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Numerical Evaluation of Fredholm Determinants and Painlevé Transcendents with applications to random matrix theory

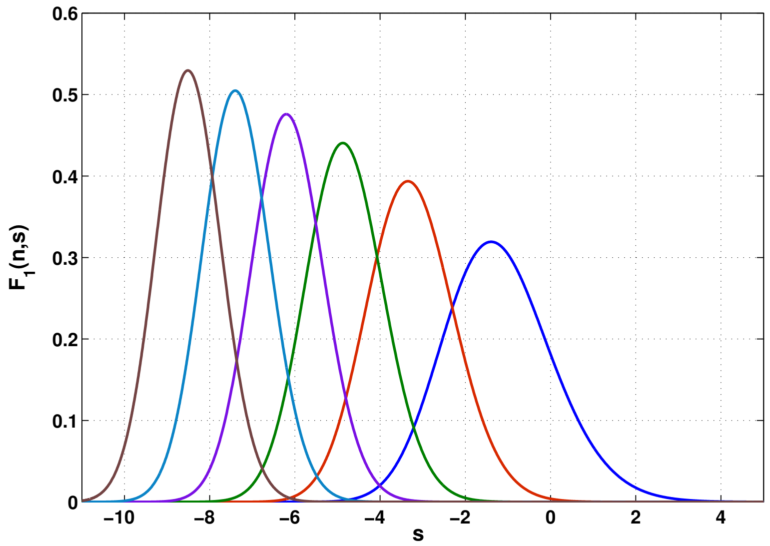
Folkmar Bornemann
Technische Universität München
Germany

e-mail:bornemann@ma.tum.de

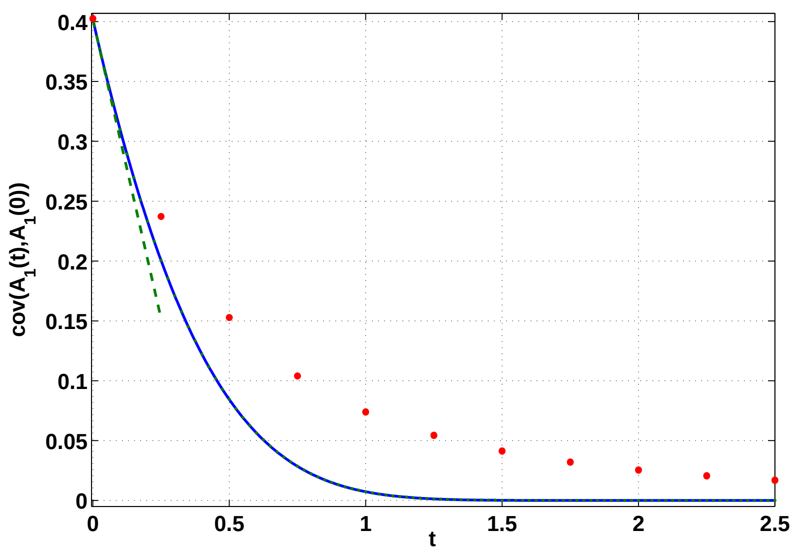
Strada Costiera 11, 34014 Trieste, Italy - Tel. +39 040 2240 111; Fax +39 040 224 163 - sci_info@ictp.it, www.ictp.it

Numerical Evaluation of Fredholm Determinants and Painlevé Transcendents

with applications to random matrix theory



Probability density of n-th largest level in edge scaled GOE



Two-point correlation function of Airy_1 process

Folkmar Bornemann

Lecture 1: Part A

Fredholm vs. Painlevé

Two tools used in integrable systems

Ivar Fredholm (1866–1927)

determinant of integral operator (1899)

$$Ku(x) = \int_a^b K(x,y)u(y)\,dy \qquad \rightsquigarrow$$

$$\det(I + zK) = \sum_{m=0}^{\infty} \frac{z^m}{m!} \int_{[a,b]^m} \det\left(K(t_i,t_j)\right)_{i,j=1}^m \,dt$$

Paul Painlevé (1863–1933)

six families of irreducible transcendental functions (1895)

$$u_{xx} = 6u^2 + x$$

$$u_{xx} = 2u^3 + xu - \alpha$$

$$u_{xx} = u^{-1}u_x^2 - x^{-1}u_x + x^{-1}(\alpha u^2 + \beta) + \gamma u^3 + \delta u^{-1}$$

$$u_{xx} = (2u)^{-1}u_x^2 + 3u^3/2 + 4xu^2 + 2(x^2 - \alpha)u + \beta u^{-1}$$

$$u_{xx} = (3u-1)(2u(u-1))^{-1}u_x^2 - x^{-1}u_x + \gamma x^{-1}u + (u-1)^2 x^{-2}(\alpha u + \beta u^{-1}) + \delta u(u+1)(u-1)^{-1}$$

$$u_{xx} = (u^{-1} + (u-1)^{-1} + (u-x)^{-1})u_x^2/2 + \cdots$$

Bulk scaling limit of Gaussian ensembles

$$E_\beta(0;s) = \mathbb{P}(\text{no levels lie in } (0,s))$$

(Gaudin '61, Mehta/des Cloizeaux '72)

$$E_2(0;s) = \det\left(I - K\restriction_{L^2(0,s)}\right)$$

$$E_1(0;s) = \det\left(I - K_+\restriction_{L^2(0,s/2)}\right)$$

$$E_4(0;s) = \cdots$$

with kernels

$$K(x,y) = \operatorname{sinc}(\pi(x-y))$$

$$K_+(x,y) = K(x,y) + K(x,-y)$$

(Jimbo/Miwa/Môri/Sato '80)

$$E_2(0;s) = \exp\left(-\int_0^{\pi s}\frac{\sigma(x)}{x}\,dx\right)$$

$$E_1(0;s) = \exp\left(-\frac{1}{2}\int_0^{\pi s}\sqrt{\frac{d}{dx}\frac{\sigma(x)}{x}}\,dx\right)E_2(0;s)^{1/2}$$

$$E_4(0;s) = \cdots$$

with σ-form of Painlevé V

$$(x\sigma_{xx})^2 = 4(\sigma - x\sigma_x)(x\sigma_x - \sigma - \sigma_x^2)$$

$$\sigma(x) \simeq \frac{x}{\pi} + \frac{x^2}{\pi^2} \qquad (x \to 0)$$

Empirical observation (Montgomery '73, Odlyzko '87)

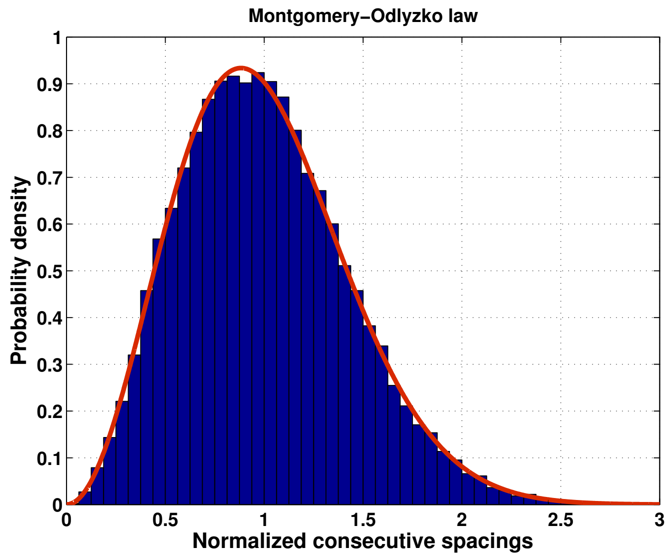
nontrivial zeros $\frac{1}{2} + i\gamma_n$ (RH: $\gamma_n > 0$)

large n statistics of spacings of

$$\frac{\gamma_n}{2\pi} \log \frac{\gamma_n}{2\pi}$$

given by bulk scaling limit of GUE

$\rightsquigarrow$ spacing density $\quad p(s) = \frac{d^2}{ds^2} E_2(0; s)$



(Gaudin '61)

$$p(s) = \frac{d^2}{ds^2} \det\left(I - K\restriction_{L^2(0,s)}\right)$$

with kernel

$$K(x,y) = \operatorname{sinc}(\pi(x-y))$$

(Jimbo/Miwa/Môri/Sato '80)

$$p(s) = \frac{d^2}{ds^2} \exp\left(-\int_0^{\pi s} \frac{\sigma(x)}{x}\, dx\right)$$

with σ-form of Painlevé V

$$(x\sigma_{xx})^2 = 4(\sigma - x\sigma_x)(x\sigma_x - \sigma - \sigma_x^2)$$

$$\sigma(x) \simeq \frac{x}{\pi} + \frac{x^2}{\pi^2} \qquad (x \to 0)$$

Gaudin's method ('61) *(bulk scaling limit of GUE)*

$$E_2(0;2s) = \mathbb{P}(\text{no levels lie in } (0,2s))$$

$$= \det\left(I - \frac{s}{2} K_s^\dagger K_s \restriction_{L^2(-1,1)}\right)$$

$$= \prod_{n=0}^{\infty} \left(1 - \frac{s}{2}|\lambda_n(K_s)|^2\right)$$

with kernel

$$K_s(x,y) = e^{i\pi s x y}$$

observe $[K_s, L_s] = 0$ for the differential operator

$$L_s u = -((1-x^2)u_x)_x + \pi^2 s^2 x^2 u$$

$$0 = (1-x^2)u(x)|_{x=\pm 1} = (1-x^2)u'(x)|_{x=\pm 1}$$

eigenfunctions of L_s are known as $u_n(x) = S_{n,0}^{(1)}(\pi s, x)$ *(radial prolate spheroidal wave functions)* $\rightsquigarrow$

$$\lambda_{2n}(K_s) = \frac{1}{u_{2n}(0)} \int_{-1}^{1} u_{2n}(\xi)\, d\xi$$

$$\lambda_{2n+1}(K_s) = \frac{i\pi s}{u'_{2n+1}(0)} \int_{-1}^{1} u_{2n+1}(\xi)\xi\, d\xi$$

*there is **no** such method for other scaling limits*

Edge scaling limit of Gaussian ensembles

$$F_\beta(s) = \mathbb{P}(\text{no levels lie in } (s,\infty))$$

(Mehta '91, Forrester '91, Ferrari/Spohn '05)

$$F_2(s) = \det\left(I - K\restriction_{L^2(s,\infty)}\right)$$

$$F_1(s) = \det\left(I - K_+\restriction_{L^2(s/2,\infty)}\right)$$

$$F_4(s) = \cdots$$

with kernels

$$K(x,y) = \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x-y}$$

$$K_+(x,y) = \mathrm{Ai}(x+y)$$

(Tracy/Widom '94/'96)

$$F_2(s) = \exp\left(-\int_s^\infty (x-s)u(x)^2\,dx\right)$$

$$F_1(s) = \exp\left(-\frac{1}{2}\int_s^\infty u(x)\,dx\right) F_2(s)^{1/2}$$

$$F_4(s) = \cdots$$

with Hastings–McLeod solution of Painlevé II

$$u_{xx} = 2u^3 + xu$$

$$u(x) \simeq \mathrm{Ai}(x) \qquad (x \to \infty)$$

n-th largest level in edge scaled GUE

$$\mathbb{P}(\text{exactly } n \text{ levels lie in } (s,\infty)) = \frac{(-1)^n}{n!} \frac{\partial^n}{\partial z^n} F_2(s;z)\Big|_{z=1}$$

