Introduction to spectral methods in Matlab III

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Finite intervals

- Periodic continuation of non-periodic functions on a finite interval: discontinuous functions on the real line, Gibbs phenomenon.
- Advantage of spectral methods lost. Therefore polynomial interpolation, not trigonometric.
- Uniform grids: Runge phenomenon

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Spectral differentiation

- Chebyshev grid: projection of equispaced points on the circle ($x \in [-1,1]$); density $\sim \frac{N}{\pi\sqrt{1-x^2}}$, $O(1/N^2)$ points for $x \sim \pm 1$, $O(1/N)$ for $x \sim 0$.
- Chebyshev (Gauss–Lobatto) points: $x_j = \cos\frac{j\pi}{N}$, $j = 0, 1, \ldots, N$
- polynomial interpolant: unique polynomial of degree $\leq N$ with $p(x_j) = v_j$, $j = 0, 1, \ldots, N$
- approximation of the derivative: $w_j = p'(x_j)$; linear relation, $\vec{w} = D_N \vec{v}$ (N even or odd).

- $N = 1$: $x_0 = 1$, $x_1 = -1$
  $p(x) = \frac{1}{2}(1+x)v_0 + \frac{1}{2}(1-x)v_1$, $p'(x) = \frac{1}{2}$

$$D_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

- $N = 2$: $x_0 = 1$, $x_1 = 0$, $x_2 = -1$

$$p(x) = \frac{1}{2}x(x+1)v_0 + (1+x)(1-x)v_1 + \frac{1}{2}x(x-1)v_2$$

$$p'(x) = \left(x + \frac{1}{2}\right)v_0 - 2xv_1 + \left(x - \frac{1}{2}\right)v_2$$

$$D_2 = \begin{pmatrix} \frac{3}{2} & -2 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 2 & -\frac{3}{2} \end{pmatrix}$$

Chebyshev differentiation matrices

- **theorem 1** *For $N \geq 1$, the Chebyshev differentiation matrix D_N is given by:*

$$(D_N)_{00} = \frac{2N^2+1}{6}, \qquad (D_N)_{NN} = -\frac{2N^2+1}{6}$$

$$(D_N)_{jj} = -\frac{x_j}{2(1-x_j^2)}, \qquad j = 1, \ldots, N-1$$

$$(D_N)_{ij} = \frac{c_i}{c_j}\frac{(-1)^{i+j}}{(x_i - x_j)} \qquad i \neq j, \quad i, j = 0, \ldots, N$$

where

$$c_i = \begin{cases} 2 & i = 0, N \\ 1 & otherwise \end{cases}$$

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Chebyshev series and FFT

- Chebyshev polynomials $T_n(x) = \cos(n \arccos x)$,
  polynomial of degree N
  unit circle: $|z| = 1$, $x = \Re z = \cos\theta$,
  $x \in [-1, 1]$, $\theta \in \mathbb{R}$, z on unit circle
  Chebyshev series, Fourier series, Laurent series
  $T_n(x) = \Re z^n = \cos n\theta = \frac{1}{2}(z^n + z^{-n})$,
  recursive relation $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x, \ldots$$

- Chebyshev polynomials form a basis in the space of polynomials, any polynomial of degree N can be written as a sum of Chebyshev polynomials: $p(x) = \sum_{n=0}^{N} a_n T_n(x)$, $x \in [-1, 1]$.
Equivalent: Laurent polynomial
$\mathrm{p}(z) = \frac{1}{2} \sum_{n=0}^{N} a_n (z^n + z^{-n})$, $|z| = 1$
trigonometric polynomial
$P(\theta) = \sum_{n=0}^{N} a_n \cos\theta$, $\theta \in \mathbb{R}$

- spectral collocation method:
$\theta_j = \frac{j\pi}{N}$, $z_j = e^{i\theta_j}$, $x_j = \cos\theta_j$, $j = 0, 1, \ldots, N$

