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Quantum mechanics and glasses

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This 3-lecture mini-course contains case studies of true quantum mechanical glasses, systems with only local interactions (and here in these cases not even disorder) that fail to reach their ground states even when assisted by the contact with a zero temperature bath.

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INTRODUCTION TO THE PROBLEM OF QUANTUM GLASSES IN NON-DISORDERED SYSTEMS

Quantum mechanical systems in contact with a reservoir are characterized by mixed ensembles of states that are best described using von Neumann's density matrix formalism [1]. At equilibrium, the density operator is given by $\hat{\rho} = e^{-\beta\hat{H}}/Z$ in the canonical ensemble, where \hat{H} is the system Hamiltonian, $\beta = 1/k_B T$ the inverse temperature of the reservoir (and of the system of interest, at equilibrium), and $Z = \text{tr}(e^{-\beta\hat{H}})$ the canonical partition function. The expectation value of any operator \hat{O} in this mixed ensemble is given by $\text{tr}(\hat{\rho}\hat{O})$. Therefore, the characterization of the properties of a quantum system at equilibrium is essentially a spectral problem. Describing the low temperature properties of a system, for example, requires the understanding of the ground state, its symmetries (or lack thereof) and its quantum orders [2], and its low lying excitations. In particular, at zero temperature, the system goes to its ground state. In the case of a quantum many-body system, if parameters in the Hamiltonian \hat{H} are tuned, the ground state can change its symmetries or its quantum orders through quantum phase transitions.

Obviously, the above picture breaks down if the quantum system resists equilibration with its environment, heat bath, or reservoir of contact. The problem of the approach to equilibrium is a formidable one. More precisely, the problem is not to prove that systems do equilibrate, because some simply do not in any experimentally accessible times, as in the case of classical glasses. In this case, the issue is how to understand and describe systems when they encounter dynamical obstructions in their path to thermal equilibration. A great deal of effort has been directed at this question by both physicists and chemists alike. In spite of much research, however, a complete theoretical description of the classical glass transition remains an open problem.

Such non-equilibrium behavior may be expected in disordered and frustrated quantum systems (like for instance quantum spin glasses [3], long-range Josephson junction arrays in a frustrating magnetic field [4], or self-generated mean-field glasses [5]). Are quantum systems that contain *no* quenched disorder and only local interactions protected

from encountering obstructions to equilibration similar to those of classical structural glass formers? The answer is **NO**: there are non-disordered systems that do not find their ground states at zero temperature, or that are not described by the equilibrium density matrix at low temperatures even though they are in contact with a heat bath [6].

These lectures are intended to introduce some of the generic problems related to non-disordered quantum glasses and a few tools that allow progress to be made in these problems.

LECTURE 1: EXAMPLES OF QUANTUM SYSTEMS WITH SIMPLE SPECTRA BUT DIFFICULTY IN EQUILIBRATING

Here we present concrete examples of quantum glasses, which are interesting because the systems, besides having local Hamiltonians with no quenched disorder, have exactly solvable spectra and topological ordered quantum ground states.

Topological order and quantum number fractionalization are some of the most remarkable properties of systems of strongly interacting particles. Some phases of matter, in contrast to common examples like crystals and magnets, are not characterized by a local order parameter and broken symmetries. Instead, as shown by Wen [7, 8], some quantum phases are characterized by their topological order, such as the degeneracy of the ground state when the system is defined on a torus or other surface of higher genus. These topological degeneracies cannot be lifted by *any* local perturbation. Topological order and quantum number fractionalization are intimately related, and much effort has recently been directed at these exotic properties, for they may be playing a role in the mechanism for high-temperature superconductivity. Also, the robustness of the topological degeneracy against local noise due to the environment is at the core of the idea behind topological quantum computation, as proposed by Kitaev [9].

Interestingly enough, strong correlations that can lead to these exotic quantum spectral properties can also impose kinetic constraints similar to those studied in the context of kinetically constrained classical glasses [10, 11, 12, 13, 14]. In quantum systems, however, tunneling is a mechanism that provides paths for a system's dynamical evolution that are not accessible to the classical kinetically constrained models. Nonetheless, in the examples that are constructed below, quantum tunneling is practically frozen out, because the size of the objects that must tunnel grow exponentially fast with inverse temperature. Hence, in these examples, not only do the classical thermal barriers grow with temperature, but also do the widths of the quantum tunneling barriers, in such a way that quantum processes are even more severely suppressed than classical ones at low temperatures, a rather different and unexpected situation. The origin of this behavior is the fact that *any* bath that couples *locally* to the physical degrees of freedom of the system can only flip large objects through virtual processes of large order in the system-bath coupling. For these quantum systems, no *physical* bath can provide a mechanism for relaxation that is not dynamically arrested at low temperatures.

Below I construct three such examples. The first is a two-dimensional (2D) quantum system with strong glass-like relaxation times when in contact with a restricted class of thermal baths; this example is used just to clarify the issue of how a given Hamiltonian requires a minimum number of degrees of freedom that the bath must locally control for the system to be able to equilibrate. The second example is a three-dimensional quantum system with strong glass-like relaxation times for any class of baths that couple locally to physical degrees of freedom of the system. The third example is a three-dimensional quantum system with fragile glass-like relaxation times.

Warmup: 2D example

The first model is constructed on a two-dimensional (2D) square lattice, shown in Fig. 1. Each site can be labeled by $i, j \in \mathbb{Z}$ that index a site in the Bravais lattice spanned by the primitive vectors $\mathbf{a}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\mathbf{a}_2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. To shorten the notation, define a superindex $I \equiv (i, j)$. At every lattice site I one defines quantum spin $S = 1/2$ operators σ_I^x , σ_I^y , and σ_I^z . The square lattice is bipartite: it contains two sets of sites, which we label A and B , and which are shown in red and blue color in Fig. 1.

Now let us define the quantum Hamiltonian in terms of the spins σ_I . Here we follow an approach similar to that of Kitaev, who constructed in a beautiful paper model quantum Hamiltonians that are exactly solvable [9]. In those models, the spins resided on links in planar lattices, but it is possible to carry out similar constructions with spins defined on vertices [15], as it is done here. Later on we show how the construction with spins on vertices can be generalized to 3D lattices.

Define a *diamond* cell P_I as the set of four lattice sites in an elementary plaquette with site I at its top. The four vertices are indexed by $J_n(I)$, for $n = 1, \dots, 4$, with one of the vertices $J_1(I) = I$. The four labels are assigned in such a way that the pairs $\{J_1, J_3\}, \{J_2, J_4\}$ are diagonally opposite sites from one another. Explicitly, $J_1(I) =$

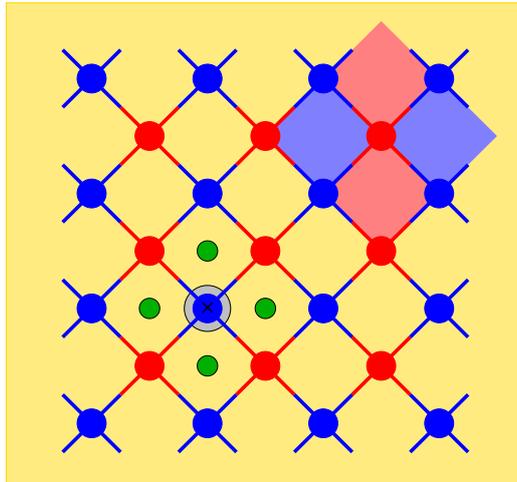


FIG. 1: Square lattice, with spin operators defined on the sites. The square lattice is bipartite, and the two sets of points A, B are shown in red and blue. A diamond contains 4 vertices in an elementary plaquette, and the diamonds can also be divided into two sets (forming a red/blue checkerboard) according to which sublattice its topmost vertex belongs to. Four-spin operators are defined on each plaquette using the σ^x and σ^y components of the spin operators, as described in the text. The green dots correspond to “defects” that are generated by applying a σ^z to the site encircled and marked with an \times .

$I \equiv (i, j), J_2(I) \equiv (i - 1, j), J_3(I) \equiv (i - 1, j - 1)$, and $J_4(I) \equiv (i, j - 1)$. It is simple to see that the total number of diamonds equals the number of spins: each lattice site I is the top vertex of a single diamond. The one-to-one relation between a site I and the diamond P_I allows us also to partition diamonds into two sets A and B (and color the corresponding diamonds red and blue, as shown in Fig. 1).

