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Summer College on Nonequilibrium Physics from Classical to Quantum Low Dimensional Systems

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Eigenstate thermalization hypothesis and quantum thermodynamics

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Eigenstate thermalization hypothesis and quantum thermodynamics

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In collaboration with Marcos Rigol and Vanja Dunjko

Support by NSF and ONR

Thermalization in classical isolated systems $\longleftarrow$ stochasticity $\longleftarrow$ instabilities $\longleftarrow$ nonlinearity

Q: What enables thermalizarion in quantum isolated systems, linear by nature?

Quantum thermalization paradox [Srednicki (1994)]

Time evolution from an initial state $|\Psi_{\text{initial}, E}>$ $\longrightarrow$ Infinite time average of the quantum expectation value an observable $\hat{A}$:

$$\lim_{T\to\infty} T^{-1} \int_0^T dt\, \langle \Psi(t)|\hat{A}|\Psi(t)\rangle$$

$$= \lim_{T\to\infty} T^{-1} \int_0^T dt \sum_{\alpha}\sum_{\alpha'} \langle \Psi(t=0)|\alpha\rangle\langle\alpha'|\Psi(t=0)\rangle A_{\alpha\alpha'} e^{i(E_\alpha - E_{\alpha'})t/\hbar}$$

$$= \sum_{\alpha} |\langle\alpha|\Psi_{\text{initial}, E}\rangle|^2 A_{\alpha\alpha}$$

Thermalization $\longrightarrow$ time average converges to the thermal average:

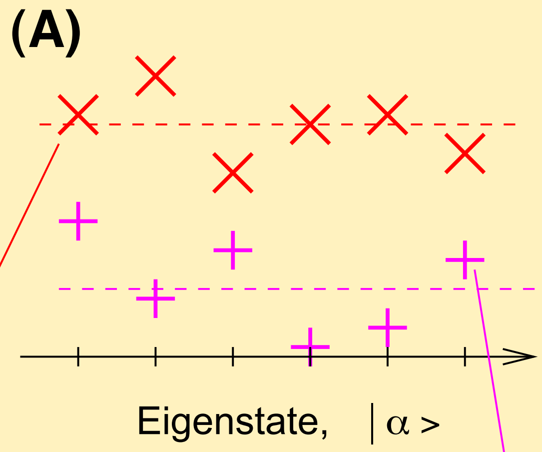
$$\sum_{\alpha} |\langle\alpha|\Psi_{\text{initial}, E}\rangle|^2 A_{\alpha\alpha} = \langle A\rangle_{\text{thermal}, T(E)}$$

Notice that while the l.h.s. is seemingly different for different initial states, r.h.s. is just a number. How could this be?

Quantum thermalization paradox, three resolutions

$$\sum_{\alpha} |\langle \alpha | \Psi_{\text{initial},E} \rangle|^2 \, A_{\alpha,\alpha} \overset{?}{=} \langle A \rangle_{\text{thermal},\ T(E)}$$

for ALL $|\Psi_{\text{initial},E}\rangle$



$A_{\alpha,\alpha}$
Diagonal matrix elements of observable $\hat{A}$

$|\langle \alpha | \Psi_{\text{initial},E} \rangle|^2$
Decomposition of $|\Psi_{\text{initial},E}\rangle$ into eigenstates $|\alpha\rangle$

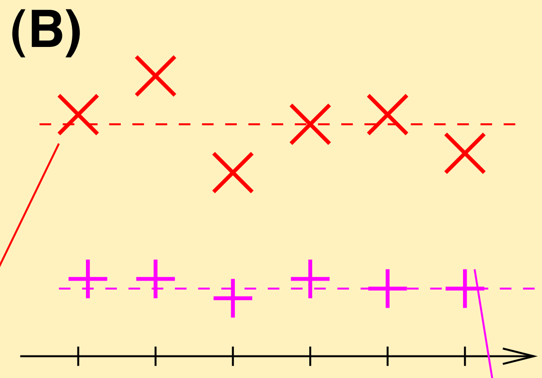
$A_{\alpha,\alpha}$ and $|\langle \alpha | \Psi_{\text{initial},E} \rangle|^2$ are decorrelated

$$\sum_{\alpha} |\langle \alpha | \Psi_{\text{initial},E} \rangle|^2 \, A_{\alpha,\alpha} \approx \sum_{\alpha} \overline{|\langle \alpha | \Psi_{\text{initial},E} \rangle|^2} \; \overline{A_{\alpha,\alpha}} = \langle A \rangle_{\text{thermal},\ T(E)}$$

Quantum thermalization paradox, three resolutions

$$\sum_{\alpha} |\langle \alpha | \Psi_{\text{initial},E} \rangle|^2 \, A_{\alpha,\alpha} \overset{?}{=} \langle A \rangle_{\text{thermal},\ T(E)}$$

for ALL $|\Psi_{\text{initial},E}\rangle$



Eigenstate, $|\alpha\rangle$

$A_{\alpha,\alpha}$

Diagonal matrix elements of observable $\hat{A}$

$|\langle \alpha | \Psi_{\text{initial},E} \rangle|^2$

Decomposition of $|\Psi_{\text{initial},E}\rangle$ into eigenstates $|\alpha\rangle$

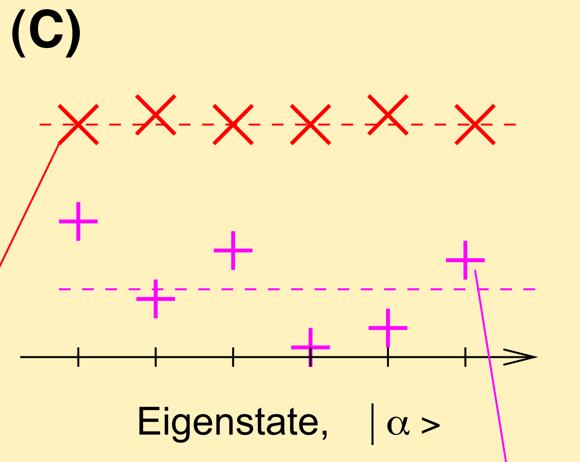
$$|\langle \alpha | \Psi_{\text{initial},E} \rangle|^2 \approx \text{const.}$$

$$\sum_{\alpha} |\langle \alpha | \Psi_{\text{initial},E} \rangle|^2 \, A_{\alpha,\alpha} \approx \sum_{\alpha} \rho_{\text{microcanon.},\ \alpha,\alpha} \, A_{\alpha,\alpha} = \langle A \rangle_{\text{thermal},\ T(E)}$$

Quantum thermalization paradox, three resolutions

$$\sum_\alpha |\langle \alpha | \Psi_{\text{initial},E}\rangle|^2 A_{\alpha,\alpha} \overset{?}{=} \langle A \rangle_{\text{thermal},\ T(E)}$$

for ALL $|\Psi_{\text{initial},E}\rangle$



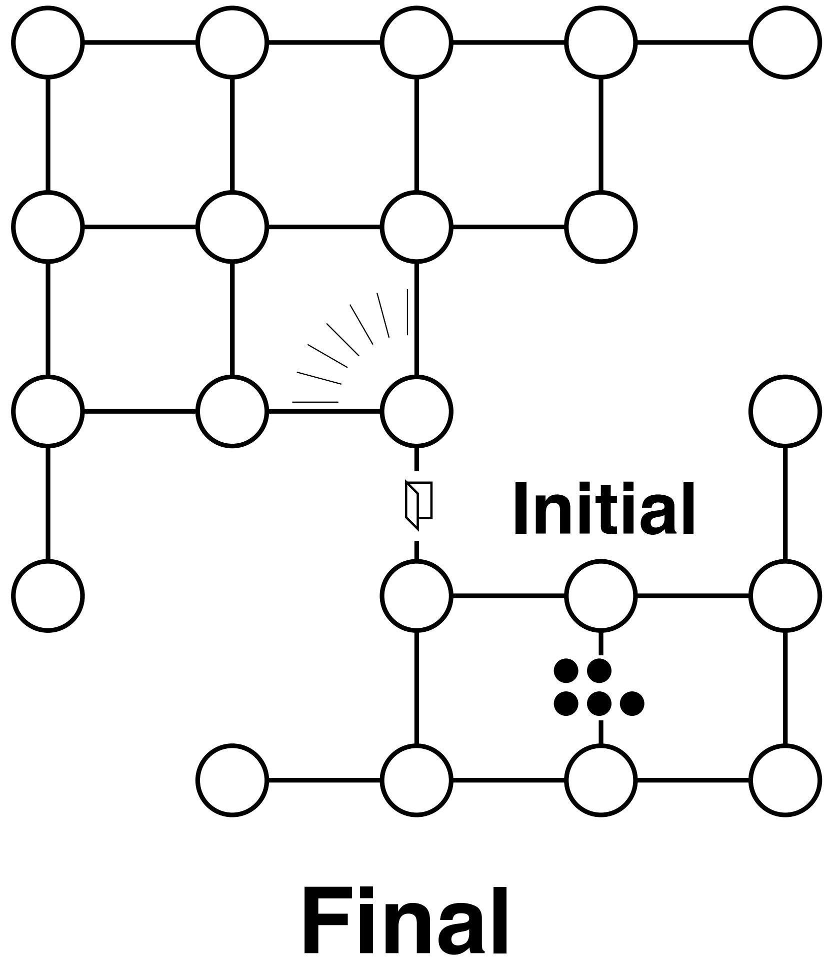
$A_{\alpha,\alpha}$
Diagonal matrix elements of observable $\hat{A}$

