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Summer School on Particle Physics in the LHC Era

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CHARGE CONJUGATION

$$(i\hat{\partial} - m + q\hat{A})\psi = 0$$

$$\psi^\dagger(-i\hat{\partial}^\dagger - m + q\hat{A}^\dagger) = 0$$

$$\bar{\psi}(-i\hat{\partial} - m + q\hat{A}) = 0$$

$$(-i\hat{\partial}^\dagger - m + q\hat{A}^\dagger)\bar{\psi}^\dagger = 0$$

NOW, CONSIDER C SUCH THAT $-\gamma_\mu^\dagger = C^{-1}\gamma_\mu C$.

$$C^{-1}(i\hat{\partial} - m - q\hat{A})C\bar{\psi}^\dagger = 0$$

$$(i\hat{\partial} - m - q\hat{A})\psi^c = 0 \quad \text{WHERE } \psi^c = C\bar{\psi}^\dagger.$$

WE SEE THAT ψ^c OBEYS THE DIRAC EQUATION WITH $q \rightarrow -q$, WHICH EXPLAINS THE NAME OF C .

LET US SHOW THAT THE DEFINITION OF C IS SOUND. CONSIDER THAT

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} \Leftrightarrow \{-\gamma_\mu^\dagger, -\gamma_\nu^\dagger\} = 2g_{\mu\nu}$$

THUS, γ_μ AND $-\gamma_\mu^\dagger$ MUST BE EQUIVALENT

γ -MATRICES; THE EQUIVALENCE IS REALIZED BY C .

(0) SHOW THAT $\gamma_5^\dagger C = C\gamma_5$.

EXERCISES: (1) SHOW THAT, IN DIRAC REPRESENTATION, ONE CAN CHOOSE $C = i\gamma^0\gamma^2$, THAT SATISFIES $C^\dagger = C^{-1}$ AND $C^t = -C$.

(2) SHOW THAT, SETTING $\gamma_\mu' = U\gamma_\mu U^\dagger$, ONE HAS $C' = UC U^\dagger$. THUS, $C^\dagger = C^{-1}$ AND $C^t = -C$ ARE ALWAYS VALID.

IT IS QUITE EVIDENT THAT C MUST MAP
SPINORS INTO SPINORS, BUT LET US CHECK IT.

$$\delta\psi = \frac{i}{2} \omega_{\mu\nu} \sigma^{\mu\nu} \cdot \psi \quad [\omega_{\mu\nu} \in \mathbb{R}]$$

$$\delta\psi^+ = -\psi^+ \frac{i}{2} \omega_{\mu\nu} \cdot \sigma^{\mu\nu +}$$

$$\left[\begin{aligned} \sigma^{\mu\nu} &= \frac{i}{4} [\gamma^\mu, \gamma^\nu] \\ \Rightarrow \sigma^{\mu\nu +} &= \frac{i}{4} [\gamma^{\mu+}, \gamma^{\nu+}] \end{aligned} \right]$$

$$\delta\bar{\psi} = -\bar{\psi} \frac{i}{2} \omega_{\mu\nu} \sigma^{\mu\nu}$$

THIS SHOWS INCIDENTALLY THAT $\bar{\psi}\psi$ IS AN
INVARIANT, SINCE ψ AND $\bar{\psi}$ TRANSFORM WITH
OPPOSITE MATRICES.

$$\delta\bar{\psi}^t = -\frac{i}{2} \omega_{\mu\nu} \sigma^{\mu\nu t} \bar{\psi}^t \quad [\sigma^{\mu\nu t} = -\frac{i}{4} [\gamma^{\mu t}, \gamma^{\nu t}]]$$

$$\begin{aligned} \delta\psi^c &= -\frac{i}{2} \omega_{\mu\nu} C \sigma^{\mu\nu t} C^{-1} \cdot C \bar{\psi}^t \\ &= \frac{i}{2} \omega_{\mu\nu} \sigma^{\mu\nu} \psi^c \end{aligned}$$

THUS, ψ^c IS A SPINOR AS ψ . ALSO NOTE

$$\delta\psi^t = \psi^t \frac{i}{2} \omega_{\mu\nu} \sigma^{\mu\nu t}$$

$$\begin{aligned} \delta\psi^t \cdot C^{-1} &= \psi^t C^{-1} \cdot \frac{i}{2} \omega_{\mu\nu} \cdot C \sigma^{\mu\nu t} C^{-1} \\ &= -\psi^t C^{-1} \frac{i}{2} \omega_{\mu\nu} \sigma^{\mu\nu} \end{aligned}$$

THUS, $\psi^t C^{-1}$ IS A SPINOR THAT TRANSFORMS AS $\bar{\psi}$.

