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Quantum dynamics near the classical limit

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Quantum dynamics near the classical limit.

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AFOSR

Epigraph

"Quantum time evolution is trivial"*

D. Huse

Notes available online http://physics.bu.edu/~asp/

Google: Polkovnikov

* taken out of context

Outline

- Quantum versus classical description of dynamics. Determinism and uncertainty. Coherent states, duality of particle and wave classical limits.
- Phase space representation of quantum mechanics through the Wigner function. Weyl ordering of operators, Moyal product.
- Quantum Liouville equation for the density matrix in the Wigner representation. Semiclassical limit (truncated Wigner approximation).
- Path integral representation of the evolution. Connection to Keldysh techniques. Causality of semiclassical description.
- Beyond semiclassical approximation: quantum jumps and quantum noise.
- Examples.

Literature:

1. A.P., **Representation of quantum dynamics of interacting systems through classical trajectories**, arXiv:0905.3384
2. M. A. Hillery, R. F. O'Connell, M. O. Scully, and E. P. Wigner, **Distribution functions in physics: Fundamentals,** *Phys. Rep.*, 106:121, 1984
3. D.F. Walls and G.J. Milburn. ***Quantum Optics***. Springer-Verlag, Berlin, 1994: C.W. Gardiner and P. Zoller. ***Quantum Noise***. Springer-Verlag, Berlin Heidelberg, third edition, 2004.
4. M. J. Steel, M. K. Olsen, L.I. Plimak, P. D. Drummond, S. M. Tan, M. J. Collett, D. F. Walls, and R.Graham, **Dynamical quantum noise in trapped Bose-Einstein condensates,** *Phys. Rev. A*, 58:4824, 1998.
5. P. B. Blakie, A. S. Bradley, M. J. Davis, R. J. Ballagh, and C. W. Gardiner. **Dynamics and statistical mechanics of ultra-cold Bose gases using c-field techniques,** *Advances in Physics*, 57:363, 2008.

From single particle physics to many particle physics.

Classical mechanics: Need to solve Newton's equation (fully deterministic given initial conditions)

Single particle

$$m\ddot{x} = f(x,t) \Rightarrow x(t+\delta t) = F(x(t),\dot{x}(t),t)$$

Many particles

$$m_i\ddot{x}_i = f(x_1,\ldots,x_n,t) \Rightarrow$$

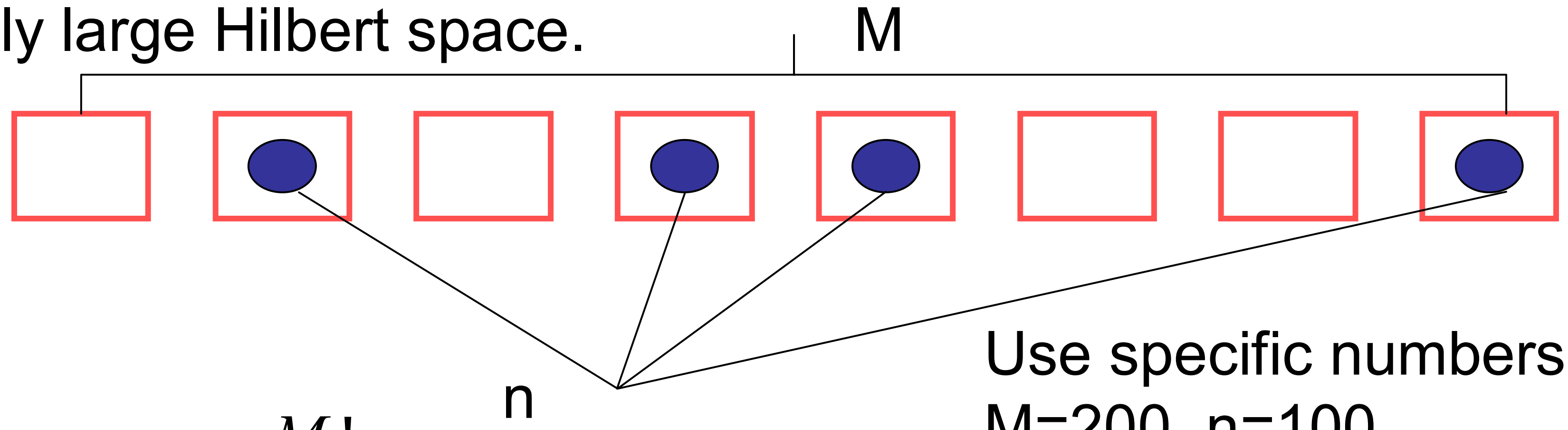
$$x_i(t+\delta t) = F_i(x_1(t),\ldots,x_n,\dot{x}_1(t),\ldots\dot{x}_n(t),t)$$

Instead of one differential equation need to solve n differential equations, not a big deal!? The only uncertainty comes from potentially unknown initial conditions. Chaos impedes our ability to make long time accurate deterministic predictions.

Quantum mechanics: Need to solve Schrödinger equation.

$$i\partial_t \psi(x,t) = H\psi(x,t) \Rightarrow \psi(x,t+\delta t) = F[\psi(x,t)]$$

Exponentially large Hilbert space.



Use specific numbers:
M=200, n=100.

Fermions: $$N_H = \frac{M!}{(M-n)!n!} \quad \Rightarrow \quad N_H \cong 10^{59}$$

Bosons: $$N_H = \frac{(M+n-1)!}{(M-1)!n!} \quad \Rightarrow \quad N_H \cong 10^{81}$$

QM gives fundamentally probabilistic description of evolution. In complex systems we deal with combination of quantum-mechanical and probabilistic uncertainty.

Expansion of quantum dynamics around classical limit.

Classical (saddle point) limit:

(i) Newtonian equations for particles,

(ii) Gross-Pitaevskii equations for matter waves,

(iii) Maxwell equations for classical e/m waves and charged particles,

(iv) Bloch equations for classical rotators, etc.

Questions:

What shall we do with equations of motion?

What shall we do with initial conditions?

What shall we do with observables?

Challenge :

How to reconcile exponential complexity of quantum many body systems and power law complexity of classical systems?

Coherent states. Dual classical corpuscular and wave limits.

Bosonic creation-annihilation operators

$$[\hat{\Psi}, \hat{\Psi}^{+}] = 1$$

Classical limit $\qquad \hat{\Psi}\hat{\Psi}^{+} \approx N \gg 1.$

$$\Psi^{\dagger} |\lambda_c\rangle = \left(\Psi^{\dagger} + \lambda (\Psi^{\dagger})^2 + \frac{\lambda^2}{2!} (\Psi^{\dagger})^3 \right) \ldots |0\rangle$$

$$= \frac{\partial}{\partial \lambda} |\lambda_c\rangle$$

$${}_c\langle \lambda | \Psi^+ \Psi | \lambda \rangle_c = |\lambda|^2 \Rightarrow |\lambda|^2 = N$$

$$\Rightarrow \lambda = \sqrt{N}\, e^{i\varphi}$$

Bose-Hubbard Hamiltonian, and classical (Gross-Pitaevskii) equations of motion.

$$\hat{H} = -J \sum_{\langle ij \rangle} \hat{\Psi}_i^{+} \hat{\Psi}_j + \frac{U}{2} \sum \hat{\Psi}_i^{+} \hat{\Psi}_i^{+} \hat{\Psi}_i \hat{\Psi}_i$$

continuum limit

$$\hat{H} = \int \Big[\frac{\hbar^2}{2\tilde{m}} (\nabla \hat{\Psi}^{+}_{(\bar{r})}) \nabla \hat{\Psi}(\bar{r}) + \frac{g}{2} \hat{\Psi}^{+}(\bar{r}) \hat{\Psi}^{+}(\bar{r}) \hat{\Psi}(\bar{r}) \hat{\Psi}(\bar{r}) \Big]$$

$$\tilde{m} \sim \frac{1}{J} \quad , \quad g \sim U$$

classical limit $\hat{\Psi} \to \Psi$

Poisson Brackets & EoM.

$$[\Psi_i^+, \Psi_j] = \delta_{ij}$$

EoM for operators

$$i\hbar \frac{\partial \hat{\Psi}_j}{\partial t} = [\hat{\Psi}_j, \hat{H}]$$

Hamiltonian dynamics.

	Particle limit	Wave limit
Phase space operators	$x,\ p$	ψ, ψ^{+}
Canonical commutation relations	$[x,\ p] = i\hbar$	$[\psi, \psi^{+}] = 1$
Classical limit - Poisson brackets	$\hbar \to 0$ $\{x,\ p\} = 1$	$\lvert\psi\rvert^{2} \to \infty$ $\{\psi, \psi^{*}\}_{c} = 1$
Classical limit – Equations of motion	$\dot{x} = \{x, H\} = p$ $\dot{p} = \{p, H\} = -\partial_{x} V$	$i\hbar\dot{\psi} = \{\psi, H\}_{c} =$ $= -\frac{\hbar^{2}}{2m}\psi + U\lvert\psi\rvert^{2}\psi$
	Newton’s equations	GP equations