Now define the operators \mathcal{O}_I as

$$\mathcal{O}_I = \sigma_{J_1(I)}^y \sigma_{J_2(I)}^x \sigma_{J_3(I)}^y \sigma_{J_4(I)}^x. \quad (1)$$

These operators commute, $[\mathcal{O}_I, \mathcal{O}_{I'}] = 0$ for all pairs I, I' . It is simple to see how: two diamonds P_I and $P_{I'}$ can share 0, 1, or, at most, 2 spins. If they share 0 spins, they trivially commute. If they share 1 spin, the component (x, y or z) of σ for that shared spin coincides for both \mathcal{O}_I and $\mathcal{O}_{I'}$ (the two diamonds touch along one of their diagonals). If they share 2 spins, the components σ used in the definition of \mathcal{O}_I and $\mathcal{O}_{I'}$ are different for both spins, there is a minus sign from commuting the x and y components of the spin operators from each of the shared spins, and the two minus signs cancel each other.

Consider the system Hamiltonian

$$\hat{H} = -\frac{\hbar}{2} \sum_I \mathcal{O}_I, \quad (2)$$

which is trivially written in terms of the \mathcal{O}_I operators, but complicated in terms of the original spins σ_I . Because the \mathcal{O}_I all commute, the eigenvalues of the Hamiltonian can be labeled by the list of eigenvalues $\{\mathcal{O}_I\}$ of all the \mathcal{O}_I . Notice that $\mathcal{O}_I^2 = \mathbb{1}$, and so each $\mathcal{O}_I = \pm 1$. In particular, the ground state corresponds to $\mathcal{O}_I = 1$ for all I .

Because the number of spins equals the number N of sites, one may naively expect that the list $\{\mathcal{O}_I = \pm 1\}$ exhausts the 2^N states in the Hilbert space, spanned by $\{\sigma_I^z = \pm 1\}$. However, there are constraints that the \mathcal{O}_I satisfy when the system is subject to periodic boundary conditions (compactified to a torus). One can show that

$$\prod_{I \in A} \mathcal{O}_I = \prod_{I \in B} \mathcal{O}_I = \mathbb{1}. \quad (3)$$

There are two constraints; therefore there are only 2^{N-2} independent $\{\mathcal{O}_I = \pm 1\}$. This implies, in particular, that there is a ground state degeneracy of $2^2 = 4$. (Notice that the ground state degeneracy is not associated with a symmetry. In particular, it is easy to show that $\langle \sigma_I^{x,y,z} \rangle = 0$.) This is a topological degeneracy, and the eigenvalues of a set of two non-local (winding or topological) operators $\mathcal{T}_{1,2}$ are needed to distinguish between the 4 degenerate ground states.

The operators $\mathcal{T}_{1,2}$ can be constructed as follows. Let $\mathcal{P}_l = \{I|i+j=l\}$ be a set of points along a horizontal line. Notice that sites on a line belong either all to sublattice A or all to sublattice B , for example $\mathcal{P}_1 \subset A$ and $\mathcal{P}_2 \subset B$. Define

$$\mathcal{T}_1 = \prod_{I \in \mathcal{P}_1} \sigma_I^y \quad (4a)$$

$$\mathcal{T}_2 = \prod_{I \in \mathcal{P}_2} \sigma_I^y. \quad (4b)$$

It is simple to check that $[\mathcal{T}_{1,2}, \mathcal{O}_I] = 0$ for all I , and the two operators $\mathcal{T}_{1,2}$ trivially commute. Hence the two eigenvalues $T_{1,2} = \pm 1$ of $\mathcal{T}_{1,2}$ can distinguish the 4 degenerate ground states.

The spectrum of the Hamiltonian Eq. (2) is that of a trivial set of $N - 2$ free spins, determined by the list of eigenvalues $\{O_I = \pm 1\}$ of all the \mathcal{O}_I , subject to the condition Eq. (3): $E_{\{O_I\}} = -\frac{\hbar}{2} \sum_I O_I$. Excitations above the ground state ($O_I = 1$ for all I) are “defects” where $O_I = -1$ in certain sites I . Because of the constraints Eq. (3), the defects appear only in pairs. These defects have non-trivial quantum statistics: they are Abelian anyons with statistical angle different from fermions or bosons [9].

The equilibrium partition function (within a topological sector) is given by $Z = \sum_{\{O_I = \pm 1\}} e^{\beta \hbar \sum_I O_I}$. At thermal equilibrium at temperature T , the thermal average $\langle O_I \rangle = \tanh \frac{\hbar}{T}$, and the concentration or density of $O_I = -1$ defects is $c = \frac{1}{2} (1 - \tanh \frac{\hbar}{T})$. Notice that we have encountered an analogous situation to that in the classical spin facilitated models [10], in particular the plaquette models displaying glassy dynamics [11, 12, 13, 14]: the thermodynamics is trivial in terms of non-interacting defect variables. But what about the dynamics of our quantum model?

Although the spectrum of the model we are discussing is the same as that of free spins in a uniform magnetic field h , the variables O_I for different diamonds I *cannot* be independently changed, as opposed to spin variables in a free spin model in a field h . The reason why is that the operators \mathcal{O}_I involve four spins; these spins are shared by neighboring operators $\mathcal{O}_{I'}$, and thus one cannot change the eigenvalue of O_I without changing the eigenvalues $O_{I'}$ of neighbors by the action of a *local* operator. What are the allowed, *physical* local dynamical evolution rules for this quantum system? Do these dynamical rules lead to equilibration?

In order to endow the system with some physical dynamics, we couple the original physical spins to *individual* baths at temperature T . Here I do not consider “Turkish” baths of multiple spins; but still, as long as the groups of spins sharing a bath are locally delimited in space, the results obtained below should be qualitatively unchanged. Moreover, allowing the bath to communicate information through long-ranged couplings (via phonons, for instance) will not change the results, as long as it operates on delimited regions of space at both ends that it acts on.

When the original individual spins are coupled to their baths, “flips” of the states of multiple \mathcal{O}_I sharing a given spin take place. Therefore, our model has a trivial spectrum but a highly correlated dynamics. It is this correlated dynamics that gives rise to non-trivial non-equilibrium behaviors.

More concretely, we introduce the bath degrees of freedom as in the Feynman-Vernon influence functional approach [16] or Caldeira-Leggett dissipative quantum mechanics formulation [17, 18], by letting the Hamiltonian of the system plus bath be

$$\hat{\mathcal{H}} = \hat{H} + \hat{H}_{\text{bath}} + \hat{H}_{\text{spin+bath}}$$

where \hat{H} is defined in Eq. (2), and

$$\hat{H}_{\text{bath}} = \sum_{I,\alpha} \int_0^\infty dx [\Pi_I^\alpha(t,x)]^2 + [\partial_x \Phi_I^\alpha(t,x)]^2, \quad (5a)$$

$$\hat{H}_{\text{spin/bath}} = \sum_{I,\alpha} g_\alpha \sigma_I^\alpha \Pi_I^\alpha(t,0), \quad (5b)$$

and where the three components ($\alpha = 1, 2, 3$) of the conjugate vector fields Φ_I and Π_I obey the equal-time commutation relation

$$[\Phi_I^\alpha(t,x), \Pi_J^{\alpha'}(t,x')] = i \delta_{IJ} \delta_{\alpha\alpha'} \delta(x-x').$$

Notice that, for each site I , the bath-spin system can be viewed as an extended bosonic string that couples to a spin at the boundary $x = 0$. The coupling amplitudes are the g_α . One can in general chose anisotropic couplings, but the most general bath should contain all of $g_{1,2,3}$. In the quantum model, acting on a site $I' \in P_I$ with one of $\sigma_{I'}^x, \sigma_{I'}^y$, or $\sigma_{I'}^z$, flips or not the eigenvalue O_I depending on whether $\sigma_{I'}^{x,y,z} \mathcal{O}_I = \mp \mathcal{O}_I \sigma_{I'}^{x,y,z}$, respectively.

If integrated out, the bath degrees of freedom away from the boundary $x = 0$ lead to a non-local in time action and to dissipation effects. Instead of working with the dissipative action, let us follow the time evolution of the system plus bath, and look at the possible evolution pathways of the quantum mechanical amplitudes of the system plus bath degrees of freedom. After evolution by time t from some initial state, the system is in some quantum mechanical superposition

$$|\Psi\rangle = \sum_{\{O_I=\pm 1\}} \Gamma_{\{O_I\}} |\{O_I\}\rangle \otimes |\Upsilon_{\{O_I\}}\rangle, \quad (6)$$

where $|\Upsilon_{\{O_I\}}\rangle$ is a state in the bath Hilbert space with norm one. (Here we focus on states in a single topological sector, although mixing sectors can be done by including the eigenvalues of the topological operators \mathcal{T} ; mixing is exponentially suppressed as the system size increases). The fact that the bath degrees of freedom couple to single quantum spins σ_I enters in the problem through the permitted channels for transferring amplitudes among the $\Gamma_{\{O_I\}}$.