EXERCISE: CHECK THAT $\overline{\psi^c} = -\psi^t C^+ = -\psi^t C^{-1}$.

WE DEFINE: $\bar{\psi}^c = -\psi^t C^{-1}$.

MAJORANA SPINOR

- A MAJORANA SPINOR IS A SPINOR THAT SATISFIES THE REALITY CONDITION:

$$\psi = \psi^c$$

IT SHOULD BE NOTED THAT THE CONDITION $\psi = \psi^*$ IS NOT INVARIANT UNDER LORENTZ TRANSFORMATION, IN GENERAL,

EXERCISE SHOW THAT IF γ^0 AND γ^i ARE PURE IMAGINARY, THE CONDITION $\psi = \psi^*$ CAN BE IMPOSED. SHOW THAT THIS IS POSSIBLE BY CONSTRUCTING SUCH γ -MATRICES. (MAJORANA)

- EXACTLY AS IT IS POSSIBLE TO DECOMPOSE A SCALAR INTO ITS REAL AND IMAGINARY PART, $\varphi = \varphi' + i \cdot \varphi''$ [WITH $\varphi' = \frac{\varphi + \varphi^*}{2}$ AND $\varphi'' = \frac{\varphi - \varphi^*}{2i}$]
- WE CAN DECOMPOSE A DIRAC SPINOR INTO

$$\psi = \psi_+ + \psi_-$$

$$\text{WHERE } \psi_{\pm} = \frac{1}{2} (\psi + \psi^c) ; (\psi_{\pm})^c = \pm \psi_{\pm} .$$

- WE CAN ~~MAJORANA SPINOR~~ FORM A MAJORANA SPINOR OUT OF A LEFT SPINOR ψ_L :

$$\chi = \psi_L + \psi_L^c$$

EXERCISE: USING $\gamma_5^c = C \gamma_5^t$ AND $\gamma_5^+ = \gamma_5$, PROVE THAT

$$P_L \psi^c = (\psi_R)^c . \text{ THUS, } P_L \chi = \psi_L .$$

MAJORANA MASS (DEFINITION)

THE DIRAC MASS TERM IN THE LAGRANGIAN

$$- m \bar{\psi} \psi$$

IS LEGITIMATE IN THE SENSE THAT IT SATISFIES
LORENTZ INVARIANCE. MORE GENERALLY WITH
VARIOUS SPINORS WE CAN CONSIDER $\bar{\psi} \lambda + \bar{\lambda} \psi$.

HOWEVER, WE JUST NOTED THAT ψ^c TRANSFORMS
JUST AS ψ , SO WE CAN FORM A SCALAR
(AN INVARIANT) AS FOLLOWS:

$$\bar{\psi} \cdot \psi^c = \bar{\psi} C \bar{\psi}^t$$

OR, MORE IN GENERAL, $\bar{\psi} \lambda^c = \bar{\psi} C \bar{\lambda}^t$.

ANOTHER WAY TO UNDERSTAND THIS, IS TO FORM
THE "USUAL" (DIRAC) MASS TERM BY USING
A MAJORANA SPINOR. NAMELY, USING

$$\chi = \psi_L + \psi_L^c$$

THAT IMPLIES $\bar{\chi} = \bar{\psi}_L + \bar{\psi}_L^c$, WE GET

$$\begin{aligned} \bar{\chi} \chi &= (\bar{\psi}_L - \psi_L C^{-1}) (\psi_L + C \bar{\psi}_L^t) \\ &= \bar{\psi}_L \psi_L - \psi_L C^{-1} \bar{\psi}_L^t \\ &\quad + \bar{\psi}_L C \bar{\psi}_L^t - \psi_L C^{-1} \psi_L \\ &= \bar{\psi}_L C \bar{\psi}_L^t - \psi_L^t C^{-1} \psi_L \end{aligned}$$

SINCE THE FIRST TWO TERMS ARE ZERO,
THIS PROCEDURE GIVES AUTOMATICALLY A REAL - i.e.,
SELF-CONJUGATED - TERM OF THE LAGRANGIAN.