Particle-wave duality in Bose-Einstein distribution

$$f(n) = \frac{1}{e^{\beta \varepsilon_n} - 1}$$

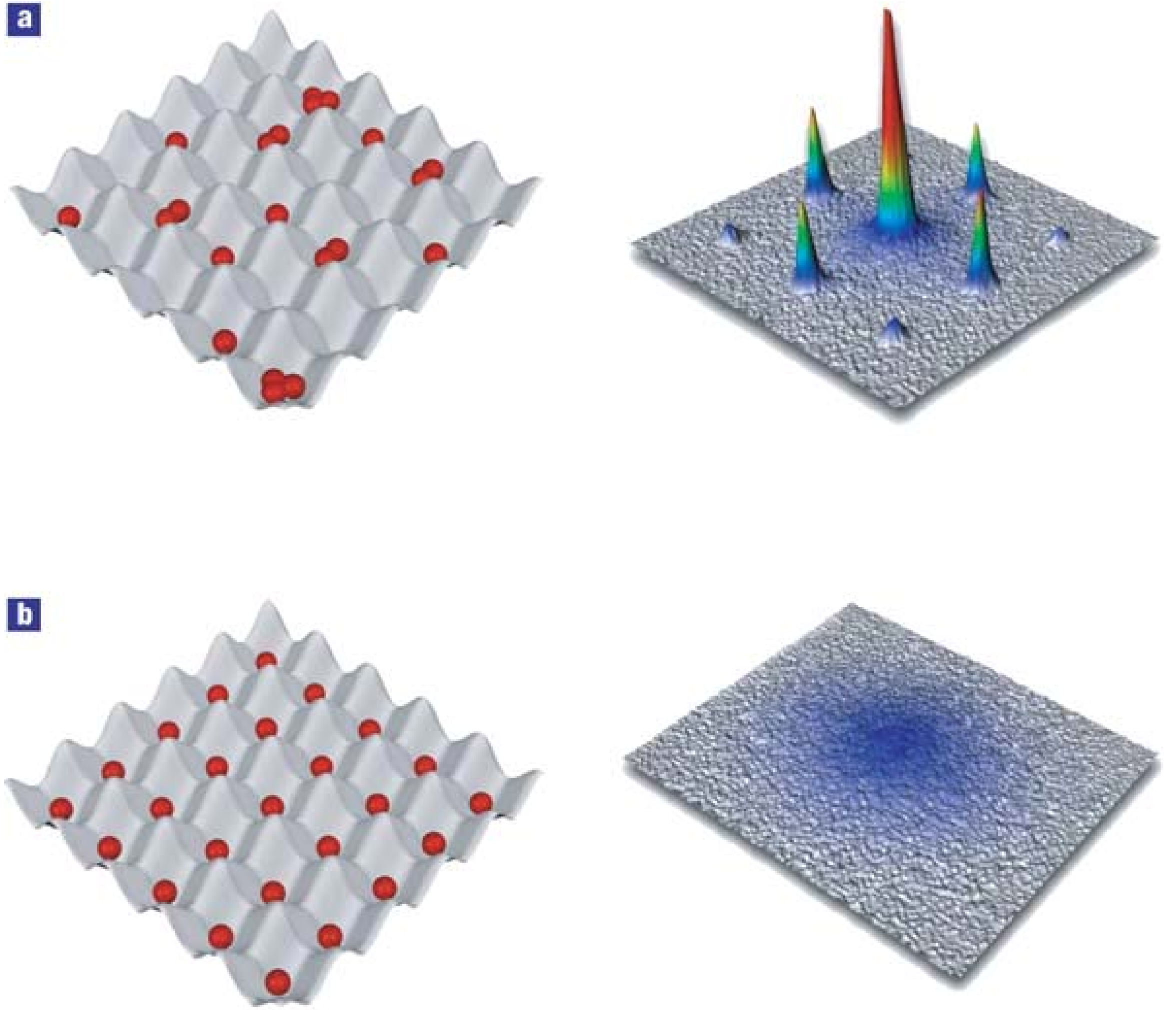
$\varepsilon_n / T \gg 1$ $\quad f(n) \propto e^{-\beta \varepsilon_n}$ - Bolzmann limit

(classical in terms of particles)

$\varepsilon_n / T \ll 1$ $\quad f(n) \simeq \frac{T}{\varepsilon_n} \Rightarrow \langle \varepsilon_n \rangle = T$ -

- equipartition : classical wave limit

Superfluid-Insulator transition as an example of particle-wave duality. (M. Greiner et. al., 2002).



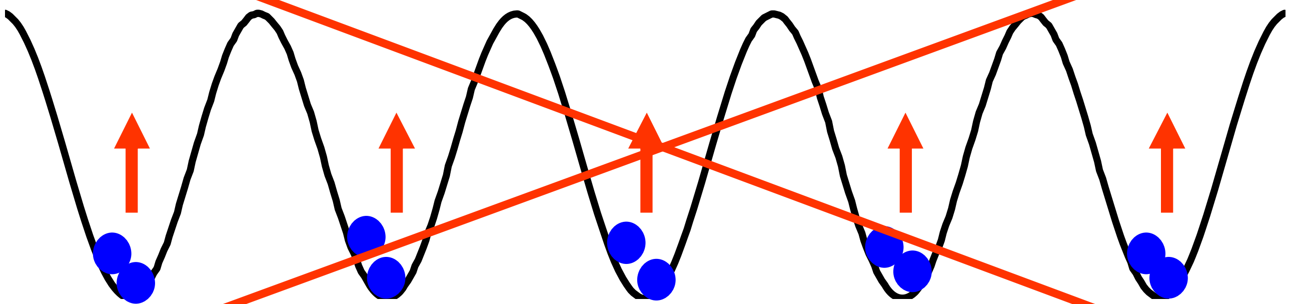
Classical phase in terms of waves.

Quantum phase transition

Classical phase in terms of particles.

Classically the ground state has a uniform density and a uniform phase.

$$H \cong -2J\sum_j \cos(\varphi_j - \varphi_{j+1}) + \frac{U}{2}\sum_j n_j^2 \qquad \psi_j \approx \sqrt{n_j}\exp[i\varphi_j]$$



However, number and phase are conjugate variables.

They do not commute: $\left[N,\varphi\right] = i \quad \rightarrow \quad \delta N \delta\varphi \geq 1$

There is a competition between the interaction leading to localization and tunneling leading to phase coherence.

How can we connect classical and quantum description?

Wigner function and Weyl ordering.

$$W(x,p) = \int d\xi\, \psi^*(x-\xi/2)\,\psi(x+\xi/2)\, e^{ip\xi/\hbar}$$

G.S. of a harmonic oscillator:

$$W(x_0,p_0) = 2\exp\left[-\frac{x_0^2}{2a_0^2} - \frac{p_0^2}{2q_0^2}\right] \quad a_0 = \sqrt{\hbar/2m\omega} \quad q_0 = \hbar/2a_0$$

Wigner function can be interpreted as a quasi probability distribution.

$$\langle \hat{\Omega}(\hat{x},\hat{p}) \rangle = \iint \frac{dx\,dp}{2\pi\hbar}\, \Omega_W(x,p)\, W(x,p)$$

$$\Omega_W(x,p) = \int d\xi \, \langle x - \frac{\xi}{2} | \hat{\Omega} | x + \xi/2 \rangle \, e^{ip\xi}$$

Wigner function is the Weyl symbol of the density matrix.

Example: Harmonic oscillator:

$$W(x_0, p_0) = 2\exp\left[-\frac{x_0^2}{2a_0^2} - \frac{p_0^2}{2q_0^2}\right] \qquad \begin{aligned} a_0 &= \sqrt{\hbar/2m\omega} \\ q_0 &= \hbar/2a_0 \end{aligned}$$

$$W(x, p, T) = 2\exp\left[-\frac{x^2}{2a_0^2\coth(\hbar\omega/2T)} - \frac{p^2}{2q_0^2\coth(\hbar\omega/2T)}\right] \rightarrow$$

$$\rightarrow \frac{\hbar\omega}{T}\exp\left[-\frac{m\omega^2 x^2 + p^2/m}{2T}\right]$$

!

Expectation value of product of operators, Moyal product.

$$\langle \hat{\Omega}(\hat{x},\hat{p}) \rangle = \iint \frac{dx\,dp}{2\pi\hbar}\, \Omega_W(x,p)\, W(x,p)$$

$$\hat{\Omega}(\hat{x},\hat{p}) = \hat{\Omega}_1(\hat{x},\hat{p})\,\hat{\Omega}_2(\hat{x},\hat{p})$$

$$\Omega_W(x,p) = ?$$

Example

$$(\hat{x}\hat{p})_W = x \exp\left[-\frac{i\hbar}{2}\left[\frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial x} - \frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p}\right]\right) p$$

$$= xp + \frac{i\hbar}{2}$$

Bopp operators

Weyl symbol can be obtained directly using Bopp operators

$$\hat{x} \to x + \frac{i}{2}\hbar \frac{\partial}{\partial \overrightarrow{p}} = x - \frac{i}{2}\hbar \frac{\partial}{\partial \overleftarrow{p}}$$

$$\hat{p} \to p - \frac{i}{2}\hbar \frac{\partial}{\partial \overrightarrow{x}} = p + \frac{i}{2}\hbar \frac{\partial}{\partial \overleftarrow{x}}$$

Coherent states

$$W(\Psi,\Psi^*) = \iint d\eta\, d\eta^* \langle \Psi - \frac{\eta}{2} | \rho | \Psi + \frac{\eta}{2} \rangle e^{-|\Psi|^2} e^{-\frac{|\eta|^2}{4}} e^{\frac{1}{2}(\eta^*\Psi - \eta\Psi^*)}$$

$W(\Psi,\Psi^*)$ is the Weyl symbol of the density matrix.

coherent state

$$|\lambda_c\rangle = e^{-|\lambda|^2/2} e^{\lambda\Psi^+} |0\rangle$$

$$W(\Psi,\Psi^*) = 2\exp\left[-2|\Psi-\lambda|^2\right]$$

Vacuum state $\lambda = 0$ $\quad W(\Psi,\Psi^*) = 2e^{-2|\Psi|^2}$

Moyal product.

$$(\Omega \Omega')_W = \Omega_W \exp\left[\frac{\Lambda_c}{2}\right] \Omega'_W$$

$$\Lambda_c = \sum_\alpha \frac{\overleftarrow{\partial}}{\partial \Psi_\alpha} \frac{\overrightarrow{\partial}}{\partial \Psi_\alpha^*} - \frac{\overrightarrow{\partial}}{\partial \Psi_\alpha} \frac{\overleftarrow{\partial}}{\partial \Psi_\alpha^*}$$

Bopp operators for coherent states

$$\hat{\underline{\Psi}}^+ \to \underline{\Psi}^* - \frac{1}{2}\frac{\overrightarrow{\partial}}{\partial\underline{\Psi}} = \underline{\Psi}^* + \frac{1}{2}\frac{\overleftarrow{\partial}}{\partial\underline{\Psi}}$$

$$\hat{\underline{\Psi}} \to \underline{\Psi} + \frac{1}{2}\frac{\overrightarrow{\partial}}{\partial\underline{\Psi}^*} = \underline{\Psi} - \frac{1}{2}\frac{\overleftarrow{\partial}}{\partial\underline{\Psi}^*}$$

Bopp operators reproduce Weyl symbol

$$\hat{\underline{\Psi}}^+\hat{\underline{\Psi}} \to \left(\underline{\Psi}^* - \frac{1}{2}\frac{\partial}{\partial\underline{\Psi}}\right)\left(\underline{\Psi} + \frac{1}{2}\frac{\partial}{\partial\underline{\Psi}^*}\right)\underline{1} =$$

$$= \underline{\Psi}^*\underline{\Psi} - \frac{1}{2}$$

Correct result.!