The processes that transfer amplitude among the $\Gamma_{\{O_I\}}$ correspond to different orders in perturbation theory on the g_α system-bath coupling. There is also a thermal probability factor coming from the bath, and that depends on the difference between the initial and final energy $E_{\{O_I\}} = -\frac{\hbar}{2} \sum_I O_I$ of the system. One class of paths is a *sequential* passage over states connected through order g_α processes; this is a “semi-classical” type trajectory.

Within this restricted class of processes, we can make a connection to the classical plaquette models, particularly the 4-spin square plaquette model whose glassy properties have been studied [13, 19, 20]. That classical model is simply obtained by defining diamond variables τ_I in place of the O_I as in Eq. (1) but using *only*, say, the z -component σ^z of spin for *all* four sites of the diamonds. In the classical four Ising spin plaquette interaction model, flipping an individual spin flips the plaquette defect variables of the four plaquettes that share the spin. This multi-defect type dynamics makes it difficult for the system to relax to equilibrium. For example, if the temperature is lowered, in order to decrease the defect density, either four defects must come together and annihilate ($4 \rightarrow 0$ decay), or three defects become one ($3 \rightarrow 1$ decay). However, the defects are not free to simply diffuse and come together. To move, an isolated single defect must first decay into three defects ($1 \rightarrow 3$ production) because of the multi-defect dynamics, then a pair can diffuse freely, and recombine with another defect through a $3 \rightarrow 1$ decay process. Because of the initial $1 \rightarrow 3$ production process, there is an energy barrier of $2h$ to be overcome. This activation barrier leads to recombination/equilibration times

$$t_{\text{seq.}} \sim \exp(2h/T)$$

that grow as temperature is lowered in an Arrhenius fashion [13].

Because there are all three components of the spin in the quantum model, and σ^x, σ^y enter in the Hamiltonian, to flip the defect variables in four neighboring plaquettes the coupling to the bath must involve *only* the σ^z component. Hence, what follows requires $g_1 = g_2 = 0$, strictly. If the coupling with the bath involves σ^x and σ^y as well, then defects can be flipped in pairs and thus are free to move diffusively across the system. This is an important difference between the classical and quantum versions of the square plaquette models, which we have more to say about later.

What about quantum tunneling? Defect annihilation can occur through virtual processes in which the number of defects is only larger in the intermediate (virtual) steps. At temperature T , the typical defect separation is $\xi = c^{-1/2} \sim e^{h/2T}$, and in perturbation theory this process has an amplitude of order $(g/h)^\xi$ (notice the energy denominator h). This amplitude leads to a recombination/equilibration times of

$$t_{\text{tun.}} \sim \exp \left[\ln(h/g) e^{h/2T} \right],$$

which grows extremely fast as the temperature is lowered. What we learn from this simple estimation is that quantum tunneling is less effective than classical sequential processes in thermalizing the system. This is counterintuitive to the notion that at low temperatures quantum tunneling under energy barriers remains an open process while classical mechanisms are suppressed due to high thermal activation costs. The reason for the particular quantum freezing in this system is simple: because the equilibration is through defect recombination, as the density of defects decrease at lower temperatures, the barrier *widths* increase with temperature, which debilitates quantum tunneling through those wide barriers. In passing, we note that in a finite system of size L , one must replace ξ by L in the estimation of the recombination/equilibration times, $t_{\text{tun.}} \sim \exp[\ln(h/g) L]$; this time scale is also of the order of that for tunneling between two topological ground states in a finite system of size L [9].

Let us return to the issue of which component of spin enters in the coupling to the bath. The minimal bath coupling involves the z -component of spin, for this component does not commute with *any* of the O_I that contain share a given spin. This minimal bath coupling is the one that leads to the interesting properties above. However, if the bath couples to the x - and y -components as well, defects now can diffuse freely, eliminating the energy $2h$ barriers for $1 \rightarrow 3$ defect production processes. Simple defect diffusion would lead to an equilibration time $t_{\text{eq}} \sim L^2$ for a system of size L (fast – polynomial in L with constant exponent). The dependence on the bath coupling is removed in the two 3D models we discuss next, one of which has even slower equilibration, as in fragile glasses.

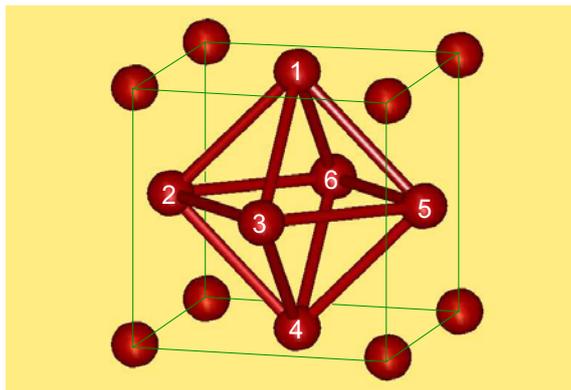


FIG. 2: Cubic cell of an fcc lattice. The centers of the six faces form an octahedron, with its sites labeled from 1 (topmost) to 6. In addition to the set of octahedra formed by the face centered sites, there are three more sets of octahedra that can be assembled from sites both on faces and on corners of the cubic cells, totaling 4 such sets. Six-spin operators are defined on these octahedra using the $\sigma^{x,y,z}$ components of spin on each vertex as described in the text.

3D quantum strong glass

Particle diffusion finds its origin in the fact that the x- or z-components of spin commute with 2 out of the 4 diamonds sharing a given spin. Therefore, the defects can be created in pairs, not quadruplets, and hence there is a mechanism for single defect diffusion through annihilation of a defect and creation on a neighboring diamond via pair flip. In the example that follows, six defect cells are shared by a single spin, in such a way that by acting with any component of the spin operators will flip 4 cells, and defect diffusion is non-existent.

The model displaying strong like glassiness is constructed on a three-dimensional (3D) face-centered cubic (fcc) Bravais lattice, spanned by the primitive vectors $\mathbf{a}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, $\mathbf{a}_2 = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, and $\mathbf{a}_3 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$. Each site can be indexed by $i, j, k \in \mathbb{Z}$, and to shorten the notation, define a superindex $I \equiv (i, j, k)$. At every lattice site I one defines quantum spin $S = 1/2$ operators σ_I^x , σ_I^y , and σ_I^z .

The fcc lattice can house sets of *octahedra*: the simplest one to visualize is the one assembled from the centers of the six faces of a cubic cell, and is shown in Fig. 2. In addition to this simple set, there are three more sets of octahedra that can be assembled from sites both on faces and on corners of the cubic cells, totaling 4 such sets, which we label by A, B, C and D .

It is simple to see that the total number of octahedra equals the number of spins: each lattice site I is the topmost vertex of a single octahedron. Define then P_I as the set of six lattice points contained by the octahedron with site I at its top. The six vertices are indexed by $J_n(I)$, for $n = 1, \dots, 6$, with one of the vertices $J_1(I) = I$. The six labels are assigned in such a way that the pairs $\{J_1, J_4\}, \{J_2, J_5\}, \{J_3, J_6\}$ are diagonally opposite sites from one another, and this number labeling is illustrated for a single octahedron in Fig. 2. From the one-to-one relation between a site I and the octahedra P_I , we can also partition the lattice sites into the four sets A, B, C and D of octahedra.

Now define the operators \mathcal{O}_I as

$$\mathcal{O}_I = \sigma_{J_1(I)}^z \sigma_{J_2(I)}^x \sigma_{J_3(I)}^y \sigma_{J_4(I)}^z \sigma_{J_5(I)}^x \sigma_{J_6(I)}^y. \quad (7)$$

These operators commute, $[\mathcal{O}_I, \mathcal{O}_{I'}] = 0$ for all pairs I, I' . It is simple to see how: two octahedra P_I and $P_{I'}$ can either share 0, 1, or, at most, 2 spins. If they share 0 spins, they trivially commute. If they share 1 spin, the component (x, y or z) of σ for that shared spin coincides for both \mathcal{O}_I and $\mathcal{O}_{I'}$ (the two octahedra touch along one of their diagonals). If they share 2 spins, the components σ used in the definition of \mathcal{O}_I and $\mathcal{O}_{I'}$ are different for both spins, there is a minus sign from commuting the spin operators from each of the shared spins, and the two minus signs cancel each other.