TRICK.
How to USE MAJORANA NEUTRINOS AS FIMACS

CONSIDER A FERMION ν_R WITH MAJORANA MASS:

$$\mathcal{L}_{\text{free}} = i \bar{\nu}_R \hat{\partial} \nu_R - \frac{M}{2} \bar{\nu}_R^c C \bar{\nu}_R^t + \frac{M}{2} \nu_R^t C^{-1} \nu_R$$

• SET

$$N = \nu_R + \nu_R^c = \nu_R + C \bar{\nu}_R^t$$

$$\Rightarrow \bar{N} = \bar{\nu}_R + \bar{\nu}_R^c = \bar{\nu}_R - \nu_R^t C^{-1}$$

• BEGIN WITH MASS TERM

$$\begin{aligned} \bar{N} N &= (\bar{\nu}_R - \nu_R^t C^{-1}) \cdot (\nu_R + C \bar{\nu}_R^t) \\ &= \bar{\nu}_R \nu_R + \bar{\nu}_R C \bar{\nu}_R^t - \nu_R^t C^{-1} \nu_R - \nu_R^t \bar{\nu}_R \\ &= 0 + \bar{\nu}_R C \bar{\nu}_R^t - \nu_R^t C^{-1} \nu_R + \bar{\nu}_R \nu_R \end{aligned}$$

↑ = 0
Fermionic anticommutation

• THEN THE KINETIC TERM

$$\begin{aligned} \bar{N} \hat{\partial} N &= (\bar{\nu}_R - \nu_R^t C^{-1}) \hat{\partial} (\nu_R + C \bar{\nu}_R^t) \\ &= \bar{\nu}_R \hat{\partial} \nu_R + \bar{\nu}_R \hat{\partial} C \bar{\nu}_R^t - \nu_R^t C^{-1} \hat{\partial} \nu_R - \nu_R^t C^{-1} \hat{\partial} C \bar{\nu}_R^t \end{aligned}$$

using $\gamma_5 C = C \gamma_5^t \Rightarrow C^{-1} \not{p}_R = \not{p}_R^t C^{-1}$ and similar, the 2nd and 3rd can be shown to be zero. The last

$$\begin{aligned} \cancel{\nu_R^t C^{-1} \gamma_\mu C \partial^\mu \bar{\nu}_R^t} - \nu_R^t C^{-1} \gamma_\mu C \cdot \partial^\mu \bar{\nu}_R^t &= + \nu_R^t \gamma_\mu^t \cdot \partial^\mu \bar{\nu}_R^t \\ &= - \partial^\mu \bar{\nu}_R^t \cdot \gamma_\mu \cdot \nu_R = + \bar{\nu}_R \hat{\partial} \nu_R + \text{total derivative} \end{aligned}$$

anticommutation
WE CONCLUDE THAT, UP TO TOTAL DERIVATIVE,

$$\mathcal{L}_{\text{free}} = \frac{i}{2} \bar{N} \hat{\partial} N - \frac{M}{2} \bar{N} N$$

APPLICATION 1: 2 MASSES AND OUR 2β

(1/3) - I

LET US SHOW AN IMPORTANT EFFECT OF THE MASS TERM:

$$- \frac{1}{2} \bar{N}_e M_{\ell\ell'} N_{\ell'} \quad , \quad \text{with} \quad \begin{cases} \ell, \ell' = e, \mu, \tau \\ N = \nu_L + \nu_L^c \end{cases}$$

CONSIDER THE FERMILAGRANGIAN FOR β DECAY

$$- \frac{G_F}{\sqrt{2}} \cos \theta_c \bar{e} \gamma^\mu (1 - \gamma_5) N_e \cdot \bar{u} \gamma_\mu (1 - \gamma_5) d$$

AT SECOND ORDER, ~~WE CONSIDER~~ WE HAVE THE PRODUCT OF 2 LEPTONIC CURRENTS:

$$\begin{aligned} & \bar{e}(x) \gamma^\mu (1 - \gamma_5) N_e(x) \cdot \bar{e}(y) \gamma^\nu (1 - \gamma_5) N_e(y) \\ &= \bar{e}(x) \gamma^\mu (1 - \gamma_5) N_e(x) \cdot \bar{e}(y) \gamma^\nu (1 - \gamma_5) C \bar{N}_e^t(y) \\ &= \quad \quad \quad \cdot \left(-\bar{N}_e(y) C^t (1 - \gamma_5)^t \gamma^{\nu t} \bar{e}(y)^t \right) \\ &= \bar{e}(x) \gamma^\mu (1 - \gamma_5) N_e(x) \cdot \left(-\bar{N}_e(y) (1 - \gamma_5) \gamma^\nu C \bar{e}(y)^t \right) \end{aligned}$$

~~THIS~~ ~~MAJORANA~~ ~~FIELD~~ IT IS POSSIBLE TO CONTRACT THE MAJORANA NEUTRINO FIELD; ~~MAJORANA~~ WITH THE SAME FORMALISM USED FOR DIRAC FIELDS.

