Asymptotic Methods for Integrable Systems in Nonlinear Wave Theory

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Outline

Introduction

The Fourier transform was invented precisely to solve linear partial differential equations of continuum mechanics. For example, consider the linear Schrödinger equation:

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We obtain a closed-form integral formula for the solution of the initial-value problem.

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- Long time (scattering, diffraction theory).
- Rough data (Gibbs phenomenon).
- Short waves (geometrical optics, weak dispersion).

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We obtain a procedure (less explicit, perhaps) for solving the initial-value problem.

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This presentation is a survey of some of these nonlinear methods. We concentrate on three example problems, and in each case we compare the procedure and results with corresponding linear problems.

Back to outline.

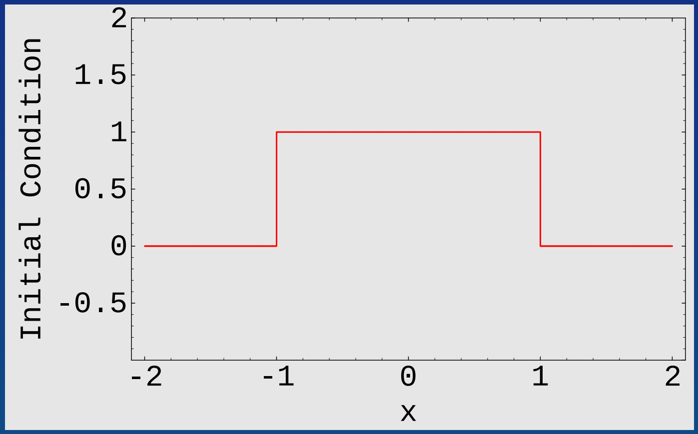
Long-Time Asymptotics for Dispersive Waves

Consider

$$i\frac{\partial\psi}{\partial t} + \frac{1}{2}\frac{\partial^2\psi}{\partial x^2} - |\psi|^2\psi = 0\,, \quad \text{(Defocusing Cubic Schrödinger)}$$

subject to the initial condition

$$\psi(x,0) = \begin{cases} 1\,, & |x| \le 1\,, \\ 0\,, & |x| > 1\,. \end{cases}$$



We are interested in the asymptotic behavior of $\psi(x,t)$ as $t \to +\infty$.

Corresponding Linear Problem

Consider for a moment instead the linear problem

$$i\frac{\partial\psi}{\partial t}+\frac{1}{2}\frac{\partial^2\psi}{\partial x^2}=0\,,\quad \text{(Linear Schrödinger)}$$

subject to the same piecewise-constant initial condition. We are interested in the asymptotic behavior of $\psi(x,t)$ as $t\longrightarrow+\infty$. By Fourier transforms, the solution of this initial-value problem is

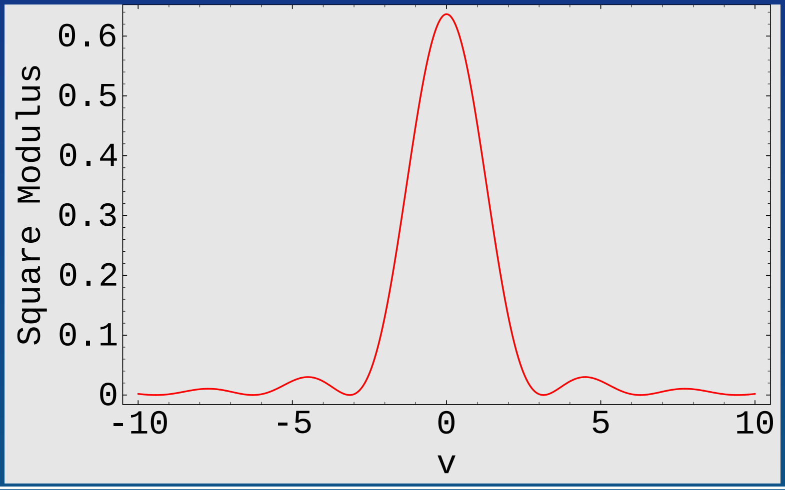
$$\psi(x,t)=\int_{-\infty}^{\infty}\frac{\sin(2z)}{\pi z}e^{-2i(zx+z^2t)}\,dz:=\lim_{L\to\infty}\int_{-L}^{L}\frac{\sin(2z)}{\pi z}e^{-2i(zx+z^2t)}\,dz\,.$$

Method of Stationary Phase

Long-time asymptotics: method of stationary phase (Stokes and Kelvin). Set $x = vt$. Phase function $I(z) := -2(zv + z^2)$. Points z_0 of stationary phase satisfy $I'(z_0) = 0$, so only one stationary phase point: $z_0 = -v/2$. Note $I(z_0) = v^2/2$ and $I''(z_0) = -4$, so the stationary phase formula is:

$$\psi(vt, t) = t^{-1/2}\sqrt{\frac{2}{\pi}}\frac{\sin(v)}{v}e^{itv^2/2-i\pi/4} + O(t^{-1}) \,, \quad \text{as } t \to +\infty.$$

A plot of the leading term of $t|\psi(vt,t)|^2$. Note that the amplitude of the long time limit is essentially a rescaled version of the Fourier transform of the initial data. This is the classical far-field diffraction phenomenon.



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   (b) Approximation of the integrand $e^{\pm is^2}g(s)$ by an analytic function $e^{\pm is^2}g(0)$. Error controlled by integration by parts.
   (c) Rotation of contour by $\pm\pi/4$ radians to turn rapid oscillation into exponential decay (steepest descent).

Nonlinear Analysis

Corresponding limit for the nonlinear problem was considered by Ablowitz and Segur, and also by Zakharov and Manakov. Formal expansion:

$$\psi(vt,t) \sim t^{-1/2}\left(\alpha(v) + \sum_{n=1}^{\infty}\sum_{k=0}^{2n}\frac{(\log(t))^k}{t^n}\alpha_{nk}(v)\right)e^{itv^2/2-i\nu(v)\log(t)}, \quad t\to\infty\,.$$

All coefficients determined in terms of α by direct substitution. Goal: rigorously determine $\alpha(v)$ in terms of initial data. Key observations:

- In (Segur & Ablowitz, 76), $|\alpha(v)|$ is related to the initial data via trace formulae.
- In (Zakharov & Manakov, 76), the leading term is used to motivate a WKB analysis of the scattering problem, and the phase $\arg(\alpha(v))$ is determined.
- In (Its, 81), a key role in the analysis is identified for an "isomonodromy" problem for parabolic cylinder functions.
- In (Deift & Zhou, 93) a universal scheme for obtaining these results is obtained as a "steepest descent method" for Riemann-Hilbert problems.

Nonlinear Analysis

The solution of the defocusing cubic Schrödinger equation is based on inverse-scattering for the self-adjoint Zakharov-Shabat operator:

1. Viewing the initial data as a "potential", obtain from the ZS problem the corresponding "reflection coefficient" $r(z)$, $z \in \mathbb{R}$. For our initial data:

$$r(z) := ie^{-2iz} \frac{e^{2i\sqrt{z^2-1}} - e^{-2i\sqrt{z^2-1}}}{(z - \sqrt{z^2-1})e^{2i\sqrt{z^2-1}} - (z + \sqrt{z^2-1})e^{-2i\sqrt{z^2-1}}}.$$

   From $r(z)$, form the "jump matrix" $\mathbf{v}(z; x, t)$, $z \in \mathbb{R}$:

$$\mathbf{v}(z; x, t) = \begin{pmatrix} 1 - |r(z)|^2 & -r(z)^* e^{-2i(xz+tz^2)} \\ r(z)e^{2i(xz+tz^2)} & 1 \end{pmatrix}.$$

2. Solve a Riemann-Hilbert problem.

Riemann-Hilbert Problem

Seek $\mathbf{M}(z;x,t)$ satisfying:

1. $\mathbf{M}(z;x,t)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$.
2. $\mathbf{M}(z;x,t) \longrightarrow \mathbb{I}$ as $z \longrightarrow \infty$.
3. For $z \in \mathbb{R}$, $\mathbf{M}_+(z;x,t) = \mathbf{M}_-(z;x,t)\mathbf{v}(z;x,t)$.

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Then

$$\psi(x,t) = 2i \lim_{z\longrightarrow\infty} zM_{12}(z;x,t)\,.$$

Deift-Zhou Steepest Descent Technique

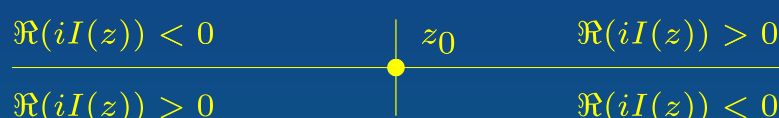
Set $x = vt$. Factorize jump matrix $\mathbf{v}(z;x,t)$ two ways:

$$\mathbf{v}(z;vt,t) = \begin{pmatrix} 1 & -r(z)^* e^{itI(z)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r(z)e^{-itI(z)} & 1 \end{pmatrix},$$

and

$$\mathbf{v}(z;vt,t) = \begin{pmatrix} 1 & 0 \\ \dfrac{r(z)e^{-itI(z)}}{1-|r(z)|^2} & 1 \end{pmatrix} \begin{pmatrix} 1-|r(z)|^2 & 0 \\ 0 & \dfrac{1}{1-|r(z)|^2} \end{pmatrix} \begin{pmatrix} 1 & -\dfrac{r(z)^* e^{itI(z)}}{1-|r(z)|^2} \\ 0 & 1 \end{pmatrix}.$$

Here $I(z) = -2(zv + z^2)$ (same phase function as in the linear theory).



