text{for any} \quad \mathbf{u} \in \mathcal{B}.$$

The following rule of thumb helps to identify systems with slow dependent variables. Rescale the independent variables

$$x \mapsto \epsilon\, x, \quad t \mapsto \epsilon\, t. \tag{1.9}$$

One has

$$\mathbf{u}_t \mapsto \epsilon\, \mathbf{u}_t, \quad \mathbf{u}_x \mapsto \epsilon\, \mathbf{u}_x, \quad \mathbf{u}_{xx} \mapsto \epsilon^2 \mathbf{u}_{xx}, \quad \text{etc.} \tag{1.10}$$

So the system will rewrite

$$\epsilon\, \mathbf{u}_t = \mathbf{f}(\mathbf{u}; \epsilon\, \mathbf{u}_x, \epsilon^2 \mathbf{u}_{xx}, \dots, \epsilon^m \mathbf{u}^{(m)}). \tag{1.11}$$

After division by ϵ and expanding the r.h.s. in Taylor series in ϵ one obtains

$$\mathbf{u}_t = \frac{1}{\epsilon}\mathbf{f}_0(\mathbf{u}) + A(\mathbf{u})\mathbf{u_x} + \epsilon\left(B(\mathbf{u})\mathbf{u}_{xx} + \frac{1}{2}L(\mathbf{u})\,(\mathbf{u}_x, \mathbf{u}_x)\right) + O(\epsilon^2) \tag{1.12}$$

where

$$\begin{aligned} \mathbf{f}_0(\mathbf{u}) &= \mathbf{f}(\mathbf{u}; 0, \dots, 0) \\ A(\mathbf{u}) &= \frac{\partial \mathbf{f}}{\partial \mathbf{u}_x}(\mathbf{u}; 0, \dots, 0) \\ B(\mathbf{u}) &= \frac{\partial \mathbf{f}}{\partial \mathbf{u}_{xx}}(\mathbf{u}; 0, \dots, 0), \end{aligned}$$

the quadratic form $L(\mathbf{u})\,(\mathbf{u}_x, \mathbf{u}_x)$ is given by the second derivatives of $\mathbf{f}$ in $\mathbf{u}_x$ etc.

In order to have all dependent variables in (1.8) slow varying in space and time one has to impose the condition

$$\mathbf{f}(\mathbf{u};0,\dots,0)=0.$$

The r.h.s. of the system then represents in the form of a formal power series in ϵ that we will now rewrite in the coordinate notations:

$$u^i_t = A^i_j(\mathbf{u})u^j_x + \epsilon\left[B^i_j(\mathbf{u})u^j_{xx} + \frac{1}{2}L^i_{jk}(\mathbf{u})u^j_x u^k_x\right] + \epsilon^2\left[C^i_j(\mathbf{u})u^j_{xxx} + M^i_{jk}(\mathbf{u})u^j_{xx}u^k_x + \frac{1}{6}N^i_{jkm}(\mathbf{u})u^j_x u^k_x u^m_x\right] + O\left(\epsilon^3\right), \quad i=1,\dots,n. \tag{1.13}$$

Summation over repeated indices will be assumed in sequel. The terms of order ϵ^k are graded homogeneous polynomials in the derivatives of the total degree $k+1$. The degrees are assigned to the derivatives according to the following rule:

$$\deg u^{(m)} = m, \quad m=1,\,2,\dots. \tag{1.14}$$

The above heuristic arguments suggest to replace the original system by its leading approximation

$$u^i_t = A^i_j(\mathbf{u})u^j_x. \tag{1.15}$$

It is expected that for small times the solution to the truncated system (1.15) gives an approximation to the solution to the full system with slow variables. In order to figure out whether such an expectation looks reasonable we have to concentrate our attention on some important properties of the first order quasilinear systems of the form (1.15). Thus we arrive at the

Dichotomy 2. Hyperbolic and elliptic systems. We say that the quasilinear system (1.15) is hyperbolic if the eigenvalues of the coefficient matrix

$$A(\mathbf{u}) = \left(A^i_j(\mathbf{u})\right)_{1\le i,j\le n}$$

are real and pairwise distinct for all $\mathbf{u}\in\mathcal{B}$.

It is known that the Cauchy problem for hyperbolic systems is well-posed for sufficiently small times in many natural classes of initial data. On the contrary the Cauchy problem with smooth initial data is ill-posed if the eigenvalues of the coefficient matrix are not real. In particular for the $n=2$ case the eigenvalues of the 2×2 coefficient matrix are or both real (the hyperbolic case) or complex conjugate (elliptic systems).

The above considerations do not provide us with any feeling about the behaviour of solutions for not necessarily small times. We attempt to get a more clear vision of these properties by using

Dichotomy 3. Hamiltonian and dissipative systems. The following two examples of solutions to Burgers equation

$$u_t + u\,u_x = u_{xx} \tag{1.16}$$

(dissipative case) and Korteweg - de Vries (KdV) equation

$$u_t + u\, u_x + u_{xxx} = 0 \tag{1.17}$$

(Hamiltonian case) suggest that the qualitative properties of solutions to Hamiltonian and dissipative systems are different. The first impression is that adding derivatives of even order to the quasilinear part yields dissipation while only odd derivatives added are allowed in Hamiltonian PDEs. The following 2×2 system with constant coefficients

$$\left.\begin{array}{rcl} u_t & = & a_{12} u_x + a_{22} v_x - u_{xx} \\ \\ v_t & = & a_{11} u_x + a_{12} v_x + v_{xx} \end{array}\right\} \tag{1.18}$$

gives an example of a Hamiltonian system with even highest derivatives.

Before proceeding with the study of Hamiltonian PDEs I will recall basics of finite dimensional Hamiltonian formalism.

In classical mechanics a Hamiltonian system with n degrees of freedom is the following system of $2n$ differential equations

$$\begin{aligned} \dot q_i &= \frac{\partial H}{\partial p_i} \\ \dot p_i &= -\frac{\partial H}{\partial q_i} \end{aligned} \tag{1.19}$$

$i = 1, \dots, n$. The function $H = H(p, q) \in \mathcal{C}^\infty(\mathbb{R}^{2n})$ is called the Hamiltonian of the system. The system can be written in a more symmetric way

$$\begin{aligned} \dot q_i &= \{q_i, H\} \\ \dot p_i &= \{p_i, H\} \end{aligned} \tag{1.20}$$

by means of Poisson brackets

$$\{f, g\} = \sum_{i=1}^n \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \tag{1.21}$$

defined for arbitrary two smooth functions on $\mathbb{R}^{2n}$. Introducing collective notations for the coordinates

$$x = (x^1, \dots, x^{2n}) = (q_1, \dots, q_n, p_1, \dots, p_n)$$

one can recast the system (1.20) into the form

$$\dot x^i = \{x^i, H\} = \pi^{ij}\frac{\partial H}{\partial x^j}, \quad i = 1, \dots, 2n. \tag{1.22}$$

Here

$$\pi^{ij} = \{x^i, x^j\} \tag{1.23}$$

is the $2n \times 2n$ matrix of Poisson brackets of coordinate functions. In the canonical coordinates $x = (q_1, \dots, q_n, p_1, \dots, p_n)$ this matrix has a constant form

$$\left(\pi^{ij}\right) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.24}$$

In terms of this matrix the operation of Poisson bracket can be rewritten as follows

$$\{f, g\} = \pi^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}. \tag{1.25}$$

It satisfies the following properties:

PB1. It is a bilinear antisymmetric map

$$\mathcal{C}^\infty(\mathbb{R}^{2n}) \times \mathcal{C}^\infty(\mathbb{R}^{2n}) \to \mathcal{C}^\infty(\mathbb{R}^{2n}).$$

PB2. It satisfies Leibnitz rule

$$\{f\, g, h\} = f\,\{g, h\} + g\{f, h\}$$

PB3. and Jacobi identity

$$\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0.$$

Thus the Poisson bracket defines on $\mathcal{C}^\infty(\mathbb{R}^{2n})$ a structure of a Lie algebra compatible with the product structure via the Leibnitz identity.