$|\langle \alpha | \Psi_{\text{initial},E}\rangle|^2$
Decomposition of $|\Psi_{\text{initial},E}\rangle$ into eigenstates $|\alpha\rangle$

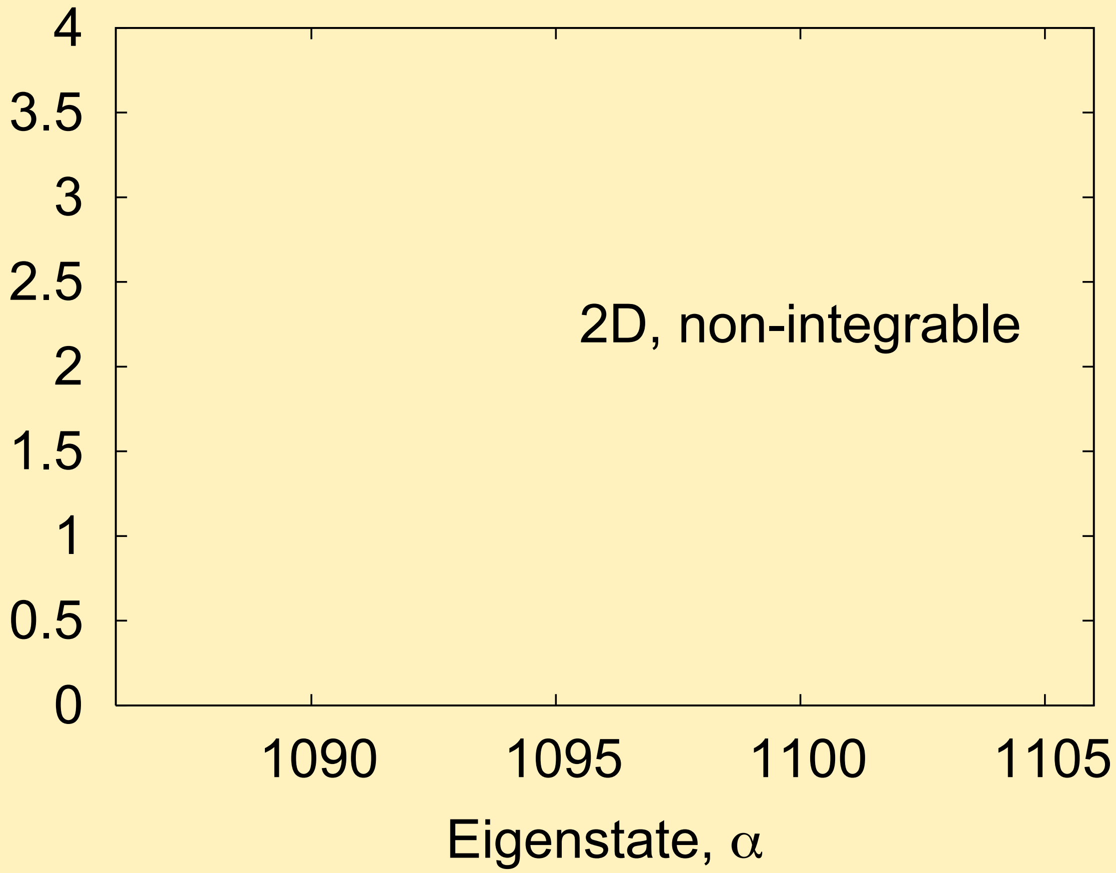
$A_{\alpha,\alpha} \approx$ const.

$$\sum_\alpha |\langle \alpha | \Psi_{\text{initial},E}\rangle|^2 A_{\alpha,\alpha} \approx \left(\sum_\alpha |\langle \alpha | \Psi_{\text{initial},E}\rangle|^2\right) A_{\text{microcan.}} = \langle A \rangle_{\text{thermal},\ T(E)}$$