Noncommutative Steepest Descent

Explicitly transform $\mathbf{M}(z;vt,t) \longrightarrow \mathbf{N}(z;vt,t)$:

$$\mathbf{N} := \mathbf{M}\begin{pmatrix} 1 & \dfrac{r(z^*)^* e^{itI(z)}}{1-r(z)r(z^*)^*} \\ 0 & 1 \end{pmatrix}$$

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$$\begin{pmatrix} 1-|r(z)|^2 & -r(z)^* e^{itI(z)} \\ r(z)e^{-itI(z)} & 1 \end{pmatrix}$$

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Noncommutative Steepest Descent

Jump conditions satisfied by $\mathbf{N}(z; vt, t)$:

$$\begin{pmatrix} 1 & -\dfrac{r(z^*)^* e^{itI(z)}}{1 - r(z)r(z^*)^*} \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ r(z)e^{-itI(z)} & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 - |r(z)|^2 & 0 \\ 0 & \dfrac{1}{1 - |r(z)|^2} \end{pmatrix}$$

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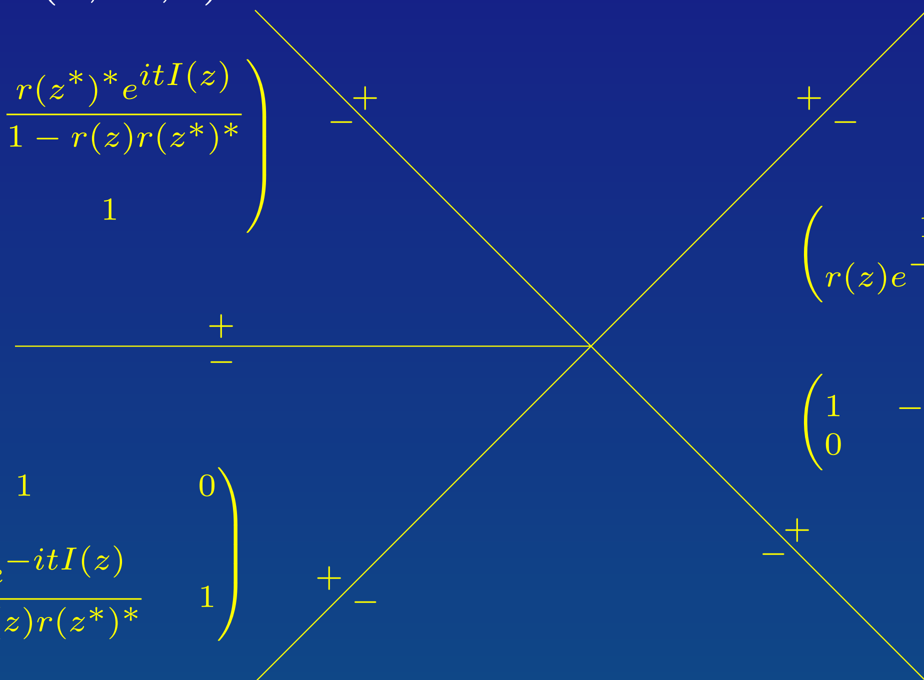
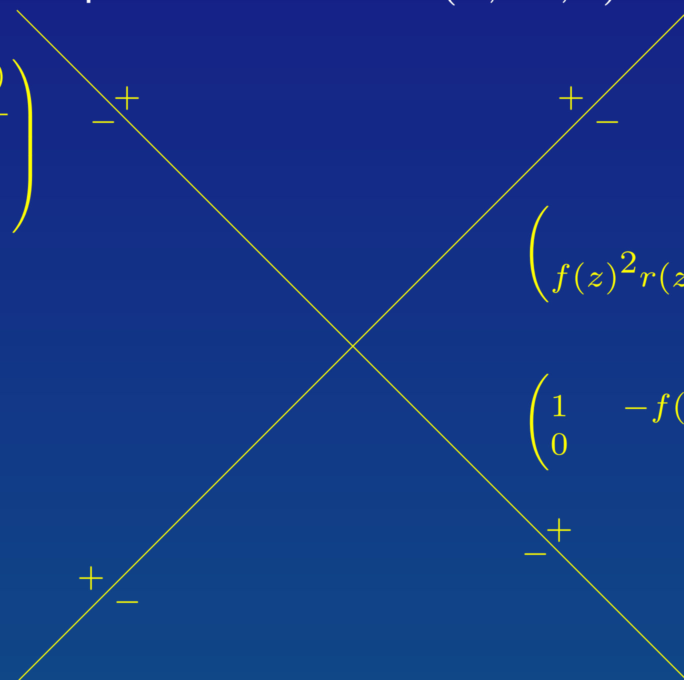


Noncommutative Steepest Descent

Set $\mathbf{O}(z;vt,t) = \mathbf{N}(z;vt,t)f(z)^{-\sigma_3}$. Jump conditions for $\mathbf{O}(z;vt,t)$:

$$\begin{pmatrix} 1 & -\dfrac{f(z)^{-2}r(z^*)^*e^{itI(z)}}{1-r(z)r(z^*)^*} \\ 0 & 1 \end{pmatrix}$$



$$\begin{pmatrix} 1 & 0 \\ f(z)^2 r(z)e^{-itI(z)} & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -f(z)^{-2}r(z^*)^*e^{itI(z)} \\ 0 & 1 \end{pmatrix}$$

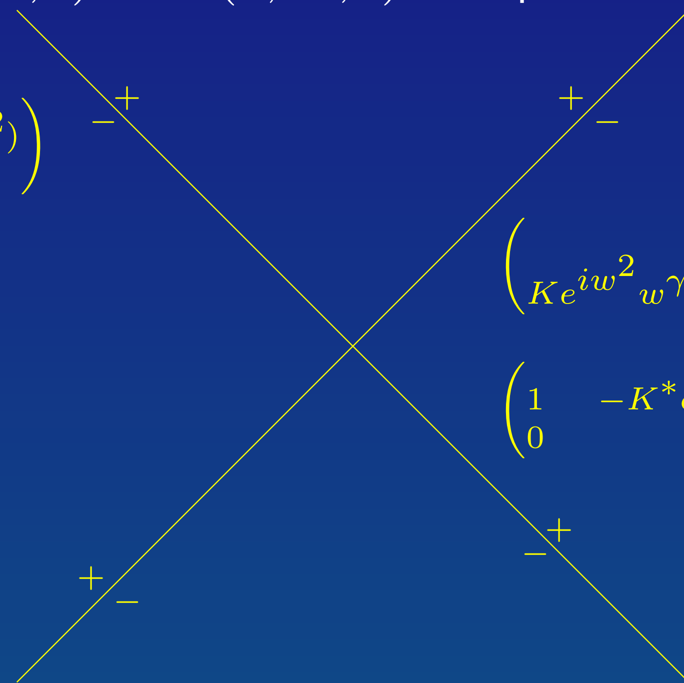
$$\begin{pmatrix} 1 & 0 \\ \dfrac{f(z)^2 r(z)e^{-itI(z)}}{1-r(z)r(z^*)^*} & 1 \end{pmatrix}$$

Note: $f(z) := \exp\left(\dfrac{1}{2\pi i}\displaystyle\int_{-\infty}^{z_0}\dfrac{\log(1-|r(s)|^2)}{s-z}\,ds\right).$

Noncommutative Steeepest Descent

Rescale: $w^2 = 2t(z - z_0)^2$. Set $\mathbf{P}(w; vt, t) := \mathbf{O}(z; vt, t)$. Jumps for $\mathbf{P}$:

$$\begin{pmatrix} 1 & -K^* e^{-i\pi\gamma} e^{-iw^2} w^{-\gamma} + O(t^{-1/2}) \\ 0 & 1 \end{pmatrix}$$



$$\begin{pmatrix} 1 & 0 \\ K e^{iw^2} w^{\gamma} + O(t^{-1/2}) & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -K^* e^{-iw^2} w^{-\gamma} + O(t^{-1/2}) \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ K e^{-i\pi\gamma} e^{iw^2} w^{\gamma} + O(t^{-1/2}) & 1 \end{pmatrix}$$

Note: $f(z)^2 = f_0^2 (z - z_0)^\gamma (1 + O(z - z_0))$, $K = f_0^2 r(z_0) e^{itv^2/2} (2t)^{-\gamma/2}$.