BEFORE PROCEEDING WE INTRODUCE THE VERY USEFUL DECOMPOSITION,

$$M_{\ell\ell'} = U_{\ell i}^* m_i U_{\ell' i}$$

WITH U = UNITARY MATRIX AND $m_i = m_1, m_2, m_3$.

EXERCISE (1) SHOW THAT THE MAJORANA MASS MATRIX M IS SYMMETRIC

(2) USE THE BI-UNITARY DECOMPOSITION TO SHOW THAT THE ABOVE DECOMPOSITION IS POSSIBLE

THIS DECOMPOSITION IMPLIES THAT $\mathbb{P}(2/3) = I$

$$-\frac{1}{2} \bar{N}_l M_{ll'} N_{l'} = +\frac{1}{2} N_l C^{-1} (U_{li}^* m_i U_{l'i}) N_{l'}$$

$$= -\frac{1}{2} \bar{N}_l m_i N_i$$

WITH:

$$N_i = \sum_{l=e,\mu,\tau} U_{li}^* N_l \Leftrightarrow N_l = \sum_{i=1,2,3} U_{li} N_i$$

THUS N_i ARE THE MASS EIGENSTATES.

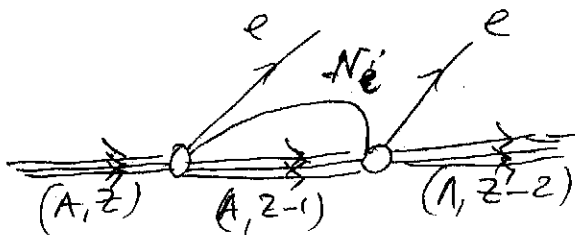
NOW WE ARE READY TO OPERATE THE CONTRACTION. THE FIELDS ARE SANDWICHED BETWEEN P_L PROJECTORS, THUS WE HAVE

$$P_L \left(\frac{i}{\hat{p} - M} \right)_{ee} P_L = P_L U_{ei}^* \frac{i}{\hat{p} - m_i} U_{ei} P_L$$

$$= P_L \frac{i(\hat{p} + m_i)}{p^2 - m_i^2} P_L U_{ei}^{*2} \simeq P_L \frac{i m_i}{p^2} U_{ei}^{*2}$$

$$= P_L \cdot \frac{i M_{ee}}{p^2}$$

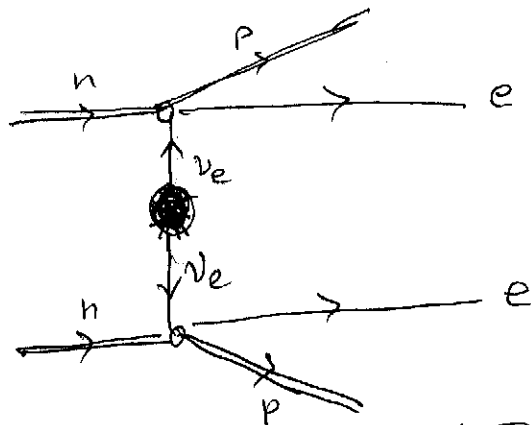
THUS, THE AMPLITUDE OF THE TRANSITION WHEN THE NEUTRINOS ARE VIRTUAL STATES, REPRESENTED BY



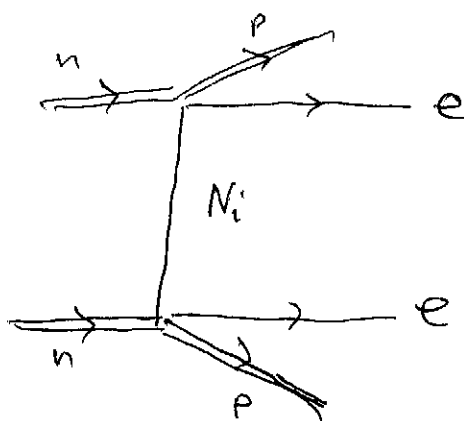
IS CONTROLLED BY M_{ee} , THE ee -ELEMENT OF THE MASS MATRIX.