Summary of phase space methods

Wigner-Weyl quantization:

$$\left\langle \hat{\Omega}(\hat{x},\hat{p}) \right\rangle = \int \frac{dxdp}{2\pi\hbar} \Omega_W(x,p) W(x,p) \quad \Big| \quad \left\langle \hat{\Omega}(\hat{\psi},\hat{\psi}^+) \right\rangle = \int d\psi d\psi^* \Omega_W(\psi,\psi^*) W(\psi,\psi^*)$$

Moyal product (basic multiplication rule)

$$\left(\hat{\Omega}_1\hat{\Omega}_2\right)_W = \Omega_{1W} \exp\left[-i\frac{\hbar}{2}\Lambda\right]\Omega_{2W} \quad \Big| \quad \left(\hat{\Omega}_1\hat{\Omega}_2\right)_W = \Omega_{1W} \exp\left[\frac{1}{2}\Lambda_c\right]\Omega_{2W}$$

$$\Lambda = \sum_\alpha \frac{\overleftarrow{\partial}}{\partial p_\alpha}\frac{\overrightarrow{\partial}}{\partial x_\alpha} - \frac{\overleftarrow{\partial}}{\partial x_\alpha}\frac{\overrightarrow{\partial}}{\partial p_\alpha} \quad \Big| \quad \Lambda_c = \sum_\alpha \frac{\overleftarrow{\partial}}{\partial \psi_\alpha}\frac{\overrightarrow{\partial}}{\partial \psi^*_\alpha} - \frac{\overleftarrow{\partial}}{\partial \psi^*_\alpha}\frac{\overrightarrow{\partial}}{\partial \psi_\alpha}$$

Bopp operators
(basic representation)

$$\hat{x} = x + i\frac{\hbar}{2}\frac{\partial}{\partial p} \quad \Big| \quad \hat{\psi} = \psi + \frac{1}{2}\frac{\partial}{\partial \psi^*}$$

$$\hat{p} = p - i\frac{\hbar}{2}\frac{\partial}{\partial x} \quad \Big| \quad \hat{\psi}^+ = \psi^* - \frac{1}{2}\frac{\partial}{\partial \psi}$$

Phase space methods and quantum dynamics

Von Neumann equation for the density matrix

$$i\hbar\dot{\hat{\rho}} = [\hat{H}, \hat{\rho}] = \hat{H}\hat{\rho} - \hat{\rho}\hat{H}$$

Expansion in $\hbar$

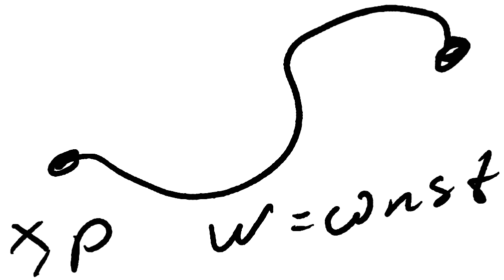
$$\dot{W} = -\frac{2}{\hbar} H_W \sin\left(\frac{\hbar}{2}\Lambda\right) W \simeq -H_W \Lambda W =$$

$$= -\sum_{\alpha} \frac{\partial H_W}{\partial p_\alpha}\frac{\partial W}{\partial x_\alpha} - \frac{\partial W}{\partial p_\alpha}\frac{\partial H_W}{\partial x_\alpha} = \{H_W, W\}$$

$\{..\}$ - is the Poisson Bracket

$$\bar{W}(t, x(t), p(t)) = const \Rightarrow \frac{dW}{dt} = 0$$

$$\frac{\partial W}{\partial t} + \frac{\partial W}{\partial x}\frac{dx}{dt} + \frac{\partial W}{\partial p}\frac{dp}{dt} = 0$$



$$\frac{\partial W}{\partial t} = \frac{\partial H_w}{\partial x}\frac{\partial W}{\partial p} - \frac{\partial H_w}{\partial p}\frac{\partial W}{\partial x}$$

Coherent states

$$i\hbar \frac{dW}{dt} = 2 H_W \sinh\left(\frac{\Lambda}{2}\right) W \simeq \frac{\partial H}{\partial \Psi}\frac{\partial W}{\partial \Psi^*} - \frac{\partial H}{\partial \Psi^*}\frac{\partial W}{\partial \Psi}$$

$$W(t, \Psi, \Psi^*) = \text{const} = 1$$

$$i\hbar \frac{\partial \Psi}{\partial t} = \frac{\partial H_W}{\partial \Psi^*} = \{\Psi, H_W\}$$ – Gross-Pitaevskii equation

Indeed

$$\underline{\psi}^{+}\underline{\psi}^{+}\,\underline{\psi}\,\underline{\psi} \to \left(\underline{\psi}^{*} - \frac{1}{2}\frac{\partial}{\partial \underline{\psi}}\right)^{2}\left(\underline{\psi} + \frac{1}{2}\frac{\partial}{\partial \underline{\psi}^{*}}\right)^{2} =$$

$$= |\underline{\psi}|^{4} - 2|\underline{\psi}|^{2} + \frac{1}{2}$$

Quantum mechanics enters through

$W(x_0, p_0)$ - can be non-positive

H_W - it is important to use Weyl ordering in the classical Hamiltonian

Ω_W - need to use Weyl symbol for the observable.

Non-equal time correlation functions.
Understanding commutation relations through response (semiclassical picture)

$$\langle \hat{x}(t_1)\ \hat{x}(t_2) \rangle = ? \qquad t_1 < t_2$$

If $t_1 > t_2$ use Bopp representation with left derivatives to preserve casuality

$$\langle \hat{x}(t_1)\, \hat{x}(t_2) \rangle = \int dx_0\, dp_0 \; x(t_1)\, x(t_2)\, W(x_0, p_0)$$

$$- i \frac{\hbar}{2} \int dx_0\, dp_0 \; \frac{\partial}{\partial p(t_2)} x(t_1) \; W(x_0, p_0)$$

This is also true for othe correlation functions.

(semi)classical interpretation of commutation relations through response

Coherent states. Same story

$$\frac{1}{2}\left(\hat{\Psi}(t_1)\,\hat{\Psi}^{+}(t_2) + \hat{\Psi}^{+}(t_2)\,\hat{\Psi}(t_1)\right) \to \Psi(t_2)\,\Psi(t_1)$$

$$\left(\hat{\Psi}(t_1)\,\hat{\Psi}^{+}(t_2) - \hat{\Psi}^{+}(t_2)\,\hat{\Psi}(t_1)\right) \to \frac{\partial}{\partial \Psi^{*}(t_1)}\,\Psi^{*}(t_2)$$

Beyond TWA (semiclassical approximation)

$$\dot{W} = -\frac{2}{\hbar} H_W \sin \frac{\Lambda \hbar}{2} W$$

$$J \; H_W = \frac{p^2}{2m} + V(x)$$

This equation can not be solved using characteristic functions.

Sketch of the path integral derivation of the time evolution.
(Very similar to Keldysh formalism)

$$\langle \hat{\Omega}(\hat{p},\hat{x},t)\rangle = \mathrm{Tr}\left[\hat{\rho}\, T_\tau\, \mathrm{e}^{i/\hbar \int_0^t \hat{\mathcal{H}}(\tau)d\tau}\hat{\Omega}(\hat{p},\hat{x},t)e^{-i/\hbar \int_0^t \hat{\mathcal{H}}(\tau)d\tau}\right]$$

$$T_\tau \mathrm{e}^{i/\hbar \int_0^t \hat{\mathcal{H}}(\tau)d\tau} \approx \prod_{i=1}^{N} \mathrm{e}^{i/\hbar \hat{\mathcal{H}}(\tau_i)\Delta\tau} \qquad \Delta\tau = t/N \text{ and } \tau_i = i\Delta\tau$$

Idea: classical limit ξ, η small, $x, p \to$
$\to$ classical trajectories. In the limit $\hbar \to 0$
forward & backward trajectories should be the same.

Exact Feynman path-integral representation of the evolution

$$S_{f,b} = i \int_0^t d\tau \; x_{f,b}(\tau) \frac{dp_{f,b}(\tau)}{d\tau} + \mathcal{H}(x_{f,b}(\tau), p_{f,b}(\tau), \tau)$$

$$\langle \Omega(p,x,t) \rangle = \int\int dx_0 dp_0 W_0(p_0, x_0) \int\int Dx(\tau) Dp(\tau) D\xi(\tau) D\eta(\tau)$$
$$\exp\left\{ \frac{i}{\hbar} \int_0^t d\tau \left[\xi(\tau) \frac{\partial p(\tau)}{\partial \tau} - \eta(\tau) \frac{\partial x(\tau)}{\partial \tau} + \mathcal{H}_W \left(p(\tau) + \frac{\eta(\tau)}{2}, x(\tau) + \frac{\xi(\tau)}{2}, \tau \right) \right.\right.$$
$$\left.\left. - \mathcal{H}_W \left(p(\tau) - \frac{\eta(\tau)}{2}, x(\tau) - \frac{\xi(\tau)}{2}, \tau \right) \right] \right\} \Omega_W(x(t), p(t), t),$$

$$W_0(p_0, x_0) = \int d\xi_0 \left\langle x_0 - \frac{\xi_0}{2} \right| \rho \left| x_0 + \frac{\xi_0}{2} \right\rangle \mathrm{e}^{ip_0\xi_0/\hbar}$$

$$\Omega_W(x, p, t) = \int d\xi \left\langle x + \frac{\xi}{2} \right| \hat{\Omega}(\hat{x}, \hat{p}, t) \left| x - \frac{\xi}{2} \right\rangle \exp\left[-\frac{ip\xi}{\hbar}\right]$$

Wigner function and Weyl oredring emerg automatically from the boundary terms at $\tau = 0$ and $\tau=t$. No special assumptions are needed. For details of the derivation see: A.P. arXiv:0905.3384, Phys. Rev. A, vol. 68 (5), 053604 (2003).

Coherent state representation:

$$\langle \hat{\Omega}(\hat{\psi},\hat{\psi}^{\star},t)\rangle = \mathrm{Tr}\left[\rho\, T_{\tau}\, \mathrm{e}^{i/\hbar\int_0^t \hat{\mathcal{H}}(\tau)d\tau}\hat{\Omega}(\hat{\psi},\hat{\psi}^{\dagger},t)e^{-i/\hbar\int_0^t \hat{\mathcal{H}}(\tau)d\tau}\right]$$

Same idea but now insert coherent states

$$\langle \hat{\Omega}(\hat{\psi},\hat{\psi}^{\dagger},t)\rangle = \int\int d\psi_0 d\psi_0^{\star} W_0(\psi_0,\psi_0^{\star})\int\int D\psi(\tau)D\psi^{\star}(\tau)D\eta(\tau)D\eta^{\star}(\tau)$$

$$\exp\left\{\int_0^t d\tau\left[\eta^{\star}(\tau)\frac{\partial\psi(\tau)}{\partial\tau} - \eta(\tau)\frac{\partial\psi^{\star}(\tau)}{\partial\tau} + i\mathcal{H}_W\left(\psi(\tau)+\frac{\eta(\tau)}{2},\psi^{\star}(\tau)+\frac{\eta^{\star}(\tau)}{2},\tau\right)\right.\right.$$

$$\left.\left.-i\mathcal{H}_W\left(\psi(\tau)-\frac{\eta(\tau)}{2},\psi^{\star}(\tau)-\frac{\eta^{\star}(\tau)}{2},\tau\right)\right]\right\}\Omega_W(\psi(t),\psi^{\star}(t),t),$$

ψ Is the classical Gross-Pitaevskii field, η is the quantum field.

