Matrix Canonical Forms

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Canonical forms for congruence and *congruence

- Congruence $A \to SAS^T$ (change variables in quadratic form $x^T Ax$)
- *Congruence $A \to SAS^*$ (change variables in Hermitian form $x^* Ax$)
- Congruence and *congruence are simpler than similarity: no inverses; identical row and column operations for congruence (complex conjugates for *congruence).
- The singular and nonsingular canonical structures are fundamentally different.

Example: *Congruence for Hermitian matrices

- Sylvester's Inertia Theorem (1852): Two Hermitian matrices are *congruent if and only if they have the same number of positive eigenvalues and the same number of negative eigenvalues (and hence also the same number of zero eigenvalues).
- Reformulate: Two Hermitian matrices are *congruent if and only if they have the same number of eigenvalues on each of the two open rays $\{te^{i0} : t > 0\}$ and $\{te^{i\pi} : t > 0\}$ in the complex plane.
- Canonical form: $(e^{i0} I_{n_+}) \oplus (e^{i\pi} I_{n_-}) \oplus 0_{n_0}$

Example: *Congruence for normal matrices

- Unitary *congruence: Two normal matrices are unitarily *congruent (unitarily similar!) if and only if they have the same eigenvalues.
- Ikramov (2001): Two normal matrices are *congruent if and only if they have the same number of eigenvalues on each open ray $\{te^{i\theta} : t > 0\}$, $\theta \in [0, 2\pi)$ in the complex plane.
- Canonical form: $(e^{i\theta_1} I_{n_{\theta_1}}) \oplus \cdots \oplus (e^{i\theta_k} I_{n_{\theta_k}}) \oplus 0_{n_0}$, $0 \leq \theta_1 < \cdots < \theta_k < 2\pi$
- What comes next? Find a theorem about *congruence of general n-by-n matrices that includes Sylvester's and Ikramov's theorems as special cases. Find an analogous theorem for congruence.

Example: Congruence of complex symmetric matrices

- Two complex symmetric matrices are congruent if and only if they have the same rank. Why?
- We know that if $A = A^T$ then there is a unitary U such that $A = U\Sigma U^T$. If $\operatorname{rank} A = r$, let $D = \operatorname{diag}(\sqrt{\sigma_1}, \ldots, \sqrt{\sigma_r}, 1, \ldots, 1)$. Then $A = (UD)(I_r \oplus 0_{n-r})(UD)^T$
- Canonical form: $I_r \oplus 0_{n-r}$
- What comes next? Find a theorem about congruence of general n-by-n matrices that includes this observation as a special case.

Nullities are *congruence and congruence invariants

- rank A = rank SAS^*
- $\dim N(A) = \dim N(SAS^*) = \dim N(SA^*S^*) = \dim N(A^*)$
- Moreover,

$$\dim(N(A) \cap N(A^*)) = \dim(N(SAS^*) \cap N(SA^*S^*))$$

- rank A = rank SAS^T
- $\dim N(A) = \dim N(SAS^T) = \dim N(SA^TS^T) = \dim N(A^T)$
- Moreover,

$$\dim(N(A) \cap N(A^T)) = \dim(N(SAS^*) \cap N(SA^TS^*))$$

- These observations are the key to the *regularization algorithm*:
- Reduce A by congruence (respectively, *congruence) to the direct sum of a nonsingular part and a singular part. Deal with each part separately.

Regularization for *congruence and congruence

- Each singular A is *congruent to $\mathcal{A} \oplus \mathcal{S}$ in which
  - $\mathcal{A}$ is nonsingular
  - $\mathcal{S} = J_{n_1}(0) \oplus \cdots \oplus J_{n_p}(0)$
  - block sizes n_i uniquely determined by the *congruence class of A.
  - *congruence class of $\mathcal{A}$ uniquely determined by the *congruence class of A
- Same for congruence.