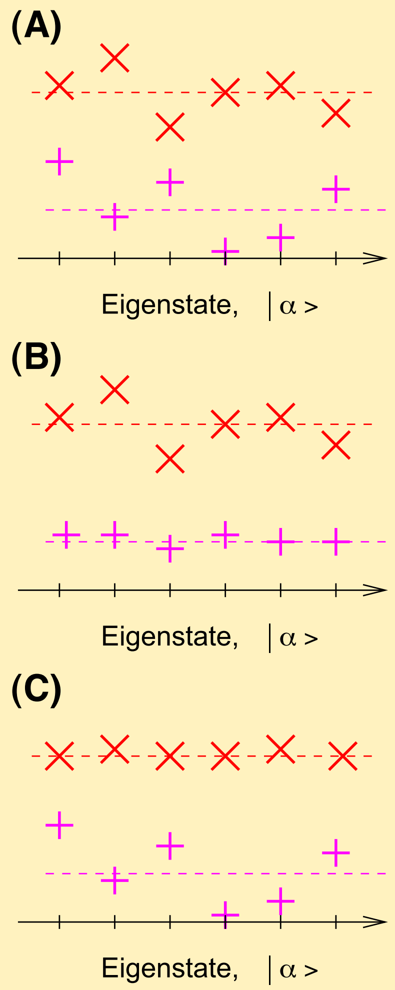
(A), (B), or (C)? Our test



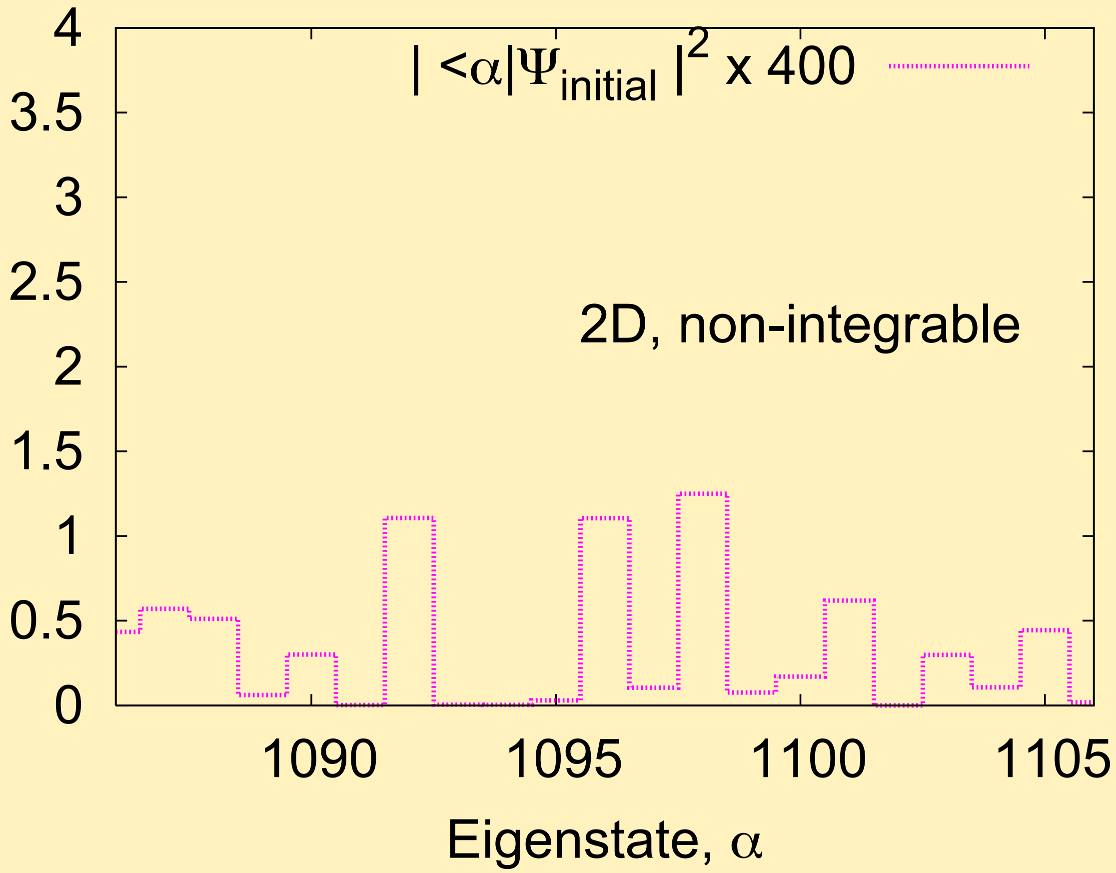
relaxation movie



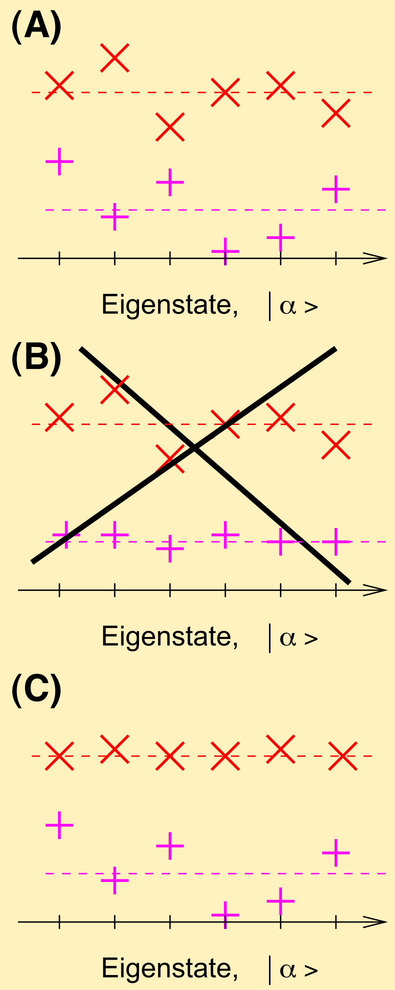
4
3.5
3
2.5
2
1.5
1
0.5
0
2D, non-integrable
1090
1095
1100
1105
Eigenstate, α



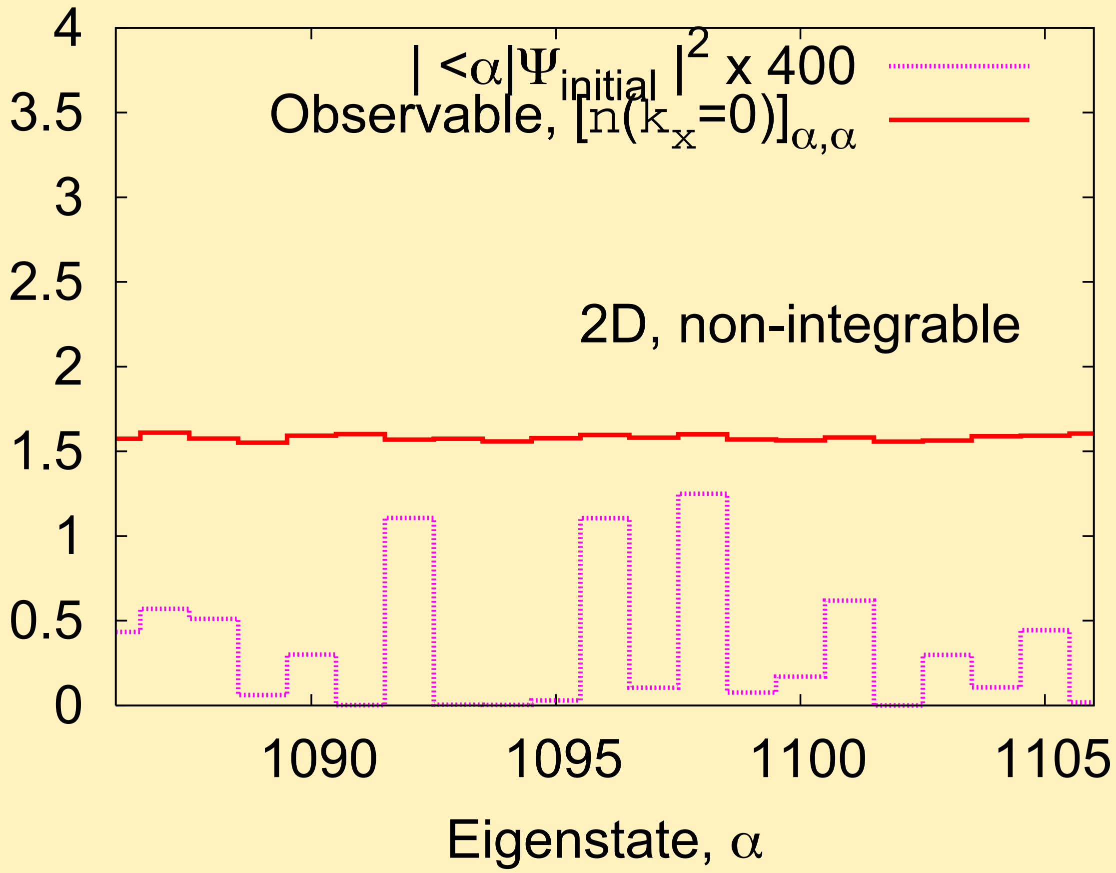
(A)
Eigenstate, |α >
(B)
Eigenstate, |α >
(C)
Eigenstate, |α >



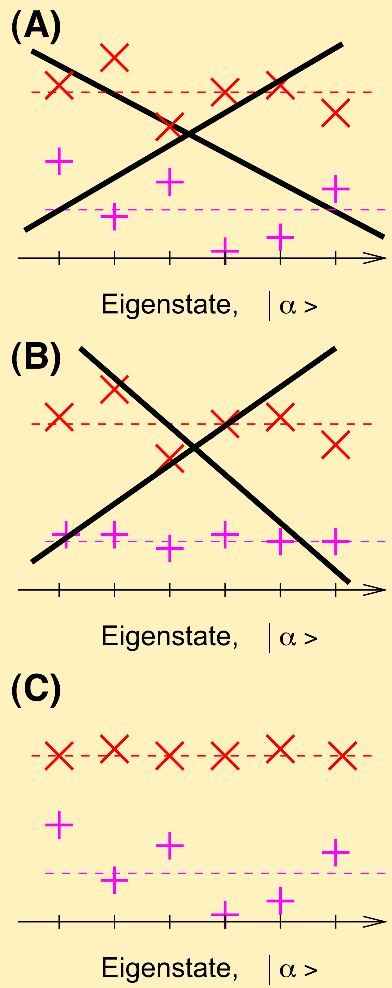
| <α|Ψinitial |2 x 400
2D, non-integrable
4
3.5
3
2.5
2
1.5
1
0.5
0
1090
1095
1100
1105
Eigenstate, α