$$\langle\Omega(p,x,t)\rangle = \int\int dx_0 dp_0 W_0(p_0,x_0) \int\int Dx(\tau)Dp(\tau)D\xi(\tau)D\eta(\tau)$$

$$\exp\left\{\frac{i}{\hbar}\int_0^t d\tau\left[\xi(\tau)\frac{\partial p(\tau)}{\partial\tau} - \eta(\tau)\frac{\partial x(\tau)}{\partial\tau} + \mathcal{H}_W\left(p(\tau)+\frac{\eta(\tau)}{2}, x(\tau)+\frac{\xi(\tau)}{2},\tau\right)\right.\right.$$
$$\left.\left. - \mathcal{H}_W\left(p(\tau)-\frac{\eta(\tau)}{2}, x(\tau)-\frac{\xi(\tau)}{2},\tau\right)\right]\right\}\Omega_W(x(t),p(t),t),$$

Recover semiclassical approximation by expanding action to the linear order in quantum fields:

$$\mathcal{H}_W\left[p(\tau)+\frac{\eta(\tau)}{2}, x(\tau)+\frac{\xi(\tau)}{2},\tau\right] - \mathcal{H}_W\left[p(\tau)-\frac{\eta(\tau)}{2}, x(\tau)-\frac{\xi(\tau)}{2},\tau\right] \approx$$
$$\approx \eta(\tau)\frac{\partial\mathcal{H}_W(p,x,\tau)}{\partial p(\tau)} + \xi(\tau)\frac{\partial\mathcal{H}_W(p,x,\tau)}{\partial x(\tau)}.$$

Then functional integration is trivial: we are getting δ-function constraints enforcing classical equations of motion:

$$\frac{dx}{d\tau} = \{x, \mathcal{H}_W\}\,, \quad \frac{dp}{d\tau} = \{p, \mathcal{H}_W\}$$

$$\frac{\partial \mathcal{H}_W(p,x,\tau)}{\partial x(\tau)} = \{\mathcal{H}_W(p,x,\tau), p\}\,,$$
$$\frac{\partial \mathcal{H}_W(p,x,\tau)}{\partial p(\tau)} = -\{\mathcal{H}_W(p,x,\tau), x\}\,.$$

Once again semiclassical – truncated Wigner – approximation

$$\langle \hat{\Omega}(\hat{x},\hat{p},t)\rangle \approx \int\int dx_0 dp_0 W_0(p_0,x_0)\Omega_W(x_{cl}(t),p_{cl}(t),t).$$

The same story happens in the coherent state basis: integrating over the quantum field in the leading order enforces Gross-Pitaevskii equations on the classical fileds:

$$i\hbar\frac{\partial \psi_i}{\partial t} = \{\psi_i, \mathcal{H}_W\}_c = \frac{\partial \mathcal{H}_W}{\partial \psi_i^\star}$$

Non-equal time correlations functions (sketch)

$$(\Omega_1(t_1)\Omega_2(t_2))_W = \int \frac{d\xi d\eta}{4\pi}\left\langle x(t_1)+\xi/2\middle|\hat{\Omega}_1(\hat{x},\hat{p},t_1)T_\tau \mathrm{e}^{i/\hbar\int_{t_1}^{t_2} H(\tau)d\tau}\right.$$

$$\left.\times\hat{\Omega}_2(\hat{x},\hat{p},t_2)\mathrm{e}^{-i/\hbar\int_{t_1}^{t_2} H(\tau)d\tau}\middle|p(t_1)-\eta/2\right\rangle \exp\left[-\frac{i}{\hbar}\left(p(t_1)x(t_1)+\frac{\xi p(t_1)-\eta x(t_1)}{2}\right)\right]$$

$$x(t_1)\to x(t_1)+\delta x_1,\quad p(t_1)\to p(t_1)+\delta p_1$$

$$W_{\Omega_1}(\delta x_1,\delta p_1)=\int\int\frac{d\xi d\eta}{(2\pi)^2}\,\mathrm{e}^{2i\delta p_1\delta x_1/\hbar}\mathrm{e}^{i\xi\delta p_1/\hbar+i\eta\delta x_1/\hbar}\Omega_1(x(t_1)+\xi/2,p(t_1)-\eta/2)$$

$$\hat{\Omega}_{nm}=\hat{x}^n\hat{p}^m$$

$$W_{1,0}(\delta x,\delta p)=\mathrm{e}^{2i\delta x\delta p/\hbar}\left(x-\frac{i\hbar}{2}\frac{\partial}{\partial\delta p}\right)\delta(\delta p)\delta(\delta x)\qquad \hat{x}(t_1)\to x(t_1)+\frac{i\hbar}{2}\frac{\partial}{\partial\delta p(t_1)}.$$

Recover Bopp operators (also automatically). Same for coherent states.

Beyond truncated Wigner approximation (TWA)

$$\langle\Omega(p,x,t)\rangle = \int\int dx_0 dp_0 W_0(p_0,x_0) \int\int Dx(\tau)Dp(\tau)D\xi(\tau)D\eta(\tau)$$

$$\exp\left\{\frac{i}{\hbar}\int_0^t d\tau \left[\xi(\tau)\frac{\partial p(\tau)}{\partial\tau} - \eta(\tau)\frac{\partial x(\tau)}{\partial\tau} + \mathcal{H}_W\left(p(\tau)+\frac{\eta(\tau)}{2}, x(\tau)+\frac{\xi(\tau)}{2},\tau\right)\right.\right.$$

$$\left.\left. - \mathcal{H}_W\left(p(\tau)-\frac{\eta(\tau)}{2}, x(\tau)-\frac{\xi(\tau)}{2},\tau\right)\right]\right\} \Omega_W(x(t),p(t),t),$$

Expand action to the third order in quantum fields (no corrections to TWA in harmonic theories)

$$\langle\hat{\Omega}(\hat{x},\hat{p},t)\rangle \approx \int\int dx_0 dp_0 W_0(x_0,p_0) \int\int Dx(\tau)D\xi(\tau)$$

$$\exp\left\{\frac{i}{\hbar}\int_0^t d\tau \left[\xi(\tau)\frac{\partial p(\tau)}{\partial\tau} + \xi(\tau)\frac{\partial H}{\partial x(\tau)} + \frac{1}{24}\xi^3(\tau)\frac{\partial^3 V(x)}{\partial^3 x(\tau)}\right]\right\} \Omega_W(x(t),p(t),t).$$

$$\exp\left(\frac{i}{24\hbar}\int_0^t d\tau\xi^3(\tau)\frac{\partial^3 V(x)}{\partial x(\tau)^3}\right) \approx 1 + \frac{i}{24\hbar}\int_0^t d\tau\xi^3(\tau)\frac{\partial^3 V(x)}{\partial x(\tau)^3} + \ldots$$

$$\int \frac{d\xi}{2\pi}\xi^3 \exp(i\alpha\xi) = \frac{1}{i^3}\frac{\partial^3}{\partial\alpha^3}\left[\delta(\alpha)\right] = -\frac{1}{i^3}\delta(\alpha)\frac{\partial^3}{\partial\alpha^3}$$

Note that α plays the role of the correction to the conjugate momentum = quantum jump

$$\langle\hat{\Omega}(\hat{(x)},\hat{p},t)\rangle \approx \int\int dx_0 dp_0 W_0(p_0,x_0)\left(1 - \int_0^t d\tau\frac{i}{24\hbar}\frac{\hbar^3}{i^3}\frac{\partial^3 V(x)}{\partial x(\tau)^3}\frac{\partial^3}{\partial\delta p(\tau)^3}\right)\Omega_W(x(t),p(t),t)$$

More generally

$$\langle\hat{\Omega}(\hat{x},\hat{p},t)\rangle = \int\int dx_0 dp_0 W_0(p_0,x_0)$$

$$\left(1 - \int_0^t d\tau\frac{1}{3!\,2^2}\frac{\hbar^2}{i^2}\frac{\partial^3 V(x)}{\partial x(\tau)^3}\frac{\partial^3}{\partial p(\tau)^3} - \int_0^t d\tau\frac{1}{5!\,2^4}\frac{\hbar^4}{i^4}\frac{\partial^5 V(x)}{\partial x(\tau)^5}\frac{\partial^5}{\partial p(\tau)^5}\right.$$
$$\left. + \int_0^t d\tau_1\int_{\tau_1}^t d\tau_2\frac{\hbar^4}{(3!\,2^2)^2 i^4}\frac{\partial^3 V(x)}{\partial x(\tau_1)^3}\frac{\partial^3}{\partial p(\tau_1)^3}\frac{\partial^3 V(x)}{\partial x(\tau_2)^3}\frac{\partial^3}{\partial p(\tau_2)^3} + \ldots\right)\Omega_W(x(t),p(t),t)$$

Quantum corrections emerge as a nonlinear response to infinitesimal jumps in classical phase space variables.

Each jump carries a factor of $\hbar^2$.

Jumps do not affect short time behavior, i.e. TWA is asymptotically exact at short times.

Equivalent representation through stochastic quantum jumps

$$\langle\hat{\Omega}(\hat{x},\hat{p},t)\rangle \approx \int\int dx_0 dp_0 W_0(x_0,p_0)$$
$$\left\{\Omega_W(x(t),p(t),t) + \frac{\hbar^2}{4}\sum_{\tau_i}\frac{\partial^3 V(x_i)}{\partial x_i^3}\int d\xi F_3(\xi)\Omega_W\left[x(t),p(t),t,\delta p_i = \xi\sqrt[3]{\Delta\tau}\right]\right\}$$

$$\int_{-\infty}^{\infty} F_3(\xi)d\xi = 0,\ \int_{-\infty}^{\infty}\xi F_3(\xi)d\xi = 0,\ \int_{-\infty}^{\infty}\xi^2 F_3(\xi)d\xi = 0,\ \int_{-\infty}^{\infty}\xi^3 F_3(\xi)d\xi = 1.$$

Proof of equivalence

$$\langle\hat{\Omega}(\hat{x},\hat{p},t)\rangle \approx \int\int dx_0 dp_0 W_0(x_0,p_0)$$

$$\left\{\Omega_W(x(t),p(t),t) + \frac{\hbar^2}{4}\sum_{\tau_i}\frac{\partial^3 V(x_i)}{\partial x_i^3}\int d\xi F_3(\xi)\Omega_W\left[x(t),p(t),t,\delta p_i = \xi\sqrt[3]{\Delta\tau}\right]\right\}$$

$$\Omega_W\left[x(t),p(t),t,\delta p_i = \xi\sqrt[3]{\Delta\tau_i}\right] \approx \Omega_W\left[x(t),p(t),t\right] + \frac{\partial\Omega_W\left[x(t),p(t),t,\delta p_i\right]}{\partial\delta p_i}\xi\sqrt[3]{\Delta\tau}$$

$$+\frac{1}{2}\frac{\partial^2\Omega_W\left[x(t),p(t),t,\delta p_i\right]}{\partial\delta p_i^2}\xi^2\sqrt[3]{\Delta\tau^2} + \frac{1}{6}\frac{\partial^3\Omega_W\left[x(t),p(t),t,\delta p_i\right]}{\partial\delta p_i^3}\xi^3\Delta\tau + O(\Delta\tau^{4/3}).$$

Integrating over ξ gives desired non-linear response.