Model Problem

Propose a model Riemann-Hilbert problem for $\tilde{\mathbf{P}}(w; vt, t)$ with jumps:

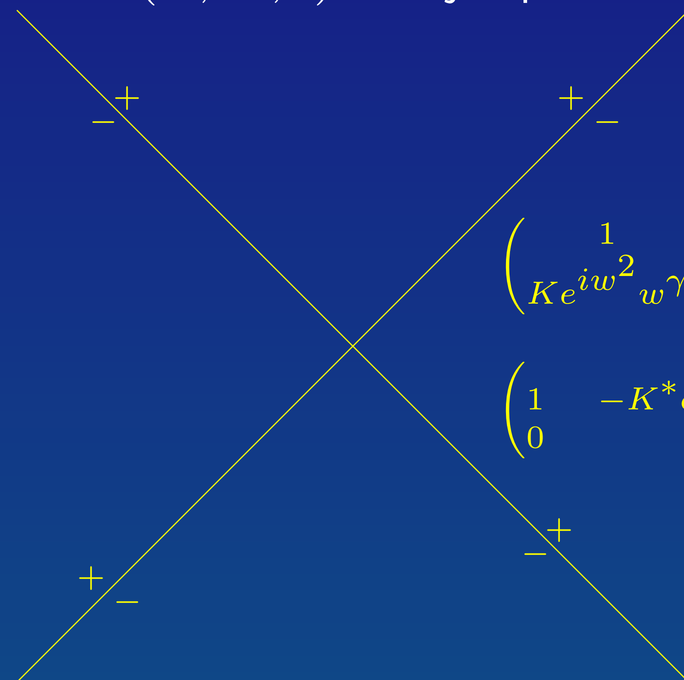
$$\begin{pmatrix} 1 & -K^* e^{-i\pi\gamma} e^{-iw^2} w^{-\gamma} \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ K e^{iw^2} w^{\gamma} & 1 \end{pmatrix}$$

$$\tilde{\mathbf{P}}(w; vt, t) \to \mathbb{I}$$

$$\begin{pmatrix} 1 & -K^* e^{-iw^2} w^{-\gamma} \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ K e^{-i\pi\gamma} e^{iw^2} w^{\gamma} & 1 \end{pmatrix}$$



Solved explicitly in terms of parabolic cylinder functions. Isomonodromy approach.

Error Analysis

Consider the error in using this model: $\mathbf{E}(w; vt, t) := \mathbf{P}(w; vt, t)\tilde{\mathbf{P}}(w; vt, t)^{-1}$. While not known explicitly, the error satisfies a Riemann-Hilbert problem:

1. $\mathbf{E}(w; vt, t)$ is analytic for $w \in \mathbb{C} \setminus X$ (X is the contour).
2. $\mathbf{E}(w; vt, t) \to \mathbb{I}$ as $w \to \infty$.
3. For $w \in \mathbb{X}$, $\mathbf{E}_+(w; vt, t) = \mathbf{E}_-(w; vt, t)\mathbf{v}_{\mathbf{E}}(w; vt, t)$.

Note that $\mathbf{v}_{\mathbf{E}} = \tilde{\mathbf{P}}_-\mathbf{v}_{\mathbf{P}}\mathbf{v}_{\tilde{\mathbf{P}}}^{-1}\tilde{\mathbf{P}}_-^{-1} = \mathbb{I} + O(t^{-1/2})$. Key fact:

$$\mathbf{v}_{\mathbf{E}} = \mathbb{I} + O(t^{-1/2}) \quad \text{implies} \quad \mathbf{E}(w; vt, t) = \mathbb{I} + O(t^{-1/2}w^{-1}) = \mathbb{I} + O(t^{-1}z^{-1})\,.$$

Error Analysis

For z along the imaginary axis (for example),

$$\begin{aligned}
\mathbf{M}(z;vt,t) &= \mathbf{N}(z;vt,t) \\
&= \mathbf{O}(z;vt,t)f(z)^{\sigma_3} \\
&= \mathbf{P}(w;vt,t)f(z)^{\sigma_3} \\
&= \mathbf{E}(w;vt,t)\tilde{\mathbf{P}}(w;vt,t)f(z)^{\sigma_3}\,.
\end{aligned}$$

With (using classical asymptotics for parabolic cylinder functions)

$$\tilde{P}_{12}(w;vt,t) = w^{-1}\tilde{p}(vt,t) + O(w^{-2}) = (2t)^{-1/2}z^{-1}\tilde{p}(vt,t) + O(t^{-1}z^{-2})\,,$$

we arrive at $\psi(vt,t) = 2i(2t)^{-1/2}\tilde{p}(vt,t) + O(t^{-1})$.

Back to outline.

Short-time Asymptotics for Dispersive Waves with Rough Data

Consider once again the nonlinear problem

$$i\frac{\partial\psi}{\partial t} + \frac{1}{2}\frac{\partial^2\psi}{\partial x^2} - |\psi|^2\psi = 0\,, \quad \text{(Defocusing Cubic Schrödinger)}$$

subject to the initial condition

$$\psi(x,0) = \begin{cases} 1\,, & |x| \leq 1\,, \\ 0\,, & |x| > 1\,. \end{cases}$$

We are now interested in the asymptotic behavior of $\psi(x,t)$ as $t \downarrow 0$.

Corresponding Linear Problem

Again we study first the linear problem

$$i\frac{\partial\psi}{\partial t}+\frac{1}{2}\frac{\partial^2\psi}{\partial x^2}=0\,,\quad \text{(Linear Schrödinger)}$$

subject to the same piecewise-constant initial condition. We are interested in the asymptotic behavior of $\psi(x,t)$ as $t\downarrow 0$. Recall the explicit solution

$$\psi(x,t)=\int_{-\infty}^{\infty}\frac{\sin(2z)}{\pi z}e^{-2i(zx+z^2t)}\,dz:=\lim_{L\to\infty}\int_{-L}^{L}\frac{\sin(2z)}{\pi z}e^{-2i(zx+z^2t)}\,dz\,.$$

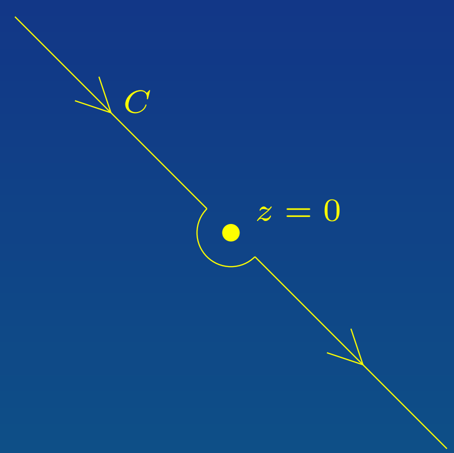
Asymptotic Analysis: Gibbs Phenomenon

The limit $t \downarrow 0$ cannot be uniform near $x = \pm 1$ because $\psi(x,t)$ is smooth for $t > 0$. Choose a contour $C : \infty e^{3\pi i/4} \longrightarrow \infty e^{-i\pi/4}$ (for $t > 0$) with $z = 0$ to the left of C. Then, with

$$x = 1 + t^{1/2}s\,,$$

we split up the integrand and get

$$\psi(1 + t^{1/2}s, t) = \int_C e^{-2i(zt^{1/2}s+z^2t)}\frac{dz}{2\pi iz} - \int_C e^{-2i(z(2+t^{1/2}s)+z^2t)}\frac{dz}{2\pi iz}\,.$$



Error Bound

We want to neglect the second term: using the change of parameter

$$z = ty - z_* \,, \quad z_* := \frac{1}{t} + \frac{s}{2t^{1/2}} \,,$$

we have

$$\int_C e^{-2i(z(2+t^{1/2}s)+z^2 t)} \frac{dz}{2\pi i z} \quad = \quad e^{2itz_*^2} \int_{y(C)} \frac{e^{-2iy^2/t}\, dy}{y - 1 - t^{1/2}s/2} \,,$$

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$$\begin{aligned} \int_C e^{-2i(z(2+t^{1/2}s)+z^2t)} \frac{dz}{2\pi i z} &= e^{2itz_*^2} \int_{y(C)} \frac{e^{-2iy^2/t}\, dy}{y - 1 - t^{1/2}s/2} \,, \\ &= e^{2itz_*^2} \int_{y(C)} O(1) e^{-2iy^2/t}\, dy \,, \end{aligned}$$

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(steepest descent over $y = 0$) as $t \downarrow 0$.