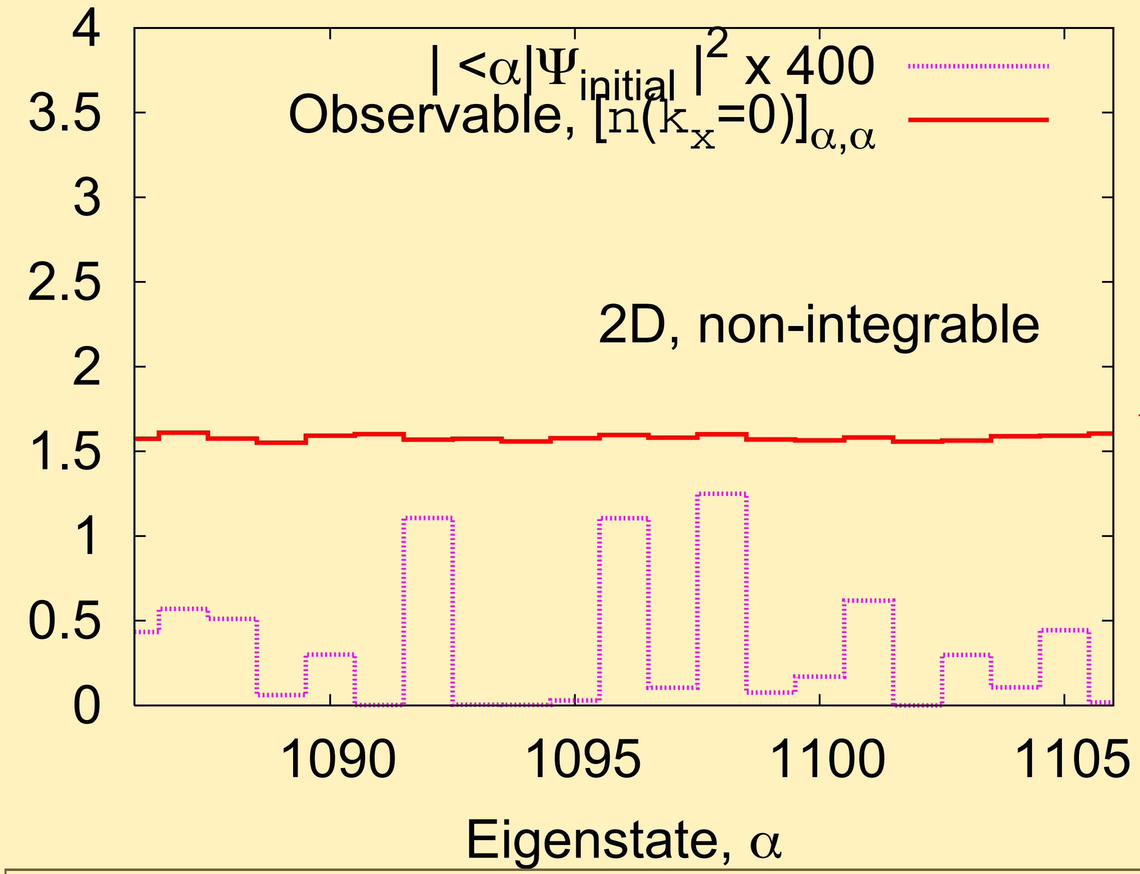
(A)
Eigenstate, |α >
(B)
Eigenstate, |α >
(C)
Eigenstate, |α >



| <α|Ψ_initial |^2 x 400
Observable, [n(k_x=0)]_α,α
2D, non-integrable
4
3.5
3
2.5
2
1.5
1
0.5
0
1090
1095
1100
1105
Eigenstate, α



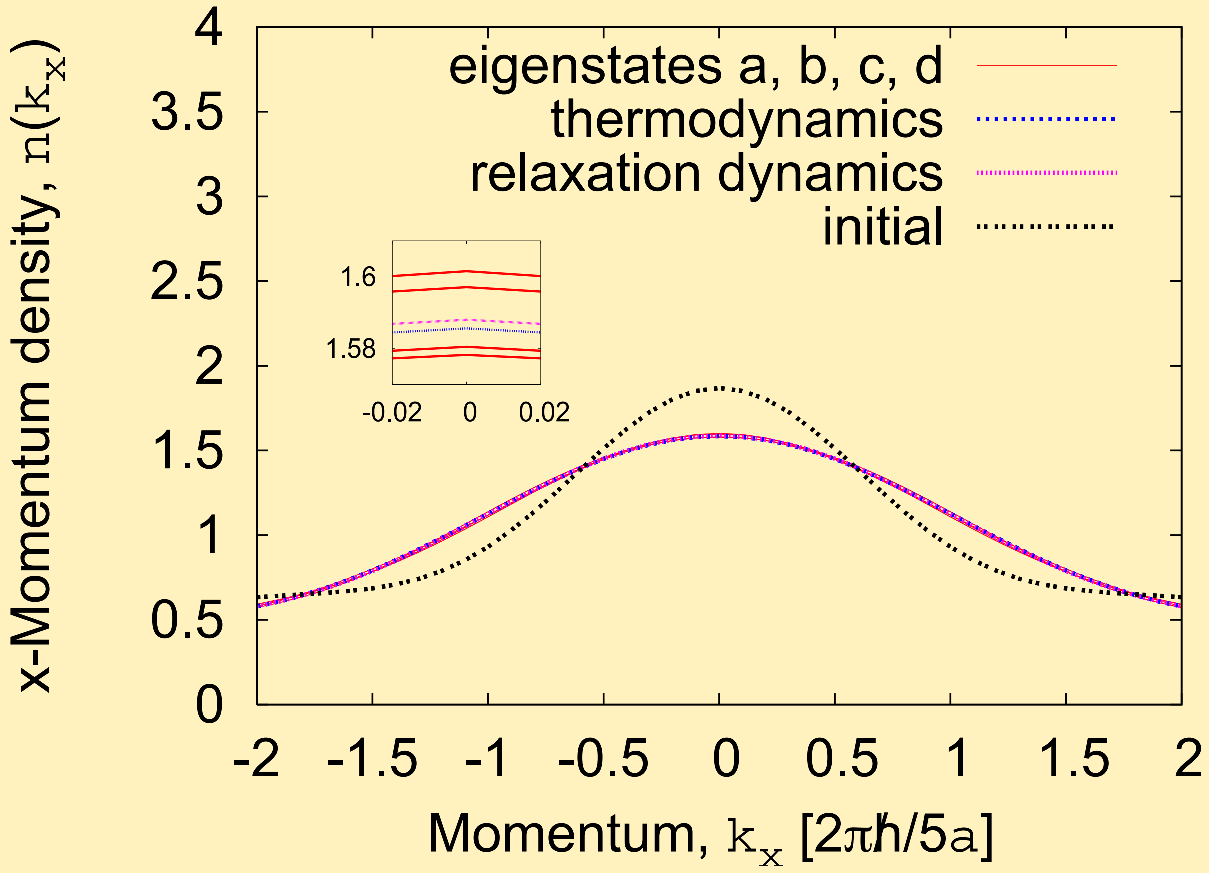
(A)
Eigenstate, |α >
(B)
Eigenstate, |α >
(C)
Eigenstate, |α >



Rigol, Dunjko, Olshanii
Nature, Apr 17 (2008)

⟵ temporal averages lose memory of initial conditions, QUANTUM ERGODICITY

Eigenstate expectation values of observables are SMOOTH functions of energy



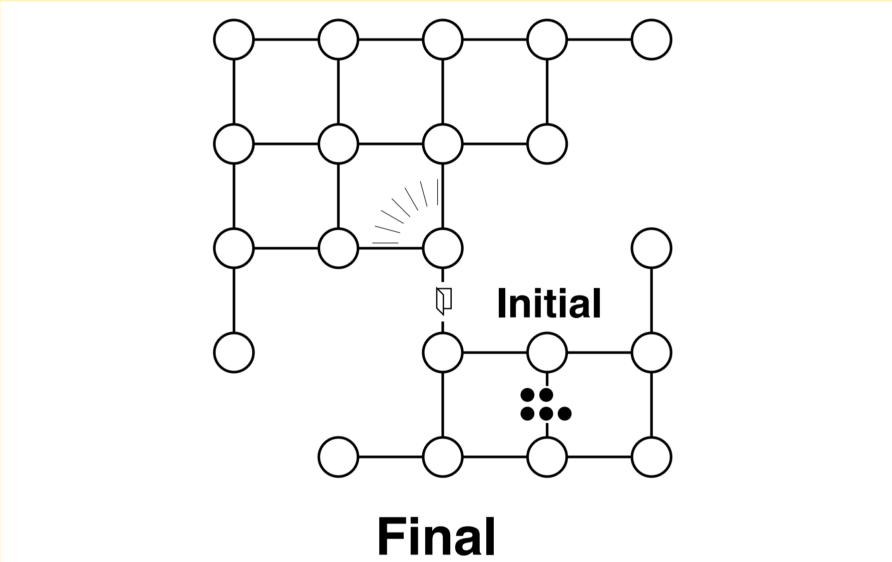
Rigol, Dunjko, Olshanii
Nature, Apr 17 (2008)

Eigenstates of close energies produce IDENTICAL momentum distributions

Comparisson with integrable case

Q: What goes wrong in the integrable case?