Possible choices of F:

$$F_3(\xi) = \frac{\delta(\xi-2)-\delta(\xi+2)-2\delta(\xi+1)+2\delta(\xi+1)}{12}$$

$$F_3(\xi) = \frac{1}{2}\left(\frac{\xi^3}{3}-\xi\right)\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\xi^2/2}$$

Many-particle generalization

$$\langle\hat{\Omega}(\hat{\boldsymbol{x}},\hat{\boldsymbol{p}},t)\rangle \approx \int\int d\boldsymbol{x}_0 d\boldsymbol{p}_0 W_0(\boldsymbol{p}_0,\boldsymbol{x}_0)$$
$$\left(1-\int_0^t d\tau \frac{1}{3!\,2^2}\frac{\hbar^2}{i^2}\frac{\partial^3 V(\boldsymbol{x})}{\partial x_\alpha \partial x_\beta \partial x_\gamma}\frac{\partial^3}{\partial p_\alpha(\tau)\partial p_\beta(\tau)\partial p_\gamma(\tau)}+\dots\right)\Omega_W(\boldsymbol{x}(t),\boldsymbol{p}(t),t)$$

$$\langle\hat{\Omega}(\hat{\boldsymbol{x}},\hat{\boldsymbol{p}},t)\rangle \approx \int\int d\boldsymbol{x}_0 d\boldsymbol{p}_0 W_0(\boldsymbol{x}_0,\boldsymbol{p}_0)$$
$$\left[1+\frac{\hbar^2}{4}\sum_j\int\int\prod_m d\xi_m \sum_{\sigma(\alpha,\beta,\gamma)}\frac{\partial^3 V(\boldsymbol{x}_j)}{\partial x_\alpha \partial x_\beta \partial x_\gamma}F_{\alpha,\beta,\gamma}(\boldsymbol{\xi})\bigg|_{\delta p_\alpha(\tau_j)=\xi_\alpha\sqrt[3]{\Delta\tau_j}}\right]\Omega_{cl}(\boldsymbol{x}(t),\boldsymbol{p}(t),t)$$

$$F_{\alpha,\alpha,\alpha}(\boldsymbol{\xi})=\frac{1}{2\sqrt{2\pi}}\left(\frac{\xi_\alpha^3}{3}-\xi_\alpha\right)\mathrm{e}^{-\xi_\alpha^2/2}\prod_{m\neq\alpha}\delta(\xi_m),$$
$$F_{\alpha,\alpha,\beta}(\boldsymbol{\xi})=\frac{\left(\xi_\alpha^2-1\right)\xi_\beta}{4\pi}\mathrm{e}^{-(\xi_\alpha^2+\xi_\beta^2)/2}\prod_{m\neq\alpha,\beta}\delta(\xi_m),$$
$$F_{\alpha,\beta,\gamma}(\boldsymbol{\xi})=\frac{\xi_\alpha\xi_\beta\xi_\gamma}{(2\pi)^{3/2}}\mathrm{e}^{-(\xi_\alpha^2+\xi_\beta^2+\xi_\gamma^2)/2}\prod_{m\neq\alpha,\beta,\gamma}\delta(\xi_m).$$

Coherent states. Same story

$$\langle\hat{\Omega}(\hat{\psi},\hat{\psi}^\dagger,t)\rangle \approx \int\int d\psi_0 d\psi_0^\star W_0(\psi_0,\psi_0^\star)$$

$$\left(1-\frac{i}{4\hbar}\int_0^t d\tau\sum_{ijkl}u_{ijkl}\left[\psi_i^\star(\tau)\frac{\partial^3}{\partial\psi_j(\tau)\partial\psi_k^\star(\tau)\partial\psi_l^\star(\tau)}-c.c.\right]\right)\Omega_W(\psi(t),\psi^\star(t),t).$$

Bose Hubbard model

$$\langle\hat{\Omega}(\hat{\psi},\hat{\psi}^\dagger,t)\rangle \approx \int\int d\psi_0 d\psi_0^\star W_0(\psi_0,\psi_0^\star)$$

$$\left[1-\frac{iU}{4\hbar}\sum_n\sum_j\int\int d\xi_j d\xi_j^\star\left(\psi_j^\star(\tau_n)F(\xi_j)-c.c.\right)\Omega_{cl}(\psi(t),\psi^\star(t),t)\right]$$

$$F(\xi_j,\xi_j,\xi_j)=\xi_j^\star\left(|\xi_j|^2-2\right)\mathrm{e}^{-|\xi_j|^2}$$

Examples

$$\hat{\mathcal{H}}_0 = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 \qquad \hat{\mathcal{H}} = \hat{\mathcal{H}}_0 - \lambda\hat{x}$$

$$\psi_0(x) = \frac{1}{(2\pi)^{1/4}\sqrt{x_0}}\,\mathrm{e}^{-x^2/4a_0^2} \qquad a_0 = \sqrt{\hbar/2m\omega}$$

$$W(x_0,p_0) = \int d\xi\,\psi^\star(x_0+\xi/2)\psi(x_0-\xi/2)\mathrm{e}^{ip_0\xi/\hbar} = 2\exp\left[-\frac{x_0^2}{2a_0^2} - \frac{p_0^2}{2q_0^2}\right]$$

Classical equations of motion

$$\frac{dp}{dt} = -m\omega^2 x + \lambda, \qquad \frac{dx}{dt} = \frac{p}{m} \qquad x_{\mathrm{cl}}(t) = \lambda/m\omega^2(1-\cos(\omega(t)))$$

$$x(t) = x_{\mathrm{cl}}(t) + x_0\cos(\omega t) + \frac{p_0}{m\omega}\sin(\omega t)$$

$$x(t) = x_{\mathrm{cl}}(t) + x_0 \cos(\omega t) + \frac{p_0}{m\omega} \sin(\omega t)$$

$$\langle \hat{x}(t) \rangle = x_{cl}(t) \qquad \langle \hat{x}^2 \rangle = \overline{x^2(t)} = x_{\mathrm{cl}}^2(t) + a_0^2$$

$$\langle \hat{x}(t)\hat{x}(t') \rangle = \overline{\left(x_{cl}(t) + x_0 \cos(\omega t) + \frac{p_0}{m\omega} \sin(\omega t) + \frac{i\hbar}{2} \frac{\partial}{\partial \delta p} \right)}$$

$$\overline{\times \left(x_{cl}(t) + x_0 \cos(\omega t') + \frac{p_0}{m\omega} \sin(\omega t') + \frac{\delta p}{m\omega} \sin(\omega(t' - t) \right)}$$

$$= x_{\mathrm{cl}}(t) x_{\mathrm{cl}}(t') + a_0^2 \cos(\omega(t - t')) + i a_0^2 \sin(\omega(t' - t)).$$

$$\left\langle \frac{\hat{x}(t')\hat{x}(t) + \hat{x}(t)\hat{x}(t')}{2} \right\rangle = x_{\mathrm{cl}}(t) x_{\mathrm{cl}}(t') + a_0^2 \cos(\omega(t - t'))$$

$$\langle \hat{x}(t)\hat{x}(t') - \hat{x}(t')\hat{x}(t) \rangle = 2 i a_0^2 \sin(\omega(t' - t)).$$

More complicated example: sine-Gordon (Frenkel-Kontorova) model

$$\hat{\mathcal{H}} = \int \frac{dx}{2}\left[\hat{n}^2(x) + (\partial_x \hat{\phi}(x))^2 - 2V\cos(\beta\hat{\phi}(x))\right]$$

$$\hat{\mathcal{H}} = \sum_j \left[\frac{\hat{n}_j^2}{2} + \frac{1}{2a^2}(\hat{\phi}_j - \hat{\phi}_{j+1})^2 - V\cos(\beta\hat{\phi}_j)\right]$$

Assume initially V=0 and the system is in the ground state

$$W(\phi_q^0, n_q^0) = \prod_q \exp\left[-\frac{|\phi_q^0|^2}{2\sigma_q^2} - 2\sigma_q^2 |n_q^0|^2\right]$$

$$\sigma_q = 1/\sqrt{2\omega_q}, \qquad \omega_q = 2\sin(q/2)$$

Semiclassical approximation (TWA) need to solve

$$\frac{d^2\phi_j}{dt^2} = (\phi_{j+1} + \phi_{j-1} - 2\phi_j) - \beta V(t)\sin(\beta\phi_j)$$

First quantum correction – have a quantum jump proportional to

$$v_{3,j,\tau_i} = -V(\tau_i)\beta^3 \sin(\beta\phi_j(\tau_i))$$

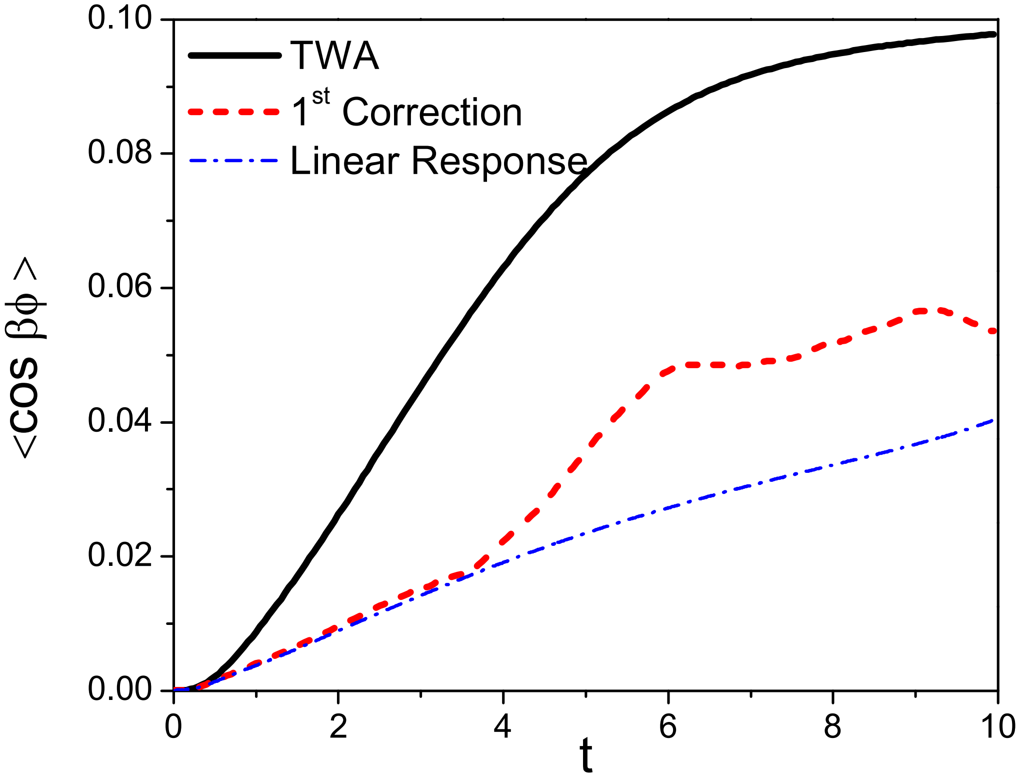
So the parameter β plays the role of $\hbar$

Take $V(t) = 0.1\tanh(0.2t)$ so that we can compare with linear response (usual perturbation theory). Note that for $\beta^2 \leq 8\pi$ the cosine potential is relevant (non-perturbative).