Leading Term

Thus, we have

$$\psi(1+t^{1/2}s,t) = \int_C e^{-2i(zt^{1/2}s+z^2t)}\frac{dz}{2\pi iz} + O(t^{1/2})\,,$$

Rescaling the integration parameter by $z = t^{-1/2}w$ we arrive at

$$\psi(1+t^{1/2}s,t) = F(s) + O(t^{1/2})\,,\quad F(s) := \int_C e^{-2i(sw+w^2)}\frac{dw}{2\pi iw}\,.$$

Steepest descent: $F(s) \longrightarrow 0$ (exponentially small) as $s \longrightarrow \infty e^{i\pi/4}$. Also,

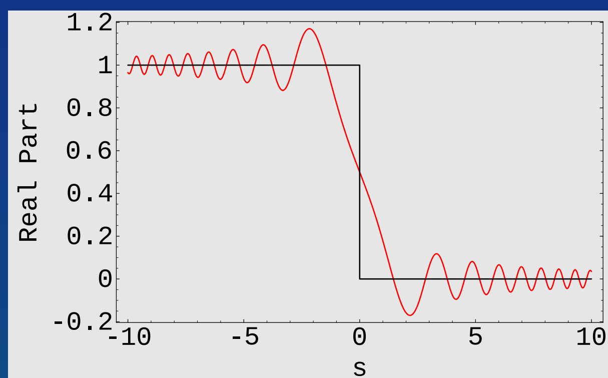
$$F'(s) = -\frac{1}{\pi}\int_C e^{-2i(sw+w^2)}\,dw = -\frac{e^{is^2/2}}{\pi}\int_C e^{-2i(w+s/2)^2}\,dw = -\frac{e^{is^2/2-i\pi/4}}{\sqrt{2\pi}}\,.$$

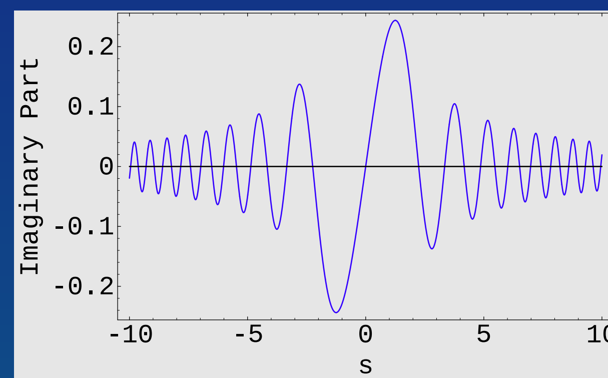
Leading Term

Consequently,

$$F(s) = \frac{e^{-i\pi/4}}{\sqrt{2\pi}} \int_s^{\infty e^{i\pi/4}} e^{iz^2/2}\, dz = \frac{1}{2}\mathrm{Erfc}\left(\frac{se^{-i\pi/4}}{\sqrt{2}}\right) ,$$

where $\mathrm{Erfc}(x) := \dfrac{2}{\sqrt{\pi}} \displaystyle\int_x^{\infty} e^{-\tau^2}\, d\tau$. Pictures of real and imaginary parts:





A kind of Gibbs phenomenon.

Nonlinear Analysis

Recall the solution technique: seek $\mathbf{M}(z; x, t)$ satisfying:

1. $\mathbf{M}(z; x, t)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$.
2. $\mathbf{M}(z; x, t) \longrightarrow \mathbb{I}$ as $z \longrightarrow \infty$.
3. For $z \in \mathbb{R}$, $\mathbf{M}_+(z; x, t) = \mathbf{M}_-(z; x, t)\mathbf{v}(z; x, t)$.

$\mathbf{M}_+(z; x, t)$

$\mathbf{M}_-(z; x, t)$

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$$\mathbf{M}_+(z;x,t)$$
$$\mathbf{M}_-(z;x,t)$$

Then

$$\psi(x,t) = 2i \lim_{z \longrightarrow \infty} z M_{12}(z;x,t)\,.$$

Nonlinear Analysis

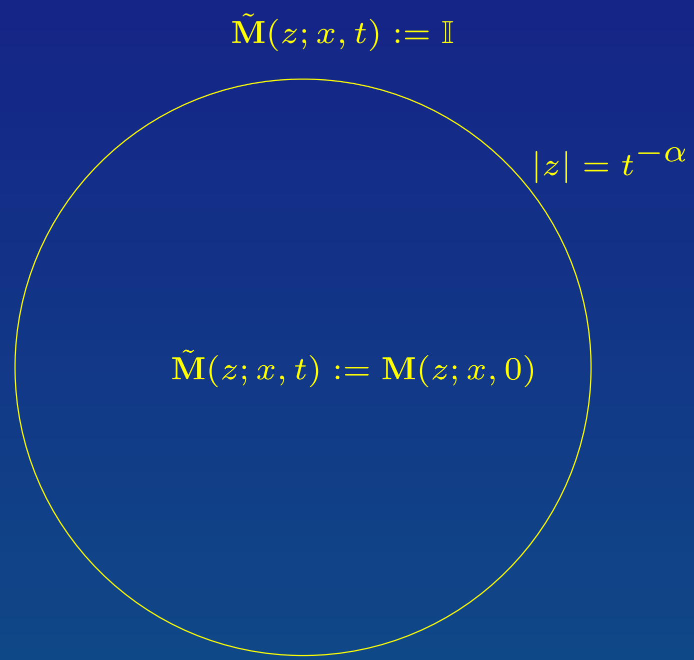
For the nonlinear analysis (DiFranco & McLaughlin, 05), the key observations are:

1. For t small, $\mathbf{v}(z; x, t)$ is close to $\mathbf{v}(z; x, 0)$ uniformly for z in bounded subsets of $\mathbb{R}$.
2. The Riemann-Hilbert problem with jump $\mathbf{v}(z; x, 0)$ has a known solution $\mathbf{M}(z; x, 0)$ (Jost functions for the initial data).
3. While approximation by $\mathbf{v}(z; x, 0)$ is good for bounded z, as z becomes large, $\mathbf{v}(z; x, 0) \longrightarrow \mathbb{I}$.

This suggests the construction of a model for $\mathbf{M}(z; x, t)$.

Model and Error

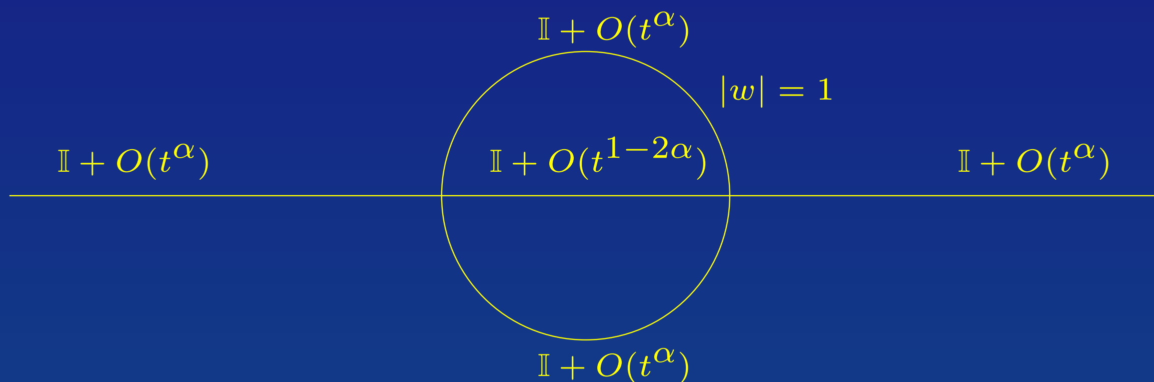
For some $\alpha > 0$ the model $\tilde{\mathbf{M}}(z; x, t)$ for $\mathbf{M}(z; x, t)$ is defined by the diagram:



The error is $\mathbf{E}(z; x, t) := \mathbf{M}(z; x, t)\tilde{\mathbf{M}}(z; x, t)^{-1}$.

Error Analysis

Set $x = 1 + t^{1/2}s$ and $w = t^{\alpha}z$. With $\mathbf{F}(w; 1 + t^{1/2}s, t) := \mathbf{E}(z; 1 + t^{1/2}s, t)$, jump conditions for $\mathbf{F}$ are (uniform for s bounded):



Optimal scaling $\alpha = 1/3$. Solve in a Hölder space by a Neumann series. Result:

$$\psi(1 + t^{1/2}s, t) = \frac{1}{2}\mathrm{Erfc}\left(\frac{se^{-i\pi/4}}{\sqrt{2}}\right) + O(t^{1/2}) \,. \quad \text{(Compare with linear case.)}$$

Back to outline.

Semiclassical Asymptotics for Modulationally Unstable Dispersive Waves

Consider now

$$i\hbar\frac{\partial\psi}{\partial t}+\frac{\hbar^2}{2}\frac{\partial^2\psi}{\partial x^2}+|\psi|^2\psi=0\,,\quad \text{(Focusing Cubic Schrödinger)}$$

with initial condition $\psi(x,0)=A(x)e^{iS(x)/\hbar}$. This is equivalent to the first-order system

$$\begin{aligned}\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u) &= 0\,,\\ \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-\frac{\partial\rho}{\partial x} &= \frac{\hbar^2}{2}\frac{\partial}{\partial x}\left[\frac{1}{2\rho}\frac{\partial^2\rho}{\partial x^2}-\left(\frac{1}{2\rho}\frac{\partial\rho}{\partial x}\right)^2\right]\,,\end{aligned}$$

for $\rho(x,t):=|\psi(x,t)|^2$ and $u(x,t):=\hbar\Im(\log(\psi))_x$, subject to the initial conditions $\rho(x,0)=A(x)^2$, $u(x,0)=S'(x)$.