Consider the system Hamiltonian as in Eq. (2), which is trivially written in terms of the \mathcal{O}_I operators, but complicated in terms of the original spins σ_I . Because the \mathcal{O}_I all commute, the eigenvalues of the Hamiltonian can be labeled by the list of eigenvalues $\{\mathcal{O}_I\}$ of all the \mathcal{O}_I . Notice that $\mathcal{O}_I^2 = \mathbb{1}$, and so each $\mathcal{O}_I = \pm 1$. In particular, the ground state corresponds to $\mathcal{O}_I = 1$ for all I .

Because the number of spins equals the number N of sites, one may naively expect that the list $\{\mathcal{O}_I = \pm 1\}$ exhausts the 2^N states in the Hilbert space. However, there are constraints that the \mathcal{O}_I satisfy when the system is subject to

periodic boundary conditions (compactified). One can show that

$$\prod_{I \in A} \mathcal{O}_I = \prod_{I \in B} \mathcal{O}_I = \prod_{I \in C} \mathcal{O}_I = \prod_{I \in D} \mathcal{O}_I = \mathbb{1}. \quad (8)$$

There are four constraints; therefore there are only 2^{N-4} independent $\{\mathcal{O}_I = \pm 1\}$. This implies, in particular, that there is a ground state degeneracy of $2^4 = 16$. This is a topological degeneracy, and the eigenvalues of a set of four non-local (topological) operators $\mathcal{T}_{1,2,3,4}$ are needed to distinguish between the 16 degenerate ground states.

The operators $\mathcal{T}_{1,2,3,4}$ can be constructed as follows. Let $\mathcal{P}_l = \{I | j + k = l\}$ be a set of points along a horizontal plane. Notice that each plane contains sites in only two of the four sublattices A, B, C, D . For example $\mathcal{P}_1 \subset A \cup B$ and $\mathcal{P}_2 \subset C \cup D$. Define

$$\mathcal{T}_1 = \prod_{I \in \mathcal{P}_1 \cap A} \sigma_I^z \quad (9a)$$

$$\mathcal{T}_2 = \prod_{I \in \mathcal{P}_1 \cap B} \sigma_I^z \quad (9b)$$

$$\mathcal{T}_3 = \prod_{I \in \mathcal{P}_2 \cap C} \sigma_I^z \quad (9c)$$

$$\mathcal{T}_4 = \prod_{I \in \mathcal{P}_2 \cap D} \sigma_I^z. \quad (9d)$$

It is simple to check that $[\mathcal{T}_{1,2,3,4}, \mathcal{O}_I] = 0$ for all I , and the $\mathcal{T}_{1,2,3,4}$ trivially commute among themselves. Hence the four eigenvalues $T_{1,2,3,4} = \pm 1$ of $\mathcal{T}_{1,2,3,4}$ can distinguish the 16 degenerate ground states.

In this model it is guaranteed that, for whichever component of spin enters in the coupling to the bath, it is impossible to flip only a pair of defects and thus there is no mechanism for defect diffusion. The reason is that any site is shared by 6 octahedra, and the operators \mathcal{O}_I for these cells are such that one can divide the 6 into 3 groups of 2 octahedra that will have in their definitions, respectively, the x,y, and z component of spin operator at the shared site. Acting with either of the three components of the spin operator on this shared site will flip at least four defects. Hence, glassiness is protected against any local thermal bath.

3D quantum fragile glass

The model displaying fragile like glassiness is constructed on a three-dimensional (3D) hexagonal close-pack lattice, shown in Fig. 3. The lattice can be viewed as two interpenetrating simple hexagonal Bravais lattices displaced from one another by $\frac{1}{3}\mathbf{a}_1 + \frac{1}{3}\mathbf{a}_2 + \frac{1}{2}\mathbf{a}_3$, where $\mathbf{a}_1 = (1, 0, 0)$, $\mathbf{a}_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right)$, and $\mathbf{a}_3 = (0, 0, 1)$ are the primitive vectors of the simple hexagonal lattice. The sites belonging to the two intercalating lattices are shown in red and blue color in Fig. 3. Each site can be labeled by $i, j, k \in \mathbb{Z}$ that index a site in the Bravais lattice spanned by $\mathbf{a}_{1,2,3}$, plus a $q = 0, 1$ that index each of the two sublattices – to shorten the notation, define a superindex $I \equiv (i, j, k; q)$. At every lattice site I one defines quantum spin $S = 1/2$ operators σ_I^x , σ_I^y , and σ_I^z .

Define now a *prism* cell P_I that contains five lattice sites $J_n(I)$, for $n = 1, \dots, 5$ as follows. For a given lattice site I , the prism P_I contains the site $J_1(I) = I$, which belongs to one sublattice of the hexagonal close-packed lattice, the three sites that belong to the other sublattice and that form an elementary triangle (sites J_2, J_3, J_4) just below the site I , and finally the site $J_5(I)$ just below that triangle, which belongs to the same sublattice as of site I . [In terms of the lattice indices, $J_5(I) \equiv (i, j, k - 1; q)$ if $I \equiv (i, j, k; q)$.] An example of two prisms is shown in Fig. 3c. Notice that the two prisms shown share a common edge, and that their tops belong to distinct (red and blue) sublattices. It is simple to see that the total number of prisms equals the number of spins: each lattice site I is the top vertex of a single prism.

Now define the operators \mathcal{O}_I as

$$\mathcal{O}_I = \sigma_{J_1(I)}^z \sigma_{J_2(I)}^x \sigma_{J_3(I)}^x \sigma_{J_4(I)}^x \sigma_{J_5(I)}^z. \quad (10)$$

The operators commute, $[\mathcal{O}_I, \mathcal{O}_{I'}] = 0$, for all pairs I, I' . Again, it is simple to see how: if I, I' belong to the same sublattice and the prisms $P_I, P_{I'}$ share a vertex, then they trivially commute as they both involve the same component (x or z) of the spin operators $\boldsymbol{\sigma}$ at the shared site. If they belong to distinct sublattices, they either share 0 spins or an edge with 2 spins, as shown in Fig. 3. If they share 2 spins, the minus signs from commuting the x and z components of spin in each of the shared sites appear an even number of times.

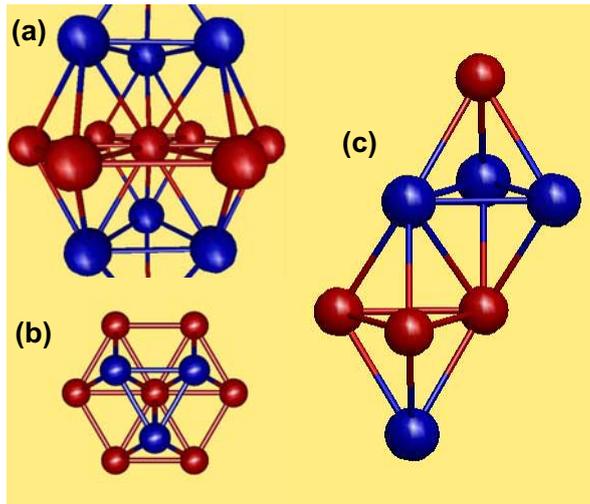


FIG. 3: Sites of an hcp lattice. (a) The hcp lattice is comprised of two interpenetrating hexagonal lattices, shown in red and blue. Prisms are defined by sets of five sites, two of which belong to one sublattice (top and bottom of the prism), and three of which belong to the other and form a triangle that lies in the layer in between the top and bottom sites of the prism. Five-spin interactions are defined on each prism as explained in the text. (b) Vertical view of the hcp lattice, which shows that the red and blue sublattices form triangular lattice planes. The blue sites stack on top of the red upward-pointing triangles, and the red sites stack on top of the downward-pointing blue triangles. (c) Two prisms with topmost sites belonging to different sublattices can share a common edge, and the five-spin operators defined on the two prisms commute because minus signs from commuting the σ^x and σ^z components appear twice, once for each shared site, and cancel.