A function $F \in \mathcal{C}^\infty(\mathbb{R}^{2n})$ commuting with the Hamiltonian H

$$\{F, H\} = 0$$

will be a *first integral* of the Hamiltonian system (1.22):

$$\dot F := \frac{\partial F}{\partial x^i} \dot x^i = 0.$$

Indeed, the r.h.s. is equal to the Poisson bracket $\{F, H\}$.

First integrals of a given Hamiltonian system define a Lie subalgebra in the Lie algebra $\mathcal{C}^\infty(\mathbb{R}^{2n})$. In particular the Hamiltonian itself is a first integral of (1.22).

A first integral F of the Hamiltonian system (1.22) generates a one-parameter group of infinitesimal symmetries of the system (1.22). That means that the commutativity $\{F, H\} = 0$ of the Hamiltonians implies commutativity

$$\frac{d}{ds}\frac{dx^i}{dt} = \frac{d}{dt}\frac{dx^i}{ds}$$

of the flows

$$\frac{dx^i}{dt} = \{x^i, H\}, \quad i = 1, \dots, 2n$$

and

$$\frac{dx^i}{ds} = \{x^i, F\}, \quad i = 1, \dots, 2n$$

Moreover these flows are infinitesimal canonical transformations of the phase space $\mathbb{R}^{2n}$, i.e., they preserve Poisson brackets. Finite canonical transformations close to identity can be represented as the time-1 shift along trajectories of a Hamiltonian system:

$$x^i \mapsto x^i + \{x^i, F\} + \frac{1}{2}\{\{x^i, F\}, F\} + \dots \tag{1.26}$$

More general Poisson brackets on a smooth manifold M are defined as a Lie algebra structure on $\mathcal{C}^\infty(M)$ satisfying the above properties **PB1** – **PB3**. In local coordinates $x^1, \dots, x^N$ the Poisson bracket is defined by a skew symmetric matrix

$$\pi^{ij}(x) = \{x^i, x^j\}. \tag{1.27}$$

All the formulae (1.22) and (3.22) remain valid also in this more general case. The matrix can be degenerate, so the dimension N of the manifold need not to be even. The skew symmetric matrix $\pi^{ij}(x)$ cannot be arbitrary. It must satisfy the following system of constraints imposed by the Jacobi identity

$$\{\{x^i, x^j\}, x^k\} + \{\{x^k, x^i\}, x^j\} + \{\{x^j, x^k\}, x^i\} = 0, \quad i,\, j,\, k = 1, \dots, N. \tag{1.28}$$

The l.h.s. of this equation reads

$$\frac{\partial \pi^{ij}}{\partial x^s}\pi^{sk} + \frac{\partial \pi^{ki}}{\partial x^s}\pi^{sj} + \frac{\partial \pi^{jk}}{\partial x^s}\pi^{si} =: \frac{1}{2}[\pi, \pi]^{ijk}. \tag{1.29}$$

The notation $[\pi, \pi]$ stands for the Schouten – Nijenhuis bracket of the bivector π with itself. In particular the bracket $[\pi, \pi]$ vanishes if the tensor π^{ij} has constant components in the coordinates $x^1, \dots, x^N$. In this case we will say that the Poisson bracket is constant (in the given coordinates $x^1, \dots, x^N$).

Let us do two remarks about perturbative techniques in the Hamiltonian formalism. There are two types of perturbations of a given Hamiltonian system: 1) a perturbation of the Hamiltonian and 2) a perturbation of the Poisson bracket. Let us first consider a perturbation of the Hamiltonian

$$H = H_0 + \epsilon H_1 + \epsilon^2 H_2 + \dots.$$

We look for a canonical transformation

$$x^i \mapsto x^i + \epsilon\{x^i, F\} + \frac{\epsilon^2}{2}\{\{x^i, F\}, F\} + \dots =: \tilde{x}^i$$

killing the perturbation. The Hamiltonian itself may depend on ϵ:

$$F = F_0 + \epsilon, F_1 + \epsilon^2 F_2 + \dots.$$

For the first correction of order $O(\epsilon)$ one arrives at the *homology equation*

$$\{H_0, F_0\} = H_1. \tag{1.30}$$

If this equation has a solution then one can kill the first order perturbation and then proceed to considering the next correction.

The second case of perturbations of the Poisson brackets will be considered for the perturbations of a constant bracket

$$\pi^{ij}(x) = \pi_0^{ij} + \epsilon\, \pi^{ij}(x) + \dots$$

where the matrix π_0^{ij} has the canonical form (1.24). The classical Darboux lemma says that any such perturbation can be eliminated by a suitable local change of coordinates. We will sketch the proof of Darboux lemma in the following form. In the linear approximation in ϵ the Jacobi identity implies vanishing of the Schouten – Nijenhuis bracket of π_0 and π_1:

$$[\pi_0, \pi_1] = 0.$$

Explicitly this equation reads

$$\pi_0^{is}\frac{\partial \pi_1^{jk}}{\partial x^s} + \pi_0^{ks}\frac{\partial \pi_1^{ij}}{\partial x^s} + \pi_0^{js}\frac{\partial \pi_1^{ki}}{\partial x^s} = 0.$$

This equation says that the first order perturbation is a 2-cocycle in the Poisson cohomology $\pi_1 \in H^2(M, \pi_0)$. To locally kill the perturbation π_1 one has to find an infinitesimal change of coordinates

$$x^i \mapsto x^i + \epsilon\, X^i(x) + \dots$$

such that, after the change of coordinates

$$\pi^{ij}(x) \mapsto \pi_0^{ij} + O(\epsilon^2).$$

For the vector field X^i one obtains the following homology equation:

$$[\pi_0, X] = \pi_1$$

or, explicitly,

$$\frac{\partial X^i}{\partial x^s}\pi_0^{sj} + \pi_0^{is}\frac{\partial X^j}{\partial x^s} = \pi_1^{ij}.$$

The 2-cocycles representable in such a form are called 2-coboundaries. In order to complete the proof of Darboux lemma it suffices to prove that any 2-cocycle is locally a coboundary. Indeed, using nondegenerateness of π_0 let us define a 2-form ω_1 by "lowering the indicies"

$$(\omega_1)_{ij} = {\pi_0^{-1}}_{ik}{\pi_0^{-1}}_{jl}\pi_1^{kl}(x).$$

The cocycle condition $[\pi_0, \pi_1] = 0$ spells out as the closedness of the 2-form:

$$d\omega_1 = 0 \quad \Leftrightarrow \quad \frac{\partial \omega_{1ij}}{\partial x^k} + \frac{\partial \omega_{1ki}}{\partial x^j} + \frac{\partial \omega_{1jk}}{\partial x^i} = 0.$$

Locally applying Poincaré lemma one obtains a 1-form $f = f_i(x)dx^i$ such that

$$\omega_1 = df.$$

The vector field

$$X^i(x) = \pi_0^{ik} f_k(x)$$

realizes the needed change of coordinates.

2. Lecture 2

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The last reminder from the finite dimensional Hamiltonian formalism is about completely integrable Hamiltonian systems. The Hamiltonian system

$$\dot{x}^i = \{x^i, H\}$$

is called *completely integrable* if the Hamiltonian H can be included into a *complete* commutative Lie algebra of Hamiltonians

$$H_1 = H, \quad H_2, H_3, \dots$$
$$\{H_i, H_j\} = 0.$$

The condition of completeness requires a precise definition. In the simplest case of a Hamiltonian system on a $2n$-dimensional phase space with the canonical Poisson bracket (1.24) one has to have n independent pairwise commuting first integrals of the system. The Hamiltonian PDEs we are going to consider will be considered as Hamiltonian systems with an infinite number of degrees of freedom. So the number of commuting Hamiltonians of a completely integrable system will necessarily be infinite. We will discuss later the precise definition of complete integrability in this case.