(Mehta '91, Forrester '91)

$$F_2(s;z) = \det\left(I - z\, K\restriction_{L^2(s,\infty)}\right)$$

with kernel

$$K(x,y) = \frac{\operatorname{Ai}(x)\operatorname{Ai}'(y) - \operatorname{Ai}'(x)\operatorname{Ai}(y)}{x-y}$$

(Tracy/Widom '94)

$$F_2(s;z) = \exp\left(-\int_s^\infty (x-s)u(x;z)^2\,dx\right)$$

with Painlevé II

$$u_{xx} = 2u^3 + xu$$

$$u(x;z) \simeq \sqrt{z}\operatorname{Ai}(x) \qquad (x \to \infty)$$

… much more involved for GOE and GSE, see later

$n \times n$-Laguerre unitary ensemble (LUE) with weight $x^\alpha e^{-x}$

$$G_{n,\alpha}(s) = \mathbb{P}(\text{no levels lie in } (0,s))$$

(Nagao/Wadati '91)

$$G_{n,\alpha}(s) = \det\left(I - K\restriction_{L^2(0,s)}\right)$$

with kernel

$$K(x,y) = \frac{\phi_{n-1}(x)\phi_n(y) - \phi_n(x)\phi_{n-1}(y)}{(n(n+\alpha))^{-1/2}(x-y)}$$

$$\phi_k(x) = \sqrt{\frac{k!}{\Gamma(k+\alpha+1)}}\, x^{\alpha/2} e^{-x/2} L_k^{(\alpha)}(x)$$

(Tracy/Widom '94)

$$G_{n,\alpha}(s) = \exp\left(-\int_0^s \frac{\sigma(x)}{x}\,dx\right)$$

with Jimbo–Miwa–Okamoto σ-form of Painlevé V

$$(x\sigma_{xx})^2 = (\sigma - x\sigma_x - 2\sigma_x^2 + (2n+\alpha)\sigma_x)^2$$
$$- 4\sigma_x^2(\sigma_x - n)(\sigma_x - n - \alpha)$$

$$\sigma(x) \simeq \frac{\Gamma(n+\alpha+1)}{\Gamma(n)\Gamma(\alpha+1)\Gamma(\alpha+2)} x^{\alpha+1} \quad (x \to 0)$$

Lecture 1: Part B

Random Matrix Theory

Universality for mathematical and physical systems

(Deift '06, Johnstone '06)

the statistics of

- neutron scattering
- multivariate statistics
- wetting & melting
- combinatorial growth models
- patience sorting
- wireless communication
- bus system in Cuernavaca
- airplane boarding
- parking gaps
- zeroes of Riemann zeta function

and many other high dimensional statistical interferences are well modelled by

Random Matrix Theory (RMT)

Gaussian Unitary Ensemble (GUE)

M_n $n \times n$-Hermitean-matrix valued random variable, coefficients i.i.d. Gaussian

Joint probability density of eigenvalues

$\lambda_1, \lambda_2, \ldots, \lambda_n$ eigenvalues of a GUE M_n $\rightsquigarrow$

$$p(\lambda_1, \ldots, \lambda_n) = c_n \prod_i e^{-\beta\lambda_i^2/2} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \qquad (\beta = 2)$$

$\prod_{i<j} |\lambda_i - \lambda_j|^\beta$ = "level repulsion": small probability of eigenvalue crossing

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- $\beta = 1$: Gaussian Orthogonal Ensemble (GOE)
- $\beta = 4$: Gaussian Symplectic Ensemble (GSE)

Level repulsion = Vandermonde determinant

$$p(\lambda_1,\ldots,\lambda_n) = c_n\, e^{-\lambda_1^2-\cdots-\lambda_n^2} \cdot \begin{vmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_n \\ \vdots & & \vdots \\ \lambda_1^{n-1} & \cdots & \lambda_n^{n-1} \end{vmatrix}^2 = \frac{1}{n!} \det\big(K_n(\lambda_i,\lambda_j)\big)_{i,j=1}^n$$

with kernel

$$K_n(x,y) = \sum_{k=0}^{n-1} \phi_k(x)\phi_k(y)$$

$\phi_0(x), \phi_1(x), \phi_2(x), \ldots$ harmonic oscillator wave functions:

$$\phi_k(x) = \frac{e^{-x^2/2} H_k(x)}{\pi^{1/4} 2^{k/2} (k!)^{1/2}} \qquad (H_k \text{ Hermite polynomial})$$

Gap probability = Fredholm determinant (Gaudin '61)

inclusion-exclusion principle $\rightsquigarrow$

$$\mathbb{P}\left(\lambda_1, \ldots, \lambda_n \notin (a,b)\right) =$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_{[a,b]^m} \det\left(K_n(t_i, t_j)\right)_{i,j=1}^{m} \, dt$$

$$= \det\Big(I - K_n\restriction_{L^2(a,b)}\Big)$$

with kernel *(by Christoffel–Darboux formula)*

$$K_n(x,y) = \sum_{k=0}^{n-1} \phi_k(x)\phi_k(y) = \frac{1}{2}\,\frac{\phi_n(x)\phi_n'(y) - \phi_n'(x)\phi_n(y)}{x-y} - \frac{1}{2}\phi_n(x)\phi_n(y)$$

double scaling asymptotics of Hermite polynomials (Plancherel/Rotach '29)

$$\phi_n(\sqrt{2n} + 2^{-1/2}n^{-1/6}x) \sim 2^{1/4}n^{-1/12}\,\mathrm{Ai}(x) \qquad (n \to \infty)$$

$\rightsquigarrow$ **Edge scaling limit of GUE** (Forrester '93)

$$\mathbb{P}\left(\frac{\lambda_{\max}(M_n) - \sqrt{2n}}{2^{-1/2}n^{-1/6}} \leqslant s\right) \to F_2(s) = \det\left(I - K\restriction_{L^2(s,\infty)}\right) \qquad (n \to \infty)$$

with Airy kernel

$$K(x,y) = \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x - y}$$

Lecture 1: Part C

From Fredholm to Painlevé

Tracy/Widom ('94)

$$F_2(s) = \exp\left(-\int_s^\infty (z-s)u(z)^2\,dz\right)$$


Craig Tracy (1945 –)


Harold Widom (1932 –)

with $u(z)$ Hastings–McLeod ('80) solution of Painlevé II:

$$u''(z) = 2u(z)^3 + z\,u(z), \qquad u(z) \sim \mathrm{Ai}(z) \quad (z \to \infty)$$

$\rightsquigarrow$ George Pólya Prize 2002

The claim

> *Without the Painlevé representations, the numerical evaluation of the Fredholm determinants is quite involved.* — Tracy/Widom '00

Given the kernel

$$K(x,y) = \frac{\phi(x)\psi(y) - \psi(x)\phi(x)}{x-y},$$

such that, with some polynomials m, A, B, C,

$$m(x)\phi'(x) = A(x)\phi(x) + B(x)\psi(x),$$

$$m(x)\psi'(x) = -C(x)\phi(x) - A(x)\psi(x).$$

Then, as Tracy and Widom (1994) obtained by operator theoretic arguments,

$$\boxed{\frac{\partial}{\partial a_k} \log\det\left(I - K\restriction_{L^2(a_0,a_1)}\right) = (-1)^k \left(p_k \frac{\partial q_k}{\partial a_k} - q_k \frac{\partial p_k}{\partial a_k}\right)} \qquad (k = 0, 1)$$

where q_k, p_k, and further functions u_i, v_i, w_i, satisfy a *contructable* system of PDEs

$$\frac{\partial q_j}{\partial a_k} = \ldots, \quad \frac{\partial p_j}{\partial a_k} = \ldots, \quad \frac{\partial u_i}{\partial a_k} = \ldots, \quad \frac{\partial v_i}{\partial a_k} = \ldots, \quad \frac{\partial w_i}{\partial a_k} = \ldots$$

with $j, k = 0, 1$ and $0 \leqslant i \leqslant \max(\deg A, \deg B, \deg C, \deg m - 1)$.

other methods for establishing

Fredholm determinant $\mapsto$ Painlevé representation

- **Adler/Shiota/van Moerbeke ('95)**: KP equation and Virasoro algebras
- **Forrester/Witte ('01)**: Okamoto's τ-function theory of Painlevé equations
- **Borodin/Deift ('02)**: Riemann–Hilbert problems

e.g., the continuous ${}_2F_1$-kernel in the representation theory of $U(\infty)$

$$K(x,y) = \frac{A(x)B(y)-B(x)A(y)}{x-y}\sqrt{\psi(x)\psi(y)}, \qquad x,y \in \left(\tfrac12, +\infty\right)$$

$$\psi(x) = \frac{\sin\pi z \sin\pi z'}{\pi^2}\cdot\left(x-\tfrac12\right)^{-z-z'}\left(x+\tfrac12\right)^{-w-w'},$$

$$A(x) = \left(\frac{x+\frac12}{x-\frac12}\right)^{w'} {}_2F_1\left[\begin{matrix} z+w',\ z'+w' \\ z+z'+w+w' \end{matrix}\middle| \frac{1}{\frac12 - x}\right],$$

$$B(x) = \frac{\Gamma(z+w+1)\Gamma(z+w'+1)\Gamma(z'+w+1)\Gamma(z'+w'+1)}{\Gamma(z+z'+w+w'+1)\Gamma(z+z'+w+w'+2)} \times \frac{1}{x-\frac12}\left(\frac{x+\frac12}{x-\frac12}\right)^{w'} {}_2F_1\left[\begin{matrix} z+w'+1,\ z'+w'+1 \\ z+z'+w+w'+2 \end{matrix}\middle| \frac{1}{\frac12 - x}\right].$$

THEOREM 7.1 *Let K_s be the restriction of the continuous ${}_2F_1$ kernel to the interval $(s,+\infty)$, $s > \frac12$. Assume that $\mathfrak{S} = z+z'+w+w' > 0$, $|z+z'| < 1$, and $|w+w'| < 1$. Then the function*

$$\sigma(s) = \left(s-\tfrac12\right)\left(s+\tfrac12\right)\frac{d\ln\det(1-K_s)}{ds} - \nu_1^2 s + \frac{\nu_3\nu_4}{2}$$

satisfies the differential equation

$$-\sigma'\left(\left(s-\tfrac12\right)\left(s+\tfrac12\right)\sigma''\right)^2 = \left(2(s\sigma'-\sigma)\sigma' - \nu_1\nu_2\nu_3\nu_4\right)^2 - \left(\sigma'+\nu_1^2\right)\left(\sigma'+\nu_2^2\right)\left(\sigma'+\nu_3^2\right)\left(\sigma'+\nu_4^2\right), \tag{7.1}$$

where

$$\nu_1 = \nu_2 = \frac{z+z'+w+w'}{2}, \quad \nu_3 = \frac{z-z'+w-w'}{2}, \quad \nu_4 = \frac{z-z'-w+w'}{2}.$$

what a triumph of dedicated men ...

