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Flavor oscillations and leptonic mixing

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LEPTONIC MIXING

THE IDEA OF LEPTONIC MIXING IS INTRODUCED BY 2 GROUPS IN JAPAN, AT TOKYO & AT NAGOYA. THE SECOND ONE, SAKATA'S, WROTE IN THE CELEBRATED MAKI - NAGAKAWA - SAKATA PAPER. THE 2-FLAVOR VERSION OF THE MODERN EQUATION

$$\nu_\ell(x) = \sum_{i=1}^{3} U_{\ell i} \cdot \nu_i(x)$$

with $\ell = e, \mu, \tau$ & $i = 1, 2, 3$

NEUTRINOS WITH GIVEN "FLAVOR"

NEUTRINOS WITH GIVEN "MASS"

IN 1963, SAKATA'S GROUP MAKES CONNECTION WITH NEUTRINO MASSES.

HOWEVER, THE METHOD TO INVESTIGATE LEPTONIC (FLAVOR) MIXING, THAT YIELDED RESULTS (i.e., OSCILLATIONS) WAS INTRODUCED ALREADY IN 1957, IN ANALOGY WITH $K^0 \bar{K}^0$ SYSTEM, AND APPLIED TO THE ABOVE CASE IN 1967 BY PONTECORVO.

UNDERGROUND AND USEFUL FORMALISM FOR OSCILLATIONS

LET US BEGIN CONSIDERING THE MEANING OF THE ~~[illegible]~~ FIELD EQUATION $\nu_\ell = U_{\ell i} \cdot \nu_i$.

THE FIELDS ON L.H.S. CAN BE EXPANDED AS USUAL:

$$\nu_i(x) = \sum_{\vec{p}\lambda} \left[b_i(\vec{p}\lambda)\, u_i(\vec{p}\lambda)\, e^{-ipx} + d_i^\dagger(\vec{p}\lambda)\, v_i(\vec{p}\lambda)\, e^{ipx} \right]$$

NOW CONSIDER THE CASE OF <u>ULTRARELATIVISTIC NEUTRINOS</u> THAT APPLY ALWAYS IN PRACTICAL SITUATIONS, AND APPROXIMATE:

$$u_i(\vec{p}\lambda) = \begin{pmatrix} \sqrt{E+m_i}\,\varphi_\lambda \\ \dfrac{\lambda p}{\sqrt{E+m_i}}\,\varphi_\lambda \end{pmatrix} \simeq \sqrt{E} \begin{pmatrix} \varphi_\lambda \\ \lambda\varphi_\lambda \end{pmatrix} + O\left(\frac{m}{E}\right)$$

WE SEE THAT AT $t=0$, IT IS POSSIBLE TO IDENTIFY:

$$\begin{cases} b_\ell \equiv U_{\ell i} \cdot b_i \\ d_\ell^\dagger \equiv U_{\ell i} \cdot d_i^\dagger \end{cases}$$

AND POSSIBLY (but not usefully):

$$\nu_\ell(t=0, \vec{x}) = \sum_{\vec{p}\lambda} \left[b_\ell(\vec{p},\lambda)\, u(\vec{p}\lambda)\, e^{i\vec{p}\vec{x}} + d_\ell^\dagger(\vec{p}\lambda)\, v(\vec{p}\lambda)\, e^{-i\vec{p}\vec{x}} \right] + O\left(\frac{m}{E}\right)$$

THUS THE RELATION BETWEEN "FLAVOR" AND MASS STATES

$$\begin{cases} |\nu_\ell\rangle = b_\ell^+ |0\rangle = U_{\ell i}^* \, b_i^+ |0\rangle = U_{\ell i}^* \, |\nu_i\rangle \\[2ex] |\bar{\nu}_\ell\rangle = d_\ell^+ |0\rangle = U_{\ell i} d_i^+ |0\rangle = U_{\ell i} \, |\bar{\nu}_i\rangle \end{cases}$$

UP TO TERMS ORDER m/E (THAT WE NEGLECT IN U.R. LIMIT, IN WHICH WE ARE INTERERESTED).