We will now proceed with the infinite dimensional generalization of Hamiltonian formalism. We will deal with systems of PDEs of the form (1.8) assuming all dependent variables to be slow (see Lecture 1 above). Let us start with considering a simple example of a Poisson bracket on the space of smooth 2π-periodic functions $u(x)$, $x \in S^1$:

$$\{u(x), u(y)\} = \delta'(x-y). \tag{2.1}$$

Here $\delta(x)$ is the Dirac delta-function on the circle defined by the condition

$$\frac{1}{2\pi}\int_0^{2\pi} f(y)\delta(x-y)\,dy = f(x) \tag{2.2}$$

for any smooth function on the circle. The derivatives of delta-function are defined in the usual way

$$\frac{1}{2\pi}\int_0^{2\pi} f(y)\delta'(x-y)\,dy = f'(x) \tag{2.3}$$

etc.

Let us now explain how to compute Poisson bracket of two functionals F and G on $\mathcal{C}^\infty(S^1)$. We consider the particular case of *local* functionals

$$F = \frac{1}{2\pi}\int_0^{2\pi} f(u; u_x, \dots, u^{(k)})\,dx, \quad G = \frac{1}{2\pi}\int_0^{2\pi} g(u; u_x, \dots, u^{(l)})\,dx. \tag{2.4}$$

Proposition 2.1. *The Poisson bracket of two local functionals is given by the formula*

$$\{F, G\} = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \frac{\delta F}{\delta u(x)}\{u(x), u(y)\}\frac{\delta G}{\delta u(y)}\,dx\,dy. \tag{2.5}$$

Here the Frechêt derivative of a local functional is represented by the Euler – Lagrange operator

$$\frac{\delta F}{\delta u(x)} = \frac{\partial f}{\partial u} - \partial_x \frac{\partial f}{\partial u_x} + \partial_x^2 \frac{\partial f}{\partial u_{xx}} - \dots, \tag{2.6}$$

the operator of the total x-derivative reads

$$\partial_x h(u; u_x, u_{xx}, \dots) = \sum_{i \geq 0} u^{(i+1)} \frac{\partial h}{\partial u^{(i)}}. \tag{2.7}$$

This formula elucidates the meaning of the symbol $\{u(x), u(y)\}$: this is just the kernel of the bilinear functional (2.5). Comparing with the expression (3.22) for the finite dimensional Poisson bracket we see that the kernel $\{u(x), u(y)\}$ is an infinite dimensional analogue of the matrix of Poisson brackets of coordinates; the novelty is that the "infinite-dimensional matrix" $\{u(x), u(y)\}$ has continuous indices x and y.

The proof of the formula (2.5) uses bilinearity of the Poisson bracket and the Leibnitz rule:

$$\begin{aligned}
\{F, G\} &= \frac{1}{(2\pi)^2} \int_0^{2\pi} dx \int_0^{2\pi} dy \, \{f(u(x); u_x, \dots), g(u(y); u_y, \dots)\} \\
&= \frac{1}{(2\pi)^2} \int_0^{2\pi} dx \int_0^{2\pi} dy \sum_{i=0}^{k} \sum_{j=0}^{l} \frac{\partial f}{\partial u^{(i)}}(x) \{u^{(i)}(x), u^{(j)}(y)\} \frac{\partial g}{\partial u^{(j)}}(y) \\
&= \frac{1}{(2\pi)^2} \int_0^{2\pi} dx \int_0^{2\pi} dy \sum_{i=0}^{k} \sum_{j=0}^{l} \frac{\partial f}{\partial u^{(i)}}(x) \partial_x^i \partial_y^j \{u(x), u(y)\} \frac{\partial g}{\partial u^{(j)}}(y) \\
&= \frac{1}{(2\pi)^2} \int_0^{2\pi} dx \int_0^{2\pi} dy \sum_{i=0}^{k} (-\partial_x)^i \frac{\partial f}{\partial u^{(i)}}(x) \{u(x), u(y)\} \sum_{j=0}^{l} (-\partial_y)^j \frac{\partial g}{\partial u^{(j)}}(y)
\end{aligned}$$

where the integration by parts was used at the very last step. The formula (2.5) is proved. □

Corollary 2.2. *The Poisson bracket* (2.1) *of two local functionals is again a local functional given by the formula*

$$\{F, G\} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\delta F}{\delta u(x)} \frac{d}{dx} \frac{\delta G}{\delta u(x)} \, dx. \tag{2.8}$$

Proof This follows from (2.5) by applying the definition (2.3) of the derivative of delta-function. □

We did not explain *why* the formula (2.1) defines a Poisson bracket. This can be done using the explicit formula (2.8). It is more instructive however to use the analogy with the finite dimensional case. The matrix $\{u(x), u(y)\}$ of the Poisson brackets of the coordinate functions is skew symmetric and constant. Indeed, the Dirac delta-function $\delta(x)$ is even, so the derivative $\delta'(x)$ is an odd function, hence

$$\{u(y), u(x)\} = \delta'(y - x) = -\delta'(x - y) = -\{u(x), u(y)\}.$$

The r.h.s. of (2.1) does not depend on u, that is, this is an infinite dimensional example of a *constant* Poisson bracket (see Lecture 1 for details). Therefore we have validity of the Jacobi identity for free.

We now give one more equivalent representation of the Poisson bracket (2.1). The Fourier coefficients

$$u(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx} \tag{2.9}$$

of the 2π-periodic function $u(x)$ can be considered as a system of coordinates on the functional space. We will use these coordinates in order to reduce the bracket (2.1) to the Darboux form.

Proposition 2.3. *The Poisson brackets of the Fourier coefficients are given by the following formla*

$$\{c_n, c_m\} = i\, n\, \delta_{n+m,0}. \tag{2.10}$$

Proof Using the expression

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} u(x)e^{-inx}dx$$

for the Fourier coefficients we obtain

$$\begin{aligned}\{c_n, c_m\} &= \frac{1}{(2\pi)^2}\int_0^{2\pi} dx \int_0^{2\pi} dy \{u(x)e^{-inx}, u(y)e^{-imy}\} = \\ &= \frac{1}{(2\pi)^2}\int_0^{2\pi} dx \int_0^{2\pi} dy e^{-inx-imy}\delta'(x-y) = \\ &= \frac{1}{2\pi}\int_0^{2\pi}(-im)e^{-i(n+m)x}dx = (-im)\,\delta_{n+m,0} = i\,n\,\delta_{n+m,0}.\end{aligned}$$

□

Corollary 2.4. *The Poisson bracket* (2.1) *is degenerate. The symplectic leaves have codimension one in the functional space, they are level surfaces of the functional*

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} u(x)\,dx. \tag{2.11}$$

The canonical Darboux coordinates on the level surfaces are given by the functionals

$$\begin{aligned} q_n &= \quad c_n, \quad n > 0 \\ p_n &= \frac{1}{in} c_{-n}, \quad n > 0. \end{aligned}$$

Observe that the mean value c_0 is a *Casimir* of the Poisson bracket (2.1): it commutes with any functional:

$$\{c_0, F\} = 0.$$

This can also be easily derived from the formula (2.8).

Let us now consider examples of Hamiltonian systems with the infinite-dimensional Poisson bracket (2.1). By definition they have the form

$$u_t(x) = \{u(x), H\} = \partial_x \frac{\delta H}{\delta u(x)} \tag{2.12}$$

where the Hamiltonian

$$H = \frac{1}{2\pi}\int_0^{2\pi} h(u; u_x, u_{xx}, \dots, u^{(m)})\, dx$$

is a local functional. We obtain an *evolutionary PDE* containing the spatial derivatives up to the order $2m+1$.

Let us consider an example of such a Hamiltonian PDE taking the Hamiltonian in the form

$$H = \frac{1}{2\pi}\int_0^{2\pi} \frac{u^3}{6}\, dx. \tag{2.13}$$

Then the Hamiltonian system of the form (2.12) coincides with Hopf equation

$$u_t = \{u(x), H\} = \partial_x \frac{u^2}{2} = u\, u_x. \tag{2.14}$$

The choice of the Hamiltonian in the form

$$H = \frac{1}{2\pi}\int_0^{2\pi} \left[\frac{1}{6}u^3 - \frac{1}{2}u_x^2\right] dx \tag{2.15}$$

generates the KdV equation (1.17) (after the time reversion $t \mapsto -t$).