Comparisson with integrable case



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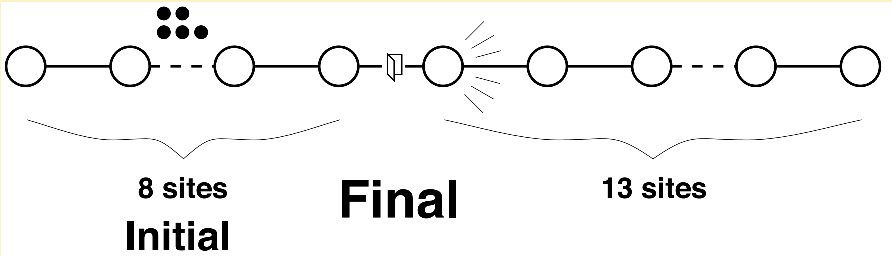
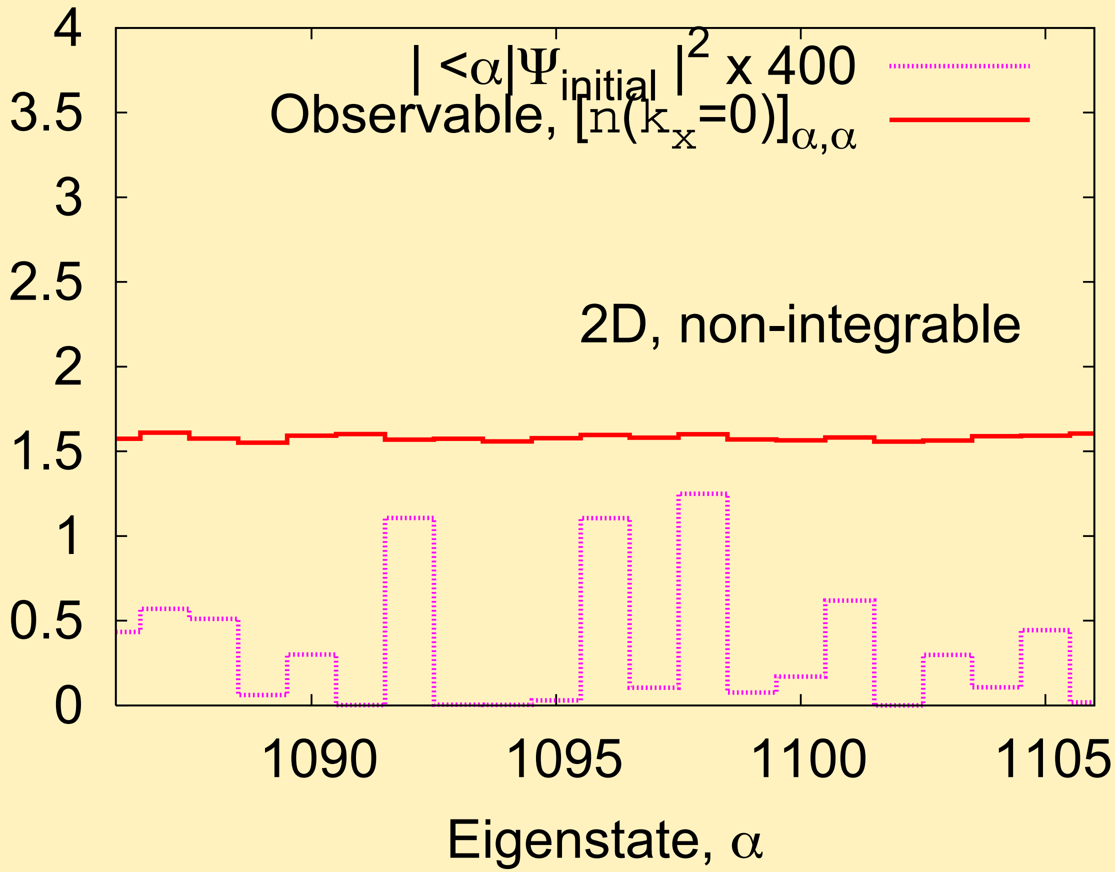


Comparisson with integrable case



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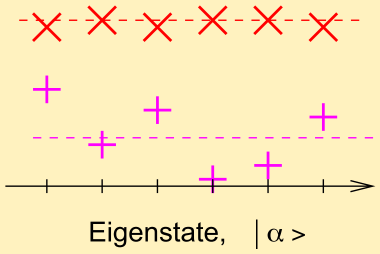
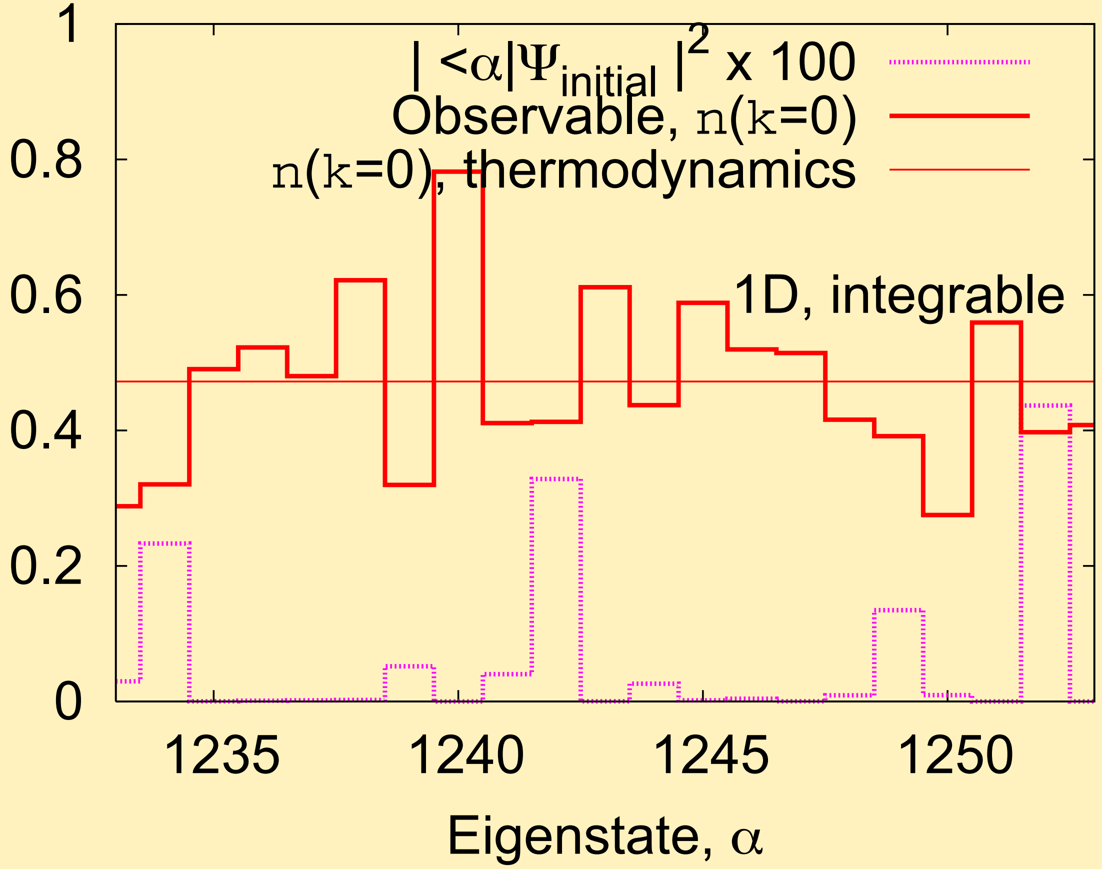
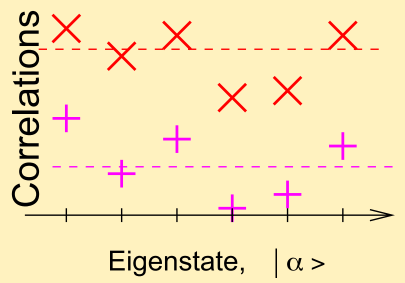


Comparisson with integrable case



Rigol, Dunjko, Olshanii
Nature, Apr 17 (2008)

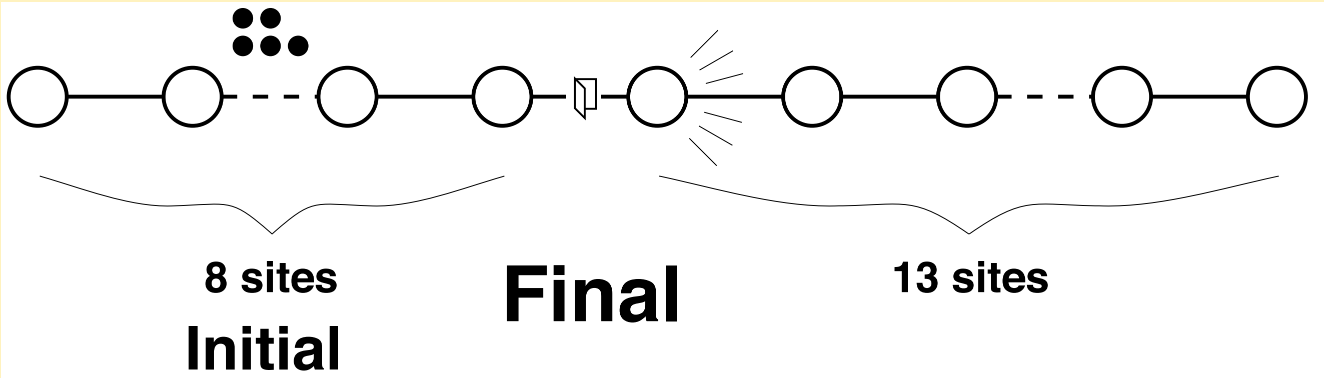
←— correlations, memory of initial conditions in temporal averages



Side remark: oscillations or equilibrium

Q: Isolated integrable systems: undamped oscillations or modified equilibrium?