EXERCISE: THE ABOVE LIMIT DOES NOT HOLD IN SOME CASES, E.G., CLOSE TO β-SPECTRUM ENDPOINT. SHOW THAT (HOW) THE ABOVE EQUATIONS HAVE TO BE CHANGED

- IT IS INTERESTING TO CONSIDER THE QUANTITIES $|U_{\ell i}^2|$. THESE CAN BE SEEN AS THE CLASSICAL PROBABILITIES TO FIND A NEUTRINO ν_i INSIDE A NEUTRINO ν_ℓ (OR, CONSIDERING $\nu_i = U_{\ell i}^* \cdot \nu_\ell$, THE CONVERSE). DENOTE BY $\bullet$ A FREE PARAMETER AND BY $*$ A PARAMETER CONSTRAINED BY UNITARITY (i.e., $\sum_i |U_{\ell i}^2| = \sum_\ell |U_{\ell i}^2| = 1$). WE FIND

$$|U_{\ell i}^2| = \begin{pmatrix} \bullet & \bullet & * \\ \bullet & \bullet & * \\ * & * & * \end{pmatrix}$$

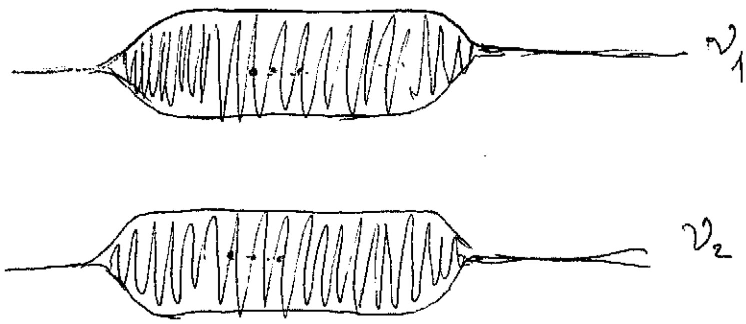
THUS THE NUMBER OF FREE PARAMETERS IS, 4, JUST AS THE PARAMETERS OF "CKM" MIXING MATRIX.

(of course these are independent of phase convention for $|\nu_\ell\rangle$ and $|\nu_i\rangle$).

EXERCISE: DERIVE THE CONSTRAINT ON THE FREE PARAMETER IMPOSED BY UNITARITY.

WHEN WE HAVE OSCILLATIONS?

IMAGINE THAT ν_e IS COMPOSED BY ν_1 & ν_2, AND BOTH ARE COMPOSED BY WAVEPACKETS OF SIZE ΔL AND THE SAME AVERAGE MOMENTUM:



THEIR VELOCITY, IN U.R. LIMIT, IS

$$v_i = \frac{p}{E_i} \simeq 1 - \frac{m_i^2}{2p^2}$$

THUS:

WHEN $(v_1 - v_2)T \gg \Delta L$, THE TWO PACKETS ARE SEPARATED; ONE NEUTRINO ARRIVES BEFORE THE OTHER ONE, NO INTERFERENCE IS POSSIBLE.

WHEN $(v_1 - v_2)T \ll \Delta L$, THE TWO PACKETS BEHAVE AS QUANTUM STATES AND OSCILLATIONS ARE POSSIBLE. ("QUANTUM REGIME").

<u>EXERCISE</u> CONSIDER A NEUTRINO OF 1 MeV, PRODUCED BY A DECAY THAT LASTS $1\mu s$ (THUS, $\Delta L = \ldots$). CALCULATE HOW MUCH TIME IT PROPAGATES IN "QUANTUM REGIME", ASSUMING THAT $m_2^2 - m_1^2 = 10^{-3}\,eV^2$.

AMPLITUDE AND PROBABILITY OF OSCILLATIONS

CONSIDER $\nu_l(\vec{p}) = U^*_{li}\,\nu_i(\vec{p})$. AFTER A DISTANCE L (FROM PRODUCTION TO DETECTION POINT) WE HAVE:

$$\nu_l(\vec{p}, L) = U^*_{li}\, e^{-i(E_i t - \vec{p}\vec{x})}\, \nu_i(\vec{p})$$

WHERE $|\vec{x}| = L$, $t \simeq L$.

CONSIDER THE OVERLAP OF THIS STATE WITH $\nu_{l'}(\vec{p}) = \nu_{l'}(\vec{p}, 0)$:

$$A_{l \to l'} = U^*_{li} \cdot e^{-i(E_i t - \vec{p}\vec{x})} \cdot U_{l'i}$$

EXPAND E_i IN UR. LIMIT: $E_i \simeq p + \frac{m_i^2}{2p}$, GETTING

$$A_{l \to l'} = U^*_{li} \cdot e^{-i\frac{m_i^2 L}{2p}} \cdot U_{l'i} \cdot e^{i(\text{IRRELEV. PHASE})}$$

NOW GET THE PROBABILITY:

$$P_{l \to l'} = |A_{l \to l'}|^2 = U_{l'i} U^*_{li} U^*_{l'j} U_{lj}\, e^{2i(\varphi_j - \varphi_i)}$$

WHERE:

$$\varphi_i = \frac{m_i^2 L}{4p} \simeq 1.27\, \frac{m_i^2}{eV^2} \cdot \frac{L}{km} \cdot \frac{GeV}{E}$$

<u>EXERCISES</u>

(1) SIMPLIFY THE EQ. FOR $P_{l \to l'}$ IN THE CASES $l = l'$ AND $l \neq l'$

(2) CALCULATE $P_{e \to \mu} - P_{\bar{e} \to \bar{\mu}}$, WHERE THE BARS DENOTE ANTINEUTRINOS.