Remark 2.5. *How one can check that an evolutionary PDE*

$$u_t = f(u; u_x, u_{xx}, \dots, u^{(m)}) \tag{2.16}$$

is a Hamiltonian system with respect to the Poisson bracket (2.1)*? This can be done by using the following two classical statements.*

Theorem 2.6. *If the differential polynomial $f(u; u_x, u_{xx}, \dots, u^{(m)})$ is a total x-derivative of another differential polynomial,*

$$f(u; u_x, u_{xx}, \dots, u^{(m)}) = \partial_x g(u; u_x, u_{xx}, \dots, u^{(m-1)}) \tag{2.17}$$

then the Euler – Lagrange derivative of the functional

$$F = \frac{1}{2\pi}\int_0^{2\pi} f(u; u_x, u_{xx}, \dots, u^{(m)})\, dx$$

identically vanishes:

$$\sum_{i=0}^{m} (-\partial_x)^i \frac{\partial f}{\partial u^{(i)}} = 0. \tag{2.18}$$

Locally this condition is also sufficient for the representation of f in the form (2.17).

Theorem 2.7. *If the differential polynomial $g(u; u_x, u_{xx}, \dots, u^{(m-1)})$ is represented as the Euler – Lagrange derivative of some local functional,*

$$g(u; u_x, u_{xx}, \dots, u^{(m-1)}) = \sum_{i\geq 0} (-\partial_x)^i \frac{\partial h}{\partial u^{(i)}} \tag{2.19}$$

then it satisfies the following system of equations

$$\frac{\partial g}{\partial u^{(j)}} = \sum_{k\geq j} (-1)^k \left(\begin{array}{c} k \\ j \end{array} \right) \partial_x^{k-j} \frac{\partial g}{\partial u^{(k)}} \tag{2.20}$$

for any $j = 0, 1, \dots, m-1$. *Locally these conditions are also sufficient for the representation of g in the form* (2.19).

The last theorem gives a solution to the inverse problem of calculus of variations found by Helmholtz (for $m - 1 = 2$).

Let us now consider a simple example of an infinite-dimensional integrable system. Given a smooth function $f(u)$ denote

$$H_f^0 = \frac{1}{2\pi} \int_0^{2\pi} f(u)\, dx. \tag{2.21}$$

Proposition 2.8. *The Hamiltonians of the form* (2.21) *commute pairwise with respect to the Poisson bracket* (2.1)*:*

$$\{H_f^0, H_g^0\} = 0 \tag{2.22}$$

for an arbitrary pair of smooth functions $f(u)$, $g(u)$.

Proof The computation using the formula (2.8) yields

$$\{H_f^0, H_g^0\} = \frac{1}{2\pi} \int_0^{2\pi} f'(u) g''(u) u_x \, dx.$$

One can find a smooth function $\Phi(u)$ such that

$$\Phi'(u) = f'(u) g''(u).$$

Then the above bracket reduces to the integral over the period of the x-derivative of the 2π-periodic function:

$$\{H_f^0, H_g^0\} = \frac{1}{2\pi} \int_0^{2\pi} \partial_x \Phi(u)\, dx = 0.$$

□

For $f = \frac{1}{6} u^3$ the Hamiltonian H_f^0 generates the Hopf equation (2.14). Thus the Hamiltonian system (2.14) can be included into an infinite-dimensional family of pairwise commuting Hamiltonians. Restricting ourselves to analytic functions only we obtain a basis in the space of commuting Hamiltonians

$$H_k = \frac{1}{2\pi} \int_0^{2\pi} \frac{u^{k+2}}{(k+2)!}\, dx, \quad k \geq -1. \tag{2.23}$$

They generate an infinite family of pairwise commuting PDEs

$$\frac{\partial u}{\partial t_k} = \{u(x), H_k\} = \frac{u^k}{k!}\frac{\partial u}{\partial x}, \quad k \geq 0. \tag{2.24}$$

For $k = 1$ one obtains Hopf equation (2.14). The family of commuting flows (2.24) is called the *dispersionless KdV hierarchy.* It is obtained from the classical KdV hierarchy by the rescaling $x \mapsto \epsilon\, x$, $t_k \mapsto \epsilon\, t_k$ and then setting ϵ to 0.

Actually the solution to the equations (2.24) can be written in a simple form. To be more specific let us consider the Cauchy problem of the form

$$u(x, \mathbf{t} = 0) = x. \tag{2.25}$$

Here $\mathbf{t}$ is the infinite vector of times

$$\mathbf{t} = (t_0, t_1, t_2, \dots).$$

This solution arises in the intersection theory of the so-called tautological classes on the Deligne – Mumford moduli spaces $\bar{\mathcal{M}}_{0,n}$.

Proposition 2.9. *For sufficiently small times the solution to the Cauchy problem* (2.25) *can be determined from the following implicit function equation*

$$u = x + \sum_{k\geq 0} t_k \frac{u^k}{k!}. \tag{2.26}$$

Proof Differentiating equation (2.26) in x and t_k yields

$$\frac{\partial u}{\partial x} = \frac{1}{1 - \sum_{k\geq 1} t_k \frac{u^{k-1}}{(k-1)!}}, \quad \frac{\partial u}{\partial t_k} = \frac{u^k/k!}{1 - \sum_{k\geq 1} t_k \frac{u^{k-1}}{(k-1)!}}.$$

This implies equation (2.24). It remains to observe that at $\mathbf{t} = 0$ the equation (2.26) reduces to the initial condition (2.25). □

Other solutions sufficiently close to (2.26) can be obtained by shifts along times t_0, t_1, ... It is clear that any analytic solution of this type can be obtained by a suitable shift. This explains the meaning of *completeness* of the family of commuting Hamiltonians (2.23).

Let us now generalize the example (2.1) of an infinite-dimensional Poisson bracket to systems of PDEs. Let η^{ij} be a symmetric $n \times n$ nondegenerate constant matrix

$$\eta^{ji} = \eta^{ij}, \quad \det\left(\eta^{ij}\right) \neq 0.$$

Introduce the following Poisson bracket on the space of 2π-periodic vector functions $\mathbf{u}(x) = (u^1(x), \dots, u^n(x))$:

$$\{u^i(x), u^j(y)\} = \eta^{ij}\delta'(x - y), \quad i, j = 1, \dots, n. \tag{2.27}$$

Repeating the above arguments one obtains the following formula for the Poisson bracket of two local functionals

$$\{F, G\} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\delta F}{\delta u^i(x)}\eta^{ij}\frac{d}{dx}\frac{\delta G}{\delta u^j(x)}\, dx. \tag{2.28}$$

The Hamiltonian system with a Hamiltonian H can be written in the form

$$u^i_t = \partial_x \eta^{ij} \frac{\delta H}{\delta u^j(x)}, \quad i = 1, \dots, n. \tag{2.29}$$

This a system of n evolutionary PDEs if the Hamiltonian is a local functional. In particular if the density of the Hamiltonian does not depend on the derivatives

$$H = \frac{1}{2\pi} \int_0^{2\pi} h(\mathbf{u})\, dx$$

then (2.29) becomes a system of the first order quasilinear PDEs

$$\begin{aligned} u^i_t &= \{u^i(x), H\} = A^i_j(\mathbf{u}) u^j_x, \quad i = 1, \dots, n \\ A^i_j(\mathbf{u}) &= \eta^{ik} \frac{\partial^2 h(\mathbf{u})}{\partial u^k \partial u^j} \end{aligned} \tag{2.30}$$

Example 2.10. *For $n = 2$ choose the matrix η in the form*

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This gives the following Poisson bracket on the space of two-component functions $(u(x), v(x))$:

$$\{u(x), v(y)\} = \{v(x), u(y)\} = \delta'(x - y),$$

other Poisson brackets vanish. The Hamiltonian systems read

$$\begin{aligned} u_t &= \partial_x \frac{\delta H}{\delta v(x)} \\ v_t &= \partial_x \frac{\delta H}{\delta u(x)}. \end{aligned}$$

Choosing the Hamiltonian in the form

$$H = \frac{1}{2\pi} \int_0^{2\pi} \left[h(u, v) + u\, v_x \right] dx$$

one obtains the Hamiltonian system of the form

$$\begin{aligned} u_t &= h_{uv} u_x + h_{vv} v_x - u_{xx} \\ v_t &= h_{uu} u_x + h_{uv} v_x + v_{xx}. \end{aligned}$$

In the particular case

$$h = \text{a quadratic form}$$

one obtains the system (1.18).