(7/5)-1

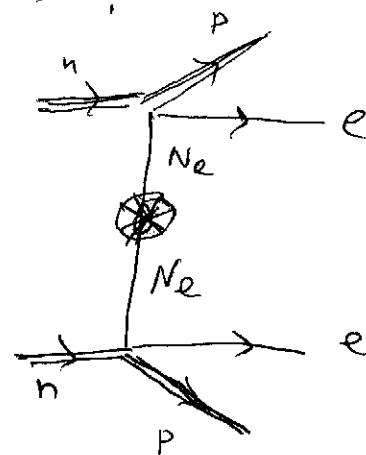
ANOTHER, QUITE USUAL DIAGRAMMATIC REPRESENTATION OF THE SAME PROCESS IS THE ONE WHERE THE INDIVIDUAL NUCLEONS AND THE USUAL NEUTRINOS ARE DISPLAYED:



IT IS UNDERSTOOD THAT THE TWO NEUTRONS COME FROM A NUCLEAR SPECIES (A, Z) AND THE TWO PROTONS MERGE INTO A FINAL NUCLEAR STATE $(A, Z-2)$. THE COMPARISON WITH THE PREVIOUS DIAGRAM, WRITTEN IN TERMS OF NUCLEONS:



or also



REVEALS AN IMPORTANT DIFFERENCE; THE MAJORANA MASS (BLACK DOT) IS A SOURCE OF LEPTON NUMBER, AS WE UNDERSTAND FROM THE 2 DIVERGING ARROWS. SIMILARLY, WE NOTE THAT THE MAJORANA SPINOR N_2 (OR N_e) DO NOT CARRY AN ARROW, WHICH MEANS THEY DO NOT CARRY ANY CONSERVED CHARGE (SUCH AS LEPTON #).

EFFECT OF RIGHT-HANDED NEUTRINOS (1/3) - II APPLICATION 2: (HEAVY)

WRITE THE LAGRANGIAN OF YUKAWA INTERACTIONS

$$\mathcal{L}(x) = - \bar{l}_L(x) Y^{(i)} N_R(x) H(x)$$

CONSIDER THE SECOND ORDER CONTRIBUTION

$$\frac{1}{2} i \int d^4x \mathcal{L}(x) \cdot i \int d^4y \mathcal{L}(y)$$

CONTRACT THE MAJORANA FIELD. THIS CAN BE DONE AS FOLLOWS:

$$\begin{aligned} \langle T [N_\alpha(x) N_\beta(y)] \rangle &= \langle T [N_\alpha(x) C_{\beta\gamma} \bar{N}_\gamma(y)] \rangle \\ &= - \langle T [N_\alpha(x) \bar{N}_\beta(y)] \rangle \cdot C_{\beta\alpha} \end{aligned}$$

THUS THE PROPAGATOR BETWEEN N and N CAN BE WRITTEN IN TERMS OF THE USUAL ONE, FROM $\bar{N}$ TO N :

$$\langle T [N_\alpha(x) \bar{N}_\beta(y)] \rangle = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \left[\frac{i}{\hat{p} - M + i\epsilon} \right]_{\alpha\beta}$$

IF $p \ll M$, THEN

$$\langle T N_\alpha(x) \bar{N}_\beta(y) \rangle \approx - \frac{i}{M} \delta_{\alpha\beta} \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} = - \frac{i}{M} \delta^4(x-y) \delta_{\alpha\beta}$$

THUS, WE GET THE EFFECTIVE CONTRIBUTION

$$i \int d^4x \mathcal{L}_{\text{eff}}(x) = + \frac{i}{2M} \int \bar{l}_a Y_{ij}^{(i)} H \cdot C_{\alpha\beta} \cdot \bar{l}_b Y_{kj}^{(i)} H$$

WHAT DOES THIS MEAN?