(3) GET THE NUMBER "1.27"!!!

2 FLAVOR CASE (the simplest and all is needed in practice)

LET US CONSIDER THE AMPLITUDE AS A MATRIX IN 2 DIM. FLAVOR SPACE, BEGIN WITH

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = U(\theta) \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}, \quad \text{WITH } U(\theta) = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$$

AND OBVIOUSLY $s = \sin\theta$, $c = \cos\theta$, WE GET.

$$A = U(\theta) \begin{pmatrix} e^{-iE_1 t} & 0 \\ 0 & e^{-iE_2 t} \end{pmatrix} U(\theta)^t$$

UP TO THE USUAL IRRELEVANT PHASE FACTOR.

FROM $A_{e\to\mu} = s \cdot c \left(e^{-iE_2 t} - e^{-iE_1 t} \right)$, WE GET

$$P_{e\to\mu} = \sin^2 2\theta \cdot \sin^2 \left(\frac{\Delta m^2_{12} L}{4E} \right), \quad \Delta m^2_{12} \equiv m_2^2 - m_1^2,$$

REMARKS:

(a) when L is small, $P_{e\to\mu} \propto \left(\frac{\Delta m^2 L}{E} \right)^2$

(b) when $\frac{\Delta m^2 L}{4E} = \frac{\pi}{2}$, $P_{e\to\mu} = \sin^2 2\theta$

(c) when $\frac{\Delta m^2 L}{4E} \gg 1$ $P_{e\to\mu} \approx \frac{\sin^2 2\theta}{2}$

(d) If $\theta \to 90° - \theta$, nothing changes.

EXERCISES (1) GET P_{ee}, $P_{\mu e}$ and $P_{\mu\mu}$ AND CHECK THAT PROBABILITY IS CONSERVED, i.e., $P_{ee} + P_{e\mu} = 1$,

(2) PLOT THE PROBABILITY AS A FUNCTION OF THE ENERGY FOR $\Delta m^2 = 2.5\ 10^{-3}\ eV^2$ AND $L = 730$ km, FOR $E_\nu = 1 - 100$ GeV.

HAMILTONIAN FORMULATION

The same amplitude derives from the "formal" Hamiltonian:

$$i\partial_t \nu(t) = H_0 \cdot \nu(t) \qquad \text{where}$$

$$H_0 = U \cdot \frac{\mathrm{diag}(m^2)}{2E} \cdot U^+$$

Indeed

$$A = e^{-iH_0 t} = U \,\mathrm{diag}\left(e^{-i2\varphi}\right) \cdot U^+$$

For antineutrinos, $U \to U^*$.

REMARK:

Indeed, the "formal" hamiltonian derives from the true, free hamiltonian

$$H_0 = :\int d^3x\, \mathcal{H}_0 : = \int d^3x\, \overline{\nu(x)} \left(\vec{\gamma}\vec{p} + m\right) \nu(x) = \sum_{\vec{p},\lambda} E(\vec{p}) \left(b^+_{(\vec{p},\lambda)} b_{(\vec{p},\lambda)} + d^+_{(\vec{p},\lambda)} d_{(\vec{p},\lambda)} \right)$$

where, obviously, $E = \sqrt{p^2 + m^2}$.

<u>EXERCISE</u> (in QFT): RE-DERIVE THE FREE HAMILTONIAN FROM THE QUANTIZED FIELD $\nu(x)$ (ASSUME IT IS A MASS EIGENSTATE).

{APPLICATION: INTERPRETING WORLD DATA}

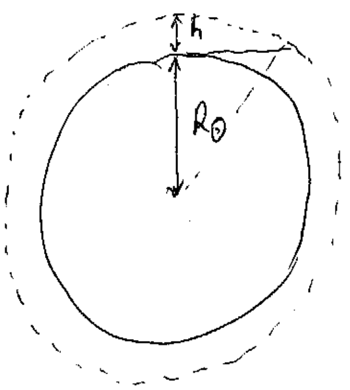
(1) ATMOSPHERIC NEUTRINOS AND REACTOR $\bar{\nu}_e$

CONSIDER ν'S PRODUCED BY COSMIC RAY INTERACTING WITH EARTH ATMOSPHERE, THROUGH

$$\pi^+ \to \mu^+ \nu_\mu \quad \text{followed by} \quad \mu^+ \to e^+ \nu_e \bar{\nu}_\mu$$

<u>EXERCISE</u>: check that the kinematics prescribe that the 3 neutrinos have almost the same mean energy (assume U.R. pions)

CONSIDER THE GEOMETRY OF THEIR PRODUCTION:



$L(\text{from above}) \equiv h \sim 10-20$ km

$L(\text{horizontal}) \sim 2 R_\oplus h \sim 500$ km

$L(\text{from below}) \sim 2R_\oplus \sim 10^4$ km.