Remark 2.11. *We do not consider here one even more "natural" class of infinite-dimensional Poisson brackets producing Hamiltonian PDEs of the form*

$$\begin{aligned} \partial_t q_i(x) &= \quad \frac{\delta H}{\delta p_i(x)} \\ \partial_t p_i(x) &= -\frac{\delta H}{\delta q_i(x)} \end{aligned} \tag{2.31}$$

$i = 1, \dots, m$. Here $n = 2m$, the nonzero brackets have the form

$$\{q_i(x), p_j(y)\} = \delta_{ij}\delta(x - y), \quad i, j = 1, \dots, m.$$

These brackets are not well adapted for describing the Hamiltonian structure of the first order quasilinear PDEs of the form (2.30). *The dependent variables in the Hamiltonian systems* (2.31) *typically are not "slow-varying" (see Lecture 1 above).*

The above Poisson brackets (2.1), (2.27) possess one important property: the Poisson bracket of two local functionals is again a local functional. Therefore a structure of Lie algebra on the space of local functionals is associated with any such bracket. For this reason we call them *local Poisson brackets*. More general local Poisson brackets can be written in the form of an infinite expansion

$$\{u^i(x), u^j(y)\} = \sum_{k\geq 0} \epsilon^k \sum_{m=0}^{k+1} A^{ij}_{k,m}(u(x); u_x, \dots, u^{(m)})\delta^{(k-m+1)}(x - y) \tag{2.32}$$
$$\deg A^{ij}_{k,m}(u; u_x, \dots, u^{(m)}) = m.$$

We emphasize that the coefficients of the bracket are polynomials in the derivatives. In particular the leading coefficients $A^{ij}_{k,0}$ must have degree zero for any k, so they do not depend on the derivatives at all.

The class of local Poisson brackets is invariant with respect to the *generalized Miura transformations* changing the dependent variables u^i to

$$\tilde{u}^i = F^i_0(u) + \sum_{k\geq 1} \epsilon^k F^i_k(u; u_x, \dots, u^{(k)}) \tag{2.33}$$
$$\deg F^i_k(u; u_x, \dots, u^{(k)}) = k$$
$$\det\left(\frac{\partial F^i_0(u)}{\partial u^j}\right) \neq 0.$$

It is again assumed that all terms of the expansion are graded homogeneous polynomials in the derivatives. It is easy to see that the transformations of the form (2.33) form a group.

Definition 2.12. *Two Poisson brackets of the form* (2.32) *are called* equivalent *if they are related by a generalized Miura transformation.*

Theorem 2.13. *Under assumption*

$$\det\left(A^{ij}_{0,0}(u)\right) \neq 0 \tag{2.34}$$

any Poisson bracket of the form (2.32) *is locally equivalent to the standard one* (2.27).

Proof The leading order term in (2.32)

$$\{u^i(x), u^j(y)\}^{[0]} = A^{ij}_{0,0}(u(x))\delta'(x - y) + A^{ij}_{0,1}(u(x); u_x(x))\delta(x - y) \tag{2.35}$$

is itself a Poisson bracket. Under the assumption (2.34) this is a so-called Poisson bracket of hydrodynamic type. The general theory of Poisson brackets of hydrodynamic type (B.Dubrovin, S.P.Novikov, 1983) says that the inverse to the matrix $\left(A^{ij}_{0,0}(u)\right)$

defines a metric of zero curvature on the ball $\mathcal{B}$. The coefficient $A^{ij}_{0,1}(u; u_x)$ linearly depending on u_x is expressed in terms of the Levi-Civita connection for the metric. Choosing flat coordinates for the metric we reduce the bracket (2.32) locally to the form

$$\{u^i(x), u^j(y)\} = \eta^{ij}\delta'(x-y) + O(\epsilon)$$

for some symmetric nondegenerate constant matrix η^{ij}. At the second step of the proof one has to kill the ϵ-perturbation of the standard Poisson bracket (2.27). This can be done using the perturbative approach to proving the Darboux lemma explained in Lecture 1 using triviality in positive degrees in ϵ of the Poisson cohomology of the bracket (2.27) proved by E.Getzler, 2002 (see also L. Degiovanni, F.Magri, V. Sciacca, 2005). □

At the end of this lecture let us give an example of reducing some less standard Poisson brackets to the form (2.27). Consider the general Fermi – Pasta – Ulam (FPU) system of an infinite number of particles on the line with a nonlinear interaction of neighbors. Denote q_n, $n \in \mathbb{Z}$ the coordinate of the n-th particle. The Hamiltonian of the FPU system reads

$$H = \sum \frac{1}{2}p_n^2 + V(q_n - q_{n-1}). \tag{2.36}$$

Here $V(q)$ is the potential of the interaction. The equations of motion

$$\begin{aligned} \dot{q}_n &= p_n \\ \dot{p}_n &= V'(q_{n+1} - q_n) - V'(q_n - q_{n-1}) \end{aligned}$$

can be recast into the form

$$\begin{aligned} \dot{u}_n &= v_{n+1} - v_n \\ \dot{v}_n &= V'(u_n) - V'(u_{n-1}) \end{aligned} \tag{2.37}$$

by the substitution

$$\begin{aligned} u_n &= q_{n+1} - q_n \\ v_n &= p_n. \end{aligned}$$

The new variables are not canonical, however:

$$\{u_n, v_m\} = \delta_{m,n+1} - \delta_{m,n}. \tag{2.38}$$

We will now introduce interpolating functions $u(x)$, $v(x)$,

$$u_n = u(n\,\epsilon), \quad v_n = v(n\,\epsilon).$$

After rescaling $t \to \epsilon\, t$ the equations of motion will be written in the form of a system of infinite order PDEs

$$\begin{aligned} u_t &= \frac{v(x+\epsilon) - v(x)}{\epsilon} = v_x + \frac{\epsilon}{2}v_{xx} + \dots \\ v_t &= \frac{V'(u(x)) - V'(u(x-\epsilon))}{\epsilon} = V''(u)u_x - \frac{\epsilon}{2}\partial_x\left(V''(u)u_x\right) + \dots. \end{aligned} \tag{2.39}$$

From (2.38) after division by ϵ one obtains the Poisson bracket

$$\{u(x), v(y)\} = \frac{\delta(x-y+\epsilon) - \delta(x-y)}{\epsilon} = \delta'(x-y) + \frac{\epsilon}{2}\delta''(x-y) + \dots \tag{2.40}$$

representing (2.39) in the Hamiltonian form with the Hamiltonian

$$H = \int \left[\frac{1}{2}v^2 + V(u)\right] dx.$$

The bracket (2.40) can be considered as a perturbation of the standard Poisson bracket (2.27). Rewriting the formula (2.40) in the form

$$\{u(x), v(y)\} = \frac{e^{\epsilon\partial_x} - 1}{\epsilon\partial_x}\delta'(x-y)$$

we immediately obtain the reducing transformation

$$\tilde{u}(x) = \frac{\epsilon\partial_x}{e^{\epsilon\partial_x} - 1}u(x) = u(x) + \sum_{k\geq 1}\frac{B_k}{k!}\epsilon^k u^{(k)}$$

$$\{\tilde{u}(x), v(y)\} = \delta'(x-y).$$

Here B_k are the Bernoulli numbers.