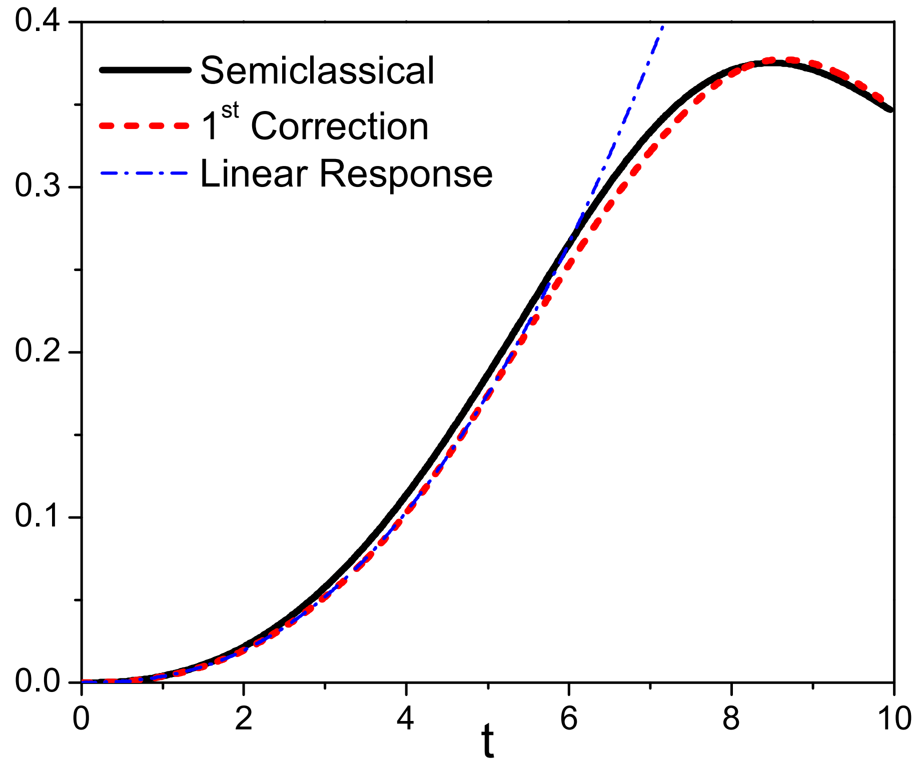
Illustration: Sine-Grodon model, β plays the role of $\hbar$

$$H = \int dx \left[\frac{1}{2} (\partial_x \varphi)^2 + \frac{1}{2} \Pi^2 - V \cos(\beta \varphi) \right]$$

$V(t) = 0.1 \tanh(0.2\, t)$



$\beta = 2\sqrt{\pi}$ –*Tonks*



$\beta = \sqrt{\pi}$

Coherent states

Initial coherent state

$$\hat{\mathcal{H}} = \frac{U}{2}\hat{\psi}^\dagger\hat{\psi}(\hat{\psi}^\dagger\hat{\psi} - 1)$$

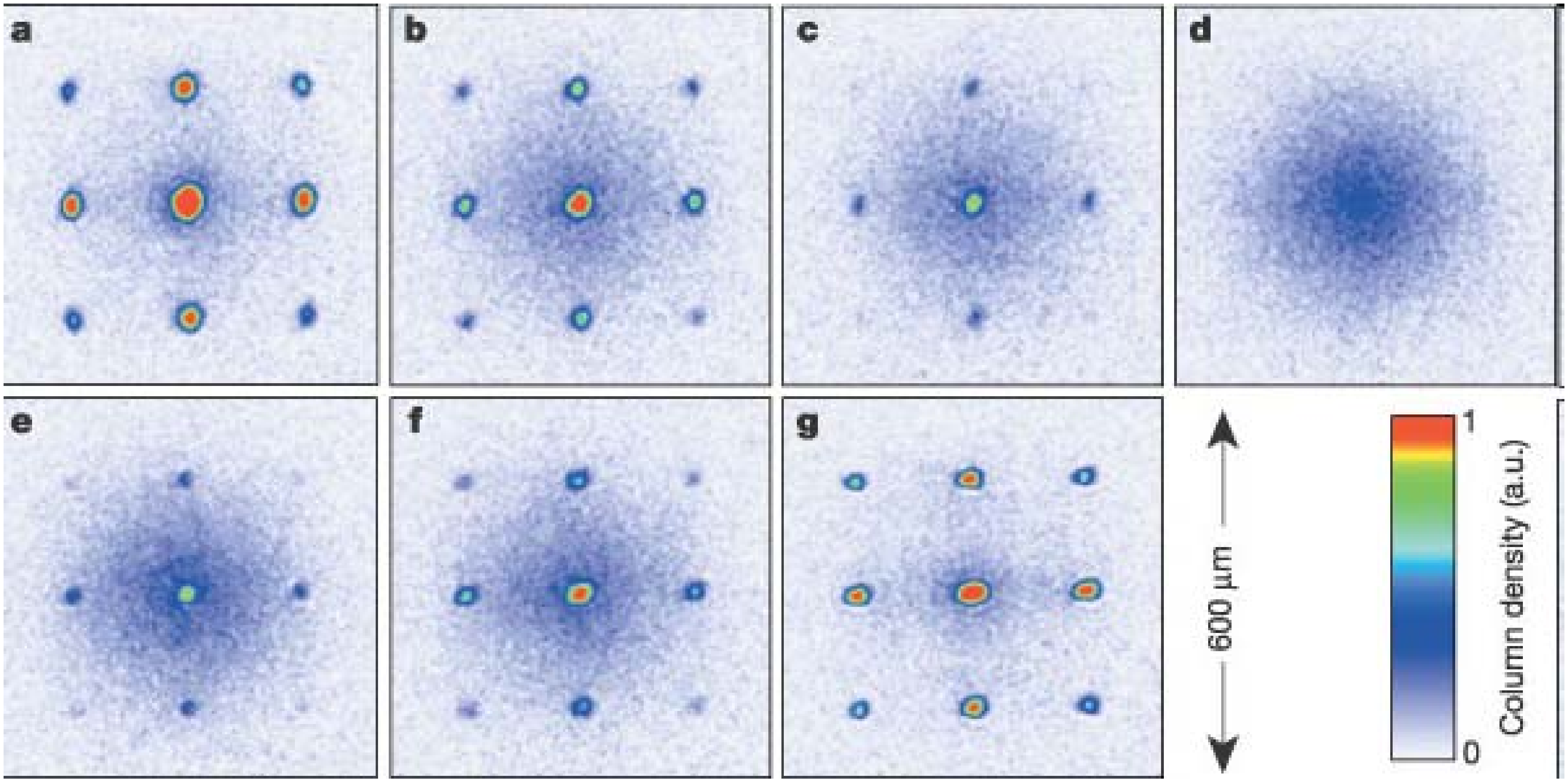
Expand the initial coherent state in the Fock basis, trivially evolve each term in time and re-sum

$$\langle\hat{\psi}(t)\rangle = \sqrt{N}\exp\left[N(\mathrm{e}^{-iUt} - 1)\right]$$

Collapse at t~2π/UN, revival at t~2π/U

Corresponds to the experiments by M. Greiner et.al. (2002)

M. Greiner, O. Mandel, T. W. Hänsch and I. Bloch, Nature 419, 51-54



$$|\alpha\rangle(t) = \mathrm{e}^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} \mathrm{e}^{-i\frac{1}{2}Un(n-1)t/\hbar} |n\rangle \qquad (2)$$

Semiclassical picture (classical limit, N→∞, U→0, UN=const)

$$\mathcal{H}_W(\psi^\star,\psi) = \frac{U}{2}|\psi|^2(|\psi|^2-2) + \frac{U}{4}$$

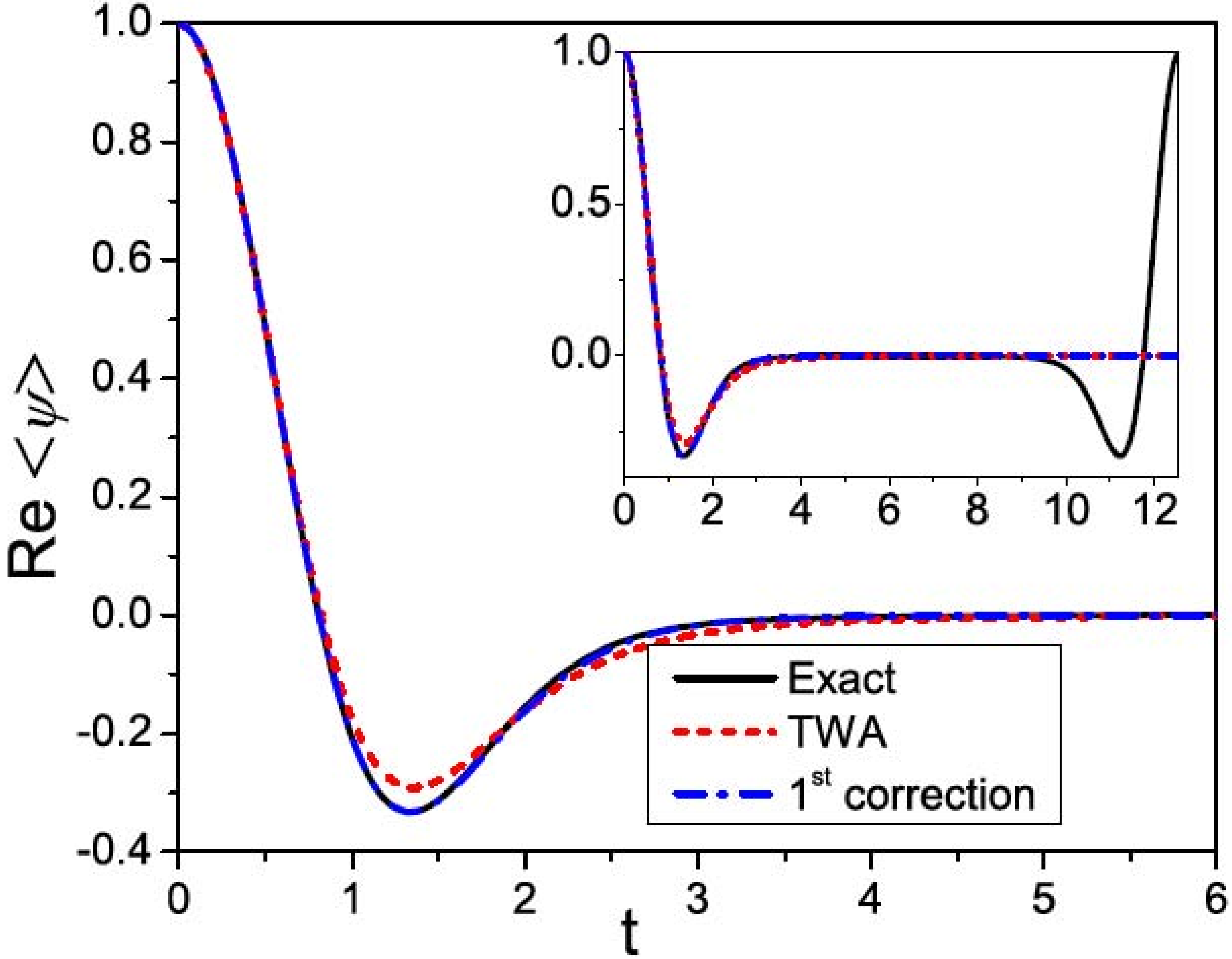
$$i\frac{\partial\psi(t)}{\partial t} = U(|\psi(t)|^2-1)\psi(t)$$