Side remark on integrable systems: oscillations or equilibrium?



Cf. 2D

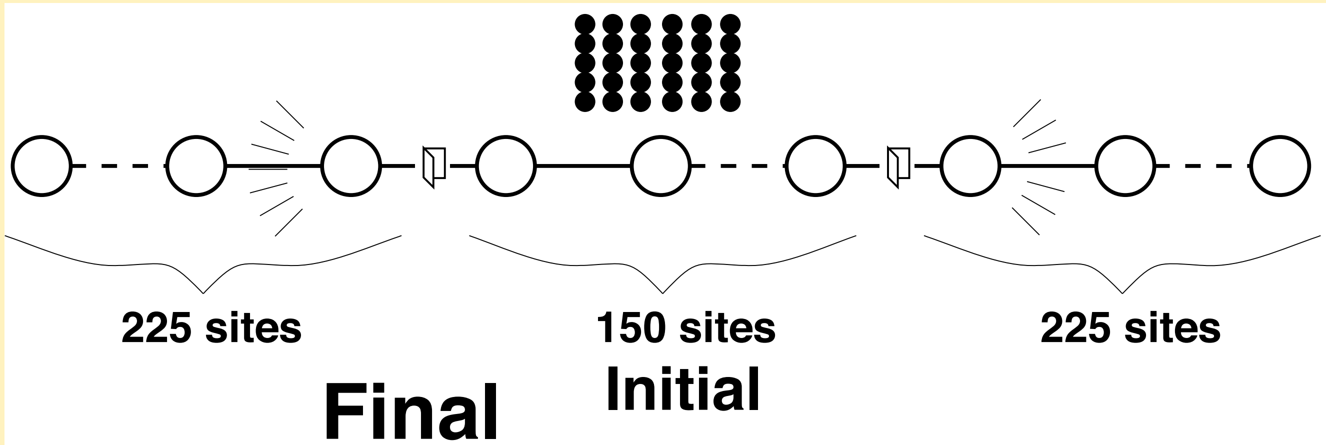
$N = 5$

init. ocpd. sites $= 8$

init. empty sites $= 13$

doors $= 1$

$\Downarrow$



Macroscopic

to test thermodynamics

Side remark on integrable systems: oscillations or equilibrium?

relaxation movie

Our integrable system does relax to some equilibrium. Which one?

Side remark on integrable systems: oscillations or equilibrium?

- All integrals of motion must be included in the Gibbs exponent

$$\exp\left[-\left(\hat{H}-\mu\hat{N}\right)/k_BT\right] \rightarrow \exp\left[\sum_i -\lambda_i\,\hat{I}_i\right]$$

$\hat{I}_i$ =(complete set of integrals of motion)

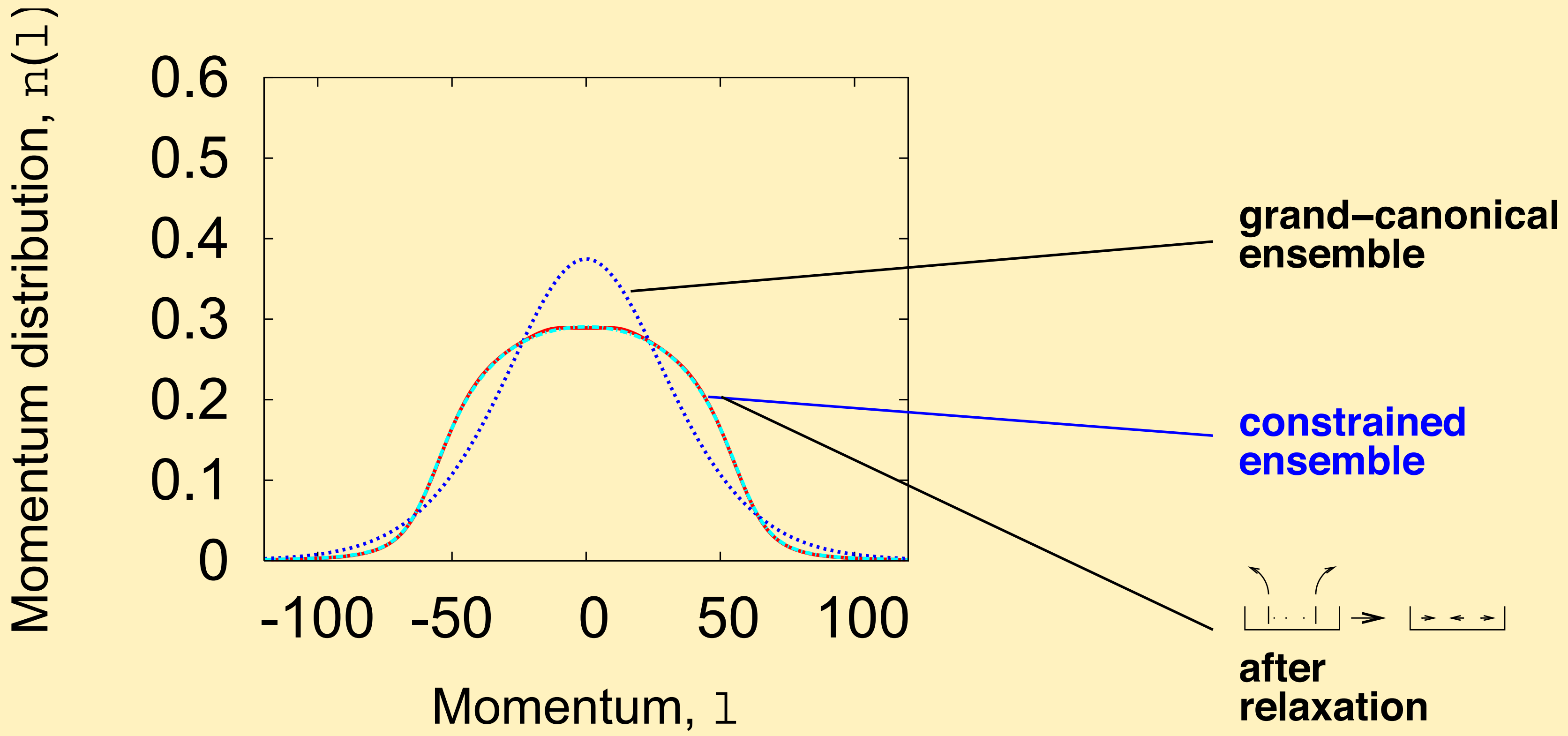
- Integrals of motion: moments of the momentum distribution of the underlying free Fermi gas = complicated bosonic operators. Example:

$$\frac{1}{4}\left(\frac{2\pi}{L}\right)^4 \hat{I}_4 = \frac{3}{2}\hat{N} + \frac{1}{J}\hat{H} + \frac{1}{4}\sum_{i=1}^{L}\left(\hat{b}_i^\dagger(1-2\hat{b}_{i+1}^\dagger\hat{b}_{i+1})\,\hat{b}_{i+2} + \text{h.c.}\right).$$

M. Rigol, V. Dunjko, V. Yurovsky, and M. Olshanii, Phys. Rev. Lett. **98**, 050405 (2007)

Side remark on integrable systems: oscillations or equilibrium?