EXP. { * NEUTRINOS $\overset{(-)}{\nu}_\mu$ FROM ABOVE OBEY EXPECTATIONS;

OBSERVATIONS {

* $\overset{(-)}{\nu}_\mu$ FROM BELOW ARE A FACTOR 2 ~~TOO~~ SUPPRESSED;

* $\overset{(-)}{\nu}_\mu$ FROM HORIZON START TO DEVIATE FROM EXPECTATION

* ν_e AGREE WITH EXPECTATIONS (checked with pure $\bar{\nu}_e$ beam)

these observations suggest $\Delta m^2 \sim 2.5\, 10^{-3}\ \text{eV}^2$, SINCE

$$\varphi = \frac{\Delta m^2 L}{4E} = \frac{\pi}{2} \times \frac{\Delta m^2}{2.5\, 10^{-3}\,\text{eV}^2} \cdot \frac{L}{500\,\text{km}} \cdot \frac{1\,\text{GeV}}{E}$$

WE NEED JUST THE 2 FLAVOR FORMULA FOR $\nu_\mu \nu_\tau$ OSCILLATIONS

$$P_{\mu\to\mu} = 1 - \sin^2 2\theta_{atm} \cdot \sin^2\left(\frac{\Delta m^2_{atm} L}{4E}\right)$$

SINCE FOR LARGE L WE HAVE $P_{\mu\to\mu} \approx \frac{1}{2}$, WE WANT $\sin^2 2\theta_{atm} \approx 1 \Rightarrow \vartheta_{atm} \approx 45°$.

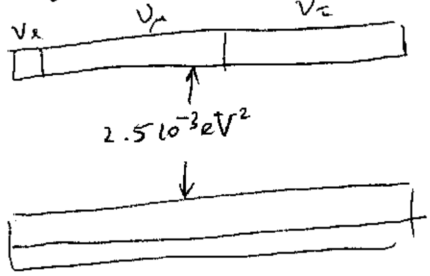
THIS FORMULA FOLLOWS FROM:

$$\begin{cases} \nu_\mu = \cos\theta_{atm}\, \nu_{LIGHT} + \sin\theta_{atm}\, \nu_{HEAVY} \\ \nu_\tau = -\sin\theta_{atm}\, \nu_{LIGHT} + \cos\theta_{atm}\, \nu_{HEAVY} \end{cases} \Leftrightarrow \begin{cases} \nu_{HEAVY} = \cos\theta_{atm}\, \nu_\tau + \sin\theta_{atm}\, \nu_\mu \\ \dots \end{cases}$$

THIS CAN BE GENERALIZED AS FOLLOWS:

$$\nu_{Heavy} = \sin\theta_R \cdot \nu_e + \cos\theta_R \left(\sin\theta_{atm}\, \nu_\mu + \cos\theta_{atm}\, \nu_\tau \right)$$

WHERE ϑ_R IS SMALL AND ACCOUNTS FOR SMALL ν_e OSCILLATIONS. GRAPHICALLY



ν_3, the heaviest

ν_1 & ν_2, effectively massless

EXERCISE: Calculate $P_{e\to e}$ from THE ABOVE ASSUMPTION, AND CHECK THAT WITH $E_\nu \sim 3$ MeV AND $L \sim 1$ km, $\bar{\nu}_e$ FROM A REACTOR COULD IN PRINCIPLE OSCILLATE WITH THE SAME $\Delta m^2 = 2.5\ 10^{-3}\ eV^2$.

(2) LOW ENERGY SOLAR ν_e AND KamLAND

THERE ARE NO (LARGE) ν_e OSCILLATIONS ON THE EXPERIMENTS WITH $\frac{L \Delta m^2_{atm}}{4E} \sim 1$.

HOWEVER SOLAR ν_e COME FROM SUCH A LARGE L, THAT WE CAN PROBE MUCH SMALLER Δm^2's.