REWRITE :

$$L_{\text{eff}} = + \frac{1}{2} \bar{L} H \cdot Y \cdot \frac{1}{M} Y^t \cdot \bar{L} H$$

LET US CONSIDER THE VACUUM VALUE BY REPLACING

$$H = \begin{pmatrix} H^0 \\ H^- \end{pmatrix} \rightarrow \begin{pmatrix} v \\ 0 \end{pmatrix}, \Rightarrow \bar{L} H = (\bar{\nu}_L \bar{e}_L) \begin{pmatrix} v \\ 0 \end{pmatrix} = v \cdot \bar{\nu}_L$$

WE FIND

$$L_{\text{eff}} \Rightarrow + \frac{1}{2} \bar{\nu}_L \left[(v Y) \cdot \frac{1}{M} \cdot (v Y)^t \right] \cdot C \cdot \bar{\nu}_L^t$$

$$\equiv - \frac{1}{2} \bar{\nu}_L m_\nu C \bar{\nu}_L^t$$

NAMELY :

$$m_\nu = - (M_{\text{Dirac}} \cdot \frac{1}{M} \cdot M_{\text{Dirac}}^t)$$

we see that the effect of heavy ν_R exchange is to ORIGINATE A MAJORANA MASS TERM FOR LIGHT (LEFT) NEUTRINOS.

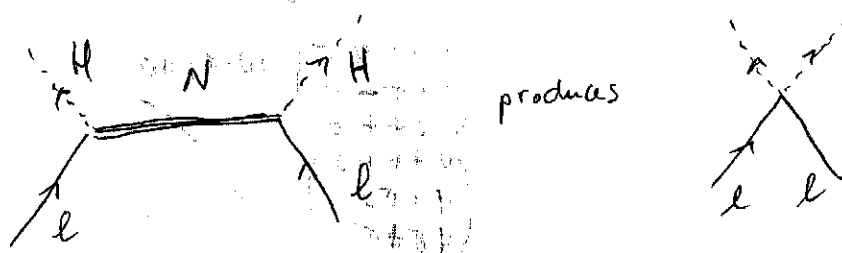
THIS IS THE WELL-KNOWN "SEE-SAW" FORMULA :

THE (GAME OF) SEE-SAW :

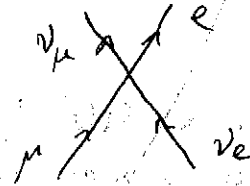
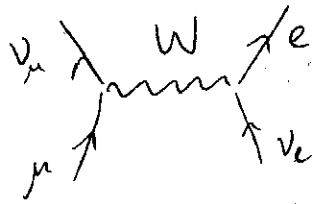


(the heavier ν_R ,
the lighter ν_L)

GRAPHICAL REPRESENTATION:



WHICH IS IN COMPLETE ANALOGY WITH THE ORIGIN OF THE FERMI LAGRANGIAN



IT IS VERY IMPORTANT TO NOTE THAT THE MASS OF ν_R , M , HAS NOTHING TO DO WITH SM MASS SCALES (e, M_W , M_Z , M_H ...). ACTUALLY, ALL OF THEM ARE PROPORTIONAL TO ν_R IN A SENSE, THERE IS JUST 1 MASS SCALE.

THUS, THE SCALE OF MASS OF ν_R HAS TO BE EXPLAINED BY PHYSICS BEYOND THE SM.

IN SHORT, RH ν ARE A VERY INTERESTING THEORETICAL POSSIBILITY. THEY GIVE RISE TO NEUTRINO MASSES FOR THE LEFT STATES. IT IS QUITE IMPORTANT THEN TO ASK WHAT WE UNDERSTAND OF ν_R .

[A MECHANISM TO PRODUCE A LEPTONIC UNBALANCE] (1/2/11)

APPLICATION 3:

- HEAVY (RIGHT HANDED) NEUTRINO DECAY CAN CAUSE EXCESS OF LIGHT LEPTONS (or ANTILEPTONS), NAMELY

$$\Gamma_{N_j \rightarrow l} \neq \Gamma_{N_j \rightarrow \bar{l}}$$

- FIRST CHECK THIS CANNOT HAPPEN AT TREE LEVEL.

$$N_j \xrightarrow{H} l = i Y_{lj} \quad N_j \xrightarrow{H} \bar{l} = i Y_{lj}^*$$

THE ABOVE VERTICES ARE THE SAME UP TO $Y \leftrightarrow Y^*$,

THUS $\Gamma_{N_j \rightarrow l}^{\text{tree}} = \Gamma_{N_j \rightarrow \bar{l}}^{\text{tree}}$

- THE INEQUALITY ARISES AT ONE LOOP. EXAMINE

$$N_j \xrightarrow{H} l + N_j \xrightarrow{H} l' \xrightarrow{H} N_k \xrightarrow{H} l + N_j \xrightarrow{H} N_k \xrightarrow{H} l$$

$$i Y_{lj} + i Y_{l'j}^* Y_{lk} Y_{lk} \cdot I_{jk}$$

LOOP FACTOR, NAMELY A FUNCTION OF M_j AND M_k .