◇ Excellent agreement with ab initio numerics

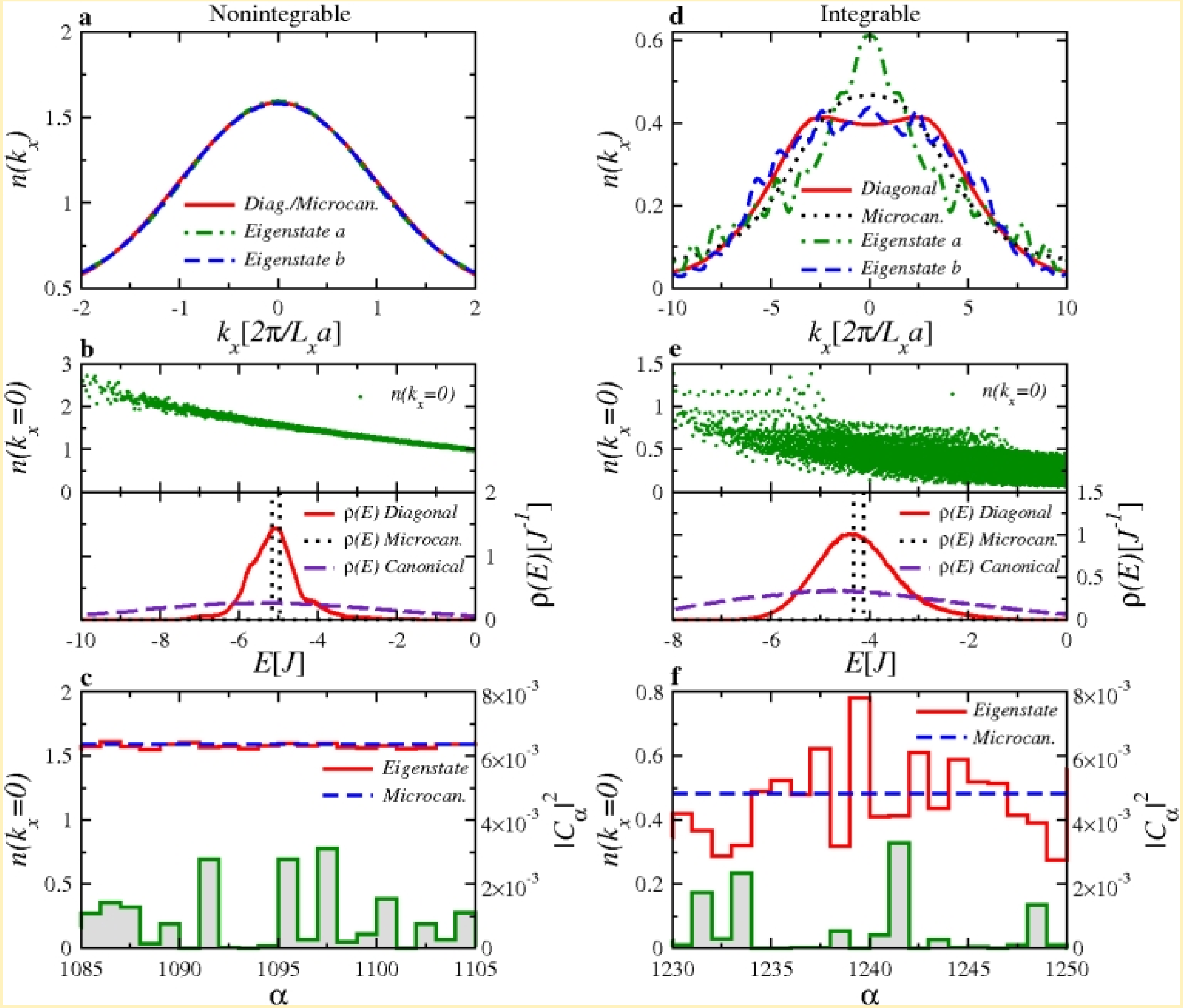


Side remark on integrable systems: oscillations or equilibrium?

Integrable gas relaxes to a *constrained* equilibrium

Follow-up: constrained equilibrium has been since recently justified for the Luttinger system (Casalilla (2006))

Overview of results: both non-integrable and integrable



Eigenstate Themalization Hypothesis

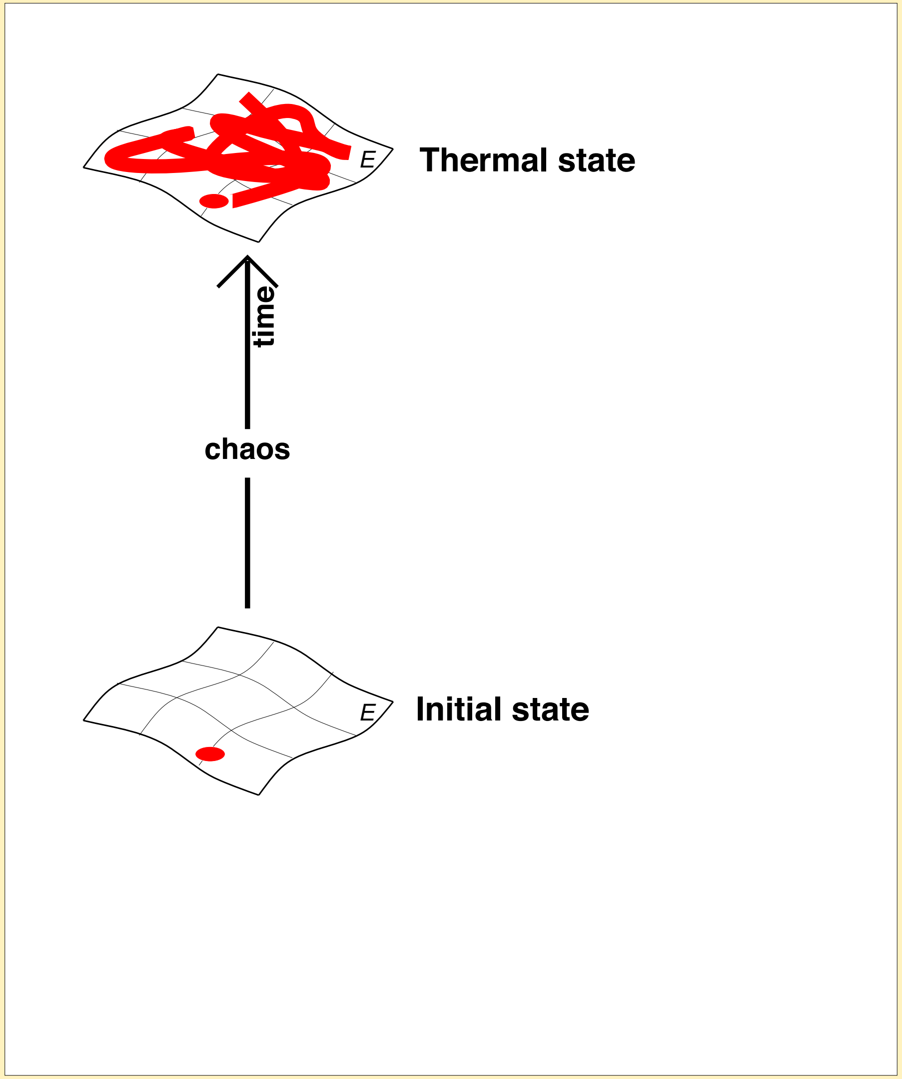
PR version: For a generic macroscopic system, quantum expectation value $\langle\alpha|\hat{A}|\alpha\rangle$ of an observable $\hat{A}$ in an eigenstate of the Hamiltonian $\hat{H}$ equals the thermal average $\langle\hat{A}\rangle_{\text{thermal},\bar{E}=E_\alpha}$ of $\hat{A}$ at the mean energy E_α:

$$\langle\alpha|\hat{A}|\alpha\rangle = \langle\hat{A}\rangle_{\text{thermal},E=E_\alpha}$$

