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Time-traveling (sterile) neutrinos

Thomas WEILER
Vanderbilt University
Department of Physics & Astronomy
P.O. Box 1807-B
Nashville 37235, U.S.A.

Time-traveling sterile neutrinos: adventures in (warped) extra dimensions

Collaborative with Heinrich Päs, James Dent, and Sandip Pakvasa

Abstract

In asymmetrically warped spacetimes different warp factors are assigned to space and to time. We discuss causality properties of these warped brane universes and argue that scenarios with two extra dimensions may allow for timelike curves which can be closed via paths in the extra-dimensional bulk. In particular, necessary and sufficient conditions on the metric for the existence of closed time-like curves are presented. We find a six-dimensional warped metric which satisfies the CTC conditions, and satisfies the null, weak and dominant energy conditions on the brane (although only the former is satisfied in the bulk). Such scenarios are interesting, since they open the possibility of experimentally testing the chronology protection conjecture by manipulating gravitons or hypothetical gauge-singlet fermions ("sterile neutrinos") on our brane, which then naturally propagate in the extra dimensions.

SYLLABUS:

Gödel, van Stockum, Tipler planes

Spacetime metrics, warped space

Closed Time-like Curves (CTCs)

From 5D to 6D

Energy and Pressure Distributions

Conclusions

GTvS Spacetime

The Gödel metric describes a pressure-free perfect fluid with negative cosmological constant and rotating matter, and the Tipler-van-Stockum (TvS) spacetime is being generated by a rapidly rotating infinite cylinder. In both cases the metric can be written as

$$ds^2 = +g_{tt}(r)\,dt^2 + 2g_{t\phi}(r)\,dtd\phi - g_{\phi\phi}(r)\,d\phi^2 - g_{rr}\,dr^2 - g_{zz}\,dz^2\,. \tag{1}$$

Here the $g_{\mu\nu}$ are complicated functions of the radial distance r from the symmetry axis, and a parameter characterizing the angular velocity of the cylinder (not shown).

Distortions of the coordinates ϕ and t in the radial direction is an example of warping.

A familiar example of time-warping is the (flat) Robertson-Walker Big Bang metric

$$ds^2 = dt^2 - a^2(t)\,d\vec{x}\cdot d\vec{x}\,.$$

Negative Time

A dynamical approach to GTvS causality examines the purely azimuthal null-curve with $ds^2 = 0$. One gets

$$\dot{\phi}_\pm = \frac{g_{t\phi} \pm \sqrt{g_{t\phi}^2 + g_{tt}\, g_{\phi\phi}}}{g_{\phi\phi}}\,, \tag{2}$$

where the $\pm$ refers to co-rotating and counter-rotating lightlike signals. The coordinate time for a co-rotating path is

$$\Delta T_+ = \Delta\phi \left(\frac{g_{\phi\phi}}{g_{t\phi} + \sqrt{g_{t\phi}^2 + g_{\phi\phi}\, g_{tt}}} \right) . \tag{3}$$

As $g_{\phi\phi}$ goes from positive to negative, the light-cone tips such that the azimuthal closed path is traversed in negative time

$$\Delta T_+ = \frac{-2\pi\, |g_{\phi\phi}|}{g_{t\phi} + \sqrt{g_{t\phi}^2 + g_{\phi\phi}\, g_{tt}}} \qquad [g_{\phi\phi} < 0]\,. \tag{4}$$

The quantum returns to its origin before it left, marking the existence of a CTC.

Note that the Lorentzian signature is maintained even as $g_{\phi\phi}$ switches sign to a negative value as long as $g_{t\phi}$ remains greater than $\sqrt{-g_{\phi\phi}\, g_{tt}}$.
In particular, $g_{t\phi}$ cannot be zero.

Lightcone (worldline) slopes

The clear discriminator of the arrows of time are the slopes of the local light-cone,

$$\frac{dt}{r\,d\phi} \equiv s_{\pm}(r) = (r\dot{\phi}_{\pm})^{-1} = \frac{1}{r}\,\frac{g_{\phi\phi}}{g_{t\phi} \pm \sqrt{-g_4}} = -\frac{1}{r}\,\frac{g_{t\phi} \mp \sqrt{-g}}{g_{tt}}\,. \tag{5}$$

Notice that if $g_{\phi\phi}$ and g_{tt} are positive, then regardless of the sign of $g_{t\phi}$, the light-cones (worldlines) remain in the first and second quadrants of the (t, ϕ) plane (as is the case of the Minkowski light-cone). Thus, for a backward flow of time, $g_{\phi\phi}$ (or, g_{tt}) must go through zero and become negative.