THE INTERFERENCE TERM BETWEEN TREE AND LOOP IN $\Gamma_{N_j \rightarrow l}$ CONTAINS

$$\text{Re}[q \cdot I_{jk}] = \text{Re}[q] \text{Re}[I_{jk}] - \text{Im}[q] \cdot \text{Im}[I_{jk}]$$

WHERE :

$$q = Y_{lj}^* Y_{l'j}^* Y_{lk} Y_{lk}$$

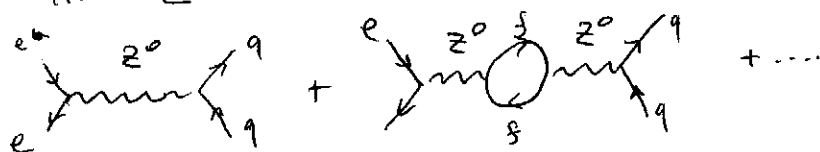
- SINCE $q \rightarrow q^*$ FOR $\Gamma_{N_j \rightarrow \bar{l}}$, WE SEE THAT $\text{Im}[I_{jk}] \neq 0$ IMPLIES THAT $\Gamma_{N_j \rightarrow l} \neq \Gamma_{N_j \rightarrow \bar{l}}$.

EXERCISES

- (1) THIS PHENOMENON MUST DEPEND ON THE VIOLATION OF LEPTONIC NUMBER ~~OF~~ OF RIGHT-HANDED NEUTRINOS. CAN YOU IDENTIFY THIS VIOLATION IN THE PREVIOUS FEYNMAN DIAGRAMS?
(HINT: CONSIDER $\nu\bar{\nu}\bar{\nu}$ DIAGRAM AS THE PROTOTYPE)

- (2) CONVINCING YOURSELF THAT IN GENERAL $\Im m [g_k] \neq 0$

(HINT: YOU MAY CONSIDER THE ANALOGY OF THE LAST DIAGRAM, THE WAVEFUNCTION DIAGRAM, WITH THE SITUATION IN Z^0 DECAY AT LEP:



THE SECOND DIAGRAM HAS TO CONTRIBUTE TO MODIFY THE TREE LEVEL EXPRESSION OF THE PROPAGATOR, $\frac{1}{s - M_Z^2}$, THAT IS INFINITE AT $s = M_Z^2$

SOME TECHNOLOGY WITH DIRAC MATRICES

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu} \mathbb{1}_{4 \times 4}$$

γ -matrices. (These usually omitted)

$$\gamma_0^\dagger = \gamma^0, \gamma_i^\dagger = -\gamma^i$$

Generate $SO(1,3)$

chirality. Anticommutator with γ -matrices, thus commutes with $\Sigma_{\mu\nu}$

Charge conjugation. Realizes the equivalence of $-\gamma^t$ & γ .

Passage between various representations. $C^\dagger = -C$ and $C^\dagger C = 1$ are universal

Fierz identity for V-A combination.

$$\Sigma_{\mu\nu} \equiv \frac{i}{4} [\gamma_\mu, \gamma_\nu]$$

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$C(-\gamma_\mu^\dagger)C^{-1} = \gamma_\mu$$

$$\gamma'_\mu = U\gamma_\mu U^{-1} \Rightarrow C' = UC U^\dagger$$

$$(\gamma_\mu P_L)_{ab} \cdot (\gamma^\mu P_L)_{cd} = -(\gamma_\mu P_L)_{ad} \cdot (\gamma^\mu P_L)_{cb}$$

DIRAC (OR NON-RELATIVISTIC) REPRESENTATION

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}$$

WEYL (OR ULTM-RELATIVISTIC) REPRESENTATION

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad C = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}$$

EXERCISES: (1) find U that realizes the equivalence of Dirac & Weyl reps

(2) Show that the above sets of matrices are OK

(3) Calculate γ^5 , $\Sigma^i = \frac{1}{2}\epsilon^{ijk}\Sigma_{jk}$, $J^i = \Sigma^{0i}$.