Consider the system Hamiltonian as in Eq. (2), which is trivially written in terms of the \mathcal{O}_I operators, but complicated in terms of the original spins σ_I . Because the \mathcal{O}_I all commute, the eigenvalues of the Hamiltonian can be labeled by the list of eigenvalues $\{O_I\}$ of all the \mathcal{O}_I . Notice that $\mathcal{O}_I^2 = \mathbb{1}$, and so each $O_I = \pm 1$. In particular, the ground state corresponds to $O_I = 1$ for all I .

Because the number of spins equals the number N of sites, one may naively expect that the list $\{O_I = \pm 1\}$ exhausts the 2^N states in the Hilbert space. However, there are constraints that the \mathcal{O}_I satisfy when the system is subject to periodic boundary conditions (compactified). Each of the two sublattices ($q = 0, 1$) of the hexagonal close-packed structure can be further subdivided into A_q, B_q or C_q according to the three sublattices of the tripartite triangular stacks of the simple hexagonal lattice. (So in all one has six sublattices $A_{0,1}, B_{0,1}$ and $C_{0,1}$.) One can show that

$$\prod_{I \in A_q \cup B_q} \mathcal{O}_I = \prod_{I \in B_q \cup C_q} \mathcal{O}_I = \prod_{I \in C_q \cup A_q} \mathcal{O}_I = \mathbb{1}. \quad (11)$$

In all, these are six constraints, but only four are independent, because the product of the three products in Eq. (11) for the same q is trivially the identity. Therefore there are only 2^{N-4} independent $\{O_I = \pm 1\}$. This implies, in particular, that there is a ground state degeneracy of $2^4 = 16$. This, again, is a topological degeneracy. The eigenvalues of a set of four non-local (topological) operators $\mathcal{T}_{1,2,3,4}$ are needed to distinguish between the 16 degenerate ground states.

The operators $\mathcal{T}_{1,2,3,4}$ can be constructed as follows. Let the plane $\mathcal{P}_{k,q}$ be the set containing sites with fixed k and q . Let

$$\mathcal{T}_1 = \prod_{I \in \mathcal{P}_{1,0} \cap (A_0 \cup B_0)} \sigma_I^z \quad (12a)$$

$$\mathcal{T}_2 = \prod_{I \in \mathcal{P}_{1,0} \cap (B_0 \cup C_0)} \sigma_I^z \quad (12b)$$

$$\mathcal{T}_3 = \prod_{I \in \mathcal{P}_{1,1} \cap (A_1 \cup B_1)} \sigma_I^z \quad (12c)$$

$$\mathcal{T}_4 = \prod_{I \in \mathcal{P}_{1,1} \cap (B_1 \cup C_1)} \sigma_I^z. \quad (12d)$$

It is simple to check that $[\mathcal{T}_{1,2,3,4}, \mathcal{O}_I] = 0$ for all I , and the $\mathcal{T}_{1,2,3,4}$ trivially commute among themselves. Hence the four eigenvalues $T_{1,2,3,4} = \pm 1$ of $\mathcal{T}_{1,2,3,4}$ can distinguish the 16 degenerate ground states.

There are relations between this 3D model and a 2D classical triangular plaquette model which has glassy behavior [11, 12, 14]. The 2D triangular plaquette model has Ising spin variables defined on the sites of a triangular lattice, and a 3-spin interaction which is the product of the Ising variables on the downward pointing triangular plaquettes only. Defect Ising variables (the 3-spin products) are defined at the center of the downward triangles, and the energy is trivial to write in terms of these defect variables: these are free Ising spins. However, the dynamics is rather non-trivial in terms of the defects, for flipping an original spin correspond to flipping all three defects on the vertices of upward pointing triangles in the dual lattice.

In our 3D model, each quantum spin σ_I is shared by 5 prisms: 3 whose centers are on the same plane, and 2 whose centers are immediately above and below site I . If the system's coupling to the bath contains the σ^y spin component, all 5 prisms are flipped. The σ^z and σ^x components flip either the eigenvalues of the 3 prisms on the plane or the 2 prisms on the vertical direction, respectively. Flipping the eigenvalues of 2 prisms in the vertical direction would lead to defect diffusion in that direction.

To connect our 3D quantum model to the triangular plaquette model, consider a compactified slab (periodic boundary conditions) in the third dimension (parallel to \mathbf{a}_3), with M layers. Because of the periodic boundary condition, the odd-even parity of the defect numbers are conserved along vertically stacked prisms regardless of the system-bath spin-flip operator, σ^x, σ^y , or σ^z . The defect number parity can be captured by defining the following operator (recall $I \equiv (i, j, k; q)$):

$$\tau_{i,j;q} = \prod_k \mathcal{O}_{(i,j,k;q)}. \quad (13)$$

It is also useful to define a similar product over the third dimension for the original spins:

$$s_{i,j;q} = \prod_k \sigma_{(i,j,k;q)}^x. \quad (14)$$

These ‘‘slab’’ operators allows us to concentrate on subspaces of the Hilbert space with a given set of $\tau_{i,j;q}$ instead of the states with given $\mathcal{O}_{i,j,k;q}$. The dynamics has process that transfer quantum mechanical amplitudes within and between these subspaces labeled by $\tau_{i,j;q}$; we can argue that the system is glassy by simply looking at the processes that transfer amplitude between the subspaces.

The variables $\tau_{i,j;q}$ and $s_{i,j;q}$ can effectively be used to relate our quantum model to two 2D systems ($q = 0$ or red, and $q = 1$ or blue) defined on sites labeled by $(i, j; q)$ of two distinct triangular lattices. The variables $s_{i,j;q}$ can be related to the original spin variables in the models of Refs. [11, 12, 14]. In particular, one can relate the $s_{i,j;q}$ and the $\tau_{i,j;q}$ using Newton's binomial coefficients through

$$s_{i,j;q} = \prod_{mn} [\tau_{n,m;q}]^{\binom{j-n}{i-m}}. \quad (15)$$

Using the Pascal triangle relation $\binom{j+1-n}{i+1-m} = \binom{j-n}{i-m} + \binom{j-n}{i+1-m}$ one can show that indeed the defect variables correspond to

$$\tau_{i,j;q} = s_{i,j;q} s_{i+1,j;q} s_{i,j+1;q}. \quad (16)$$

We can focus again on two classes of processes, sequential passage over states connected through order g_α processes (‘‘semi-classical’’ type trajectories), and on quantum tunneling process. The analysis of the sequential process is similar to that of the classical 2D models [11, 12, 14], and goes as follows. Single flips of s correspond to concomitant flips of three τ defects. Defects can only be annihilated in triplets. The defects are not free to diffuse and come together; instead, they move through the production of more defects. For example, a defect can decay into two more defects, by flipping one s variable. Now, in order to bring three defects separated by a distance ξ together, one has to go through intermediate steps with a large number of defects that are created. There is an hierarchical organization for these intermediate processes; equilateral triangles of size $\xi = 2^\ell$ require the creation of ℓ extra intermediate defects. Hence there is an energy barrier of order ℓh to be overcome. For a typical equilibrium separation $\xi = c^{-1/2} \sim e^{h/2T}$, the barriers to be overcome in the equilibration process are of order $h/2 \ln 2Th$. Hence, the equilibration time scales as

$$t_{\text{seq.}} \sim \exp[(h/T)^2 / 2 \ln 2],$$

a much slower relaxation than the Arrhenius one for the 2D quantum model.

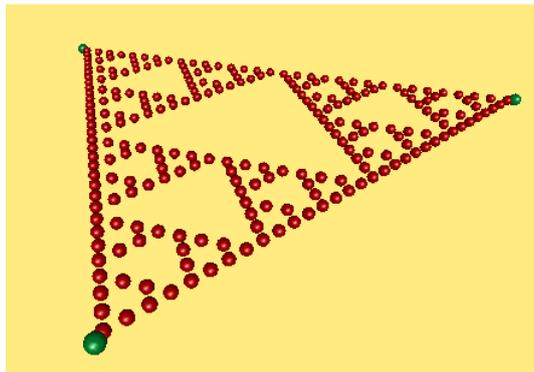


FIG. 4: To annihilate three defects (shown in green) at the corners of an equilateral triangle, one must flip the spins in a “fractal” membrane (containing sites shown in red) that stretches between the defects. For a triangle of size 2^ℓ , there are 3^ℓ sites in the membrane. The annihilation of the three defects through quantum tunneling is a virtual process of order the number of sites that are involved (number of red sites). Hence, the amplitude for the quantum tunneling process vanishes exponentially with the “volume” of the membrane.