3. Lecture 3

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Our starting point was in looking at the solutions to the system

$$\begin{aligned} u_t^i = A_j^i(\mathbf{u})u_x^j &+ \epsilon\left[B_j^i(\mathbf{u})u_{xx}^j + \frac{1}{2}L_{jk}^i(\mathbf{u})u_x^j u_x^k\right] \\ &+\epsilon^2\left[C_j^i(\mathbf{u})u_{xxx}^j + M_{jk}^i(\mathbf{u})u_{xx}^j u_x^k + \frac{1}{6}N_{jkm}^i(\mathbf{u})u_x^j u_x^k u_x^m\right] + O\left(\epsilon^3\right), \\ & i = 1, \dots, n. \end{aligned} \tag{3.1}$$

considering it as a perturbation of the first order quasilinear part

$$u_t^i = A_j^i(\mathbf{u})u_x^j. \tag{3.2}$$

The full system is assumed to be written in the Hamiltonian form

$$u_t^i = \{u^i(x), H\} = \partial_x\eta^{ij}\frac{\delta H}{\delta u^j(x)} \tag{3.3}$$

with respect to the Poisson bracket (2.27) with the local Hamiltonian written in the form of an expansion

$$H = H_0 + \epsilon\, H_1 + \epsilon^2 H_2 + \dots, \quad H_k = \int h_k(u; u_x, \dots, u^{(k)})\, dx \tag{3.4}$$

where every term h_k in the expansion of the Hamiltonian density is a graded homogeneous differential polynomial of the degree k. In particular the Hamiltonian density $h_0 = h_0(\mathbf{u})$ gives the Hamiltonian formulation of the leading approximation

(3.2). Moreover the leading order system (3.2) will be assumed to be hyperbolic for $\mathbf{u} \in \mathcal{B} \subset \mathbb{R}^n$.

The natural questions in the theory of such perturbations are:

• to classify perturbations (3.1) – (3.4) modulo canonical transformations

$$u^i \mapsto \tilde{u}^i = u^i + \epsilon\{u^i(x), K\} + \frac{\epsilon^2}{2}\{\{u^i(x), K\}, K\} + \dots \tag{3.5}$$

(the time-ϵ shift) generated by local Hamiltonians K;

• assuming the leading order system (3.2) to be integrable to classify all integrable perturbations of this system;

• to compare the properties of solutions to the perturbed and unperturbed systems.

Example 3.1. *Arbitrary Hamiltonian perturbations of Hopf equation up to the order $O(\epsilon^4)$ are described by the Hamiltonians*

$$H = \int \left[\frac{1}{6}u^3 - \frac{\epsilon^2}{24}c(u)u_x^2 + \epsilon^4 p(u)u_{xx}^2\right] dx \tag{3.6}$$

depending on two arbitrary functions $c = c(u)$ and $p = p(u)$.

Remarkably all these perturbations remain integrable in this approximation! Namely, extending the commuting Hamiltonians of the form H_f^0 (see (2.21) above) according to the following formula

$$\begin{aligned} H_f &= H_f^0 + \epsilon^2 H_f^2 + \epsilon^4 H_f^4 = \int h_f(u; u_x, u_{xx})\, dx \\ h_f &= f - \frac{\epsilon^2}{24} c\, f''' u_x^2 + \epsilon^4 \left[\left(p\, f''' + \frac{c^2\, f^{(4)}}{480}\right) u_{xx}^2 \right. \\ &\left. - \left(\frac{c\, c''\, f^{(4)}}{1152} + \frac{c\, c'\, f^{(5)}}{1152} + \frac{c^2\, f^{(6)}}{3456} + \frac{p'\, f^{(4)}}{6} + \frac{p\, f^{(5)}}{6}\right) u_x^4\right] \end{aligned} \tag{3.7}$$

one obtains commutativity

$$\{H_f, H_g\} = O(\epsilon^6) \tag{3.8}$$

for an arbitrary pair of functions $f = f(u)$, $g = g(u)$.

Open Problem. *For arbitrary functional parameters $c = c(u)$, $p = p(u)$ prove existence and uniqueness, modulo canonical transformations* (3.5)*, of an extension of the Hamiltonians* (3.7) *to all orders in ϵ preserving commutativity.*

So far existence of such an extension is known ony for three particular cases:

1) $c(u) = \text{const}$, $p(u) = 0$ (KdV equation).

2) $c(u) = \text{const} \cdot u$, $p(u) = 0$ (Camassa – Holm equation).

3) $c(u) = \text{const}$, $p(u) = \text{const}$ (Volterra lattice equation).

Let us now proceed to studying the solutions to the perturbed system, beginning with the case of small times. One can expect that for small times the solutions of the perturbed and unperturbed systems are sufficiently close. In order to write the perturbative solution to (3.1) one can use the following remarkable *quasitriviality transformation* we are going to explain now.

First, the Hamiltonian perturbation (3.1) of (3.2) is called *trivial* if the two systems are equivalent modulo canonical transformations of the form (3.5) generated by a local Hamiltonian K polynomially depending on u_x, u_{xx}, etc. in every order in ϵ. Thus the perturbed Hamiltonian can be represented in the form

$$H = H_0 + \epsilon H_1 + \epsilon^2 H_2 + \cdots = H_0 + \epsilon\{H_0, K\} + \epsilon^2\{\{H_0, K\}, K\} + \dots \tag{3.9}$$

The perturbation is called *quasitrivial* if the representation (3.9) exists but the density of the generating Hamiltonian $K$is not polynomial in the derivatives.

Example 3.2. *The quasitriviality transformation for the perturbation* (3.6) *is generated by the Hamiltonian*

$$K = \int \left[\frac{\epsilon}{24} c(u) u_x \log u_x + \epsilon^3 \left(\frac{c^2(u)}{5760}\frac{u_{xx}^3}{u_x^3} - \frac{p(u)}{4}\frac{u_{xx}^2}{u_x}\right)\right] dx. \tag{3.10}$$

This statement implies that the substitution

$$\begin{aligned}
v \mapsto u = v &+ \frac{\epsilon^2}{24}\partial_x\left(c\frac{v_{xx}}{v_x} + c' v_x\right) + \epsilon^4 \partial_x\left[c^2\left(\frac{v_{xx}^3}{360\, v_x^4} - \frac{7\, v_{xx} v_{xxx}}{1920\, v_x^3} + \frac{v_{xxxx}}{1152\, v_x^2}\right)_x\right.\\
&+ c\,c'\left(\frac{47\, {v_{xx}}^3}{5760\, {v_x}^3} - \frac{37\, v_{xx}\, v_{xxx}}{2880\, {v_x}^2} + \frac{5\, v_{xxxx}}{1152\, v_x}\right) + c'^2\left(\frac{v_{xxx}}{384} - \frac{{v_{xx}}^2}{5760\, v_x}\right) + c\,c''\left(\frac{v_{xxx}}{144} - \frac{{v_{xx}}^2}{360\, v_x}\right)\\
&+ \frac{1}{1152}\left(7\, c'\, c''\, v_x\, v_{xx} + c''^2\, {v_x}^3 + 6\, c\, c'''\, v_x\, v_{xx} + c'\, c'''\, {v_x}^3 + c\, c^{(4)}\, {v_x}^3\right)\\
&\left.+ p\left(\frac{{v_{xx}}^3}{2\, {v_x}^3} - \frac{v_{xx}\, v_{xxx}}{{v_x}^2} + \frac{v_{xxxx}}{2\, v_x}\right) + p' v_{xxx} + p''\frac{v_x\, v_{xx}}{2}\right]
\end{aligned} \tag{3.11}$$

generated by the Hamiltonian (3.10) transforms solutions $v = v(x,t)$ of the Hopf equation

$$v_t = v\, v_x$$

to the solutions of the perturbed equation

$$u_t = \partial_x \frac{\delta H}{\delta u(x)}, \quad H = \int \left[\frac{1}{6}u^3 - \frac{\epsilon^2}{24}c(u) u_x^2 + \epsilon^4 p(u) u_{xx}^2\right] dx$$