Regularization algorithm: Step 1

- **Step 1:** Choose nonsingular S so the top rows of SA are independent and the bottom m_1 rows are zero, then form SAS^*; partition it so that the upper left block is square:

$$A \longmapsto SA = \begin{bmatrix} A' \\ 0 \end{bmatrix} \quad \leftarrow \text{independent rows}$$

$$\longmapsto SAS^* = \begin{bmatrix} A'S^* \\ 0 \end{bmatrix} = \left[\begin{array}{c|c} M & N \\ \hline 0 & 0_{m_1} \end{array}\right] \quad (M \text{ is square})$$

- $m_1 = \dim N(A) = \dim N(A^*) =$ the *nullity of* A
- $m_2 = \operatorname{rank} N$
- $m_1 - m_2 =$ nullity of $N = \dim(N(A) \cap N(A^*)) =$ the *normal nullity* of A
- $m_2 =$ the *non-normal nullity* of $A =$ the nullity of M
- Same for congruence.

Regularization algorithm: Step 2

- **Step 2:** Choose nonsingular R so the top rows of RN are zero and the bottom m_2 rows are independent:

$$RN = \begin{bmatrix} 0 \\ E \end{bmatrix} \quad \leftarrow m_2 \text{ independent rows}$$

- $$\left[\begin{array}{c|c} M & N \\ \hline 0 & 0_{m_1} \end{array}\right] \longmapsto (R \oplus I) \left[\begin{array}{c|c} M & N \\ \hline 0 & 0_{m_1} \end{array}\right] (R \oplus I)^*$$

$$= \left[\begin{array}{c|c} RMR^* & RN \\ \hline 0 & 0_{m_1} \end{array}\right] = \left[\begin{array}{cc|c} A_{(1)} & B & 0 \\ C & D & E \\ \hline \multicolumn{2}{c|}{0} & 0_{m_1} \end{array}\right] \begin{array}{l} \}m_2 \\ \}m_1 \end{array}$$

D is m_2-by-m_2 and $A_{(1)}$ is strictly smaller than A.

- Same for congruence.

The regularizing decomposition

- If $A_{(1)}$ is nonsingular or missing, stop.
- If $A_{(1)}$ is present and singular, repeat steps 1 and 2 on $A_{(1)}$ to obtain m_3 (the nullity of $A_{(1)}$) and m_4 (the non-normal nullity of $A_{(1)}$). Repeat (for τ steps, say) until either a nonsingular block ($m_{2\tau+1} = 0$) or an empty block is obtained.
- A singular $A \in M_n$ is *congruent to $\mathcal{A} \oplus \mathcal{S}$ in which the *regular part* $\mathcal{A}$ is nonsingular and

$$\mathcal{S} = J_1^{[m_1 - m_2]} \oplus J_2^{[m_2 - m_3]} \oplus \cdots \oplus J_{2\tau-1}^{[m_{2\tau-1} - m_{2\tau}]} \oplus J_{2\tau}^{[m_{2\tau}]}$$

- $J_k^{[p]} := J_k \oplus \cdots \oplus J_k$ (p direct summands)
- The integers $m_1 \geq m_2 \geq \cdots \geq m_{2\tau} \geq 0$, as well as the *congruence class of $\mathcal{A}$, are uniquely determined by the *congruence class of A.

Example 1

- $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} M & N \\ [0\ 0] & 0 \end{bmatrix}$; $M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $N = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- $R = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $RN = \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ E \end{bmatrix}$
- $RMR^* = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} A_{(1)} & B \\ C & D \end{bmatrix}$
- $A_{(1)} = [-1]$, $m_1 = 1$, $m_2 = 1$, $m_3 = 0$.
- Same for congruence
- A is *congruent to $[-1] \oplus J_2$ and ^{T}congruent to $[1] \oplus J_2$:
- $[i]\,[-1]\,[i] = [+1]$

Example 2

- $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} M & N \\ [0\ 0] & 0 \end{bmatrix}$; $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $N = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- $R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $RN = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ E \end{bmatrix}$
- $RMR^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A_{(1)} & B \\ C & D \end{bmatrix}$
- $A_{(1)} = [1]$, $m_1 = 1$, $m_2 = 1$, $m_3 = 0$.
- Same for congruence
- A is both *congruent and ^{T}congruent to $[1] \oplus J_2$.

Examples 1 & 2

- The matrices in Examples 1 and 2

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are congruent since they are both congruent to $[1] \oplus J_2$.