Through quantum tunneling processes, defect annihilation can again occur via virtual processes in which the number of defects is only larger in the intermediate (virtual) steps. The order in perturbation theory in g grows very fast with defect separation. An example is shown in Fig. 4; basically, to annihilate three defects at the edges of an equilateral triangle of size $\xi = 2^\ell$, one must flip 3^ℓ original spins laying on a mold defined by a Sierpinski gasket. (Notice that here the hierarchy is built starting from the microscopic scale.) So in perturbation theory the quantum recombination process has an amplitude of order $(g/h)^{3^\ell}$, which leads to recombination/equilibration times

$$t_{tun.} \sim \exp \left[\ln(h/g) e^{\frac{\ln 3}{2 \ln 2}} h/T \right],$$

which grows extremely fast as the temperature is lowered. Again, we learn from this simple estimation is that quantum tunneling is less effective than classical sequential processes in thermalizing the system.

Summary

We have shown that there are *local* quantum Hamiltonians for which it is possible to show that the system resists equilibration with a thermal bath. Indeed in the models studied, there is a “protection” against thermal equilibration with *any* bath, as long as the *physical* degrees of freedom of the system only couple to the bath *locally*.

It is often believed that quantum tunneling provides an escape route against dynamical slowdown caused by heightened thermal energy barriers as the temperature is lowered. However, for the systems here presented, classical sequential processes are more effective than quantum tunneling processes in thermalizing the system. The reason for the freezing of quantum tunneling is that equilibration is through defect recombination, and as the density of defects decrease at lower temperatures, the barrier *widths* increase with temperature.

The widening of the barrier widths is another way to say the following: as the defects grow apart, the size of the object that has to quantum flip states to annihilate the defects grows. The order in perturbation theory scales as a power of this object that extends between defects, and hence the amplitude for annihilating the defects via quantum tunneling grows exponentially with the object size. Of course, the perturbative argument assumes that the recombination is through processes with intermediate virtual higher energies. For this to happen, simple defect diffusion must not occur, and this is precisely what happens in the the 3D models we discussed in this lecture.

LECTURE 2: MAPPING BETWEEN CLASSICAL DYNAMICS AND QUANTUM IMAGINARY TIME EVOLUTION

In this lecture we shall discuss a useful tool to study certain types of quantum glasses, here in the sense of systems with many ground states, exploring a mapping between certain types of classical dynamics and quantum Hamiltonians. More details than what is here presented can be found in two papers: Ref. 21, in collaboration with Claudio Castellano, Christopher Mudry, and Pierre Pujol, and Ref. 22, in collaboration with Giulio Biroli and Francesco Zamponi.

This mapping will allow us to make concrete statements about particular quantum glassy systems backed by knowledge obtained from the studies of the classical counterparts. For example, we can construct classes of local quantum Hamiltonians without disorder which we can argue are glassy because the mapped classical systems are known to be.

The general approach is based on a mapping between quantum Hamiltonians and (classical) Fokker-Planck operators which allows us to obtain results on ground state properties and time dependent correlation functions from the analysis of the stochastic dynamics of a classical equilibrium system. This connection, already well-known for a few decades [23, 24] for particle systems moving in real space, has been generalized recently to discrete systems with localized degrees of freedom, such as spins and dimers, and much used in the context of the so called Rokhsar-Kivelson points [21, 25, 26]. The connection is far-reaching since it allows one to obtain controlled and highly non-trivial results on the phase diagram and dynamical properties of a quantum many body problem [21]. Moreover, it has been used to construct an efficient algorithm for Quantum Monte Carlo at zero temperature [27].

Langevin dynamics: the Fokker-Planck and Schrödinger operators

Here we shall explore this classical-quantum connection for the particular case of bosonic point particles, following the standard route for mapping classical Langevin dynamics for a many-particle system and its associated Fokker-Planck operator to a Schrödinger operator [23, 24]. The particular derivation below was originally presented in Ref. 22 and applied to the study of the superglass state of hard spheres.

Consider N particles whose evolution is determined by the following Langevin equations:

$$\gamma_i \frac{d\mathbf{x}_i}{dt} = - \frac{\partial}{\partial \mathbf{x}_i} U_N(\mathbf{x}_1, \dots, \mathbf{x}_N) + \boldsymbol{\eta}_i(t) , \quad i = 1, \dots, N , \quad (17)$$

where γ_i are friction coefficients, $\eta_i^\alpha(t)$ is a Gaussian white thermal noise with variance $\langle \eta_i^\alpha(t) \eta_j^\beta(t') \rangle = 2T \gamma_i \delta_{ij} \delta_{\alpha\beta} \delta(t - t')$. Furthermore, T is the temperature (with $k_B = 1$) and α and β run from 1 to the spatial dimension d (henceforth the boldface notation indicates vectors). The potential will eventually be assumed to be the sum of (symmetric) pair potentials,

$$U_N(\{\mathbf{x}\}) \equiv U_N(\mathbf{x}_1, \dots, \mathbf{x}_N) = \frac{1}{2} \sum_{i \neq j} V_{ij}(\mathbf{x}_i - \mathbf{x}_j) , \quad (18)$$

with $V_{ij} = V_{ji}$, and $\frac{\partial}{\partial \mathbf{x}_i} U_N \equiv \nabla_i U_N = \sum_{j(\neq i)} \nabla V_{ij}(\mathbf{x}_i - \mathbf{x}_j)$.

It is well known [23, 24] that the evolution equation for the probability distribution $P(\{\mathbf{x}\})$ can be written as a Schrödinger equation in imaginary time:

$$\partial_t P = -H_{FP} P \quad (19)$$

where the Fokker-Planck operator reads:

$$H_{FP} = - \sum_i \frac{1}{\gamma_i} \frac{\partial}{\partial \mathbf{x}_i} \left[\nabla_i U_N + T \frac{\partial}{\partial \mathbf{x}_i} \right] \quad (20)$$

The Fokker-Planck operator is non-Hermitian and can be proven to have all eigenvalues larger than or equal to zero [23, 24]. The zero eigenvalue corresponds –as it can be readily checked– to the stationary distribution which is the equilibrium Gibbs probability measure:

$$P_G(\{\mathbf{x}\}) = \frac{1}{Z_N} e^{-\frac{1}{T} U_N(\{\mathbf{x}\})} = \frac{1}{Z_N} e^{-\frac{1}{2T} \sum_{i \neq j} V_{ij}(\mathbf{x}_i - \mathbf{x}_j)} . \quad (21)$$

Setting for simplicity $\hbar = 1$, the Fokker-Planck operator can be mapped into a Hermitian quantum Hamiltonian by a similarity transformation

$$H = e^{\frac{1}{2T}U_N} H_{FP} e^{-\frac{1}{2T}U_N}, \quad (22)$$

that leads to

$$H = \sum_i \frac{1}{\gamma_i} \left[-T \frac{\partial^2}{\partial \mathbf{x}_i^2} - \frac{1}{2} \nabla_i^2 U_N + \frac{1}{4T} (\nabla_i U_N)^2 \right] = \sum_i \frac{\mathbf{p}_i^2}{2m_i} + \mathcal{V}_N(\{\mathbf{x}\}) \quad (23)$$

This expression corresponds to a Hamiltonian for particles with mass $m_i = \gamma_i/(2T)$ and an effective potential which is the sum of a two body and three-body interaction:

$$\begin{aligned} \mathcal{V}_N(\{\mathbf{x}\}) &= \sum_i \frac{1}{\gamma_i} \left[-\frac{1}{2} \nabla_i^2 U_N + \frac{1}{4T} (\nabla_i U_N)^2 \right] \\ &= -\frac{1}{2} \sum_{j \neq i} \frac{1}{\gamma_i} \nabla^2 V_{ij}(\mathbf{x}_i - \mathbf{x}_j) + \frac{1}{4T} \sum_{i;j(\neq i);j'(\neq i)} \frac{1}{\gamma_i} \nabla V_{ij}(\mathbf{x}_i - \mathbf{x}_j) \cdot \nabla V_{ij'}(\mathbf{x}_i - \mathbf{x}_{j'}). \end{aligned} \quad (24)$$

The eigenfunctions of the quantum Hamiltonian and of the Fokker Planck operator are in a one to one correspondence. Indeed, by applying the similarity transformation introduced above one finds:

$$\Psi_E(\{\mathbf{x}\}) \propto e^{\frac{U_N}{2T}} P_E(\{\mathbf{x}\}) \quad (25)$$

where P_E indicates the right eigenfunction of the Fokker-Planck operator with eigenvalue E , and Ψ_E its counterpart associated to the quantum Hamiltonian. This also implies that all the eigenvalues E corresponding to the Fokker-Planck operator are identical to the ones of the quantum Hamiltonian.