IF WE START FROM:

$$\nu_e = \cos\theta_R \left(\cos\theta_{sol} \nu_1 + \sin\theta_{sol} \nu_2 \right) + \sin\theta_R \nu_3$$

AND, AGAIN, SET $\theta_R \to 0$, WE REDUCE TO 2 FLAVOR CASE. THE FORMULA THAT APPLIES IS JUST:

$$P_{e\to e} = 1 - \sin^2 2\theta_{sol} \cdot \sin^2\left[\frac{\Delta m^2_{sol} L}{4E}\right]$$

THE ν_e SUPPRESSION FACTOR, $P_{e\to e} \sim 0.6$ INDICATES THAT $\sin^2 2\theta_{sol} \sim 0.8$. FURTHER INDEPENDENT TEST WITH REACTOR $\bar{\nu}_e$ ($E \sim 3$ MeV) AT A DISTANCE OF $L \sim 100$ km (!) LEAD TO PRECISE DETERMINATION $\Delta m^2_{sol} \sim 8 \cdot 10^{-5}$ eV^2 $\ll$ Δm^2_{atm} (KamLAND)

(c) SUMMARY AND STANDARD PARAMETERIZATION

$$U = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12}-s_{13}s_{23}c_{12}e^{i\delta}, & +c_{23}c_{12}-s_{13}s_{23}s_{12}e^{i\delta} & c_{13}s_{23} \\ +s_{23}s_{12}-s_{13}c_{23}c_{12}e^{i\delta}, & -s_{23}c_{12}-s_{13}c_{23}s_{12}e^{i\delta} & c_{13}c_{23} \end{pmatrix}$$

with $s_{12} = \sin\theta_{12}$ $\quad c_{23} = \cos\theta_{23}$ etc.

The 4 parameters are those that control oscillation. δ CONTROLS CP violating oscillation, it is at present unknown.

It is possible to identify (with previous notation)

$$\vartheta_{12} = \vartheta_{sol}, \quad \theta_{23} = \theta_{atm}, \quad \theta_{13} = \vartheta_R.$$

At present, we know that $\vartheta_{13} < 10^\circ$, $\vartheta_{23} \sim 45^\circ$, $\vartheta_{12} \sim 30^\circ$, $\Delta m^2_{23} \sim 2{,}5\ 10^{-3}$ $\Delta m^2_{12} \sim 8\ 10^{-5}\ eV^2$.

BUT THERE ARE SOME PENDING POINT IN THE DISCUSSION:

1 – WHAT ABOUT HIGH ENERGY SOLAR NEUTRINOS?

2 – HOW WE DISTINGUISH $\theta_{12} = 30^\circ$ FROM $\vartheta_{12} = 60^\circ$ WHEN $P_{e\to e}$ DEPENDS ON $\sin^2 2\theta_{12}$?

3 – HOW DO WE KNOW THE <u>SIGN</u> OF Δm^2_{ij}?

ALL THESE REQUIRE TO GO A BIT DEEPER.

THE MATTER TERM

ν_i's RECEIVE THE PHASES $e^{-iE_i t}$ AND THIS LEAD TO OSCILLATION. ν_e's IN MATTER RECEIVE A SIMILAR PHASE AND THIS MODIFIES OSCILLATIONS.

THIS IS DUE TO USUAL WEAK INTERACTION THAT PRODUCE FORWARD SCATTERING $\nu_e(\vec{p})\, e^-$(at rest) $\longrightarrow$ $\nu_e(\vec{p})\, e^-$(at rest). [no momentum change]

INDEED CONSIDER THE HAMILTONIAN DENSITY

$$\mathcal{H} = \frac{G_F}{\sqrt{2}}\, \bar{\nu}_e \gamma^a (1-\gamma_5) e \;\bar{e}\, \gamma_a (1-\gamma_5) \nu_e = \frac{G_F}{\sqrt{2}}\, \bar{\nu}_e \gamma^a (1-\gamma_5) \nu_e \cdot \bar{e}\, \gamma_a (1-\gamma_5) e$$

WHERE THE SECOND FORM (THE USEFUL ONE) IS THE SAME AS THE FIRST THANKS TO FIERZ IDENTITY (see Appendix).