Painlevé property

$$u_{xx} = F(x, u, u_x) \qquad (F \text{ rational in } u, u_x \text{ and analytic in } x)$$

s.t. $u(x)$ has *no* movable singularities other than poles

Paul Painlevé (1863–1933)

$\rightsquigarrow$ *six new families of irreducible transcendental functions (1895)*

(1) $u_{xx} = 6u^2 + x$

(2) $u_{xx} = 2u^3 + xu - \alpha$

(3) $u_{xx} = u^{-1}u_x^2 - x^{-1}u_x + x^{-1}(\alpha u^2 + \beta) + \gamma u^3 + \delta u^{-1}$

(4) $u_{xx} = (2u)^{-1}u_x^2 + 3u^3/2 + 4xu^2 + 2(x^2 - \alpha)u + \beta u^{-1}$

(5) $u_{xx} = (3u-1)(2u(u-1))^{-1}u_x^2 - x^{-1}u_x + \gamma x^{-1}u + (u-1)^2 x^{-2}(\alpha u + \beta u^{-1}) + \delta u(u+1)(u-1)^{-1}$

(6) $u_{xx} = (u^{-1} + (u-1)^{-1} + (u-x)^{-1})u_x^2/2 - (x^{-1} + (x-1)^{-1} + (u-x)^{-1})u_x$
$\qquad + u(u-1)(u-x)x^{-2}(x-1)^{-2}\left(\alpha + \beta x u^{-2} + \gamma(x-1)(u-1)^{-2} + \delta x(x-1)(u-x)^{-2}\right)$

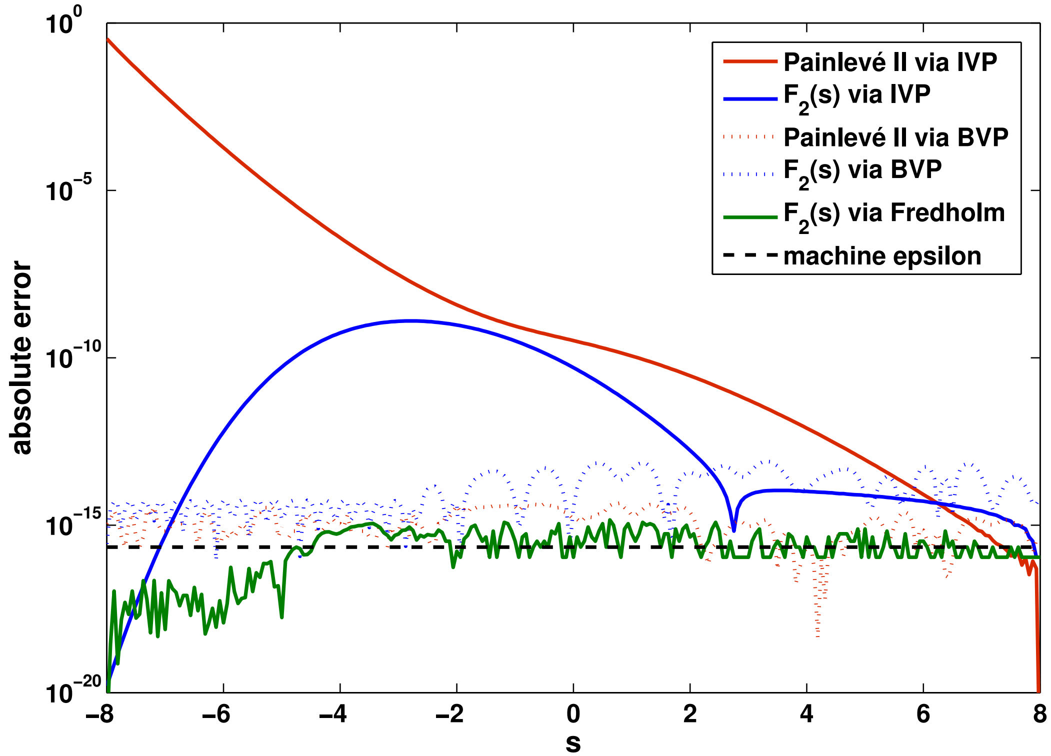
all of them appear in RMT

Lecture 1: Part D

Solving Painlevé Numerically as ODEs

Numerical evaluation of the Tracy–Widom distribution $F_2(x)$

. . . there is yet no library software for the Painlevé transcendents



Absolute error of various numerical approaches using IEEE double precision

- via Painlevé II as IVP *(backwards)*

  Prähofer ('04): 16 digits (1500 internally!)

  Bejan ('05): 3 digits

  Edelman/Persson ('05): 8 digits @ 8.9 sec

- via Painlevé II as BVP

  Tracy/Widom ('94): 10 digits (75 internally!)

  Dieng ('05): 9 digits @ 3.7 sec

  Driscoll/B./Trefethen ('08): 13 digits @ 1.3 sec

- via Fredholm determinant

  B. ('08): 15 digits @ 0.69 sec

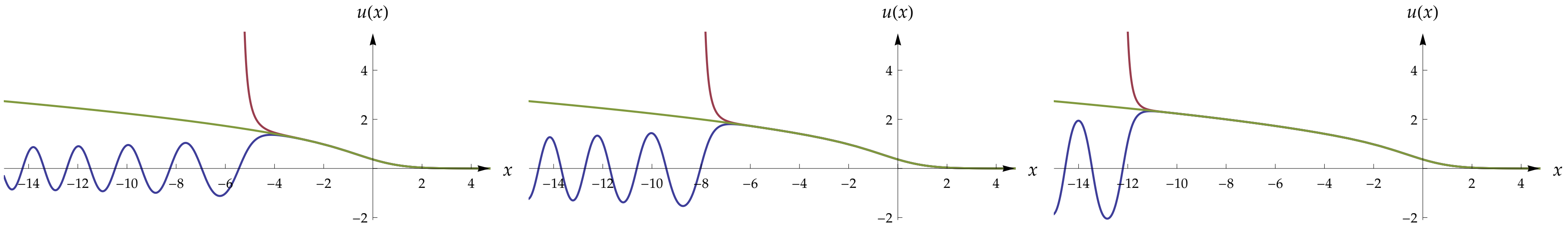
solution via Fredholm determinant: *much simpler, more efficient, and more accurate*

Explanation

solution of Painlevé II,

$$u_{xx} = 2u^3 + xu, \qquad u(x) \simeq \vartheta \,\mathrm{Ai}(x) \quad (x \to \infty),$$

is *separatrix* for $\vartheta = 1 \quad \rightsquigarrow$ IVP highly unstable



$u(x)$ with $\vartheta = 1 - 10^{-4}, 1, 1 + 10^{-4}$ $\quad$ $u(x)$ with $\vartheta = 1 - 10^{-8}, 1, 1 + 10^{-8}$ $\quad$ $u(x)$ with $\vartheta = 1 - 10^{-16}, 1, 1 + 10^{-16}$

consequences

- F_2 via IVP solution of Painlevé II $\rightsquigarrow$ not more than 8 digits in IEEE arithmetic
- calculate F_2 via a BVP solution $\rightsquigarrow$ *connection formula* needed:

$$u(x) \simeq \vartheta \,\mathrm{Ai}(x) \quad (x \to \infty) \qquad \Rightarrow \qquad u(x) \simeq \,? \quad (x \to -\infty)$$

Painlevé II (Ablowitz/Segur '77, Hastings/McLeod '80, Its/Kapaev '89)

$$u_{xx} = 2u^3 + xu, \qquad u(x) \simeq \vartheta \operatorname{Ai}(x) \qquad (x \to \infty)$$

$\Rightarrow$ for $x \to -\infty$

- $0 < \vartheta < 1$ $\quad u(x) \simeq c_\vartheta (-x)^{1/4} \cos\left(\frac{2}{3}(-x)^{3/2} + c'_\vartheta \log(8(-x)^{3/2}) + \phi_\vartheta\right)$
- $\vartheta = 1$ $\quad u(x) \simeq \sqrt{-x/2}$
- $\vartheta > 1$ $\quad u(x) \simeq (x - x_\vartheta)^{-1} \qquad (x \to x_\vartheta)$

σ-form of Painlevé V (McCoy/Tang '86, Basor/Tracy/Widom '92, Widom '94)

$$(x\sigma_{xx})^2 = 4(\sigma - x\sigma_x)(x\sigma_x - \sigma - \sigma_x^2), \qquad \sigma(x) \simeq \frac{\vartheta}{\pi} x + \frac{\vartheta^2}{\pi^2} x^2 \quad (x \to 0)$$

$\Rightarrow$ for $x \to \infty$

- $0 < \vartheta < 1$ $\quad \sigma(x) \simeq -\log(1 - \vartheta) x / \pi$
- $\vartheta = 1$ $\quad \sigma(x) \simeq x^2/4$
- $\vartheta > 1$ $\quad \sigma(x) \simeq c_\vartheta (x - x_\vartheta)^{-1} \qquad (x \to x_\vartheta)$

general advice from numerical dynamical systems:

stable calculation of connecting orbits requires the solution as a BVP

Vanilla version of BVP approach to Hastings–McLeod: mix & match

1. choose $a < 0$, $b > 0$
2. solve BVP on $x \in [a, b]$:

$$u''(x) = 2u(x)^3 + x\,u(x)$$

$$u(a) = \sqrt{\frac{-a}{2}}\left(1 + \frac{1}{8}a^{-3} - \frac{73}{128}a^{-6} + \frac{10657}{1024}a^{-9}\right)$$

$$u(b) = \mathrm{Ai}(b)$$

3. decrease a, increase b, repeat until accurate enough

Remark: $a = -30$ and $b = 8$ is good for 14 digits accuracy *nonlinear BVP to be solved*

$$u''(x) = f(x, u(x)), \qquad u(a) = \alpha, \quad u(b) = \beta$$

Newton iteration in function space

- start value: some function $u_0(x)$ with $u_0(a) = \alpha$, $u_0(b) = \beta$
- iterate for $k = 0, 1, 2, \ldots$ until numerical convergence:

$$\Delta u_k'' = f_u(x, u_k)\Delta u_k + f(x, u_k) - u_k'', \qquad \Delta u_k(a) = 0, \quad \Delta u_k(b) = 0$$

$$u_{k+1} = u_k + \Delta u_k$$

Remarks:

- quadratic convergence: in each step the number of correct digits is doubled
- highly accurate solution of linear problems: spectral collocation

$\rightsquigarrow$ code with “Chefuns” and “Chebops”

Trefethen/Battles '04: *Numerically* manipulate smooth univariate functions

IEEE arithmetic for $\mathbb{R}$ $\rightsquigarrow$ "Chebfun" representation of smooth functions

by polynomial interpolation in Chebyshev points *to about machine precision*

Background theory

- barycentric formula (Salzer '72)

$$p_N(x) = \frac{\sum_{k=0}^{\prime\prime N} (-1)^k f(x_k)/(x - x_k)}{\sum_{k=0}^{\prime\prime N} (-1)^k/(x - x_k)}$$

- exponential convergence for analytic functions (Bernstein '12): $\exists\, \rho > 1$

$$\|f - p_N\|_\infty = O(\rho^{-N})$$

- numerically stable (Higham '04)

$\rightsquigarrow$ realized in object oriented Matlab: e.g., `sum(f)` calculates $\int_a^b f(x)\,dx$

The Chebop system (Driscoll/B./Trefethen ’08)

linear operators $\rightsquigarrow$ spectral collocation matrices of “automatic dimension”

Newton iteration, once more

$$(D^2 - f_u(x,u_k))\Delta u_k = f(x,u_k) - D^2 u_k, \qquad \Delta u_k(a) = 0, \quad \Delta u_k(b) = 0$$

$$u_{k+1} = u_k + \Delta u_k$$

```
function u = BVP(f,fu,a,b,alpha,beta,tol)
    [d,x] = domain(a,b);
    D = diff(d);
    u = ((b-x)*alpha+(x-a)*beta)/(b-a);
    while true
        L  = D^2 - diag(fu(x,u)) & 'dirichlet'; L.scale = norm(u);
        du = L\(f(x,u)-D^2*u);
        u  = u+du;
        if norm(du) <= tol, break; end;
    end
end
```

Lecture 1: Part E

Resurrecting Fredholm Determinants

Fredholm (1899)


Ivar Fredholm (1866–1927)

integral equation of the 2nd kind

$$u(x) + z \int_a^b K(x,y)u(y)\,dy = f(x) \qquad (x \in (a,b))$$

uniquely solvable iff

$$\det\left(I + zK\restriction_{L^2(a,b)}\right) = \sum_{m=0}^{\infty} \frac{z^m}{m!} \int_{[a,b]^m} \det\left(K(t_i,t_j)\right)_{i,j=1}^{m} dt \quad \neq \quad 0$$

⇝ theory of compact operators (Hilbert, Schmidt, Carleman, Riesz, Schauder, ...)

⇝ of historical interest only?

> *But the qualitative insight that the theory gave could also be achieved in a simpler way. The significance of Fredholm's work was more the qualitative insight than the explicit formulas.* — Gårding '98

Many applications, though:

- atomic collision theory
- inverse scattering
- Floquet theory
- Feynman path integrals
- autocorrelation of the Ising model
- renormalization in QFT
- random matrix theory
- combinatorial growth processes

Unlike the new, abstract theories, Fredholm dealt with integral operators, and his central notion was the determinant associated with such operators. Since this determinant appears in some modern theories (inverse scattering, integrable systems), it is time to resurrect it. — Lax '02

Transverse Ising chain (McCoy/Perk/Shrock '83)

autocorrelation function ($T = 0$ gives $\rightsquigarrow$ sine kernel and Painlevé V)

$$\chi(t) = e^{-t^2/2} \det\left(I - iK_t\restriction_{L^2(-1,1)}\right)$$

$$K_t(x,y) = \tanh(\beta\sqrt{1-x^2})\,\frac{\sinh(t(x-y))}{\pi(x-y)}, \qquad \beta^{-1} = kT$$

Korteweg–de Vries equation (Dyson '76, Oishii '79, Pöppe '84)

$$u_t + u_{xxx} + 6uu_x = 0$$

solved by

$$u(x,t) = 2\,\partial_x^2 \log\det\left(I + zK_t\restriction_{L^2(-\infty,x)}\right) \qquad (z \in \mathbb{C})$$

$$K_t(x,y) = v(x+y,t)$$

$$0 = v_t + 8v_{xxx}$$

Schrödinger operator

$$-\partial_x^2 + u(x)$$

potential u *uniquely* characterized by

- point spectrum:

  $-\kappa_n^2$ eigenvalue with normalized eigenfunction $\psi_n(x) \simeq c_n e^{-\kappa_n x} \quad (x \to \infty)$

- continuous spectrum: reflection coefficient $r(k)$ to the right (skew-symmetric)

Dyson's formula (1976)

$$\boxed{u(x) = -2\frac{\partial^2}{\partial x^2} \log\det\left(I - K\restriction_{L^2(x,\infty)}\right)}$$

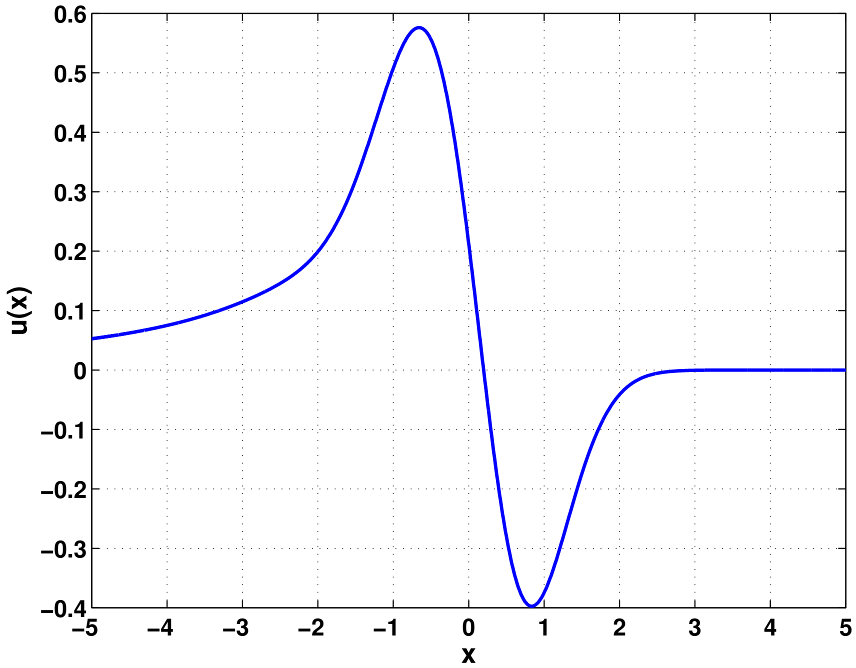
with kernel

$$K(\xi,\eta) = F(\xi+\eta), \quad F(\xi) = \sum_{n=1}^{N} c_n^2 e^{-\kappa_n \xi} + \frac{1}{2\pi}\int_{-\infty}^{\infty} r(k) e^{ik\xi}\, dk$$

Two examples

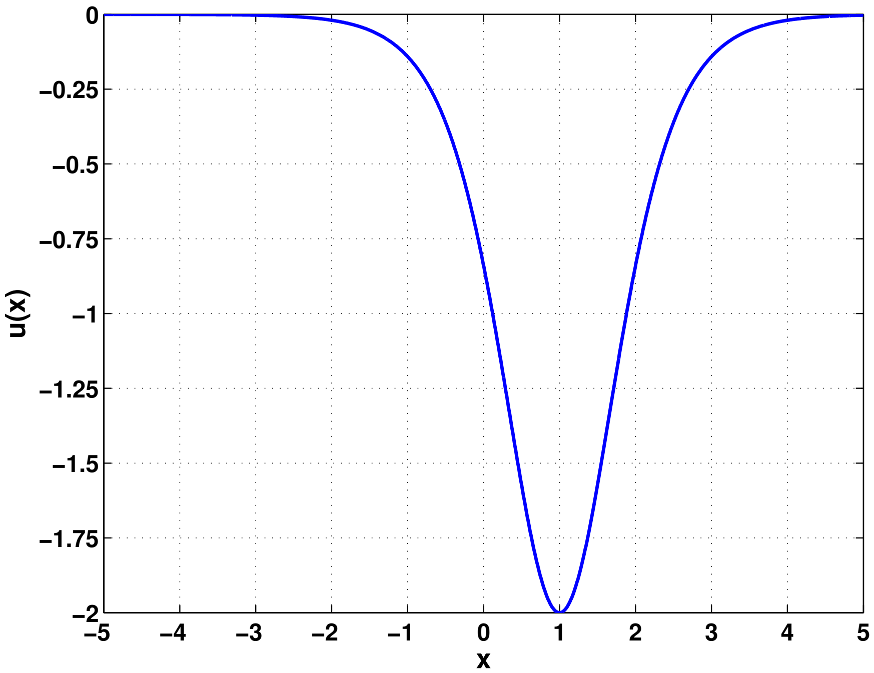
- no point spectrum
- $r(k) = \exp(-k^2)$

$$\rightsquigarrow F(\xi) = \frac{1}{2\sqrt{\pi}} \exp(-\xi^2/4)$$



- $\kappa_1 = 1, c_1 = \sqrt{2}\, e$
- reflectionless: $r(k) = 0$

$$\rightsquigarrow F(\xi) = 2e^2 \exp(-\xi)$$



Lecture 2: Part A

Numerical Evaluation of Fredholm Determinants

Determinants of trace class operators K in a Hilbert space $\mathcal{H}$

(Grothendieck '56, Gohberg/Kreĭn '59, Dunford/Schwarz '63, Simon '77)

there are several equivalent definitions of the *entire* function $d(z) = \det(I + zK)$

1. F_n finite rank, $F_n \to K$ in trace class norm

$$d(z) = \lim_{n\to\infty} \det(I + zF_n)$$

2. $\lambda_1(K), \lambda_2(K), \lambda_3(K), \ldots$ eigenvalues of K (accounting multiplicities)

$$d(z) = \prod_{j=1}^{\infty}(1 + z\lambda_j(K))$$

3. analytic continuation

$$d(z) = \exp(\operatorname{tr}\log(I + zK))$$

4. generalizing Fredholm's classical power series

$$d(z) = \sum_{m=0}^{\infty} z^m \operatorname{tr} \bigwedge^m K$$

Galerkin idea

- n-dimensional subspace $V_n \subset \mathcal{H}$ with ON basis ϕ_j and ON-projection

$$P_n : \mathcal{H} \to V_n$$

- calculate $n \times n$-determinant

$$d_n(z) = \det(I + zP_nKP_n) = \det(\delta_{ij} + z\langle\phi_i, K\phi_j\rangle)_{i,j=1}^{n}$$