$$W(\psi_0,\psi_0^\star) = 2\exp[-2|\psi_0-\sqrt{N}|^2]$$

$$\psi_0(t) = \sqrt{N}\exp\left[-\frac{iUNt}{1+iUt/2}\right]\exp[iUt]\frac{1}{(1+iUt/2)^2},$$

$$\psi_1(t) = \psi_0(t)\frac{U^2t^2}{4}\left(1-\frac{iUNt}{3\,(1+iUt/2)^2}-\frac{iUt}{3\,(1+iUt/2)}\right)$$

Semiclassical expansion reproduces expansion of exact quantum result in series in 1/N: ψ_0 up to $1/N^2$, ψ_1 up to $1/N^4$



N=4

Semiclassics accurately reproduces collapse but not revival.

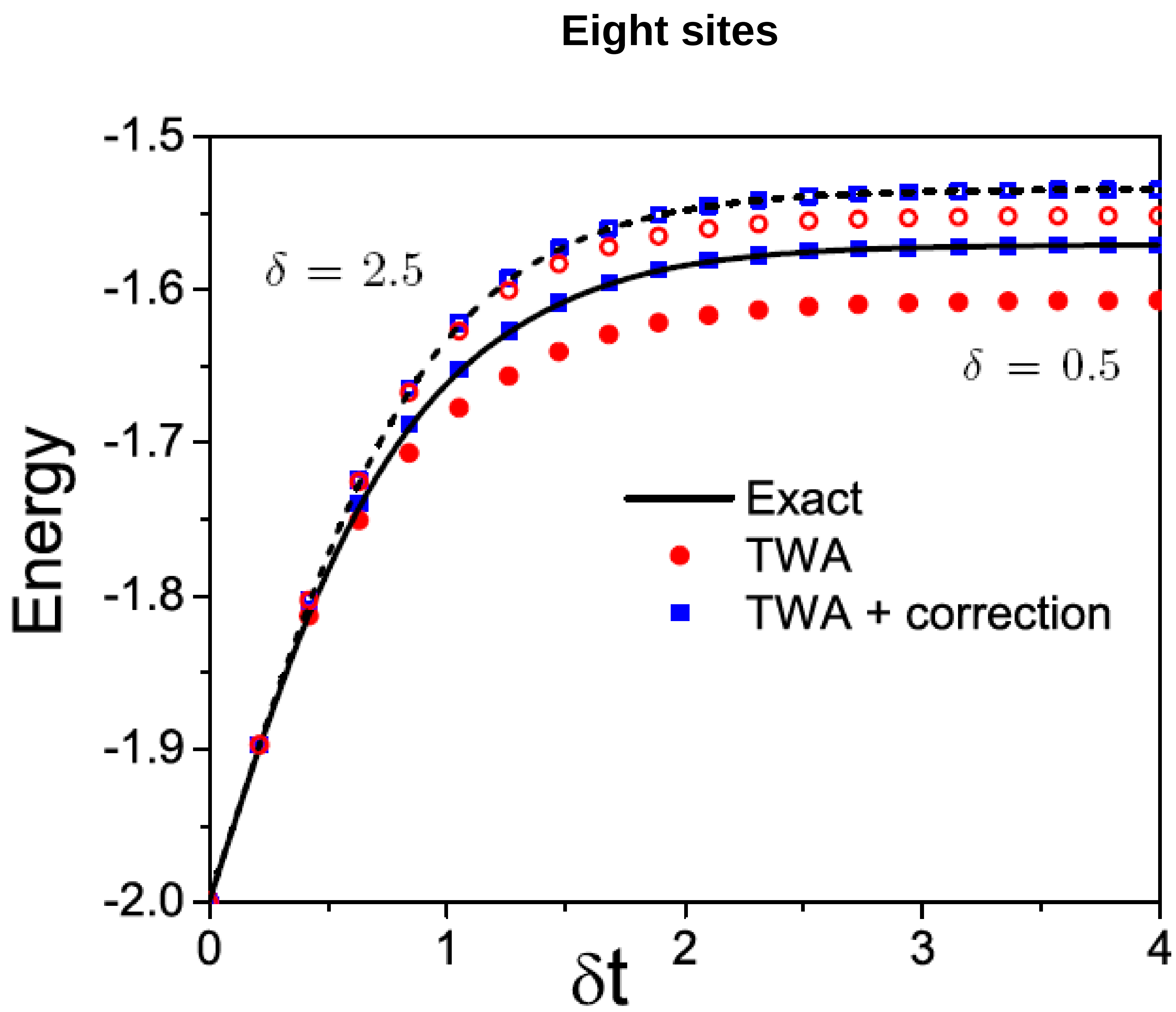
Turning on interactions in a system of interacting bosons

$$\hat{\mathcal{H}} = -J \sum_j \left(\hat{\psi}_j^\dagger \hat{\psi}_{j+1} + \hat{\psi}_{j+1}^\dagger \hat{\psi}_j \right) + \frac{U(t)}{2} \hat{\psi}_j^\dagger \hat{\psi}_j (\hat{\psi}_j^\dagger \hat{\psi}_j - 1)$$

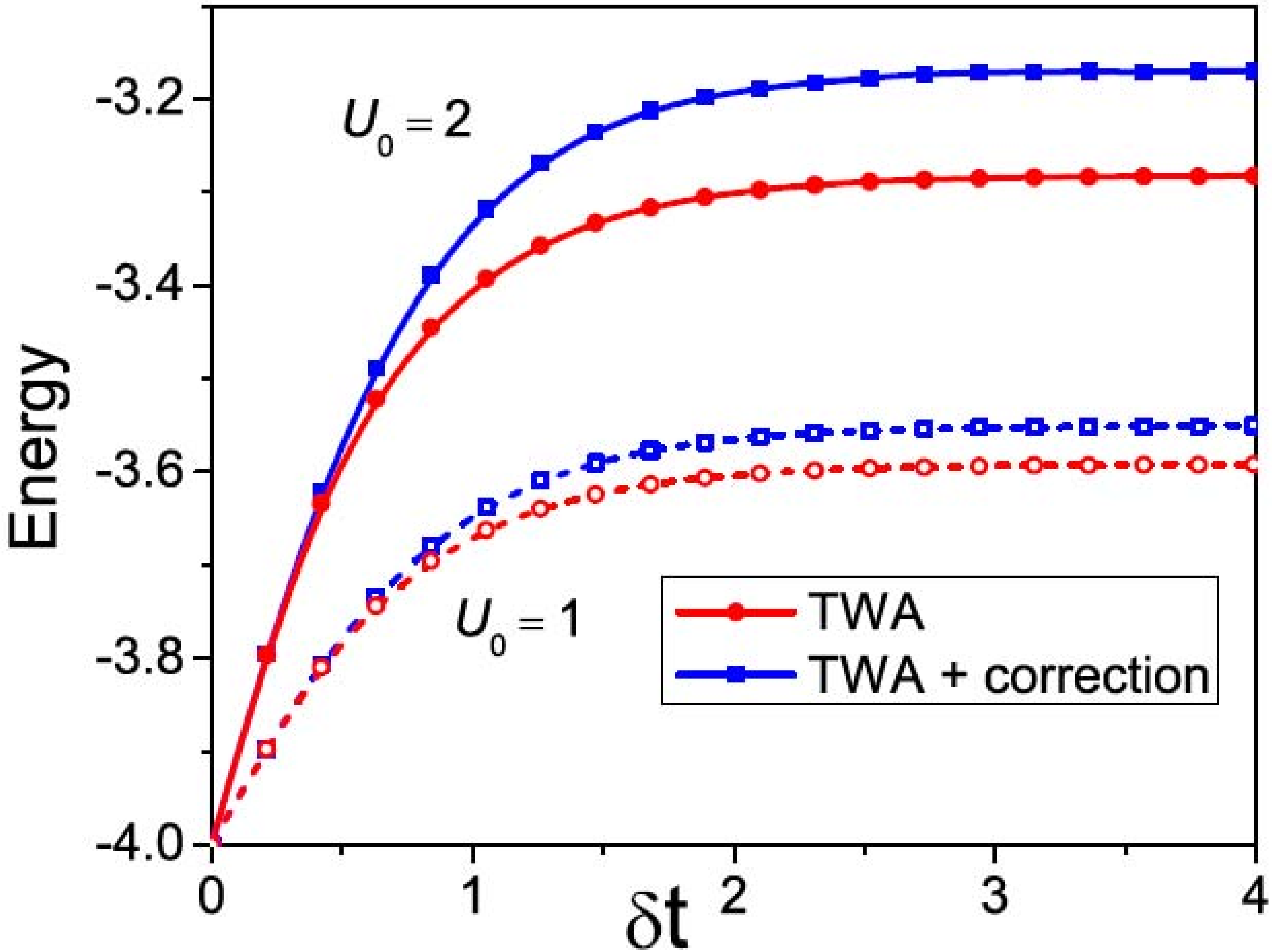
$$U(t) = U_0 \tanh(\delta t)$$

$$W(\psi_j, \psi_j^\star) = \prod_j 2 \exp\left[-2 |\psi_j - \sqrt{N} \exp(i\phi)|^2 \right]$$

Choose N=1 (per site), J=1, U_0=1. Follow energy in the system.