FOR NON-RELATIVISTIC ELECTRONS, ALL TERMS OF $\langle e^-$ at rest $|\, \bar{e}(x) \gamma_a (1-\gamma_5) e(x)\, | e^-$ at rest$\rangle$ give zero except THE ONE AT $a=0$ and WITH VECTOR CHARACTER; WHEN WE NOTE THAT $\int \bar{e}(x) \gamma^0 e(x)\, d^3x$ IS THE OPERATOR THAT COUNTS ELECTRONS, WE UNDERSTAND THAT

$$\langle e^- \text{ at rest} |\, \bar{e}(x) \gamma_0\, e(x)\, | e^- \text{ at rest} \rangle = \rho_e(x)$$

SIMILARLY FOR RELATIVISTIC NEUTRINOS:

$$\begin{cases} \langle \nu_e |\, \bar{\nu}_e(x) \gamma^0 P_L\, \nu_e(x)\, | \nu_e \rangle = \rho_{\nu_e}(x) \\ \langle \bar{\nu}_e |\, \bar{\nu}_e(x) \gamma^0 P_L\, \nu_e(x)\, | \bar{\nu}_e \rangle = -\rho_{\bar{\nu}_e}(x) \end{cases}$$

THUS, ON A SINGLE NEUTRINO STATE

$$\begin{cases} \langle \nu_e, e^- \text{ at rest} |\, \int d^3x\, \mathcal{H}\, | \nu_e, e^- \text{ at rest} \rangle = \sqrt{2}\, G_F \cdot \rho_e(x) \\ \langle \bar{\nu}_e\; e^- \text{ at rest} |\, \int d^3x\, \mathcal{H}\, | \nu_e, e^- \text{ at rest} \rangle = -\sqrt{2}\, G_F\, \rho_e(x) \end{cases}$$

[~~APPENDIX~~ REMARKS ON DENSITIES OF PARTICLES]

(*) $$\hat{\Psi}(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} \left(b_p u_p e^{-ipx} + d_p^\dagger v_p e^{ipx} \right) \qquad \text{(FIELD)}$$

$$|\psi\rangle = \int \frac{d^3q}{(2\pi)^3} \psi(\vec{q}) \, |\vec{q}\rangle \qquad \text{(VECTOR)}$$

where $|\vec{q}\rangle = \dfrac{b^\dagger_{\vec{q}}}{\sqrt{2E_{\vec{q}}}} |0\rangle$, $\langle q' | \vec{q} \rangle = (2\pi)^3 \delta^3(\vec{q} - \vec{q}\,')$.

$$\psi(\vec{x}) = \langle x | \psi \rangle = \int \frac{d^3q}{(2\pi)^3} \psi(\vec{q}) \, e^{i\vec{q}\vec{x}} \qquad \text{(WAVEFUNCTION)}$$

since $\langle \vec{x} | \vec{q} \rangle = e^{i\vec{q}\vec{x}}$ with above normalization

(*) Now consider:

$$\hat{\psi}(x)|\psi\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{u_p}{\sqrt{2E_p}} e^{-ipx} \psi(\vec{p})$$

For an UR neutrino, $u_p/\sqrt{E_p} = (\varphi_-, -\varphi_-)$ (where "$-$" means negative helicity). For a NR electron, $u_p/\sqrt{2E_p} = (\varphi, 0)$. Thus:

$$\hat{\psi}(x)|\psi\rangle = \begin{cases} \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_- \\ -\varphi_- \end{pmatrix} \psi(x) \, e^{-iEt} & \text{neutrino UR} \\ \begin{pmatrix} \varphi \\ 0 \end{pmatrix} \psi(x) \, e^{-iEt} & \text{electron NR} \end{cases}$$

and we conclude that, in both cases:

$$\langle \psi | \hat{\psi}^\dagger(x) \hat{\psi}(x) | \psi \rangle = |\psi(x)|^2$$

which is the local density of particles.

THE TOTAL HAMILTONIAN OF PROPAGATION OF NEUTRINOS

$$H = U \cdot \mathrm{diag}(k) \cdot U^\dagger + V(x)\ \mathrm{diag}(1,0,0) \quad \text{with} \begin{cases} k_i = m_i^2/2E \\ V(x) = \sqrt{2}\, G_F\, \rho_e(x) \end{cases}$$

$$\frac{V}{k} = \frac{\sqrt{2}\, G_F \cdot \rho_e}{\Delta m^2/2E} = \left(\frac{\rho\, Y_e/N_A}{100\ \mathrm{moles/cm^3}}\right)\left(\frac{8\cdot 10^{-5}\ \mathrm{eV}^2}{\Delta m^2}\right)\left(\frac{E}{5\ \mathrm{MeV}}\right)$$

Thus we see that for low energy solar ν_e, $E \sim 1$ MeV, the new term is small and one can rely on vacuum oscillations. For high energies, instead, it must be included.

EXERCISE : Show that even if $\vartheta_{13} = \vartheta_{12} \neq 0$, the new ~~oscillations~~ (matter) term is irrelevant for oscillations with Δm^2_{atm} in the sun, where $\rho/N_A \sim 150\ g/cm^3$.