This is a problem with a strong (for small $\hbar$) modulational instability.

Linearized Problem

Linearize about a constant state. For $\rho^{(0)} > 0$ and $u^{(0)}$ constants, set

$$\rho = \rho^{(0)} \cdot (1 + A)\,, \quad u = u^{(0)} + B\,,$$

and drop all nonlinear terms in A and B to obtain

$$\begin{aligned}
\frac{\partial A}{\partial t} + u^{(0)}\frac{\partial A}{\partial x} + \frac{\partial B}{\partial x} &= 0\,, \\
\frac{\partial B}{\partial t} + u^{(0)}\frac{\partial B}{\partial x} - \rho^{(0)}\frac{\partial A}{\partial x} &= \hbar^2\frac{\partial^3 A}{\partial x^3}\,.
\end{aligned}$$

Linearized Problem

Go to a moving frame: $y = (x - u^{(0)}t)/\sqrt{\rho^{(0)}}$:

$$\begin{aligned} \frac{\partial A}{\partial t} + \frac{1}{\sqrt{\rho^{(0)}}}\frac{\partial B}{\partial y} &= 0\,, \\ \frac{\partial B}{\partial t} - \sqrt{\rho^{(0)}}\frac{\partial A}{\partial y} &= \frac{\hbar^2}{\rho^{(0)}\sqrt{\rho^{(0)}}}\frac{\partial^3 A}{\partial y^3}\,. \end{aligned}$$

Cross differentiate, set $\epsilon = \hbar/\rho^{(0)}$:

$$\begin{aligned} \frac{\partial^2 A}{\partial t^2} + \frac{\partial^2 A}{\partial y^2} + \epsilon^2\frac{\partial^4 A}{\partial y^4} &= 0\,, \\ \frac{\partial^2 B}{\partial t^2} + \frac{\partial^2 B}{\partial y^2} + \epsilon^2\frac{\partial^4 B}{\partial y^4} &= 0\,. \end{aligned}$$

Classical Steepest Descent Analysis for Linearized Problem

Study $u^\epsilon_{tt} + u^\epsilon_{xx} + \epsilon^2 u^\epsilon_{xxxx} = 0$ with $u^\epsilon(x,0) = f(x)$ and $u^\epsilon_t(x,0) = 0$:

$$\begin{aligned} u^\epsilon(x,t) &= \int_{\epsilon^2k^2\le 1} \hat{f}(k)\cosh(kt\sqrt{1-\epsilon^2k^2})e^{ikx}\,dk \\ &\quad + \int_{\epsilon^2k^2>1} \hat{f}(k)\cos(kt\sqrt{\epsilon^2k^2-1})e^{ikx}\,dk\,. \end{aligned}$$

Assume $\hat{f} \in L^1(\mathbb{R})$. Then by dominated convergence, for each real x and t fixed,

$$\lim_{\epsilon\to 0}\int_{\epsilon^2k^2>1} \hat{f}(k)\cos(kt\sqrt{\epsilon^2k^2-1})e^{ikx}\,dk = 0\,.$$

Therefore, the interesting asymptotics of $u^\epsilon(x,t)$ are carried by

$$U^\epsilon(x,t) := \int_{\epsilon^2k^2\le 1} \hat{f}(k)\cosh(kt\sqrt{1-\epsilon^2k^2})e^{ikx}\,dk\,.$$

Classical Steepest Descent Analysis for Linearized Problem

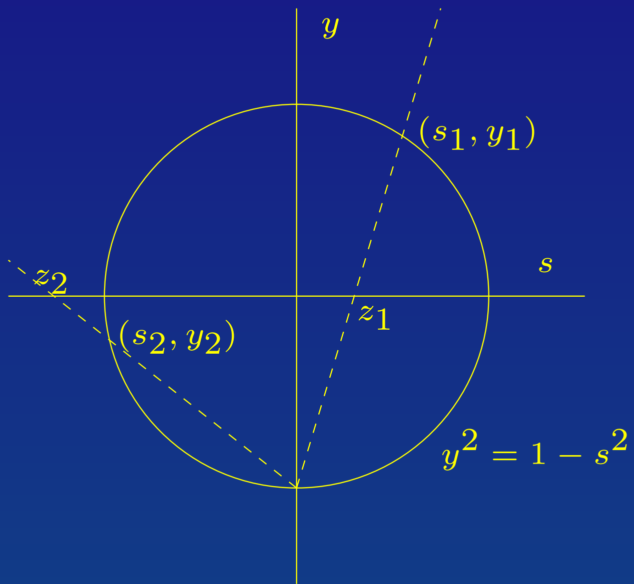
Change the variable of integration by $k = s/\epsilon$. Thus

$$U^\epsilon(x,t) = \frac{1}{2}\int_{-1}^{1} e^{th_+(s;c)/\epsilon} F(s;\epsilon)\, ds + \frac{1}{2}\int_{-1}^{1} e^{th_-(s;c)/\epsilon} F(s;\epsilon)\, ds\,,$$

where

$$c = \frac{x}{t}\,, \quad h_\pm(s;c) := ics \pm s\sqrt{1-s^2}\,, \quad F(s;\epsilon) := \frac{1}{\epsilon}\hat{f}\left(\frac{s}{\epsilon}\right)\,.$$

Classical Steepest Descent Analysis for Linearized Problem



Remove square roots by stereographic projection:

$$z = \frac{s}{1+y}\,, \quad s^2 + y^2 = 1\,,$$

$$s = \frac{2z}{1+z^2}\,, \quad y = \frac{1-z^2}{1+z^2}\,.$$

Therefore, $U^\epsilon(x,t) = \frac{1}{2}\int_{-1}^{1} e^{tH_+(z;c)/\epsilon} G(z;\epsilon)\,dz + \frac{1}{2}\int_{-1}^{1} e^{tH_-(z;c)/\epsilon} G(z;\epsilon)\,dz,$

where

$$H_\pm(z;c) := \frac{2(ic \pm 1)z + 2(ic \mp 1)z^3}{(1+z^2)^2}\,, \quad G(z;\epsilon) := \frac{2}{\epsilon}\cdot\frac{1-z^2}{(1+z^2)^2}\hat{f}\left(\frac{2}{\epsilon}\cdot\frac{z}{1+z^2}\right)\,.$$

Classical Steepest Descent Analysis for Linearized Problem

Assume $f(x) = e^{-|x|}$ and thus $\hat{f}(k) = \frac{1}{2\pi i}\left[\frac{1}{k-i} - \frac{1}{k+i}\right]$. Then,

$$G(z;\epsilon) = \frac{1}{\pi i} \cdot \frac{1-z^2}{1+z^2}\left[\frac{1}{2z - i\epsilon(1+z^2)} - \frac{1}{2z + i\epsilon(1+z^2)}\right] .$$

Note the symmetries:

$$G(z^*;\epsilon) = G(z;\epsilon)^* , \quad G(-z;\epsilon) = G(z;\epsilon) ,$$

$$H_\pm(z^*;-c) = H_\pm(z;c)^* , \quad H_\pm(-z;c) = -H_\pm(z;c) , \quad H_\pm(z;-c) = -H_\mp(z;c) .$$

From these it follows that

$$U^\epsilon(-ct,t) = U^\epsilon(ct,t) , \quad U^\epsilon(ct,t) = \Re\left(\int_{-1}^{1} e^{tH_+(z;c)/\epsilon} G(z;\epsilon)\, dz\right) .$$

Thus, it suffices to analyze the exponent $H_+(z;c)$ for $c \geq 0$.

Classical Steepest Descent Analysis for Linearized Problem

Poles of $F(s;\epsilon)$ at $s=\pm i\epsilon$: $\displaystyle\operatorname*{Res}_{s=\pm i\epsilon} F(s;\epsilon)\,ds = \pm\frac{1}{2\pi i}$.