Eight sites
-1.5
-1.6
-1.7
-1.8
-1.9
-2.0
Energy
0
1
2
3
4
δt
δ = 2.5
δ = 0.5
Exact
TWA
TWA + correction

2D lattice 32x32 sites



Dicke model (many-level Landau-Zener problem)

$$\hat{\mathcal{H}} = -\lambda \hat{b}^\dagger \hat{b} + \frac{g}{\sqrt{2S}}\left(\hat{b}^\dagger \hat{S}^- + \hat{b}\hat{S}^+\right)$$

Consider $\lambda(t)=-\delta t$. Start in the with spin pointing up and no bosons.

$$W(b^\star, b, S_x, S_y, S_z) \approx 2\mathrm{e}^{-2|b|^2}\frac{1}{\pi S}\mathrm{e}^{-(S_x^2+S_y^2)/S}\delta(S_z - S)$$

$$i\frac{\partial b}{\partial t} = -\lambda(t) b + \frac{g}{\sqrt{2S}} S^-$$

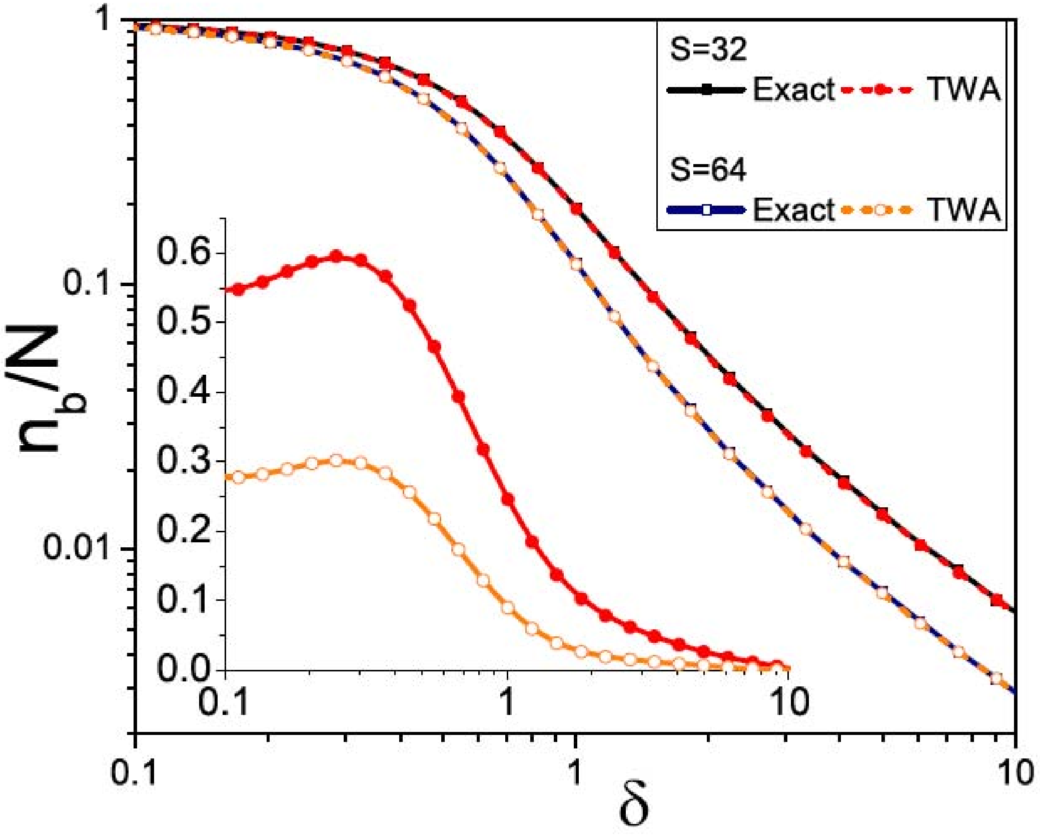
$$\frac{\partial \mathbf{S}}{\partial t} = \frac{2g}{\sqrt{2S}}\mathbf{B}\times\mathbf{S},$$

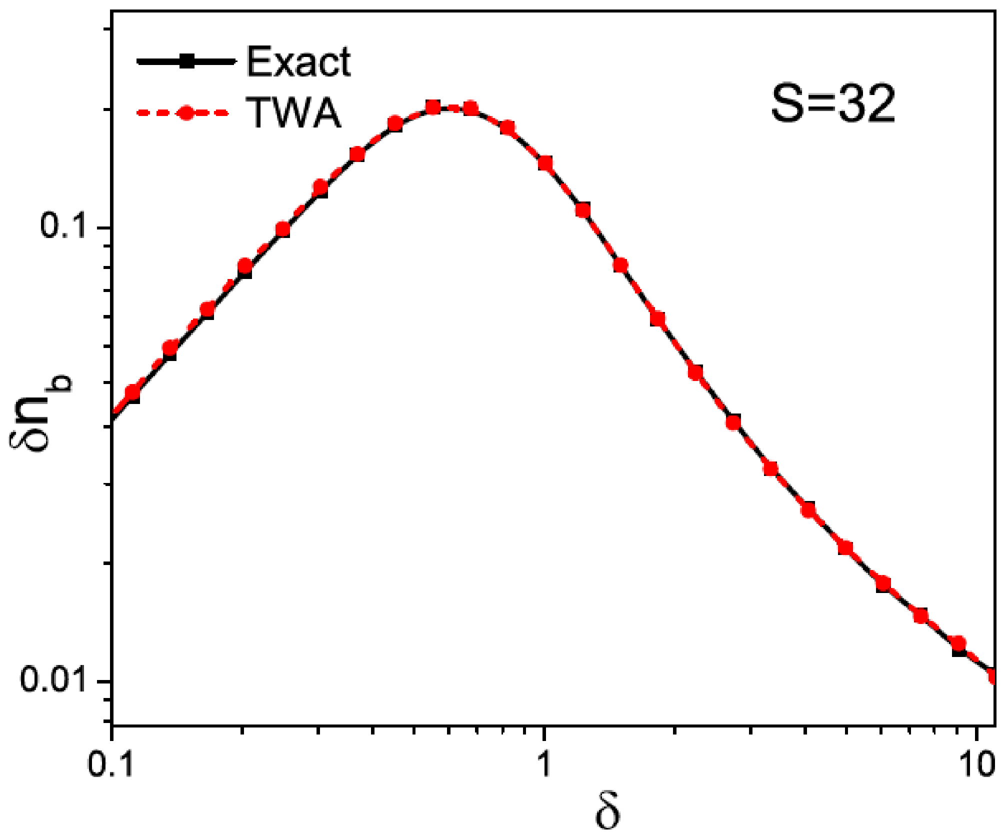
Classical limit: have exact solution $b(t)=0$, $S_z(t)=S$, $S_x(t)=S_y(t)=0$.
Quantum mechanically expect that at $\delta\to 0$ – adiabatically follow the ground state: $\langle b^+ b\rangle \to 2S$, $S_z \to -S$.

The problem can be solved analytically using adiabatic invariants:

A. Altland, V. Gurarie, T. Kriecherbauer, AP, PRA 79, 042703 (2009) , A.P. Itin, P. Törmä, arXiv:0901.4778.

$$n_B/N \simeq 1 - \frac{\delta}{\pi} \ln \frac{\pi N}{\delta} \quad \text{for } \delta \to 0.$$





Almost perfect agreement with the exact result in the whole range of δ

Key points of this lecture. 1) Hamiltonian dynamics.

	Particle limit	Wave limit
Phase space operators	$x,\ p$	ψ, ψ^{+}
Canonical commutation relations	$[x,\ p] = i\hbar$	$[\psi, \psi^{+}] = 1$
Classical limit - Poisson brackets	$\hbar \to 0$ $\{x,\ p\} = 1$	$\lvert\psi\rvert^2 \to \infty$ $\{\psi, \psi^{*}\}_c = 1$
Classical limit – Equations of motion	$\dot{x} = \{x, H\} = p$ $\dot{p} = \{p, H\} = -\partial_x V$	$i\hbar\dot{\psi} = \{\psi, H\}_c =$ $= -\dfrac{\hbar^2}{2m}\psi + U\lvert\psi\rvert^2\psi$
	Newton's equations	GP equations

2) Phase space representation of QM (naturally emerges from Feynman path interal)

Wigner-Weyl quantization:

$$\left\langle \hat{\Omega}(\hat{x},\hat{p}) \right\rangle = \int \frac{dxdp}{2\pi\hbar} \Omega_W(x,p) W(x,p) \quad \Bigg| \quad \left\langle \hat{\Omega}(\hat{\psi},\hat{\psi}^+) \right\rangle = \int d\psi d\psi^* \Omega_W(\psi,\psi^*) W(\psi,\psi^*)$$

$$W_0(p_0,x_0) = \int d\xi_0 \left\langle x_0 - \frac{\xi_0}{2} \right| \rho \left| x_0 + \frac{\xi_0}{2} \right\rangle e^{ip_0\xi_0/\hbar}$$

$$\Omega_W(x,p,t) = \int d\xi \left\langle x + \frac{\xi}{2} \right| \hat{\Omega}(\hat{x},\hat{p},t) \left| x - \frac{\xi}{2} \right\rangle \exp\left[-\frac{ip\xi}{\hbar}\right]$$

Bopp operators:
generate Weyl symbol.
Provide natural interpretation of commutation relations through jumps in the classical phase space

$$\hat{x} = x + i\frac{\hbar}{2}\frac{\partial}{\partial p} \quad \Bigg| \quad \hat{\psi} = \psi + \frac{1}{2}\frac{\partial}{\partial \psi^*}$$

$$\hat{p} = p - i\frac{\hbar}{2}\frac{\partial}{\partial x} \quad \Bigg| \quad \hat{\psi}^+ = \psi^* - \frac{1}{2}\frac{\partial}{\partial \psi}$$

3) Representation of quantum dynamics. Semiclassical approximation:

$$\langle \hat{\Omega}(\hat{x}, \hat{p}, t) \rangle \approx \int\int dx_0 dp_0 W_0(p_0, x_0) \Omega_W(x_{cl}(t), p_{cl}(t), t).$$

Quantum corrections: nonlinear response or stochastic quantum jumps with non-positive probability.

$$\langle \hat{\Omega}(\hat{x}, \hat{p}, t) \rangle \approx \int\int dx_0 dp_0 W_0(x_0, p_0)$$

$$\left\{ \Omega_W(x(t), p(t), t) + \frac{\hbar^2}{4} \sum_{\tau_i} \frac{\partial^3 V(x_i)}{\partial x_i^3} \int d\xi F_3(\xi) \Omega_W \left[x(t), p(t), t, \delta p_i = \xi \sqrt[3]{\Delta \tau} \right] \right\}$$

$$F_3(\xi) = \frac{1}{2} \left(\frac{\xi^3}{3} - \xi \right) \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\xi^2/2}$$

These methods are very useful to analyze various quantum (coherent) dynamical problems with initial conditions. Many applications to cold atoms. Open new possibilities.