Same transformation works also for solutions of the dispersionless hierarchy

$$v_s = \partial_x f'(v)$$

transforming them to solutions of the perturbed hierarchy

$$u_s = \partial_x \frac{\delta H_f}{\delta u(x)}.$$

Open Problem. *Find a construction of the Hamiltonian K generating the quasitriviality transformation for the KdV hierarchy.*

Recall that the KdV hierarchy describes isospectral deformations of the Schroedinger operator

$$L = \frac{\epsilon^2}{2}\partial_x^2 + u(x).$$

The equations of the hierarchy can be represented in the Lax form

$$\frac{\partial L}{\partial t_k} = [A_k, L], \quad A_k = c_k \left(L^{\frac{2k+1}{2}}\right)_+.$$

Here $(\quad)_+$ is the differential part of the pseudodifferential operator,

$$c_k = \frac{2^{\frac{2k+1}{2}}}{(2k+1)!!}$$

a suitable normalization constant. The dispersionless limit $\epsilon \to 0$ of the KdV hierarchy is described by the symbol

$$\lambda(p, x) = \frac{1}{2}p^2 + v(x)$$

of the Lax operator replacing the commutators of the operators L, A by the Poisson brackets of their symbols λ, α

$$[A, L] \to \{\alpha, \lambda\} = \frac{\partial \alpha}{\partial p}\frac{\partial \lambda}{\partial x} - \frac{\partial \alpha}{\partial x}\frac{\partial \lambda}{\partial p}.$$

In order to return back to the full KdV hierarchy one has to quantize the symbols. The canonical quasitriviality transformation (3.11) does the same work in a different way.

We will now continue the comparative study of solution to the perturbed and non-perturbed systems. Solutions to hyperbolic systems typically have a finite life span, $0 \leq t < t_0$. As $t \to t_0 - 0$ the solution $u(x,t)$ tends to a finite limit but the derivatives u_x, u_t blow up at some point $x = x_0$. This phenomenon is called *gradient catastrophe* of solutions to hyperbolic systems.

Our goal is to compare the solutions to perturbed and unperturbed systems near the point of gradient catastrophe of the latter. Such a comparison will lead us to formulation of an important *universaity conjecture* saying that, loosely speaking the shape of the perturbed solution near the point of gradient catastrophe of the unperturbed one essentially is independent of the choice of generic solution and, moreover of the choice of generic perturbation.

Let us explain the main motivations and the precise formulation of the universality conjecture on the simplest example of Hamiltonian perturbations (3.6) of Hopf equation. Let us first give more details about the behaviour of solutions to Hopf equation

$$v_t + v\, v_x = 0$$

near the point of gradient catastrophe.

Lemma 3.3. *Near the point of catastrophe the solution to Hopf equation is approximately equal to the function $v = v(x,t)$ determined by the cubic equation*

$$x = v\, t - \frac{v^3}{6},$$

up to shifts, rescalings and Galilean transformations.

Proof Let (x_0, t_0) be the point of catastrophe, $v_0 = \lim_{t\to t_0-0} v(x_0, t)$. We represent the solution to Hopf equation in the implicit form using the method of characteristics

$$x = v\,t + f(v).$$

The imlicit function theorem is applicable as soon the v-derivative is different from 0 for all x:

$$t + f'(v) \neq 0.$$

The first moment of time for which the above condition fails to be true is the moment of gradient catastrophe. Thus at the point (x_0, t_0, v_0) one must have

$$\begin{aligned}
&x_0 = v_0 t_0 + f(v_0)\\
&t_0 + f'(v_0) = 0\\
&f''(v_0) = 0.
\end{aligned}$$

The last condition holds true since the graph of the solution has an inflection at the point of catastrophe. Impose the genericity assumption saying that this inflection point does not degenerate, i.e.,

$$f'''(v_0) \neq 0.$$

Introduce shifted variables

$$\bar{x} = x - x_0, \quad \bar{t} = t - t_0, \quad \bar{v} = v - v_0.$$

After the rescaling

$$\begin{aligned}
&\bar{x} \mapsto k\,\bar{x}\\
&\bar{t} \mapsto k^{2/3}\bar{t}\\
&\bar{v} \mapsto k^{1/3}\bar{v}
\end{aligned} \tag{3.12}$$

with $k \to 0$ and a Galilean transformation

$$\bar{x} \mapsto \bar{x} - v_0\bar{t}$$

the implicit function equation will read

$$\bar{x} = \bar{v}\,\bar{t} + \frac{1}{6} f'''(v_0)\bar{v}^3 + O(k^{1/3}).$$

In the limit $k \to 0$ one obtains a cubic equation for $\bar{v}$.

In a similar manner one can describe the local structure of solution to a two-component hyperbolic system near the point of catastrophe. Let $r_\pm = r \pm (u, v)$ be the Riemann invariants of a two-component system, $x_\pm$ the characteristic directions at the point of catastrophy. Then, after shifts and rescalings a generic solution to a hyperbolic system

$$\begin{aligned}
u_t &= a_{11}(u, v)u_x + a_{12}(u, v)v_x\\
v_t &= a_{21}(u, v)u_x + a_{22}(u, v)v_x
\end{aligned}$$

is the following standard Whitney singularity:

$$\begin{aligned}
x_+ &= r_+\\
x_- &= r_+ r_- - \frac{1}{6} r_-^3.
\end{aligned} \tag{3.13}$$

The idea of the proof is similar to the above: at the generic point of catastrophe only one of the Riemann invariants breaks down ((3.13) describes the catastrophe of the invariant r_-).

Open Problem. *Describe the local structure near the point of gradient catastrophe of generic solutions to a hyperbolic Hamiltonian system* (3.2) *with* $n \geq 3$ *components.*

Recall that for $n \geq 3$ a generic hyperbolic system does not possess Riemann invariants, so the above arguments do not work. The Hamiltonian hyperbolic systems possessing Riemann invariants are all integrable; the local structure near the critical point in this case is similar to (3.13).

Let us return to solutions to the perturbed Hopf equation. In order to describe the local structure near the point of gradient catastrophe (also called *critical point*) let us introduce an appropriate special function. This function is determined from the following ordinary differential equation

$$X = T\,U - \left[\frac{1}{6}U^3 + \frac{1}{24}(U'^2 + 2\,U\,U'') + \frac{1}{240}U^{IV}\right] \tag{3.14}$$

depending on the real parameter T. The needed solution is selected by smoothness for all $X \in \mathbb{R}$ along with the asymptotics

$$U \sim \sqrt[3]{-6\,X}, \quad |X| \to \infty.$$

Theorem 3.4. *(T.Claeys, M.Vanlessen) The special solution to* (3.14) *exists and is unique for all real* T.

We will denote $U(X,T)$ this solution to (3.14) depending on the parameter T.

We ar enow ready to formulate the universality conjecture describing critical behaviour of solutions to generic Hamiltonian perturbations of Hopf equation.

Universality Conjecture. *Let* $v = v(x,t)$ *be a solution to Hopf equation smooth for*

$$|x - x_0| < r, \quad 0 \leq t < t_0.$$

Let it have a generic gradient catastrophe at the point $x = x_0$, $t = t_0$*; denote* $v_0 = v(x_0, t_0)$. *Denote* $u = u(x,t;\epsilon)$ *the solution to the perturbed Hopf equation satisfying*

$$u(x, 0; \epsilon) = v(x, t_0).$$

1) There exists $\rho < r$ *such that the solution* $u(x,t;\epsilon)$ *is defined on the domain*

$$|x - x_0| < \rho, \quad 0 \leq t < t_0.$$

Moreover as $\epsilon \to 0$ *it converges to* $v(x,t)$ *for* $t < t_0 - \Delta$ *for any positive* Δ.