This implies four poles in the z-plane, two near $z=\infty$ and two near $z=0$: $z=\pm i\epsilon/2+O(\epsilon^3)$. By conformal invariance of residues,

$$\operatorname*{Res}_{z=\pm i\epsilon/2+O(\epsilon^3)} G(z;\epsilon)\,dz = \operatorname*{Res}_{s=\pm i\epsilon} F(s;\epsilon)\,ds = \pm\frac{1}{2\pi i}\,.$$

Furthermore, if $\delta<|z|<M$ is any fixed annulus, the asymptotic behavior

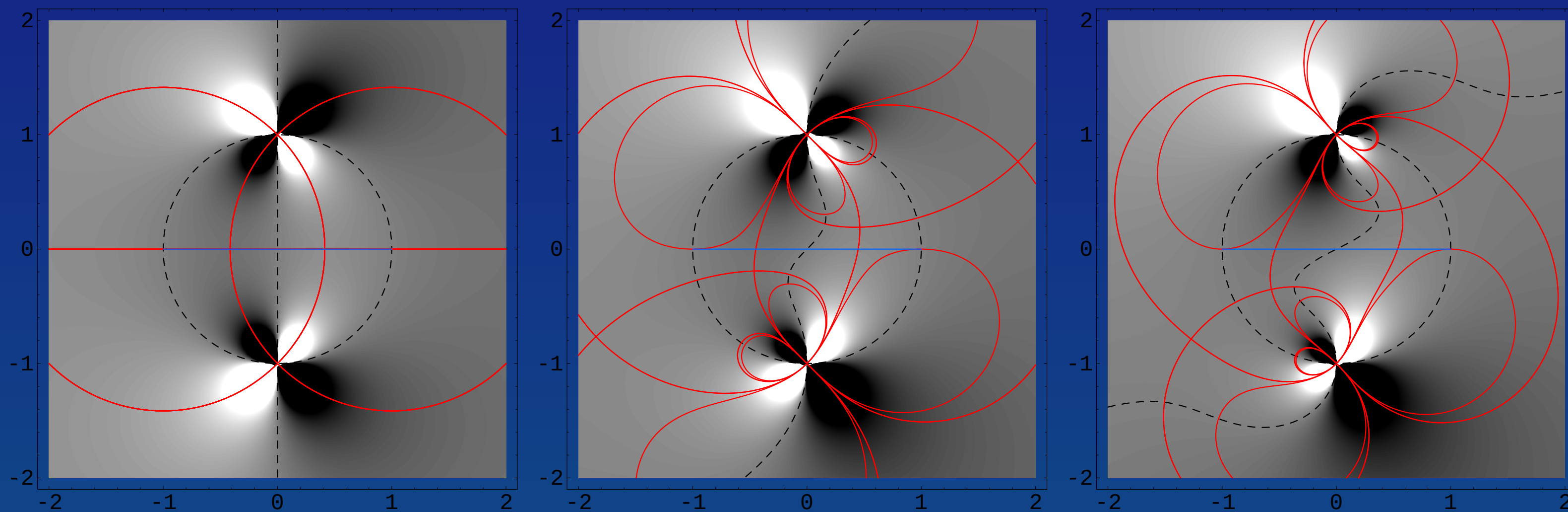
$$G(z;\epsilon)=\epsilon\frac{1-z^2}{2\pi z^2}+O(\epsilon^3)$$

holds uniformly as $\epsilon\to 0$. Deformation of the contour from $z=0$ thus will produce residues, but we may then use the asymptotic formula for $G(z;\epsilon)$ in the integrand. Note also:

$$H_+(\pm i\epsilon/2+O(\epsilon^3)) = \mp\epsilon(c\mp i)+O(\epsilon^3)\,.$$

Steepest Descent Contours

Determine the best location for the contour of integration. Landscape plots for $\Re(H_+(z;c))$:



$c = 0$ $\qquad$ $c = 1$ $\qquad$ $c = 2$

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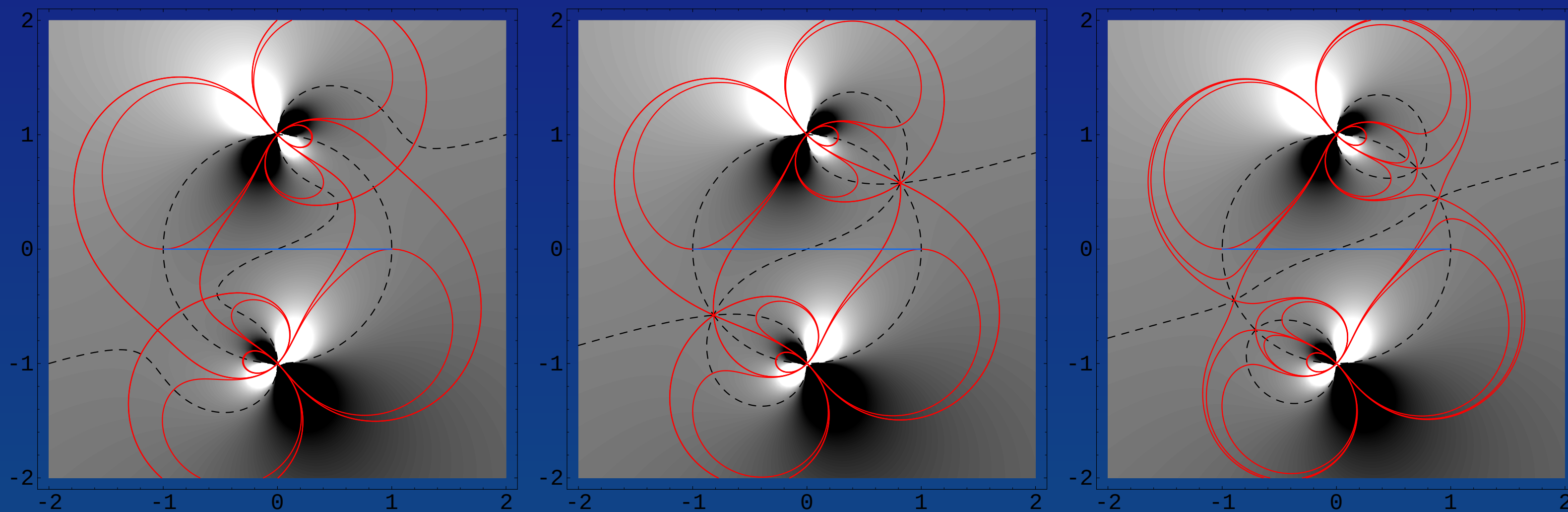

$c = 5/2$ $\qquad$ $c = 2\sqrt{2}$ $\qquad$ $c = 3$

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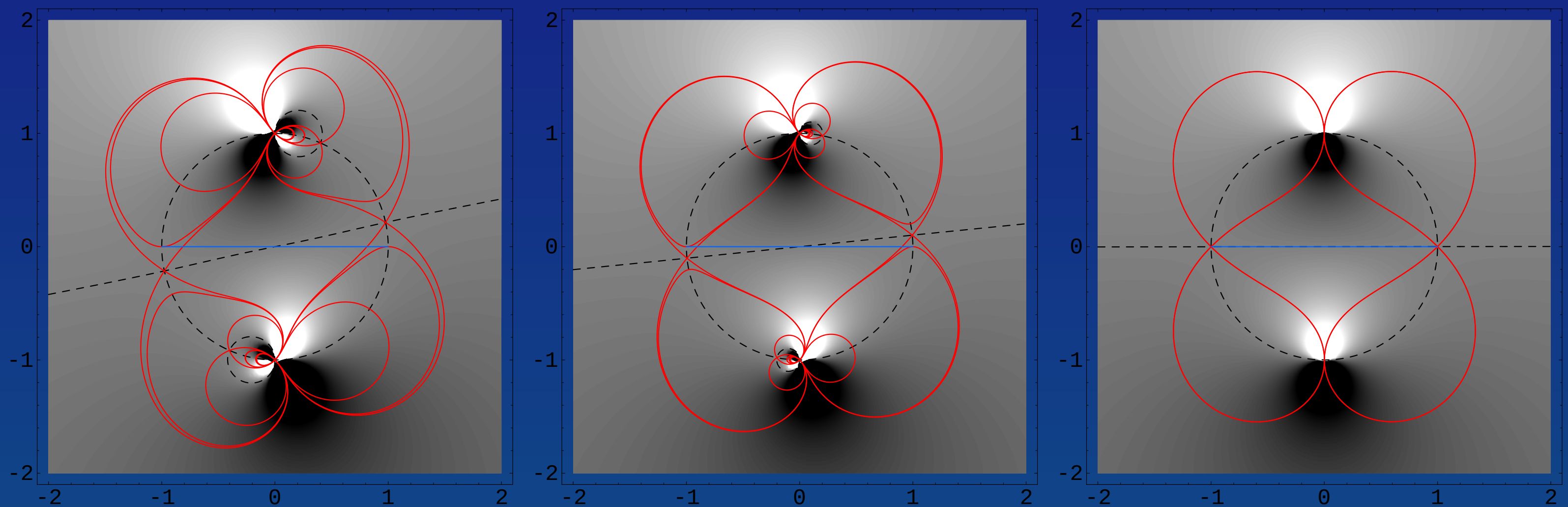


$c = 5$ $c = 10$ $c = 1000$

Results of Linear Analysis

Steepest descent analysis of the Fourier integral representation for $U^\epsilon(ct,t)$ shows that:

- If $t > 0$ and $0 \le c < 2\sqrt{2}$, then a saddle point z_c dominates (angle of passage is θ_c) and the solution is rapidly oscillatory and exponentially large:

$$U^\epsilon(ct,t) = \epsilon^{3/2}\left|\frac{1-z_c^2}{z_c^2}\right|\frac{e^{t\Re(H_+(z_c;c))/\epsilon}}{\sqrt{2\pi t|H_+''(z_c;c)|}}$$

$$\times \left[\cos\left(\frac{t}{\epsilon}\Im(H_+(z_c;c)) + \theta_c + \arg\left(\frac{1-z_c^2}{z_c^2}\right)\right) + O(\epsilon)\right].$$

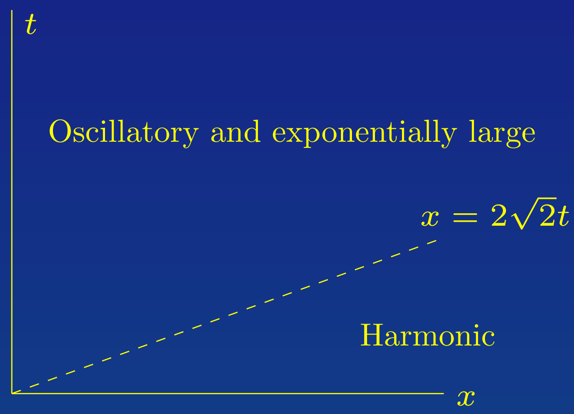
- If $t > 0$ and $c > 2\sqrt{2}$, then a residue dominates, and

$$U^\epsilon(ct,t) = e^{-ct}\cos(t) + O(\epsilon) = e^{-x}\cos(t) + O(\epsilon).$$

Note that the function $e^{-x}\cos(t)$ is an exact solution of Laplace's equation in x and t.