Theorem 1 (B. '08)
If the sequence V_n satisfies the consistency condition

$$\bigcup_{n=1}^{\infty} V_n \text{ dense in } \mathcal{H}$$

then, uniformly for bounded z,

$$d_n(z) = \det(I + zP_nKP_n) \to d(z) = \det(I + zK) \qquad (n \to \infty).$$

K integral operator on $\mathcal{H} = L^2(a,b)$ with *continuous* kernel

Theorem 2 (B. '08)

K selfadjoint and V_n spanned by first n eigenfunctions (Ritz–Galerkin method)

- if kernel is $C^{k-1,1}([a,b]^2)$,

$$d_n(z) - d(z) = O(n^{\frac{1}{2}-k}) \qquad (n \to \infty).$$

- if kernel is bounded analytic in a neighborhood of $[a,b]^2$, there is a $\rho > 1$ with

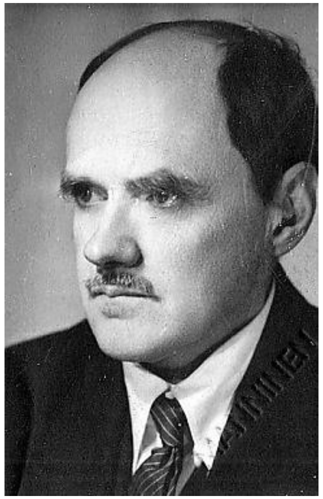
$$d_n(z) - d(z) = O(\rho^{-n}) \qquad (n \to \infty).$$

Idea of proof: perturbation bound and eigenvalue decay

$$|\det(I + zP_nKP_n) - \det(I + zK)| \underset{\text{'75}}{\overset{\text{Seiler/Simon}}{\leqslant}} |z|e^{1+|z|\cdot\|K\|_{\mathcal{J}_1}} \cdot \|P_nKP_n - K\|_{\mathcal{J}_1}$$

$$= c(z,K) \cdot \sum_{j=n+1}^{\infty} |\lambda_j(K)| \underset{\text{'31}}{\overset{\text{Hille/Tamarkin}}{=}} O(\cdots)$$

Nyström (1930)



Evert Nyström (1895–1960)

solved a Fredholm equation $(I + zK)u = f$ of the 2nd kind, i.e.

$$u(x) + z \int_a^b K(x,y)u(y)\,dy = f(x) \qquad (x \in (a,b))$$

using an n-point quadrature formula Q (weights w_j, nodes x_j)

$$u(x_i) \approx u_i : \qquad u_i + z \sum_{j=1}^n w_j K(x_i, x_j) u_j = f(x_i) \qquad (i = 1, \ldots, n)$$

$\rightsquigarrow$ **straightforward idea** (B. '08)

approximate $d(z) = \det(I + zK)$ simply by the corresponding $n \times n$ determinant

$$\boxed{d_Q(z) = \det(\delta_{ij} + z\, w_j K(x_i, x_j))_{i,j=1}^n}$$

Matlab code to evaluate $\det\left(I - K\restriction_{L^2(a,b)}\right)$

```
function d = FredholmDeterminant(K,a,b,m)
    [w,x] = QuadratureRule(a,b,m);
    w = sqrt(w);
    [xi,xj] = ndgrid(x,x);
    d = det(eye(m)-(w'*w).*K(xi,xj));
end
```

---

example: GUE bulk spacing distribution $E_2(0;s)$ *for* $s = 0.1$ *and* $s = 1$

```
> FredholmDeterminant(@(x,y)sinc(pi*(x-y)),0,1,7)
```

$$E_2(0;0.1) \doteq 0.90002\,72717\,98259$$

```
> FredholmDeterminant(@(x,y)sinc(pi*(x-y)),0,1,17)
```

$$E_2(0;1.0) \doteq 0.17021\,74213\,79185$$

CPU times: 0.5 ms and 0.6 ms, resp.

Using m quadrature points

machine precision $2.22044 \cdot 10^{-16}$

m	d_m	$\lvert d_m - d\rvert$	$\lvert d_{2m} - d_m\rvert$
4	0.17374 23519 58664	$3.52493 \cdot 10^{-3}$	$3.52477 \cdot 10^{-3}$
8	0.17021 75721 03969	$1.50724 \cdot 10^{-7}$	$1.50724 \cdot 10^{-7}$
16	0.17021 74213 79186	$2.77555 \cdot 10^{-16}$	$3.05311 \cdot 10^{-16}$
32	0.17021 74213 79185	$2.77555 \cdot 10^{-17}$	

(super-) exponential convergence $\rightsquigarrow$

- simple a posteriori error estimates
- automatic choice of m = #quadrature points

long term arithmetic (using, e.g., Mathematica)

$$d = E_2(0;1) = 0.17021\,74213\,79185\,23073\,26530\,52561\,54896\,98724$$

K integral operator on $\mathcal{H} = L^2(a, b)$ with *continuous* kernel

Theorem 1′ (B. ’08)

If the family Q of quadrature formulae converges for continuous functions, then

$$d_Q(z) \to d(z)$$

uniformly for bounded z.

Theorem 2′ (B. ’08)

For a family of quadrature formulae Q of order ν with positive weights:

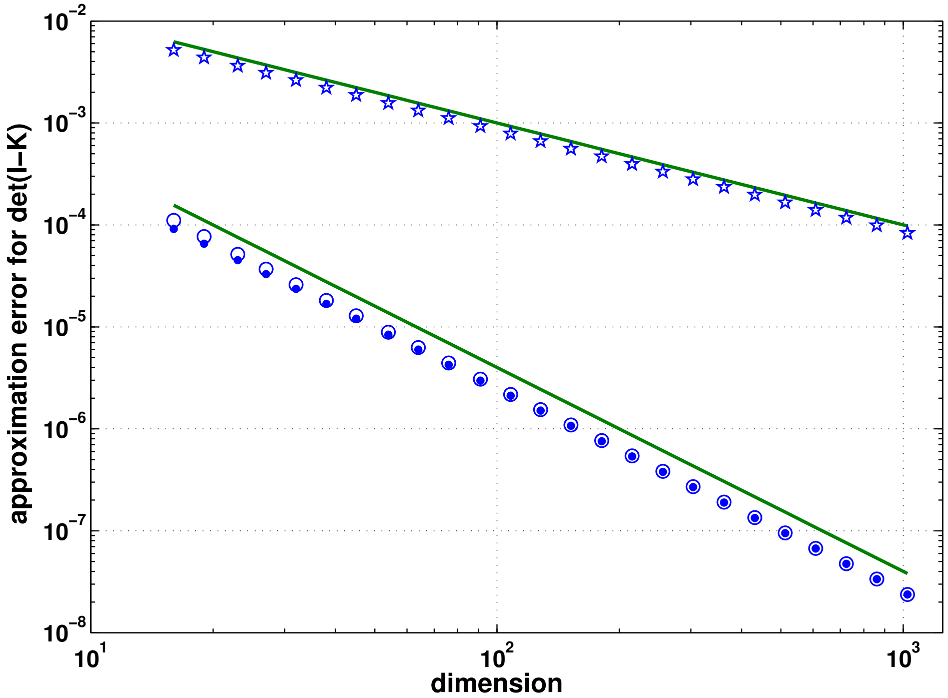
- if kernel is $C^{k-1,1}([a, b]^2)$,

$$d_Q(z) - d(z) = O(\nu^{-k}) \qquad (\nu \to \infty).$$

- if kernel is bounded analytic in a neighborhood of $[a, b]^2$, there is a $\rho > 1$ with

$$d_Q(z) - d(z) = O(\rho^{-\nu}) \qquad (\nu \to \infty).$$

A Green's kernel on $L^2(0,1)$



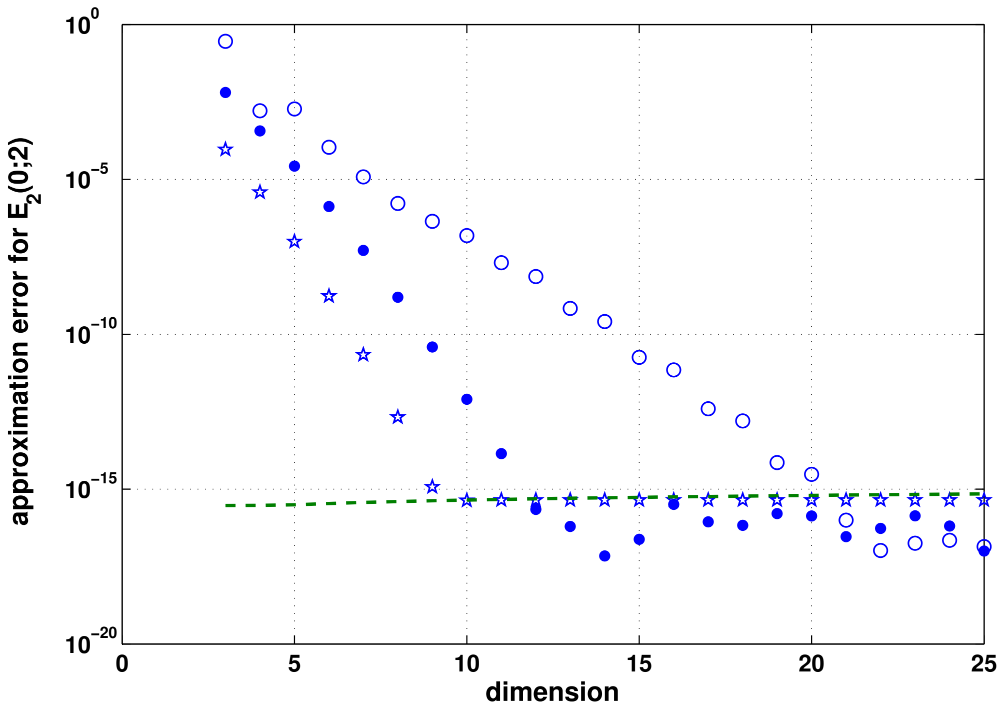
$$K(x,y) = \begin{cases} x(1-y) & x \leqslant y, \\ y(1-x) & x > y. \end{cases}$$

$$\det(I - zK\restriction_{L^2(0,1)}) = \frac{\sin\sqrt{z}}{\sqrt{z}}$$

- Ritz–Galerkin (stars) using *exact* eigenvalues: $O(n^{-1})$
- Gauss–Legendre (dots) or Clenshaw–Curtis (circles): $O(n^{-2})$

Gap probability of GUE *(bulk scaling limit)*

$$E_2(0;s) = \det\left(I - K\restriction_{L^2(0,s)}\right), \qquad K(x,y) = \operatorname{sinc}(\pi(x-y))$$



stars: Ritz–Galerkin (Gaudin's method), dots: Gauss–Legendre, circles: Clenshaw–Curtis