2) There exists a positive $\delta = \delta(\epsilon) > 0$ *such that the solution* $u(x,t;\epsilon)$ *can be extended onto a bigger domain*

$$|x - x_0| < \rho, \quad 0 \leq t < t_0 + \delta(\epsilon).$$

Near the critical point (x_0, t_0) *it has the following asymptotics:*

$$u(x,t;\epsilon) \simeq v_0 + \epsilon^{2/7}\left(\frac{v_0}{\kappa^2}\right)^{1/7} U\left(\frac{x - x_0 - v_0(t - t_0)}{(\kappa\, v_0^3)^{1/7}\epsilon^{6/7}}, \frac{t - t_0}{(\kappa^3 v_0^2)^{1/7}\epsilon^{4/7}}\right) + O\left(\epsilon^{4/7}\right). \tag{3.15}$$

Here

$$\kappa = -f'''(v_0).$$

There are two main motivations fo the Universality Conjecture. First, one can accompany the above rescaling

$$\begin{aligned} \bar x &\mapsto k\,\bar x \\ \bar t &\mapsto k^{2/3}\bar t \\ \bar u &\mapsto k^{1/3}\bar u \end{aligned} \tag{3.16}$$

with the rescaling of the parameter ϵ:

$$\epsilon \mapsto k^{7/6}\epsilon.$$

After such a rescaling and the Galilean transformation

$$\bar x \mapsto \bar x - v_0 \bar t$$

the perturbed equation

$$\begin{aligned} &u_t + u\,u_x + \frac{\epsilon^2}{24}\left[2c\,u_{xxx} + 4c' u_x u_{xx} + c'' u_x^3\right] + \epsilon^4\left[2p\,u_{xxxxx}\right. \\ &\left. +2p'(5u_{xx}u_{xxx} + 3u_x u_{xxxx}) + p''(7u_x u_{xx}^2 + 6u_x^2 u_{xxx}) + 2p''' u_x^3 u_{xx}\right] = 0. \end{aligned} \tag{3.17}$$

will tend to the KdV equation

$$u_t + u\,u_x + \frac{\epsilon^2}{12}\,c(v_0) u_{xxx} = O(k^{1/3}) \tag{3.18}$$

assuming that

$$c(v_0) \neq 0$$

(one more genericity assumption).

The second step is used in order to specify the needed solution to the KdV equation. This solution will be specified by the so-called *string equation*. Let us introduce a primitive $g(u)$ of the function f:

$$g'(u) = f(u).$$

Observe that the equation of the method of characteristics can be written as the equation for the critical points of the function

$$\Phi_{x,t}(v) := t\,\frac{v^2}{2} - x\,v - g(v)$$

depending on the parameters x, t. We replace now this function by the functional

$$\hat\Phi_{x,t}[u] = \int \left[t\,\frac{u^2}{2} - x\,u - h_g(u; u_x, \dots; \epsilon)\right] dx = \int \left[t\,\frac{u^2}{2} - x\,u\right] dx - H_g$$

where h_g is defined by the formula (3.7) specialized at the KdV case

$$c(u) = c(v_0) =: c_0, \quad p(u) = 0,$$

i.e.

$$h_g = g - \frac{\epsilon^2}{24} c_0\, g''' u_x^2 + \epsilon^4 \left[\frac{{c_0}^2\, g^{(4)}}{480}\, u_{xx}^2 - \frac{{c_0}^2\, g^{(6)}}{3456}\, u_x^4 \right] + O(\epsilon^6).$$

Proposition 3.5. *The solutions to the Euler – Lagrange equations*

$$\frac{\delta \hat{\Phi}_{x,t}}{\delta u(x)} \equiv u\, t - x - \frac{\delta H_g}{\delta u(x)} = 0 \tag{3.19}$$

satisfy KdV equation (3.18)*. The unique formal solution to* (3.19) *that for* $\epsilon \to 0$ *admits a regular expansion in* ϵ *is obtained from the solution* $v = v(x,t)$ *to Hopf equation by the quasitriviality transformation* (3.11)*.*

Applying the same rescaling of x, t, u and ϵ we see that the "string equation" (3.19) tends to the P_I^2 equation (3.14).

After all these heuristic arguments let us formulate the first rigorous result supporting the universality conjecture.

Theorem 3.6. *(T.Claeys, T.Grava). The Universality Conjecture holds true for generic solutions to the KdV equation with analytic initial data.*

At the end we will consider an example of a two-component Hamiltonian perturbation of a hyperbolic PDE obtained from the Toda lattice equations (cf. the example of general FPU systems above)

$$\begin{array}{rcccl} u_t & = & \frac{1}{\epsilon}\left[v(x+\epsilon) - v(x)\right] & = & v_x + \frac{1}{2}\epsilon\, v_{xx} + \dots \\ v_t & = & \frac{1}{\epsilon}\left[e^{u(x)} - e^{u(x-\epsilon)}\right] & = & e^u u_x - \frac{1}{2}\epsilon\, \left(e^u\right)_{xx} + \dots \end{array} \tag{3.20}$$

The system (3.20) is integrable; it can be included into an infinite family of pairwise commuting Hamiltonian flows described in terms of isospectral deformations of difference Lax operator

$$L = e^{\epsilon \partial_x} + v(x) + e^{u(x)} e^{-\epsilon \partial_x}.$$

Denote $\mathbf{t}$ the infinite vector of times of the Toda hierarchy. Consider the solution specified by the initial data

$$e^{u(x,0;\epsilon)} = x, \quad v(x,0;\epsilon) = 0. \tag{3.21}$$

As before the solution is understood in the form of a formal ϵ expansion.

Proposition 3.7. *The tau-function of the particular solution* (3.21) *to Toda hierarchy coincides with the formal asymptotic expansion of the GUE partition function*

$$\begin{aligned} \tau(x, \mathbf{t}; \epsilon) &= \frac{1}{\mathrm{Vol}(N)} \int_{N\times N} e^{-\frac{1}{\epsilon}\mathrm{Tr} V(M)} dM \\ V(M) &= \frac{1}{2} M^2 - \sum_{k\geq 3} t_k M^k, \quad N = \frac{x}{\epsilon} \\ \mathrm{Vol}(N) &= \frac{2^{N/2} \pi^{\frac{N^2}{2}} \epsilon^{-\frac{N^2}{2} + \frac{1}{12}}}{\prod_{k=0}^{N-1} k!}. \end{aligned} \tag{3.22}$$

Recall that the formal expansion of the free energy of the GUE ensemble is written in the form

$$\log \tau(x, \mathbf{t}; \epsilon) \sim \frac{x^2}{2\epsilon^2}\left(\log x - \frac{3}{2}\right) - \frac{1}{12}\log x + \zeta'(-1) + \sum_{g\geq 2}\left(\frac{\epsilon}{x}\right)^{2g-2}\frac{B_{2g}}{2g(2g-2)} + \sum_{g\geq 0}\epsilon^{2g-2}F_g(x; t_3, t_4, \dots) \tag{3.23}$$

where B_{2g} are Bernoulli numbers, $\zeta(s)$ the Riemann zeta-function,

$$F_g(x; t_3, t_4, \dots) = \sum_n \sum_{k_1,\dots,k_n} a_g(k_1, \dots, k_n) t_{k_1} \dots t_{k_n} x^h,$$

$$h = 2 - 2g - \left(n - \frac{|k|}{2}\right), \quad |k| = k_1 + \dots + k_n,$$

generate the numbers of fat graphs

$$a_g(k_1, \dots, k_n) = \sum_\Gamma \frac{1}{\#\,\mathrm{Sym}\,\Gamma}$$

where

$$\Gamma \;=\; \text{a connected fat graph of genus } g \text{ with } n \text{ vertices of the valencies } k_1, \dots, k_n,$$

$\mathrm{Sym}\,\Gamma$ is the symmetry group of the graph.

The above proposition puts at least some part of universality results of the theory of random matrices into the more general setting of the problem of universality of critical behaviour in Hamiltonian PDEs.

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