- However, they are not *congruent:

$$A \text{ is } {}^*\text{congruent to } [-1] \oplus J_2,$$

$$B \text{ is } {}^*\text{congruent to } [+1] \oplus J_2, \text{ and}$$

$$[s]\,[-1]\,[\bar{s}] = |s|^2\,[-1] \neq [+1]$$

The regular part

- Define: $A^{-*} := \left(A^{-1}\right)^*$. The matrix $A^{-*}A$ is the *cosquare of A.
- If $A \to S^*AS$, then

$$A^{-*}A \to \left(S^{-1}A^{-*}S^{-*}\right)\left(S^*AS\right) = S^{-1}\left(A^{-*}A\right)S$$

- If $A^{-*}A \to S^{-1}\left(A^{-*}A\right)S$, then

$$A^{-*}A \to S^{-1}\left(A^{-*}S^{-*}S^*A\right)S^{-1} = \left(S^*AS\right)^{-*}\left(S^*AS\right)$$

- *congruence of A corresponds to similarity of $A^{-*}A$
- The Jordan Canonical Form of the *cosquare of A is a *congruence invariant of A.
- Same for congruence and cosquares $A^{-T}A$

The JCF of *cosquares and cosquares

- $$\left(A^{-*}A\right)^{-1} = A^{-1}A^{*} \overset{s}{\sim} A^{*}A^{-1} = \left(A^{-*}A\right)^{*} \overset{s}{\sim} \overline{A^{-*}A}$$
- The Jordan Canonical Form of $A^{-*}A$ is a direct sum of blocks of two types:

  $$\begin{bmatrix} J_k(\mu) & 0 \\ 0 & J_k(1/\bar{\mu}) \end{bmatrix} \text{ with } 0 < |\mu| < 1, \text{ and } J_k(\lambda) \text{ with } |\lambda| = 1$$
- $$\left(A^{-T}A\right)^{-1} = A^{-1}A^{T} \overset{s}{\sim} A^{T}A^{-1} = \left(A^{-T}A\right)^{T} \overset{s}{\sim} A^{-T}A$$
- The Jordan Canonical Form of $A^{-T}A$ is a direct sum of blocks of two types:

  $$\begin{bmatrix} J_k(\mu) & 0 \\ 0 & J_k(1/\mu) \end{bmatrix} \text{ with } 0 \neq \mu \neq (-1)^{k+1}, \text{ and } J_k((-1)^{k+1})$$

Canonical blocks for *congruence and congruence

- $$\Gamma_k = \begin{bmatrix} & & & & & \iddots \\ & & & & \iddots & \iddots \\ & & & -1 & -1 & \\ & & 1 & 1 & & \\ & -1 & -1 & & & \\ 1 & 1 & & & & \end{bmatrix}_{k \times k} \qquad \text{(real)}$$

$$\Gamma_1 = [1], \quad \Gamma_2 = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

- $\Gamma_k^{-T}\Gamma_k$ is similar to $J_k((-1)^{k+1})$, so Γ_k is indecomposable under congruence or *congruence.

Canonical blocks for *congruence and congruence

- $$\Delta_k = \begin{bmatrix} & & & 1 \\ & & \diagup & i \\ & 1 & \diagup & \\ 1 & i & & \end{bmatrix}_{k \times k} \qquad \text{(symmetric)}$$

  $$\Delta_1 = [1], \quad \Delta_2 = \begin{bmatrix} 0 & 1 \\ 1 & i \end{bmatrix}$$

- $\Delta_k^{-*}\Delta_k$ is similar to $J_k(1)$, so Δ_k is indecomposable under *congruence.