(a) SOLUTION WITH CONSTANT MATTER DENSITY

UP TO AN IRRELEVANT CONSTANT, THE 2×2 CASE IS

$$H = \frac{k}{2}\, U\, \mathrm{diag}(-1,1)\, U^t + \frac{V}{2}\, \mathrm{diag}(1,-1) = \frac{1}{2}\begin{pmatrix} V - k c_2 , & k s_2 \\ -k s_2 , & k c_2 - V \end{pmatrix}$$

WHERE

$$U = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \quad \text{AND} \quad k = k_2 - k_1 = \frac{\Delta m^2_{12}}{2E}.$$

THIS IS JUST THE SAME AS VACUUM CASE DEFINING

$$\begin{cases} k \cdot s_2 \equiv k_m \cdot s_{2m} \\ k c_2 - V \equiv k_m \cdot c_{2m} \end{cases}$$

EXERCISE: CALCULATE ϑ_m (mixing in matter) if $V \ll k$, $V \gg k$, $V = k \cdot c_2$.

(b) SLOWLY VARYING MATTER DENSITY (adiabatic solution)

WE SHOWED THAT FOR CONSTANT MASS DENSITY

$$i\partial_t \nu = \frac{1}{2} U_m \begin{pmatrix} -k_m & 0 \\ 0 & k_m \end{pmatrix} U_m^t \nu$$

IN OTHER WORDS, WE REDUCE TO VACUUM OSCILLATIONS SIMPLY REDEFINING $\theta \to \theta_m$, $k \to k_m$.

IT IS USEFUL TO CONSIDER THE PROPAGATION OF LOCAL EIGENSTATES OF THE HAMILTONIAN n:

$$\boldsymbol{\nu} = U_m \cdot \boldsymbol{n}$$

INDEED, THEIR PROPAGATION IS:

$$i\partial_t n = \begin{pmatrix} -k_m/2 & i\dot{\vartheta}_m \\ -i\dot{\vartheta}_m & k_m/2 \end{pmatrix} n$$

AND IF $\dot{\vartheta}_m = \partial_t \vartheta_m$ IS MUCH SMALLER THAN k_m, THE SOLUTION IS TRIVIAL.

LUCKILY, THIS IS THE CASE OF SOLAR NEUTRINOS, WHEN $\rho(x) \propto e^{-10x/R_\odot}$.

EXERCISE CALCULATE $|k_m/2\dot{\theta}_m|$ FOR THE SUN; ASSUME $\Delta m^2 = 8\,10^{-5}\,eV^2$ & $\vartheta = 30°$. (THE CORRECT ORDER OF MAGNITUDE IS $k \cdot R_\odot/10$) SHOW THAT IT IS QUITE LARGE, THUS $\dot{\vartheta}_m$ IS NEGLIGIBLE.

INTERPRETATION OF HIGH ENERGY SOLAR NEUTRINOS

CONSIDER A ν_e PRODUCED IN THE CENTER OF THE SUN. AT HIGH ENERGY,

$$H \approx \begin{pmatrix} V/2 & 0 \\ 0 & -V/2 \end{pmatrix}$$

THE $\nu_e = (1, 0)$ is APPROXIMATIVELY THE LOCAL EIGENSTATE n_2 (THE HEAVIER ONE). THIS MEANS $\vartheta_m \approx 90°$, NAMELY:

$$n = U_m^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \approx \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

ν_e IN FLAVOR BASE

IN THE CENTER OF THE SUN

n_2 IN LOCAL EIGENSTATE ("MASS") BASE

AS WE SAW, n_2 REMAINS SUCH ALL THE WAY TILL THE EXIT, AND EVENTUALLY REACHES THE DETECTOR.

IF WE ASK WHICH IS THE PROBABILITY THAT WE FIND A ν_e, WE SEARCH THE OVERLAP WITH

$$n = U^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c \\ s \end{pmatrix}$$

ν_e in flavor base.

IN VACUUM

A SUPERPOSITION OF n_1 & n_2

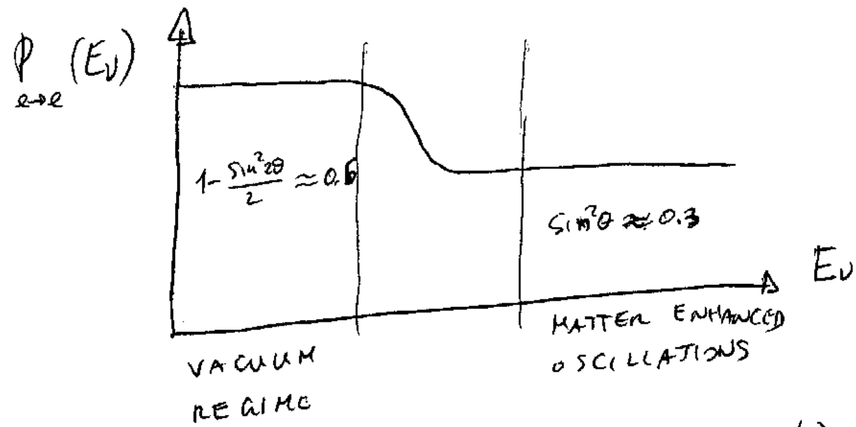
THUS WE FIND:

$$P_{e\to e} = \left| (c, s) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right|^2 = \sin^2\theta,$$