In particular, this relation, together with Eq. (21), allows one to obtain straightforwardly the ground state wavefunction of the quantum problem, which is of the Jastrow form [28, 29, 30]:

$$\Psi_G(\{\mathbf{x}\}) = \sqrt{P_G(\{\mathbf{x}\})} = \frac{1}{\sqrt{Z_N}} \exp \left[-\frac{1}{4T} \sum_{i \neq j} V_{ij}(\mathbf{x}_i - \mathbf{x}_j) \right]. \quad (26)$$

The logic of the approach we pursue using this mapping is the following: we take as starting point Hamiltonians with many-body potentials of the form Eq. (24), for which the Jastrow form Eq. (26) is exact. In general, wavefunctions of this form lead to more than two-body interactions $\mathcal{V}_N(\{\mathbf{x}\})$ (note that also He^4 has weak higher order interactions). The important point is that if the two-body potentials $V_{ij}(\mathbf{x}_i - \mathbf{x}_j)$ are short-ranged (*i.e.* local) potentials, then $\mathcal{V}_N(\{\mathbf{x}\})$ is also local, and thus the many-body Hamiltonians on which we focus are *local* (non-local Hamiltonian may lead to pathological behaviors).

Because we know exactly the ground state wavefunction, and it is related to a Boltzmann-Gibbs measure for a classical system, quantum static correlation functions can be computed in terms of classical static correlation functions [30]. Furthermore, as we shall show and noticed by Henley [26], the mapping generalizes also to dynamical correlation functions. Hence, we will obtain quantum dynamical correlation functions at zero temperature by analytic continuation of classical (stochastic) dynamical correlation functions.

Identical bosons

Let us first consider the simplest case of N identical *bosons* characterized by the Hamiltonian H in Eq. (23) with $\gamma_i \equiv \gamma = 1$ and $V_{ij} \equiv V$. It is important to remark that since the particles are bosons one has to consider only many-body states that are completely symmetric under permutation of particles, and study *only* observables that are invariant under permutation of particles (e.g. the density-density correlator). This is clearly not a difficult constraint to handle since the Jastrow form (26) with $V_{ij} = V$ is completely symmetric. Furthermore, even in the study of dynamical correlations this will not be a problem because if one starts from a probability law completely symmetric under permutation of particles, the symmetrization carries over to all later times. This follows trivially from the Fokker-Planck evolution Eq. (19), since if the state P and the operator H_{FP} are both symmetric under exchange of particles, so is the time derivative $\partial_t P$ and thus the many-body state thereafter.

For a given classical isotropic 2-body potential $V(\mathbf{x}) = V(|\mathbf{x}|)$, the resulting quantum potential energy will have 2-body and 3-body interactions:

$$\begin{aligned} \mathcal{V}_N(\{\mathbf{x}\}) &= \sum_{i>j} v^{\text{pair}}(\mathbf{x}_i - \mathbf{x}_j) + \sum_{i \neq j \neq j' \neq i} v^{\text{3-body}}(\mathbf{x}_i - \mathbf{x}_j, \mathbf{x}_i - \mathbf{x}_{j'}) \\ v^{\text{pair}}(\mathbf{x}) &= -\nabla^2 V(\mathbf{x}) + \frac{1}{2} [\nabla V(\mathbf{x})]^2 = -\frac{d-1}{r} V'(r) - V''(r) + \frac{1}{2} [V'(r)]^2 \\ v^{\text{3-body}}(\mathbf{x}, \mathbf{x}') &= \frac{1}{4} \nabla V(\mathbf{x}) \cdot \nabla V(\mathbf{x}') = \frac{1}{4} \frac{\mathbf{x}}{r} \cdot \frac{\mathbf{x}'}{r'} V'(r) V'(r'), \end{aligned} \quad (27)$$

where d is the spatial dimension and $r \equiv |\mathbf{x}|$.

From quantum to classical: Hamiltonians which are sums of projectors in terms of classical dynamics

Just above we discussed how one starts from a classical system with Langevin dynamics, or equivalently a the Fokker-Planck description, and goes to quantum Hamiltonian that shares the excitation spectrum with the Fokker-Planck operator. The ground state of the quantum system is related to the Boltzmann-Gibbs measure of the classical system. This construction, carried above for point particles diffusing in the continuum in presence of interactions (which has been known for long) [23, 24], can be generalized to other systems, starting generically from a Master equation describing the evolution of classical probabilities, and then constructing an associated quantum Hamiltonian; this procedure was followed by Henley [26] in constructing Rokhsar-Kivelson type Hamiltonians. One can also make a connection between quantum and classical coming the other way [21], starting from a generic Rokhsar-Kivelson type Hamiltonian, *i.e.* a Hamiltonian that can be written as a sum of projectors sharing a common null vector, and showing that it can always be related to the stochastic dynamics of a classical system.

Consider the Hilbert space \mathcal{H} given by the span of all orthogonal and normalized (orthonormal) states $|\mathcal{C}\rangle$, labeled by the index $\mathcal{C} \in \mathcal{S}$ with \mathcal{S} a countable set,

$$\langle \mathcal{C} | \mathcal{C}' \rangle = \delta_{\mathcal{C}\mathcal{C}'}, \quad \mathbb{1} = \sum_{\mathcal{C} \in \mathcal{S}} |\mathcal{C}\rangle \langle \mathcal{C}|. \quad (28)$$

Consider a generic (Hermitian) Hamiltonian \hat{H} acting on this Hilbert space, and define the subset $\mathcal{L} \subset \mathcal{S} \times \mathcal{S}$ to be the set of all the pairs $(\mathcal{C}, \mathcal{C}')$ with $\mathcal{C} \neq \mathcal{C}'$ such that the off-diagonal matrix elements of the $|\mathcal{S}| \times |\mathcal{S}|$ Hermitian matrix

$$(H_{\mathcal{C}\mathcal{C}'} := \langle \mathcal{C} | \hat{H} | \mathcal{C}' \rangle) \quad (29)$$

are non-vanishing. For simplicity, we will make the two technical assumptions that the Hilbert space is fully connected under the time-evolution operator, *i.e.*, any two states in \mathcal{H} have a non-vanishing matrix element for some power of \hat{H} , and finite-dimensional. These two conditions will be needed when using the Perron-Frobenius theorem to establish the non-degeneracy of the GS. Of course, dropping these assumptions result in disconnected sectors and many possible true equilibrium states in the classical equivalent model and many ground states in the quantum system. Notice that broken connectivity implies broken ergodicity, but not the converse. Basically, ergodicity is a statement on time scales to connect two sectors, while disconnectivity means the sectors remain trully disconnected no matter how long one waits.

Consider Hamiltonians

$$\hat{H} = \frac{1}{2} \sum_{(\mathcal{C}, \mathcal{C}') \in \mathcal{L}} w_{\mathcal{C}, \mathcal{C}'} \hat{Q}_{\mathcal{C}, \mathcal{C}'} \quad (30a)$$

where $w_{\mathcal{C}, \mathcal{C}'} > 0$ and

$$\begin{aligned} \hat{Q}_{\mathcal{C}, \mathcal{C}'} &= e^{-K\varepsilon_{\mathcal{C}, \mathcal{C}'/2}} |\mathcal{C}\rangle \langle \mathcal{C}| + e^{+K\varepsilon_{\mathcal{C}, \mathcal{C}'/2}} |\mathcal{C}'\rangle \langle \mathcal{C}'| \\ &\quad - e^{-iK\phi_{\mathcal{C}, \mathcal{C}'/2}} |\mathcal{C}'\rangle \langle \mathcal{C}| - e^{+iK\phi_{\mathcal{C}, \mathcal{C}'/2}} |\mathcal{C}\rangle \langle \mathcal{C}'| \end{aligned} \quad (30b)$$

are projectors:

$$(\hat{Q}_{\mathcal{C}, \mathcal{C}'})^2 = 2 \cosh(K\varepsilon_{\mathcal{C}, \mathcal{C}'/2}) \hat{Q}_{\mathcal{C}, \mathcal{C}'} \quad (31)$$

with eigenvalues 0 (null state) and $2 \cosh(K\varepsilon_{\mathcal{C},\mathcal{C}'}/2)$. We now require that there exists a simultaneous zero mode for all the Hermitian operators $\widehat{Q}_{\mathcal{C},\mathcal{C}'}$ in Eq. (30b). One verifies that the nodeless wavefunction