Canonical form for *congruence

- Each square complex A is *-congruent to a direct sum, determined uniquely up to permutation of summands, of matrices of three types

$$J_k(0), \quad e^{i\theta}\Delta_k, \quad \begin{bmatrix} 0 & I_k \\ J_k(\mu) & 0 \end{bmatrix}$$

  in which $0 \le \theta < 2\pi$ and $0 < |\mu| < 1$.
- There is one block $\pm e^{i\theta}\Delta_k$ for each block $J_k(\lambda)$ of $\mathcal{A}^{-*}\mathcal{A}$ with $\lambda = e^{2i\theta}$. The $\pm$ is determined by the inertias of certain Hermitian matrices (2 algorithms).
- There is one block $\begin{bmatrix} 0 & I_k \\ J_k(\mu) & 0 \end{bmatrix}$ for each pair of blocks $J_k(\mu) \oplus J_k(\bar{\mu}^{-1})$ of $\mathcal{A}^{-*}\mathcal{A}$ with $|\mu| > 1$.
- The angles θ in the coefficients of the Δ_k blocks are the *canonical angles of A of order k*

Canonical form for congruence

- Each square complex A is congruent to a direct sum, determined uniquely up to permutation of summands, of matrices of the three types

$$J_k(0), \quad \Gamma_k, \quad \begin{bmatrix} 0 & I_k \\ J_k(\mu) & 0 \end{bmatrix}$$

  in which $\mu \neq 0$, $\mu \neq (-1)^{k+1}$, and μ is determined up to replacement by $1/\mu$.
- There is one block $\begin{bmatrix} 0 & I_k \\ J_k(\mu) & 0 \end{bmatrix}$ for each pair of blocks $J_k(\mu) \oplus J_k(\mu^{-1})$ of $\mathcal{A}^{-T}\mathcal{A}$ with $\mu \neq (-1)^{k+1}$.
- There is one block Γ_k for each block $J_k((-1)^{k+1})$ of$\mathcal{A}^{-T}\mathcal{A}$.
- Nonsingular matrices are congruent if and only if their cosquares are similar; this is NOT true for *congruence.

*Congruence to a diagonal matrix

- Matrices that are diagonalizable by *congruence must have *congruence canonical blocks that are all 1-by-1, so only blocks of the form $J_1(0) = [0]$ and $e^{i\theta}\Delta_1 = [e^{i\theta}]$ can occur in their *canonical forms: Nullity = normal nullity, and $\mathcal{A}^{-*}\mathcal{A}$ is diagonalizable with all eigenvalues of modulus 1.
- For a normal matrix, *congruence to a diagonal matrix can be achieved with a unitary matrix. The (only) *congruence invariants are the rays that its eigenvalues lie on, and the multiplicity of eigenvalues on each ray. (Ikramov's theorem)
- For a Hermitian matrix, only two rays can occur: $(0, \infty)$ and $(-\infty, 0)$. So the (only) *congruence invariants are the respective multiplicities, that is, the number of positive and the number of negative eigenvalues. (*Sylvester's Inertia Theorem*)
- The *congruence canonical form is the desired generalization of Sylvester's Inertia Theorem from Hermitian matrices to all complex square matrices.

Canonical forms for congruence: Applications

- A is congruent to A^T via S such that $S^2 = I$
- A is *congruent to A^T via S such that $S\bar{S} = I$
- Canonical pairs for any Hermitian pair via $A = H + iK$
- Canonical pairs for any symmetric/skew-symmetric pair
- Canonical form for A such that $A + A^*$ is positive (semi)definite
- Squared normal matrices: A^2 is normal, e.g., $A^2 = I$
- Zero and the field of values $F(A) = \{x^*Ax : x^*x = 1\}$
- Convexity of the rank-k numerical range (quantum error correction; Li & Sze)
- Characterization of matrices A such that $SAS^T = A \Rightarrow \det S = +1$
  - The congruence canonical form of A contains no blocks of odd size, that is, no blocks $J_k(0)$ or Γ_k with odd k. (T. Gerasimova et al.)

References

T. Gerasimova, R. Horn, and V. Sergeichuk, Matrices that are self-congruent only via matrices of determinant one.

R. Horn and V. Sergeichuk, A regularizing algorithm for matrices of bilinear and sesquilinear forms, *Linear Algebra Appl.* 412 (2006) 380-395.

R. Horn and V. Sergeichuk, Canonical matrices of bilinear and sesquilinear forms, *Linear Algebra Appl.* 428 (2008) 193-223.

C.-K. Li and N.-S. Sze, Canonical forms, higher rank numerical ranges, totally isotropic subspaces, and matrix equations, *Proc. Amer. Math. Soc.* 136 (2008), 3013-3023.

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