Results of Linear Analysis

The positive quadrant of the (x, t)-plane:



Note that:

- The harmonic limit is a strong limit.
- The exponentially large and oscillatory behavior precludes even a weak limit.

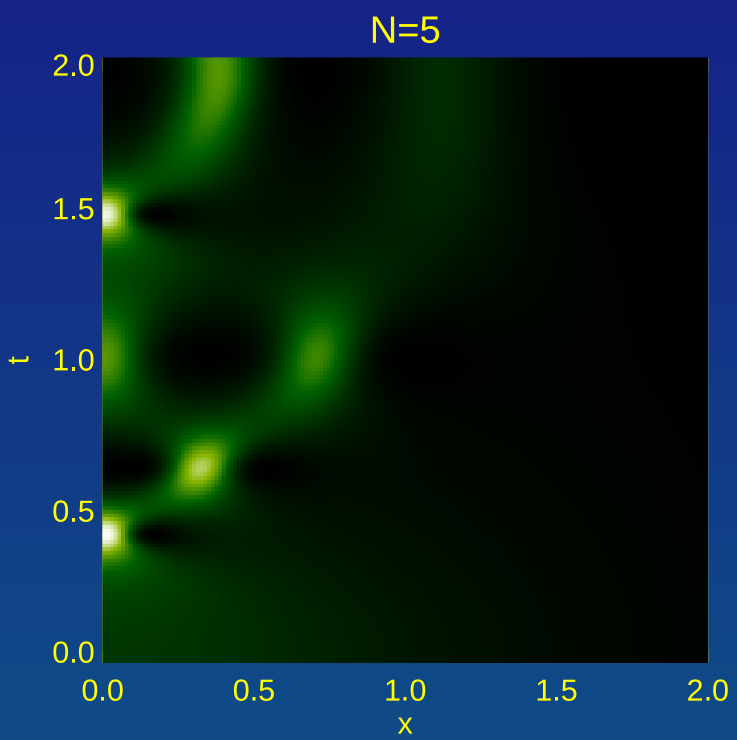
Nonlinear Problem

Consider the nonlinear problem with initial data $\psi(x,0) = 2\,\mathrm{sech}(x)$.

Equivalent to the Satsuma-Yajima "N-soliton" with $N = 2/\hbar$.

Images from (Miller & Kamvissis, 98) and (Lyng & Miller, 07).

Weak limits evidently exist in oscillatory regions (bounded amplitude).



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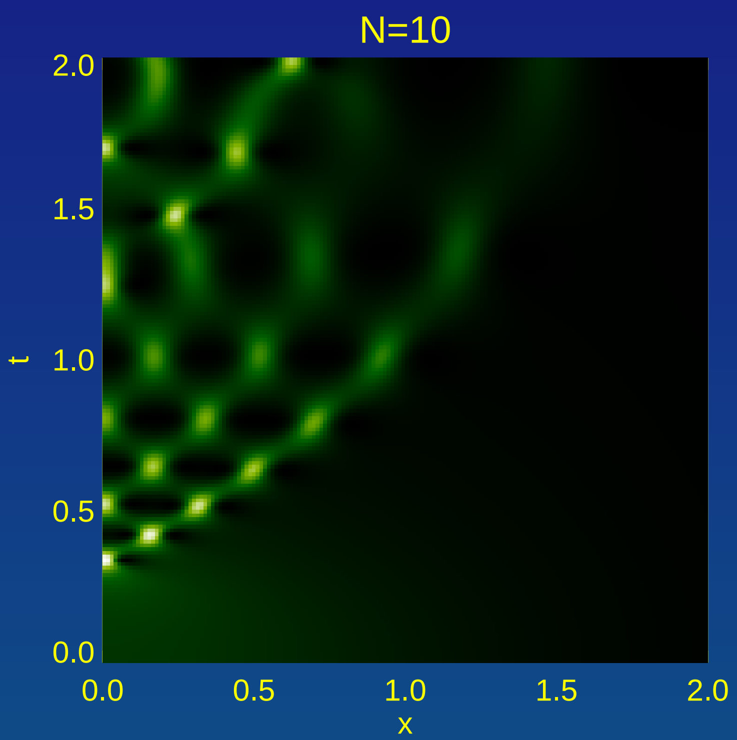

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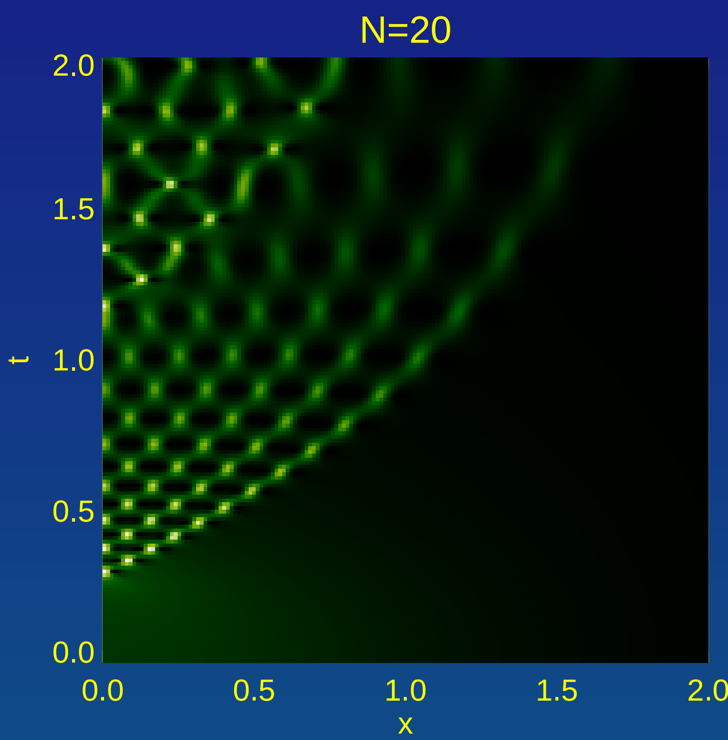

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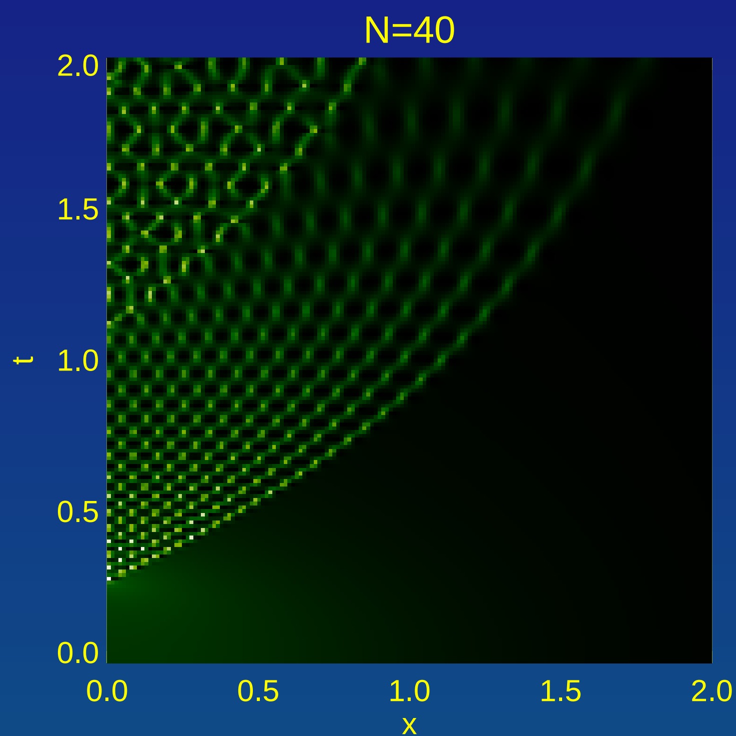
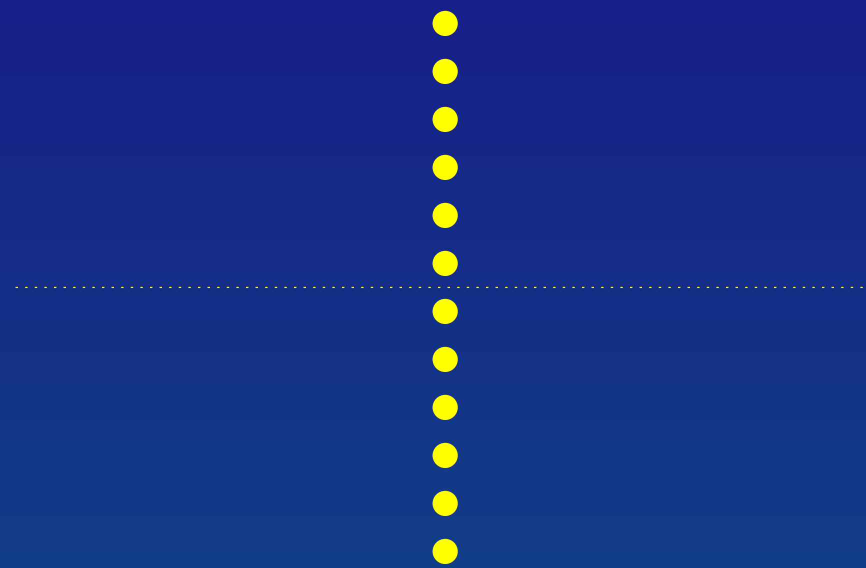