Idea of proof of Thm. 1′: change the order of limits *(read the masters!)*

$$\det(I + zK) \overset{\text{Fredholm}}{\underset{1903}{=}} \sum_{m=0}^{\infty} \frac{z^m}{m!} \int_a^b \cdots \int_a^b \det\left(K(t_i, t_j)\right)_{i,j=1}^{m} \, dt_1 \cdots dt_m$$

$$\overset{\text{Pólya}}{\underset{1933}{=}} \sum_{m=0}^{\infty} \lim_{n\to\infty} \frac{z^m}{m!} \sum_{k_1=1}^{n} \cdots \sum_{k_m=1}^{n} w_{k_1} \cdots w_{k_m} \cdot \det\Big(K(x_{k_i}, x_{k_j})\Big)_{i,j=1}^{m}$$

$$\overset{\text{Hilbert}}{\underset{1904}{=}} \lim_{n\to\infty} \sum_{m=0}^{\infty} \frac{z^m}{m!} \sum_{k_1=1}^{n} \cdots \sum_{k_m=1}^{n} w_{k_1} \cdots w_{k_m} \cdot \det\Big(K(x_{k_i}, x_{k_j})\Big)_{i,j=1}^{m}$$

$$= \lim_{n\to\infty} \sum_{m=0}^{n} \frac{z^m}{m!} \sum_{k_1=1}^{n} \cdots \sum_{k_m=1}^{n} \det\Big((K_n)_{k_i,k_j}\Big)_{i,j=1}^{m} \overset{\text{v. Koch}}{\underset{1892}{=}} \lim_{n\to\infty} \det(I + zK_n)$$

with the $n \times n$-matrix

$$K_n = \left(w_j K(x_i, x_j)\right)_{i,j=1}^{n}$$

NOTE ON THE FREDHOLM DETERMINANT.

BY DR. W. A. HURWITZ.

(Read before the American Mathematical Society, December 30, 1913.)

THE theory of linear integral equations presents many analogies with the theory of linear algebraic equations; in fact the former may be regarded in a quite definite and accurate sense as a limiting case of the latter. As in various other mathematical theories concerned with limiting cases, two methods suggest themselves for the proof of theorems: one may go through the process of taking limits in the results of the algebraic theory—this method is typified by the early work of Hilbert* on integral equations; or one may use the algebraic theory merely to suggest theorems, which one then proves independently—this is the procedure in the fundamental paper of Fredholm.† While any sweeping statement comparing the two plans would be unwise, it seems reasonably clear that the second method will ordinarily be the more elegant.

indeed, "*merely suggesting*" is the prevailing Modus Operandi, e.g.:

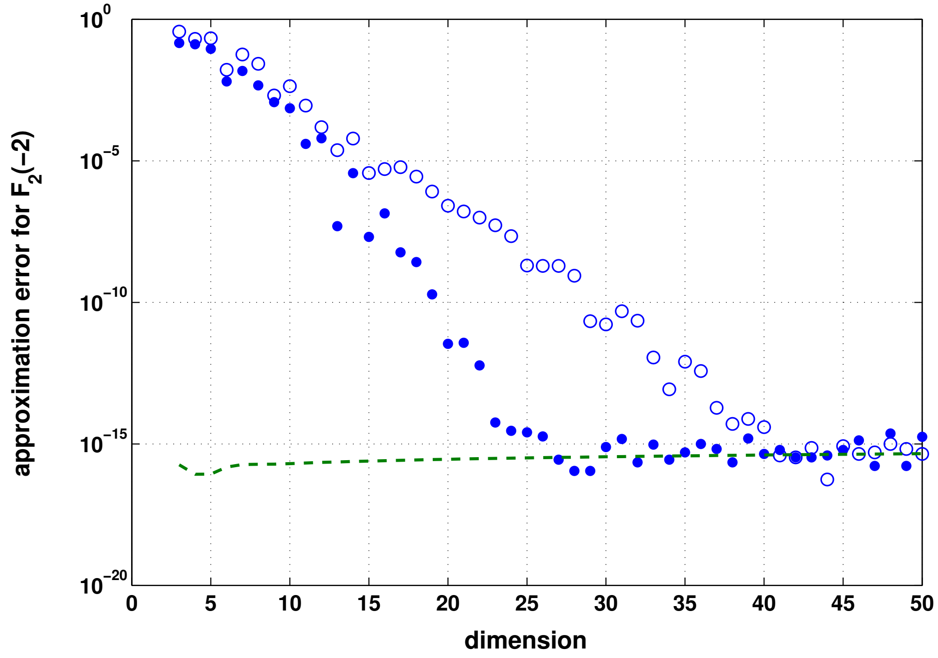
- "*cause de l'analogie*" (Fredholm 1909)
- "*so far . . . purely tentative, we now start ab initio*" (Whittaker/Watson 1927)
- "*calcolato senza scrupoli . . . abbandonando le considerazioni euristiche*" (Tricomi 1954)
- "*we first operate heuristically to guess the solution*" (Widom 1969)

Lecture 2: Part B

Replacing Painlevé and beyond

Tracy–Widom distribution *(edge scaling limit of GUE)*

$$F_2(s) = \det\left(I - K\restriction_{L^2(s,\infty)}\right), \qquad K(x,y) = \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x-y}$$



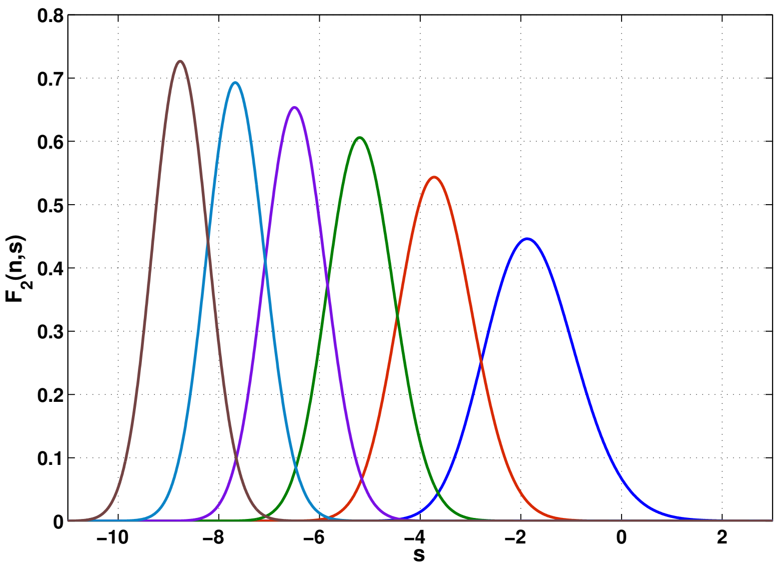
dots: Gauss–Legendre, circles: Clenshaw–Curtis

Perturbation bound for n-dimensional determinants: (B. '08)

$$\text{round-off error} \leqslant \sqrt{n}\,\|K_n\|_F \cdot u_{\text{machine}}$$

n-th largest level in edge scaled GUE

$$\mathbb{P}(\text{exactly } n \text{ levels lie in } (s,\infty)) = \frac{(-1)^n}{n!} \frac{\partial^n}{\partial z^n} \det\left(I - zK\restriction_{L^2(s,\infty)}\right)\Big|_{z=1}$$



probability density of n-th largest level in edge scaled GUE

Numerical method

$f(z) = \det(I + zK)$ is *entire* of order 0

$\rightsquigarrow$ Cauchy's formula applies

$$\boxed{\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi r^n}\int_0^{2\pi} e^{-in\theta} f(z + re^{i\theta})\, d\theta}$$

- Trapezoidal rule *exponentially* convergent
- numerical stability: judicious choice of $r > 0$

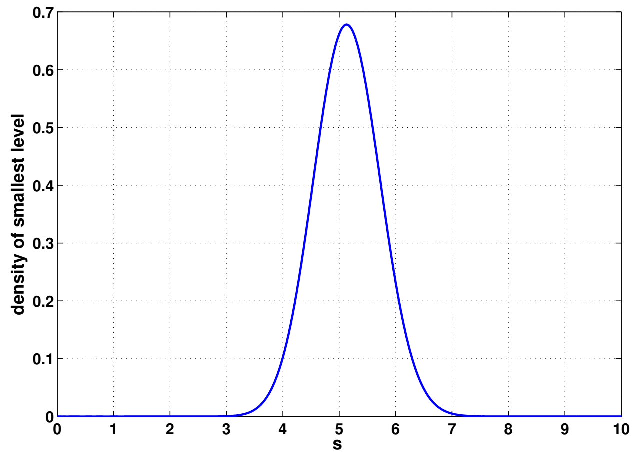
  Example: (B. '09)

  f entire of order $\rho > 0$ and type $\sigma > 0$

$$\boxed{r = (n/\rho\sigma)^{1/\rho}}$$

$n \times n$-LUE with weight $x^\alpha e^{-x}$ ($n = 80$, $\alpha = 40$, to be concrete)

$$\mathbb{P}(\text{no levels lie in } (0,s)) = \det\left(I - K\restriction_{L^2(0,s)}\right)$$



probability density of minimal level in 80×80-LUE with $\alpha = 40$

$$K(x,y) = \frac{\phi_{n-1}(x)\phi_n(y) - \phi_n(x)\phi_{n-1}(y)}{(n(n+\alpha))^{-1/2}(x-y)}$$

$$\phi_k(x) = \sqrt{\frac{k!}{\Gamma(n+\alpha+1)}}\, x^{\alpha/2} e^{-x/2} L_k^{(\alpha)}(x)$$

$\mu = 5.14156\,81318\cdots$ [5.141 *from* 10^4 *samples*]

$\sigma^2 = 0.34347\,52478\cdots$ [0.339 —″—]

compare with Painlevé V approach:

$$(x\sigma_{xx})^2 = (\sigma - x\sigma_x - 2\sigma_x^2 + (2n+\alpha)\sigma_x)^2 - 4\sigma_x^2(\sigma_x - n)(\sigma_x - n - \alpha)$$

$$\mathbb{P}(s) = \exp\left(-\int_0^s \frac{\sigma(x)}{x}\,dx\right) \qquad \sigma(x) \simeq \frac{\Gamma(n+\alpha+1)}{\Gamma(n)\Gamma(\alpha+1)\Gamma(\alpha+2)} x^{\alpha+1} \quad (x \to 0)$$

connection formula (Forrester/Witte '02): $\sigma(x) = nx - \alpha n + \alpha n^2 x^{-1} + O(x^{-2}) \quad (x \to \infty)$