It is useful to consider the product of slopes

$$s_{+}(r)\, s_{-}(r) = \frac{-1}{r^2}\frac{g_{\phi\phi}}{g_{tt}}. \tag{6}$$

For time to move backwards one of the world lines defining the light-cone must move into the lower half of the $t - \phi$ plane. From (6) one can see that (i) this happens smoothly if $g_{\phi\phi}$ goes through zero; (ii) happpens discontinuously if g_{tt} goes through zero; (iii) that a smooth change in the sign of $g_{t\phi}$ cannot move either slope through zero to the domain of negative time.

With the focus here on a smooth change of sign for $g_{\phi\phi}$, it is useful to examine the slopes at small $g_{\phi\phi}$. One finds

$$s_{\pm}(\text{leading order in } g_{\phi\phi}) = \begin{cases} \frac{1}{2r}\,\frac{g_{\phi\phi}}{g_{t\phi}} \\ \\ -\frac{2}{r}\,\frac{g_{t\phi}}{g_{tt}} \end{cases} \tag{7}$$

It is clear that the slope s_+ goes through zero with $g_{\phi\phi}$, leaving the first quadrant and moving into the fourth quadrant. With increasing ϕ, time for the associated co-rotating world line runs backwards. On the other hand, the sign of s_- remains unchanged, and time for the associated counter-rotating world line continues to run forward.

In the following we will apply a similar argument to different scenarios of asymmetrically warped spacetimes.

GTvS, the Good, the Bad, the Ugly

It is instructive to mention the visceral arguments against the relevance of the Gödel and TvS metrics. First of all, they are not asymptotically flat, and so presumably cannot occur within our Universe; rather, they must be our Universe, which contradicts observation. Secondly, the initial conditions from which they can evolve are either non-existent (Gödel) or sick (TvS). Furthermore, the TvS metric assumes an infinitely-long cylinder of matter, which is unphysical. On the positive side, literally, the Einstein equation endows $\rho = T^0_{\ 0} = (R^0_{\ 0} - \frac{1}{2}R)/8\pi\, G_N$ (with the geometric RHS determined by the metric) with a positive value everywhere; there is no need for "exotic" $\rho < 0$ matter. A further positive feature is the simplicity of finding the CTC by travel along the periodic variable ϕ.

A linear path off the brane

We may replace the periodic coordinate of GTvS with the unbounded x coordinate, and omit the y and z coordinates for brevity. Then one obtains

$$ds^2 = g_{tt}(u,v)\,dt^2 + 2g_{tx}(u,v)\,dxdt - g_{xx}(u,v)\,dx^2 - du^2 - dv^2\,. \tag{8}$$

(Notice in particular the sign convention on the coefficient of dx^2.)

The speed of light at any point will depend on (u,v) through the metric elements. The restriction to Lorentzian signature implies that

$$-g_6 \equiv -\mathrm{Det}(g_{\mu\nu}) = g_{tt}(u,v)\,g_{xx}(u,v) + g_{tx}^2(u,v) > 0\,. \tag{9}$$

World lines for lightlike travel (null lines) satisfy

$$0 = g_{tt}(u,v) + 2g_{tx}(u,v)\,\dot{x} - g_{xx}(u,v)\,\dot{x}^2 - \dot{u}^2 - \dot{v}^2\,. \tag{10}$$

The solutions to (10) for the analogs of co-rotating and counter-rotating light speed at fixed (u,v) are

$$\dot{x}_\pm = \frac{g_{tx}(u,v) \pm \sqrt{-g_6}}{g_{xx}(u,v)}\,. \tag{11}$$

On the brane, $\dot{x}$ must equal $c = 1$, so we again choose $g_{tt}(0,0) = g_{xx}(0,0) = 1$ and $g_{tx}(0,0) = 0$.

Causal properties of Eq. (11)

We assume that g_{tt} is everywhere positive, so that (i) coordinate time t is everywhere timelike, and (ii) no singularities are introduced in s_+s_- or in g_{tt}. As shown in Eq. (6), the sign of g_{tx} does not influence the causal structure, and for definiteness we take it to be positive semidefinite. It is the sign of the metric element g_{xx} that has smooth causal significance.

Similar to the causal analysis of the GTvS model, we write the two slopes of the light-cone as

$$\frac{dt}{dx} \equiv s_\pm(u,v) = (\dot{x}_\pm)^{-1} = \frac{g_{xx}(u,v)}{g_{tx}(u,v) \pm \sqrt{g_{tx}^2(u,v) + g_{xx}(u,v)g_{tt}(u,v)}}\,. \tag{12}$$

From this, one readily gets

$$s_+s_- = \frac{-g_{xx}}{g_{tt}}\,. \tag{13}$$

It is easily seen that when g_{xx}, g_{tt}, and g_{tx} are all positive, the slopes are of opposite sign, and are connected to the Minkowski metric in the smooth limit $g_{tx} \to 0$. Thus, with g_{tt} and g_{tx} assumed positive, time flows in the usual manner if g_{xx} is positive. Furthermore, with $g_{xx} > 0$, we have $\text{sign}(g_{tx} \pm \sqrt{-g_6}) = \pm$, so that from Eq. (11), one has $\dot{x}_+ > 0$, and $\dot{x}_- < 0$. Thus, a positive g_{xx} (as in the Lorentz metric) offers the standard situation with time flowing forward and velocity $\dot{x}$ having either sign.