Nonlinear Analysis

Once again, we have a Riemann-Hilbert problem of inverse scattering. No jump discontinuity — instead $2N$ poles (soliton eigenvalues) with homogeneous conditions on residues.

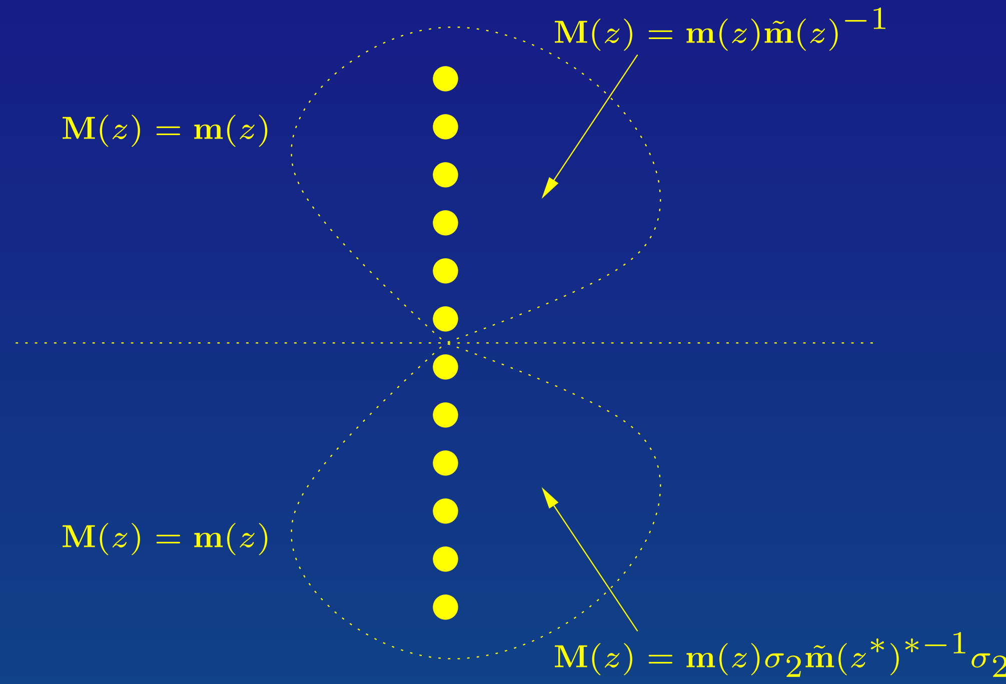


Riemann-Hilbert problem: seek $\mathbf{m}(z)$ meromorphic with simple poles at the soliton eigenvalues such that

1. $\mathbf{m}(z) \longrightarrow \mathbb{I}$ as $z \longrightarrow \infty$.
2. At pole z_k in the upper half-plane, the residue of the first column of $\mathbf{m}(z)$ is proportional to the value of the second column of $\mathbf{m}(z)$.
3. $\mathbf{m}(z^*)^* = \sigma_2 \mathbf{m}(z) \sigma_2$.

Nonlinear Analysis

Interpolation method (Kamvissis, McLaughlin, & Miller, 03): find an explicit simultaneous local solution of all the residue conditions in the upper half-plane: $\tilde{\mathbf{m}}(z)$. Then introduce a new unknown $\mathbf{M}(z)$ by:



- Local solution $\tilde{\mathbf{m}}(z)$ constructed by analytic interpolation of auxiliary spectral data (norming constants) at the soliton eigenvalues.
- Poles of $\mathbf{m}(z)$ swapped for jump discontinuities of $\mathbf{M}(z)$.

Nonlinear Analysis

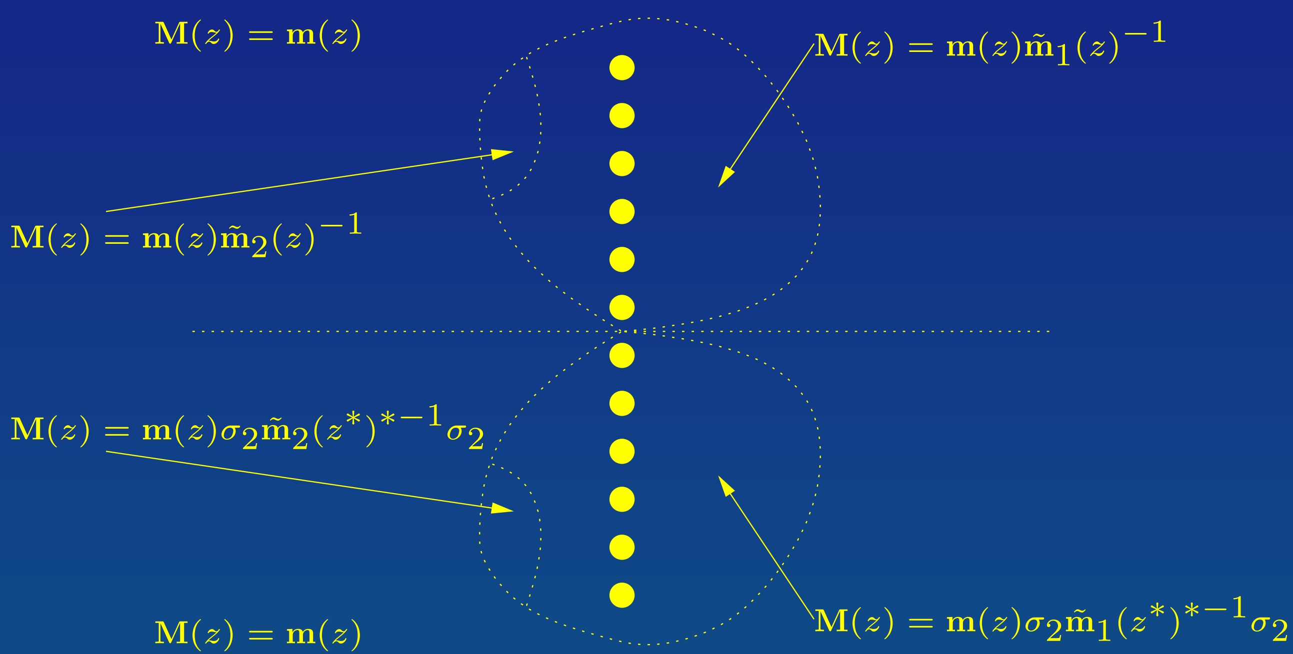
Next step: "stabilize" the Riemann-Hilbert problem for $\mathbf{M}(z)$ in the limit $N \longrightarrow \infty$. Consider a problem of potential theory: choose the curve surrounding the eigenvalues to have "minimal weighted capacity":

- The choice of this special curve is again a nonlinear analogue of the classical method of steepest descent for integrals.
- The extremal measure realizing the minimal capacity has arcs of support, and gaps. The number of support arcs is related to the genus of Riemann theta functions involved in the asymptotic solution.

This leads to a complete theory of the primary caustic curve, a phase transition from genus zero to genus two.

Nonlinear Analysis

Surprisingly, further analysis of the secondary caustic curve (Lyng & Miller, 07) requires the use of multiple local solutions:



Nonlinear Analysis

Important challenges remain:

1. Making the determination of the caustic curves computationally efficient.
2. Extending the procedure to more general initial data. Some generalized Satsuma-Yajima potentials considered by (Tovbis, Venakides, & Zhou, 04).
3. Greatest challenge: analysis with nonanalytic initial data.

Return to outline.

Conclusion

Here are the main points to keep in mind:

- There is a growing toolbox of nonlinear asymptotic methods in direct analogy with the classical methods for approximation of exponential or oscillatory integrals.

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Thank You!

Back to outline.