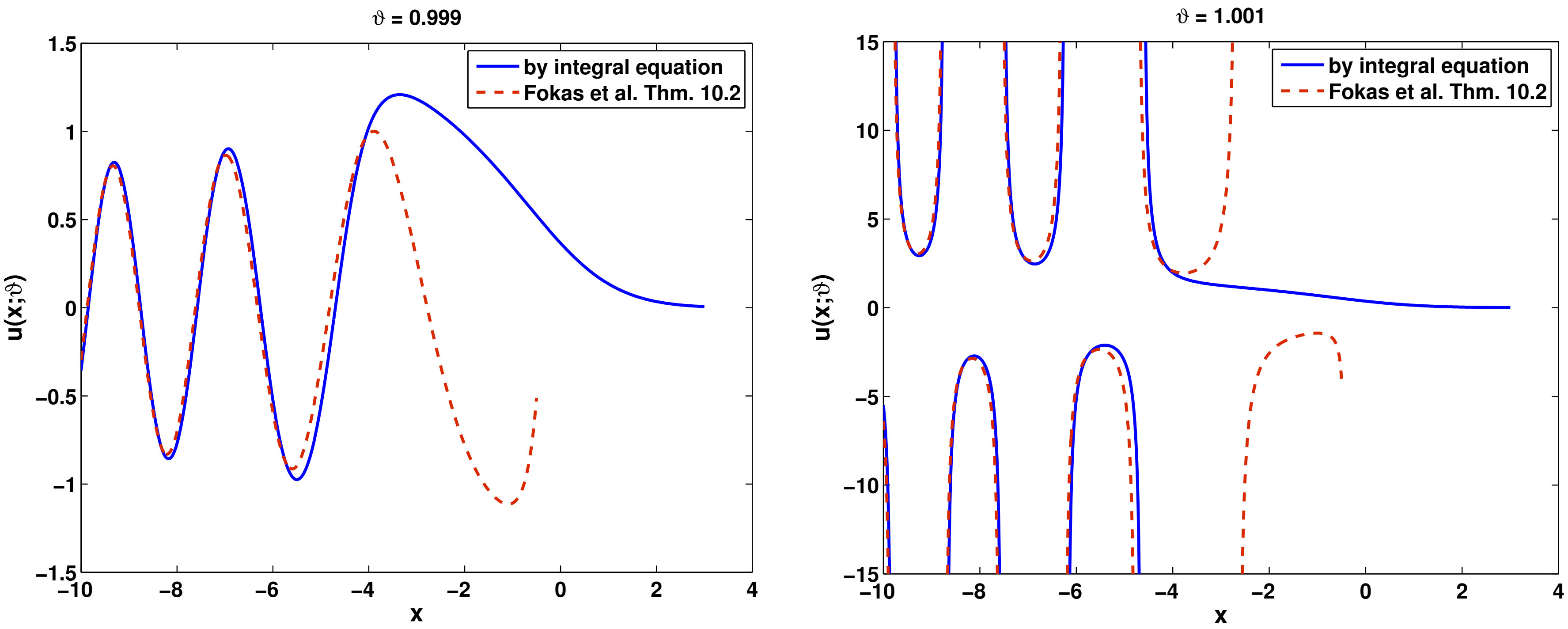
Solving Painlevé by Fredholm integral equations

Painlevé II

$$u_{xx} = 2u^3 + xu, \qquad u(x) \simeq \vartheta\,\mathrm{Ai}(x) \quad (x \to \infty),$$

is solved by a generalization of Dyson's formula (Dyson '76, Ablowitz/Ramani/Segur '80)

$$u(x;\vartheta) = \sqrt{-\frac{d^2}{ds^2}\log\det\left(I - \vartheta^2 K\restriction_{L^2(x,\infty)}\right)} = \vartheta\left(I - \vartheta^2 K\restriction_{L^2(x,\infty)}\right)^{-1}\mathrm{Ai}(x)$$



Is there a general method to establish "Painlevé $\mapsto$ Fredholm determinant"?

Lecture 2: Part C

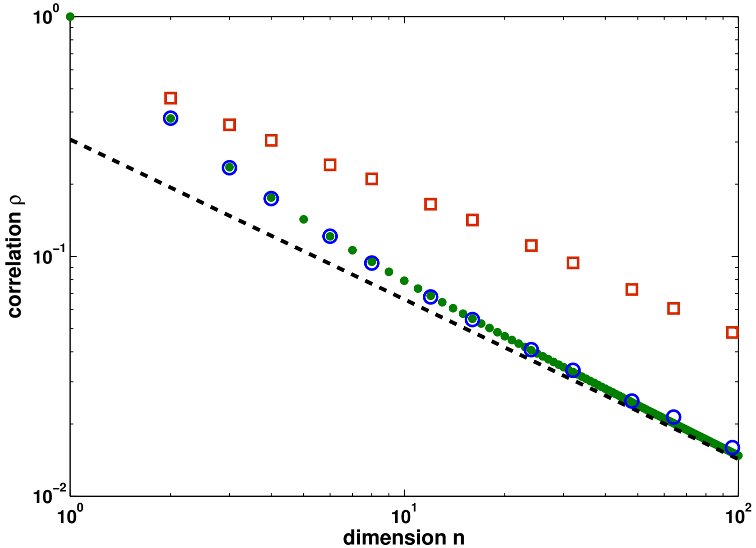
Matrix Kernel Determinants

Joint probability distribution of the extreme eigenvalues of $n \times n$ GUE

$$\mathbb{P}(x \leqslant \lambda_{\min}^{(n)} \leqslant \lambda_{\max}^{(n)} \leqslant y) = \det\left(I - K_n \restriction_{L^2((-\infty,x)\cup(y,\infty))}\right)$$

$$= \det\left(I - \begin{pmatrix} K_n & K_n \\ K_n & K_n \end{pmatrix} \restriction_{L^2(-\infty,x)\oplus L^2(y,\infty)}\right)$$

Non-universality of nonlinear statistics (B. '09)



correlation coefficient $\rho(\lambda_{\min}^{(n)}, \lambda_{\max}^{(n)})$

- green dots: GUE

  with asymptotics (dashed line)

  $$\rho(\lambda_{\min}^{(n)}, \lambda_{\max}^{(n)}) = \frac{n^{-2/3}}{4\sigma^2} + O(n^{-4/3})$$

  $\sigma^2 = 0.81319\,47928$ variance of F_2

- red squares: hermitean matrices with entries uniformly i.i.d. on $[-1, 1]$

Systems of integral operators = integral operator on coproduct

$$K = \begin{pmatrix} K_{11} & \cdots & K_{1N} \\ \vdots & & \vdots \\ K_{N1} & \cdots & K_{NN} \end{pmatrix} \quad \text{on} \quad \bigoplus_{k=1}^{N} L^2(I_k) \quad \text{with matrix kernel } K_{ij}(x,y)$$

representable as a *single* integral operator on

$$L^2\left(\coprod_{k=1}^{N} I_k\right) \cong \bigoplus_{k=1}^{N} L^2(I_k), \qquad \coprod_{k=1}^{N} I_k = \bigcup_{k=1}^{N} I_k \times \{k\},$$

with *scalar* kernel (Fredholm 1903)

$$K(x,y) = \sum_{i,j=1}^{N} \mathbb{1}_{I_i}(x) K_{ij}(x,y) \mathbb{1}_{I_j}(y)$$

$\rightsquigarrow$ straightforward extension of projection and quadrature method

Edge scaling limit of GUE matrix diffusion = Airy$_2$ process

$M_n(t)$ $n \times n$-Hermitean-matrix valued process, coefficients Ornstein–Uhlenbeck

$$\mathcal{A}_2(t) = \lim_{n\to\infty} \frac{\lambda_{\max}\left(M_n(n^{-1/3}t)\right) - \sqrt{2n}}{2^{-1/2}n^{-1/6}}$$

relation to PNG (polynuclear growth) droplet model

$\rightsquigarrow$ joint probablity distribution (Prähofer/Spohn '02)

$$\mathbb{P}(\mathcal{A}_2(t) \leqslant s_1, \mathcal{A}_2(0) \leqslant s_2) = \det\left(I - \begin{pmatrix} K_0 & K_t \\ K_{-t} & K_0 \end{pmatrix}\restriction_{L^2(s_1,\infty)\oplus L^2(s_2,\infty)}\right)$$

with kernel

$$K_t(x,y) = \begin{cases} \displaystyle\int_0^\infty e^{-\xi t}\,\mathrm{Ai}(x+\xi)\,\mathrm{Ai}(y+\xi)\,d\xi & t \geqslant 0 \\ \displaystyle -\int_{-\infty}^0 e^{-\xi t}\,\mathrm{Ai}(x+\xi)\,\mathrm{Ai}(y+\xi)\,d\xi & t < 0 \end{cases}$$

Adler/van Moerbeke ('05)

$G(t,x,y) = \log \mathbb{P}(\mathcal{A}_2(t) \leqslant x, \mathcal{A}_2(0) \leqslant y)$ satiesfies nonlinear 3rd order PDE

$$
\begin{aligned}
t\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) G = {} & \frac{\partial^3 G}{\partial x^2 \partial y}\left(2\frac{\partial^2 G}{\partial y^2} + \frac{\partial^2 G}{\partial x \partial y} - \frac{\partial^2 G}{\partial x^2} + x - y - t^2\right) \\
& - \frac{\partial^3 G}{\partial y^2 \partial x}\left(2\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial x \partial y} - \frac{\partial^2 G}{\partial y^2} - x + y - t^2\right) \\
& + \left(\frac{\partial^3 G}{\partial x^3}\frac{\partial}{\partial y} - \frac{\partial^3 G}{\partial y^3}\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) G
\end{aligned}
$$

$\rightsquigarrow$ asymptotic expansions, e.g.:

$$
\mathbb{P}(\mathcal{A}_2(t) \leqslant x, \mathcal{A}_2(0) \leqslant y) = F_2(x)F_2(y) + \frac{F_2'(x)F_2'(y)}{t^2} + O(t^{-4})
$$

aside: useful for numerical calculations? most probably not ...

Edge scaling limit of GOE matrix diffusion = Airy$_1$ process ?