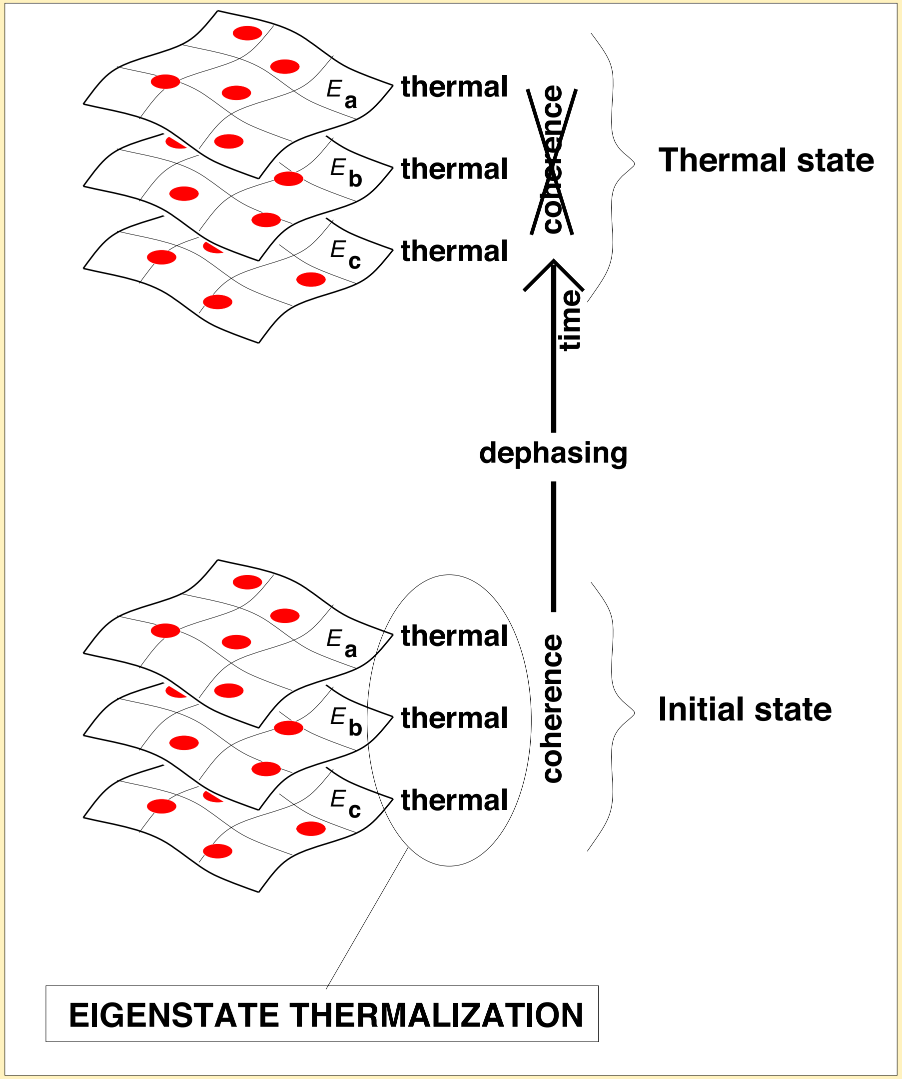
For a generic macroscopic system, expectation value $\langle\alpha|\hat{A}|\alpha\rangle$ of an observable $\hat{A}$ in an eigenstate of the Hamiltonian $\hat{H}$ with energy E_α are smooth functions of energy E_α

Deutsch, 1991; Srednicki, 1994

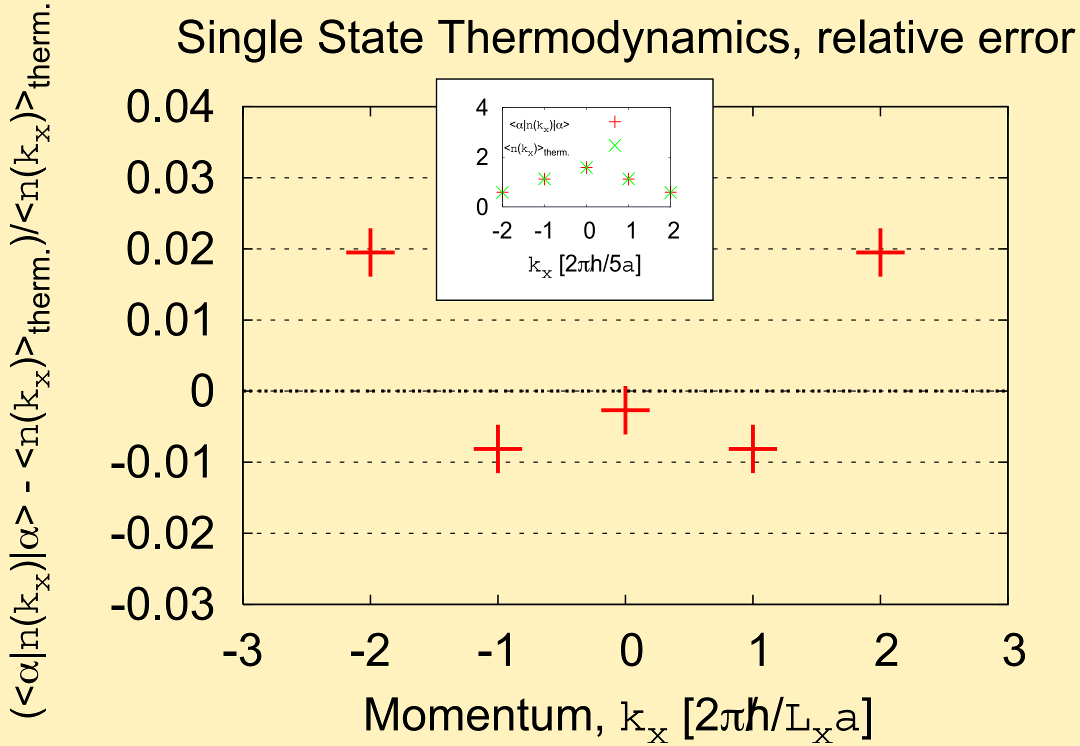
Bird's view: While classical thermalization originates from chaos, ...



… quantum thermalization follows from Eigenstate Thermalization



Implications. A new computational method: Single State Thermodynamics



Thanks to the Eigenstate Thermalization phenomenon, it is sufficient to calculate a single eigenstate of a given energy to determine the thermal expectation values of all the observables at this energy. A 10^5-fold increase in numerically accessible Hilbert space size expected.

How much has been known previosly?

"Nothing is ever discovered for the first time."

Sir Michael Berry

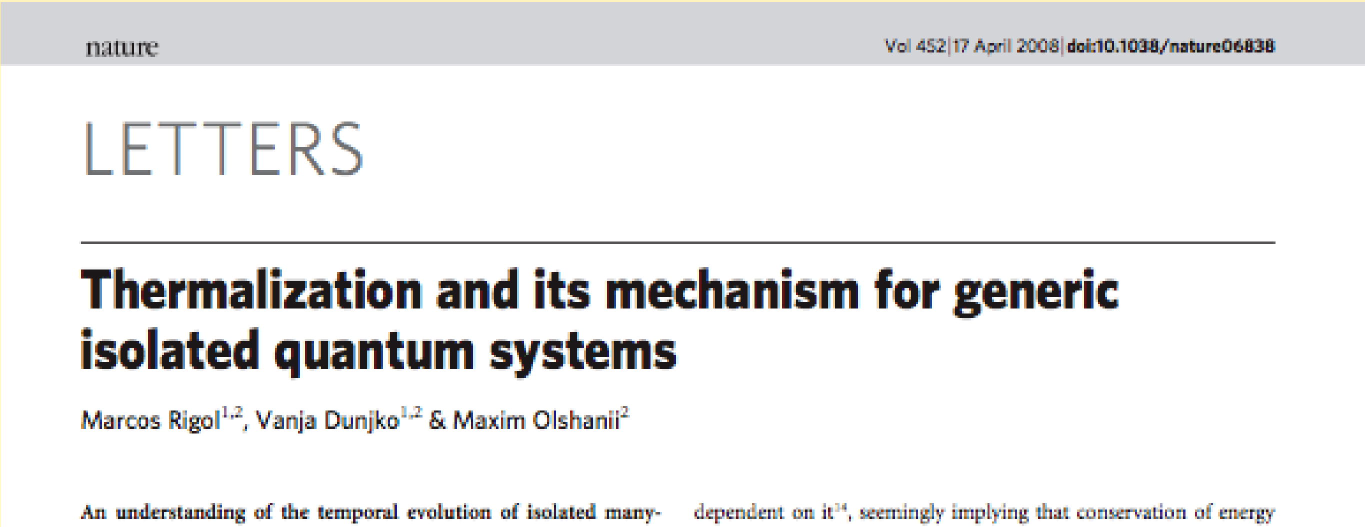
How much has been known previosly?

Quantum chaos: quantum systems with chaotic classical counterparts, semi-classical regime

Schnirelman, Zelditch, Colin de Verdière, Zworski: Suppression of the eigenstate-by-eigenstate fluctuations of the density in classically ergodic billiards at high energies

Feingold-Peres, Wilkinson: Suppression of the eigenstate-by-eigenstate fluctuations of any simple observable at high energy. Related to decay of classical auto-correlation functions

Summary and Future


nature   Vol 452 | 17 April 2008 | doi:10.1038/nature06838

LETTERS

Thermalization and its mechanism for generic isolated quantum systems

Marcos Rigol[1,2], Vanja Dunjko[1,2] & Maxim Olshanii[2]

An understanding of the temporal evolution of isolated many- dependent on it[3,4], seemingly implying that conservation of energy

Summary

◇ The first numerical demonstration of the Eigenstate Thermalization effect in a system large enough to support it.

Future plans

◇ Explain and quantify the Eigenstate Thermalization phenomenon itself.
- Systems with no classical counterparts and/or far from semi-classics