THAT IS NOT ANYMORE SYMMETRIC IN THE EXCHANGE $\theta \to 90°-\theta$, AS WAS THE VACUUM FORMULA:

$$P_{e\to e} = 1 - \frac{\sin^2 2\theta}{2}.$$

THE EXPERIMENTAL OBSERVATIONS CORRESPOND TO THE FOLLOWING (SLIGHTLY COMPLICATED) PICTURE:



IN SUMMARY, HIGH ENERGY (few MeV) SOLAR ν_e PERMIT US TO MEASURE $\vartheta = \vartheta_{12}$.

EXERCISE: The effect of ν_3 can be formally included considering that it receives a big propagation phase, so that

$$\nu_e = c_{13}(c_{12}\nu_1 + s_{12}\nu_2) + s_{13}\nu_3 e^{i\infty},$$

$$A_{e\to e}(3 \text{ flavors}) = c_{13}^2 A_{e\to e}(2 \text{ flavors}) + s_{13}^2 e^{i\infty}.$$

(1) OBTAIN THE FORMULAE FOR $P_{e\to e}$(3 flavors) AT LOW AND AT HIGH ENERGIES.

(2) SHOW THAT A PRECISE MEASUREMENT OF SOLAR ν_e FLUX AT LOW AND AT HIGH ENERGIES PERMITS TO LEARN SOMETHING ON ϑ_{13}.

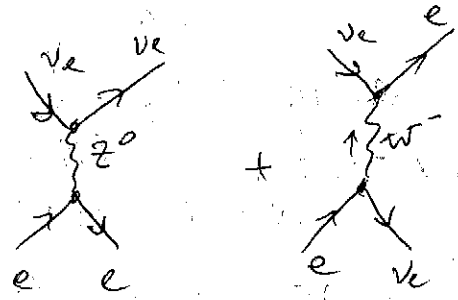
APPENDIX ON:

"HOW TO REMEMBER FIERZ IDENTITY FOR V−A"

FOR EXAMPLE, THE FOLLOWING ONE:

$$\bar{\mu}\gamma^a(1-\gamma_5)\nu_\mu \cdot \bar{\nu}_e\gamma_a(1-\gamma_5)\,e = \bar{\mu}\gamma^a(1-\gamma_5)e \cdot \bar{\nu}_e\gamma_a(1-\gamma_5)\nu_\mu$$

(0) It involves only LEFT FIELDS.

(1) 

$= NC + CC =$ NC with modified couplings at low energies

(2) $\left|\mathcal{M}_{(\mu\ \text{decay})}\right|^2 \propto (p_\mu\, p_{\nu_e})(p_{\nu_\mu}\, p_e)$

μ decay averaged amplitude squared symmetric in this exchange.

$- + -$

(3) CC-caused MSW effect:

$$\mathcal{L} = -\frac{G_F}{\sqrt{2}}\,\bar{e}\gamma^a(1-\gamma_5)\nu_e \cdot \bar{\nu}_e\gamma_a(1-\gamma_5)e \overset{\text{Fierz}}{=} -\frac{G_F}{\sqrt{2}}\,\bar{e}\gamma^a(1-\gamma_5)e \cdot \bar{\nu}_e\gamma_a(1-\gamma_5)\nu_e$$

in NR limit, non-polarized matter, $\bar{e}\gamma^a e = J^a_{em} \to (\rho_e, \vec{0})$

(4) In chiral representation:

$$\gamma^a P_L \cdot \gamma_a P_L = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \vec{\sigma} \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & \vec{\sigma} \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0\cdot 0 & 1\cdot 1 - \vec{\sigma}\cdot\vec{\sigma} \\ 0\cdot 0 & 0\cdot 0 \end{pmatrix}$$

REDUCES TO $\quad 1_{ab}\cdot 1_{cd} - \vec{\sigma}_{ab}\cdot\vec{\sigma}_{cd} = -\left(1_{ad}\cdot 1_{cb} - \vec{\sigma}_{ad}\cdot\vec{\sigma}_{cb}\right)$, easy to prove.