$M_n(t)$ $n \times n$-symmetric-matrix valued process, coefficients Ornstein–Uhlenbeck

$$\mathcal{A}_1(t) = \lim_{n\to\infty} \frac{\lambda_{\max}\left(M_n(2n^{-1/3}t)\right) - \sqrt{n}}{n^{-1/6}}$$

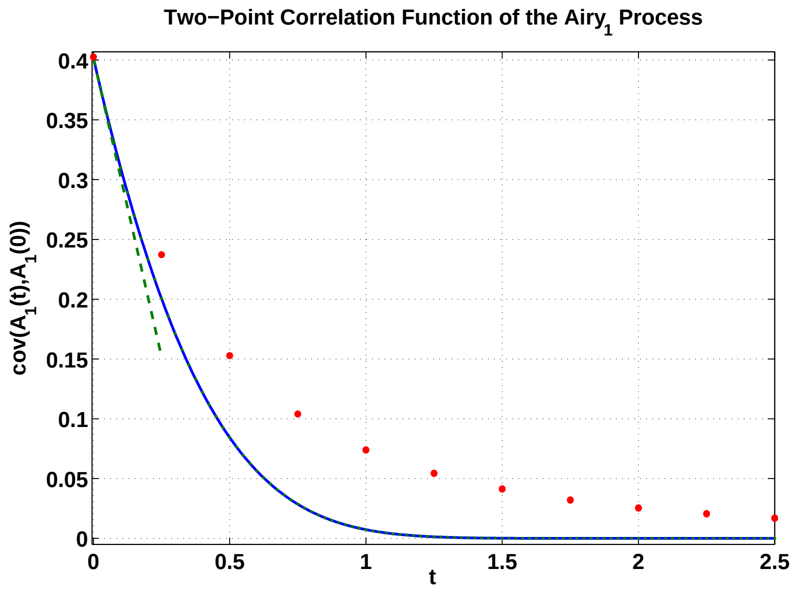
(Sasamoto '05, Borodin/Ferrari/Prähofer/Sasamoto '07)

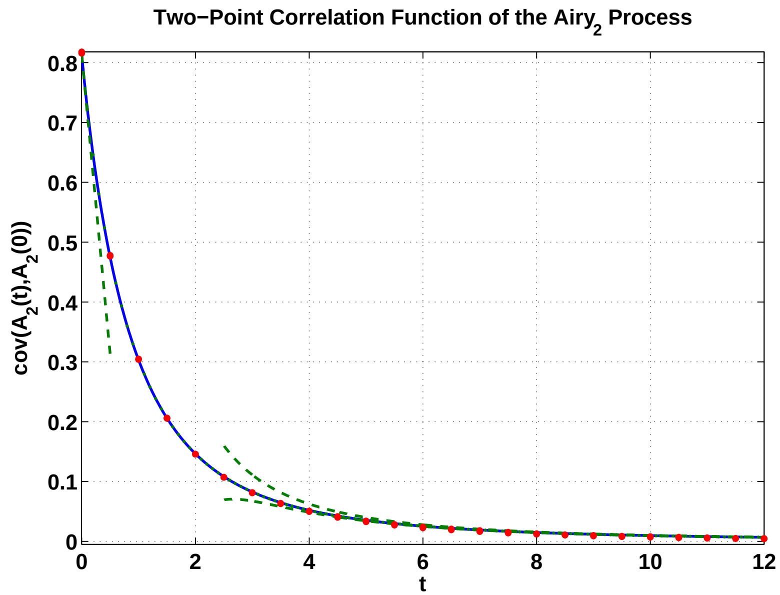
conjectured relation to the flat PNG model (universality!) $\rightsquigarrow$

$$\mathbb{P}(\mathcal{A}_1(t) \leqslant s_1, \mathcal{A}_1(0) \leqslant s_2) \overset{?}{=} \det\left(I - \begin{pmatrix} K_0 & K_t \\ K_{-t} & K_0 \end{pmatrix}\restriction_{L^2(s_1,\infty)\oplus L^2(s_2,\infty)}\right)$$

with kernel

$$K_t(x,y) = \begin{cases} \operatorname{Ai}(x+y+t^2)e^{t(x+y)+2t^3/3} - \dfrac{\exp(-(x-y)^2/(4t))}{\sqrt{4\pi t}} & t > 0 \\ \operatorname{Ai}(x+y+t^2)e^{t(x+y)+2t^3/3} & t \leqslant 0 \end{cases}$$





red: Monte–Carlo for matrix size $N = 128$ and $N = 256$; green: known asymptotics; blue: numerical calculation with absolute precision $5 \cdot 10^{-11}$

$$\mathrm{cov}(\mathcal{A}_k(t), \mathcal{A}_k(0)) = \mathbb{E}(\mathcal{A}_k(t)\mathcal{A}_k(0)) - \mathbb{E}(\mathcal{A}_k(t))\mathbb{E}(\mathcal{A}_k(0))$$

$$= \int_{\mathbb{R}^2} s_1 s_2 \frac{\partial^2 \mathbb{P}(\mathcal{A}_k(t) \leqslant s_1, \mathcal{A}_k(0) \leqslant s_2)}{\partial s_1 \partial s_2}\, ds_1 ds_2 - \mathbb{E}(F_k)^2$$

Conclusion: (B./Ferrari/Prähofer ’08)

limit of GOE matrix diffusion $\neq$ Airy$_1$ process *(empirically, so)*

n-th largest level in edge scaled GSE

$$\mathbb{P}(\text{exactly } n \text{ levels lie in } (s,\infty)) = E_4(n;s) = \frac{(-1)^n}{n!}\,\frac{\partial^n}{\partial z^n} F_4(s;z)\Big|_{z=1}$$

(Forrester/Nagao/Honner '99, Tracy/Widom '04)

$$F_4(s;z) = \sqrt{\det\left(I - \frac{z}{2}\begin{pmatrix} S(x,y) & SD(x,y) \\ IS(x,y) & S(y,x)\end{pmatrix}\restriction_{L^2(s,\infty)\oplus L^2(s,\infty)}\right)}$$

$$S(x,y) = K_{\mathrm{Ai}}(x,y) - \tfrac{1}{2}\,\mathrm{Ai}(x)\int_y^\infty \mathrm{Ai}(\eta)\,d\eta$$

$$SD(x,y) = -\partial_y K_{\mathrm{Ai}}(x,y) - \tfrac{1}{2}\,\mathrm{Ai}(x)\,\mathrm{Ai}(y)$$

$$IS(x,y) = -\int_x^\infty K_{\mathrm{Ai}}(\xi,y)\,d\xi + \tfrac{1}{2}\int_x^\infty \mathrm{Ai}(\xi)\,d\xi \int_y^\infty \mathrm{Ai}(\eta)\,d\eta$$

$$K_{\mathrm{Ai}}(x,y) = \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x-y}$$

A new formula from numerical experiments (B. '09)

$$F_{\pm}(s;z) = \det\left(I \mp \sqrt{z}\, K\restriction_{L^2(s/2,\infty)}\right), \qquad K(x,y) = \mathrm{Ai}(x+y)$$

yields *(Ferrari/Spohn '05)*

$$F_4(s;1) = \tfrac{1}{2}\left(F_+(s;1) + F_-(s;1)\right)$$

$$F_2(s;z) = F_+(s;z) \cdot F_-(s;z)$$

How about

$$\boxed{F_4(s;z) = \tfrac{1}{2}\left(F_+(s;z) + F_-(s;z)\right),}$$ then?

first, there was no obvious reason for it . . . ,

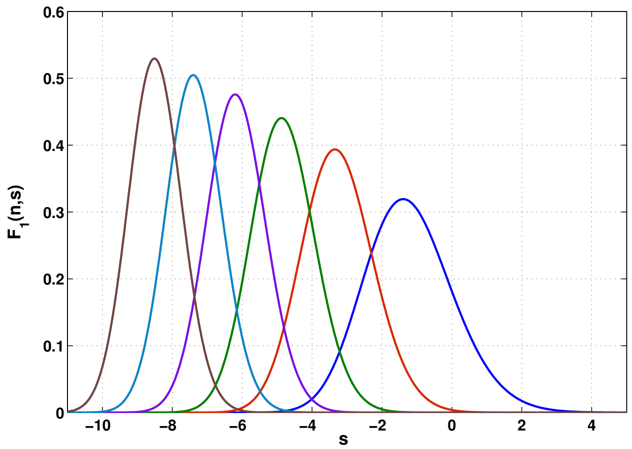
. . . but numerical tests with random s and z indicated the formula to be *true*

later, proof via Painlevé II representation (B. '09, Forrester '06)

n-th largest level in edge scaled GOE

$$E_\pm(n;s) = \frac{(-1)^n}{n!}\,\frac{\partial^n}{\partial z^n} F_\pm(s;z)\Big|_{z=1}, \qquad K(x,y) = \mathrm{Ai}(x+y)$$

yields



Probability density of n-th largest level in edge scaled GOE

$$E_1(2n;s) = E_+(n;s) - \sum_{k=0}^{n-1} \frac{\binom{2k}{k} E_1(2n-2j-1;s)}{2^{2k+1}(k+1)}$$

$$E_1(2n+1;s) = \frac{E_+(n;s)+E_-(n;s)}{2} - E_1(2n;s)$$

technique: elimination process using interrelations (Forrester/Rains '01) between GOE, GUE, and GSE

$$\mathrm{GUE}_n = \mathrm{even}(\mathrm{GOE}_n \cup \mathrm{GOE}_{n+1}), \qquad \mathrm{GSE}_n = \mathrm{even}(\mathrm{GOE}_{2n+1})$$

Compare with matrix kernel determinant for edge scaled GOE

$$E_1(n;s) = \frac{(-1)^n}{n!} \frac{\partial^n}{\partial z^n} F_1(s;z)\Big|_{z=1}$$

(Tracy/Widom '04)

$$F_1(s;z) = \sqrt{\det\left(I - z\begin{pmatrix} S(x,y) & SD(x,y) \\ IS(x,y) & S(y,x)\end{pmatrix}\restriction_{X_1(s,\infty)\oplus X_2(s,\infty)}\right)}$$

$$S(x,y) = K_{\mathrm{Ai}}(x,y) + \tfrac{1}{2}\left(1 - \tfrac{1}{2}\,\mathrm{Ai}(x)\int_y^\infty \mathrm{Ai}(\eta)\,d\eta\right)$$

$$SD(x,y) = -\partial_y K_{\mathrm{Ai}}(x,y) - \tfrac{1}{2}\,\mathrm{Ai}(x)\,\mathrm{Ai}(y)$$

$$IS(x,y) = -\tfrac{1}{2}\mathrm{sgn}(x-y) - \int_x^\infty K_{\mathrm{Ai}}(\xi,y)\,d\xi + \tfrac{1}{2}\left(\int_y^x \mathrm{Ai}(\xi)\,d\xi + \int_x^\infty \mathrm{Ai}(\xi)\,d\xi \int_y^\infty \mathrm{Ai}(\eta)\,d\eta\right)$$

$$K_{\mathrm{Ai}}(x,y) = \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x-y}$$

Hilbert–Schmidt operator with trace class diagonal

$\rightsquigarrow$ determinant to be understood as Hilbert–Carleman regularized determinant

A further consequence of our recursion formula for GOE

(B. '09, Forrester '06)

$$F_1(s;z) = \frac{1}{2}\left(\det\left(I - \sqrt{z(2-z)}\,K\restriction_{L^2(s/2,\infty)}\right)\left(1 + \sqrt{\frac{z}{2-z}}\right) + \det\left(I + \sqrt{z(2-z)}\,K\restriction_{L^2(s/2,\infty)}\right)\left(1 - \sqrt{\frac{z}{2-z}}\right)\right)$$

with $K(x,y) = \operatorname{Ai}(x+y)$

this determinantal expression is amenable to our numerical method

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