Chebychev differentiation via FFT

- Chebyshev differentiation via FFT: data $\{v_j\}$ at $\{x_j\}$
  - construct vector $\vec{V}$ of length $2N$: $\vec{V}(0:N) = \vec{v}$, $V_{2N-j} = v_j$, $j = 1, \ldots, N$
  - FFT: $\hat{V}_k = \frac{\pi}{N} \sum_{j=1}^{2N} e^{-ik\theta_j} V_j$, $k = -N+1, \ldots, N$
  - Put $\hat{W}_k = ik\hat{V}_k$, $\hat{W}_N = 0$
  - FFT: $W_j = \frac{1}{2\pi} \sum_{k=-N+1}^{N} e^{ik\theta_j} \hat{W}_k$, $j = 1, \ldots, 2N$
  - Compute $w_j = -\frac{W_j}{\sqrt{1-x_j^2}}$, $j = 1, \ldots, N-1$ and

$$w_0 = \frac{1}{2\pi} \sum_{n=1}^{N-1} n^2 \hat{v}_n + \frac{1}{4\pi} N^2 \hat{v}_N, \quad w_N = \sum_{n=1}^{N-1} (-1)^{n+1} n^2 \hat{v}_n + \frac{1}{4\pi} (-1)^{N+1} N^2 \hat{v}_N$$

- realization via DCT (discrete cosine transformation), Matlab Signal Processing Toolbox

Boundary value problems

- Example: $u_{xx} = e^{4x}$, $x \in [-1, 1]$, $u(\pm 1) = 0$ (Dirichlet)
solution: $u(x) = (e^{4x} - x \sinh 4 - \cosh 4)/16$

- numerically: ∂_{xx} replaced by D_N^2 (direct computation possible)

- boundary conditions: interior points $x_1, \ldots, x_{N-1}$ considered, $\vec{v} = (v_1, \ldots, v_{N-1})^T$ vector of unknowns

- spectral differentiation: $p(x)$ unique polynomial of degree $\leq N$ with $p(\pm 1) = 0$ and $p(x_j) = v_j$, $j = 1, \ldots, N-1$; $w_j = p''(x_j)$.

- Thus v_0 and v_N are fixed and w_0 and w_N can be ignored; the reduced matrix $\tilde{D}_N^2 = D_N^2(1 : N-1, 1 : N-1)$ is important

- The differential equation is approximated by the system of linear equations

$$\tilde{D}_N^2 v = f$$

- non-linear problems, for instance $u_{xx} = e^u$:
  fixed point iteration: $u_{n+1,xx} = e^{u_n}$, convergence with initial guess u_0 sufficiently close to the solution.

- spectral resolution: $u_{xx} = \lambda u$, $|x| \leq 1$, $u(\pm 1) = 0$
  solution: $u_n = \sin n\pi(x+1)/2$, $\lambda_n = -n^2\pi^2/4$, $n = \pm 1, \pm 2, \ldots$
  numerically: eigenvalues of $\tilde{D}_N^2$ via *eig*
  points per wavelength (ppw) near $x = 0$ (coarsest part of the grid):
  'reasonable' resolution for at least 2ppw

- two-dimensional problem: example $u_{xx} + u_{yy}$
  2d-Chebyshev grid, density of points close to the center of order $(2/\pi)^d$ with respect to the boundary
  1d: $w_i = \sum_{j=0}^{N} (D_N^2)_{ij} v_j$
  2d: $w_{nm} = \sum_{i,j=0}^{N} (D_N^2)_{ni} (D_N^2)_{mj} v_{ij}$
  tensor product $\Delta \to D_N^2 \times \hat{1} + \hat{1} \times D_N^2$, in Matlab *kron*

τ-method

- approaches to solve boundary value problems with spectral methods:
  - interpolants that satisfy the boundary conditions
  - add supplementary equations corresponding to the boundary conditions: τ-method (more flexible for general boundary conditions); these equations replace the ones deleted in the previous approach