$$|\Psi_0\rangle := \sum_{\mathcal{C} \in \mathcal{S}} e^{-KE_{\mathcal{C}}/2} |\mathcal{C}\rangle, \quad (32)$$

is annihilated by $\widehat{Q}_{\mathcal{C},\mathcal{C}'}, \forall (\mathcal{C}, \mathcal{C}') \in \mathcal{L}$, provided the integrability conditions

$$\varepsilon_{\mathcal{C},\mathcal{C}'} = E_{\mathcal{C}'} - E_{\mathcal{C}} \quad (33a)$$

and

$$\phi_{\mathcal{C},\mathcal{C}'} = \Phi_{\mathcal{C}} - \Phi_{\mathcal{C}'} \quad (\text{i.e. removable via a pure gauge transformation: } |\mathcal{C}\rangle \rightarrow e^{-iK\Phi_{\mathcal{C}}/2} |\mathcal{C}\rangle) \quad (33b)$$

on the real-valued parameters $\varepsilon_{\mathcal{C},\mathcal{C}'}$ and $\phi_{\mathcal{C},\mathcal{C}'}$ are satisfied for some real-valued function $E_{\mathcal{C}}$ defined on \mathcal{S} , and for all $(\mathcal{C}, \mathcal{C}') \in \mathcal{L}$. With the assumption that all couplings $w_{\mathcal{C},\mathcal{C}'}$ are positive, we see that all off-diagonal matrix elements of \widehat{H} are negative, so by the Perron-Frobenius theorem [32] the ground state wavefunction has positive amplitudes, which is precisely the case for the eigenstate (32), which is annihilated by all projectors $\widehat{Q}_{\mathcal{C},\mathcal{C}'}$.

The types of Hamiltonians above, which are sums of projectors with positive coefficients, and such that the projectors annihilate a common state, are the generalizations of the Rokhsar-Kivelson Hamiltonians constructed for the quantum dimer models. We denote these generalized Hamiltonians as of the RK type. For the full correspondence between quantum Hamiltonians of the RK type and classical statistical systems to be established, we need also to account for the approach of thermodynamic equilibrium in the classical system as we now explain.

From quantum dynamics to classical stochastics

There exists a correspondence between quantum Hamiltonians \widehat{H} of the RK type (30) and classical statistical systems endowed with time stochastics through a Master equation of the matrix type. The correspondence can be easily seen as follows.

Choose any two \mathcal{C} and \mathcal{C}' in \mathcal{S} and define the matrix elements

$$H_{\mathcal{C}\mathcal{C}'} := \langle \mathcal{C} | \widehat{H} | \mathcal{C}' \rangle \quad (34a)$$

and

$$W_{\mathcal{C}\mathcal{C}'} := -e^{-K(E_{\mathcal{C}} - E_{\mathcal{C}'})/2} H_{\mathcal{C}\mathcal{C}'}, \quad (34b)$$

respectively. Hermiticity and time-reversal symmetry of \widehat{H} imply the condition of symmetry

$$H_{\mathcal{C}\mathcal{C}'} = H_{\mathcal{C}'\mathcal{C}} \quad (35a)$$

which, in turn, implies the condition of detailed balance

$$W_{\mathcal{C}\mathcal{C}'} e^{-KE_{\mathcal{C}'}} = W_{\mathcal{C}'\mathcal{C}} e^{-KE_{\mathcal{C}}} \quad (35b)$$

for any pair $\mathcal{C}, \mathcal{C}' \in \mathcal{S}$.

We can write the Schrödinger equation

$$\dot{\psi}_{\mathcal{C}}(\tau) = \sum_{\mathcal{C}' \in \mathcal{S}} H_{\mathcal{C}\mathcal{C}'} \psi_{\mathcal{C}'}(\tau) \quad \forall \mathcal{C} \in \mathcal{S}, \quad (36)$$

and defining

$$p_{\mathcal{C}} = \Psi_{0\mathcal{C}} \times \psi_{\mathcal{C}} = \frac{1}{\sqrt{Z(K/2)}} e^{-KE_{\mathcal{C}}/2} \psi_{\mathcal{C}}, \quad (37)$$

where $|\Psi_0\rangle$ is the ground state of \widehat{H} , we establish the Master equation

$$\dot{p}_{\mathcal{C}}(\tau) := \sum_{\mathcal{C}' \in \mathcal{S}}^{\mathcal{C}' \neq \mathcal{C}} \left[W_{\mathcal{C}\mathcal{C}'} p_{\mathcal{C}'}(\tau) - W_{\mathcal{C}'\mathcal{C}} p_{\mathcal{C}}(\tau) \right], \quad \forall \mathcal{C} \in \mathcal{S} \quad (38)$$

TABLE I: Correspondences between quantum Hamiltonians of the RK type \hat{H} and classical systems with stochastic in time captured by the transition matrix $(W_{CC'})$. (From Ref. 21.)

Quantum system at RK point	Classical system
Hilbert space with basis \mathcal{B} labeled by \mathcal{S}	Configuration space \mathcal{S}
Ground state wavefunction	Boltzmann distribution
Quantum phase transitions	Classical phase transitions
Hamiltonian matrix: $(\langle \mathcal{C} \hat{H} \mathcal{C}' \rangle)$	Transition matrix: $(W_{CC'})$
$\left. \begin{array}{l} \text{Positive-semidefinite} \\ \text{decomposition conditions} \end{array} \right\}$	$\left\{ \begin{array}{l} \text{Positive transition rates} \\ \text{Conservation of probabilities} \end{array} \right.$
Integrability conditions	
Energy eigenvalues	Relaxation rates
Eigenfunctions	Right/Left eigenfunctions

whose properly normalized solution can be interpreted as the instantaneous probability (a number between 0 and 1) for the classical system to be in configuration \mathcal{C} . Equation (38) defines in a natural and unique way the classical stochastics at the reduced temperature K induced by the quantum Hamiltonian of the RK type (30) on the associated classical system in thermodynamic equilibrium. On the other hand, the special balance between the kinetic and the potential terms characteristic of an RK Hamiltonian guarantees that the transition matrix $(W_{CC'})$ satisfies the conservation of normalization condition $W_{CC} = -\sum_{\mathcal{C}' \in \mathcal{S}}^{\mathcal{C}' \neq \mathcal{C}} W_{\mathcal{C}'\mathcal{C}} \forall \mathcal{C} \in \mathcal{S}$, as can be verified directly using Eqs. (30a,30b) and (34b).

We thus have established a correspondence between generalized RK Hamiltonians of the form (30) and stochastic classical systems that can be represented by the Master equation of the matrix type (38). The dictionary between the RK type Hamiltonians and the classical stochastic systems is summarized in table I (from Ref. 21).

LECTURE 3: THE CASE STUDY OF THE SUPERGLASS

This lecture presents a “solid” example, constructed with Giulio Biroli and Francesco Zamponi, of a superglass, a bosonic system without diagonal (density) long range order, but off-diagonal (nonzero condensation) long range order. The lecture is presented using slides (see appendix), including a discussion of the experimental results of Kim and Chan on supersolidity, and recent experiments by Davis’s group. A brief summary of the presentation is as follows.

Summary of slides (see attached .pdf file)

We make use of the relationship between the classical and quantum models of lecture 2, and study the zero temperature phases of the quantum system. We work out the properties of the quantum phases by using the classical Gibbs measure defined by the square of the Jastrow wavefunction [29, 30, 31]. We focus on a system of N identical bosons with a particular interaction that corresponds to a Jastrow state with an hard sphere potential. We remark that this is the hard sphere *wavefunction* model, not the hard sphere Hamiltonian model. The control parameter is the particle density $\rho = N/V$ of the particles, or the *packing fraction* $\phi = \pi\rho/6$. We use known results for the phase diagram of classical hard spheres [33, 34, 35] to investigate the quantum model. At low density, the system is liquid, and upon increasing the density it undergoes a first order phase transition to a crystalline state, that is arranged in a face centered cubic (FCC) lattice. A metastable glassy phase can be obtained in the classical problem if the density is increased fast enough or in presence of small bi-dispersity (binary mixtures). This glass phase can be compressed until the random close packing (RCP) fraction $\phi \sim 0.64$. These three classical phases have their corresponding counterparts in the mapped quantum system. These are superfluid, supercrystal and superglass phases.

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