On the other hand, if g_{xx} is negative, then Eq. (13) shows that one light-cone slope has changed sign. The small g_{xx} limit of the slopes

$$s_\pm(\text{leading order in } g_{xx}) = \begin{cases} \frac{g_{xx}}{2g_{tx}} \\ \\ -\frac{2g_{tx}}{g_{tt}} \end{cases} \tag{14}$$

reveals that it is the positive slope which has passed through zero to become negative, signifying a world line moving from the first quadrant, through the x-axis, into the fourth quadrant where times flows backwards for increasing x. With both slopes negative, one has that $\dot{x}_\pm(g_{xx} < 0) < 0$. Thus, travel with increasing time is in the negative x direction, while travel with decreasing time is in the positive x direction. We summarize the causal properties of the metric (8) in a Table:

	$g_{xx} > 0$	$g_{xx} < 0$
$\Delta T > 0$	$\dot{x}_+ > 0$ $(\Delta x > 0)$	
	$\dot{x}_- < 0$ $(\Delta x < 0)$	$\dot{x}_- < 0$ $(\Delta x < 0)$
$\Delta T < 0$		$\dot{x}_+ < 0$ $(\Delta x > 0)$

Table 1: Solution types for metric (8), and their casual properties. In particular, note that no solution exists for motion backwards in time along the negative-x direction.

Causality Table

CTC "construction"

The world line which we investigate is the following: the signal travels first from the brane at $(u, v) = (0, 0)$ to the hyperslice at (u_1, v_1), then from (u_1, v_1) to the hyperslice at (u_2, v_2), and finally back from (u_2, v_2) to the point of origin $(0, 0)$ on the brane (see Fig. 2). While on the (u_1, v_1) hyperslice, the signal travels a distance ΔX in the positive x-direction over a negative time $\Delta T_1 = -|\Delta T_1|$. While on the (u_2, v_2) hyperslice, the signal travels back an equal negative distance $-\Delta X$ in time ΔT_2 to close the spatial projection of the worldline on the brane. To close the worldline on the brane, it is necessary that $T_2 + T_1 < 0$. (But not equal to zero, as we allow for small positive travel times from the brane at $(u, v) = (0, 0)$ to (u_1, v_1), from (u_1, v_1) to (u_2, v_2), and back from (u_2, v_2) to $(0, 0)$.)

Figure 1: A spacetime map of a closed timelike curve in an asymmetrically warped universe: (i) A signal leaves the brane and arrives at the $v > 0$ with $u = 0$ hyperslice. The transit time is negligible compared to the next step.
(ii) As viewed from the brane, the signal takes a spacelike shortcut at constant $v > 0$, $u = 0$, from the origin O to point $P1$ with $t_1 < 0$. The dt/dx slope $s1$ is outside of the light-cone.
(iii) The signal travels from $(v, u) = (> 0, 0)$ to $(v, u) = (0, > 0)$ in negligible time compared to step (ii).
(iv) As viewed from the brane, the signal then travels back along a path of constant $u > 0$, with $v = 0$, from $P1$ to $P2 = (t, 0)$. The dt/dx slope $s2$ is more outside of the light-cone, such that elapsed time t remains negative.
(v) The signal returns to the brane at $(v, u) = (0, 0)$ in negligible time compared to (ii).
A Lorentz boost transforms $P1$ and $P2$ along their respective brane-space hyperbolas to $P1'$, and to $P2' = (t', x')$. It can be shown that time t' is again negative, and $|x'| < |t'|$ so that a world-line inside the light-cone closes the curve. The Lorentz-transformed paths are related to shortcuts via $u > 0$ with $v = 0$ first, and $v > 0$ with $u = 0$ second.

Causality Table

The transit time $(\Delta T_1)_\pm$ for light to travel a positive distance $\Delta X > 0$ at constant (u_1, v_1), as viewed from the brane, is

$$\begin{aligned}(\Delta T_1)_\pm &= \int_0^{\Delta T_1} dt = \int_0^{\Delta X} dx \, \frac{g_{xx}(u_1, v_1)}{g_{tx}(u_1, v_1) \pm \sqrt{-g_6(u_1, v_1)}} \\ &= \Delta X \left(\frac{g_{xx}(u_1, v_1)}{g_{tx}(u_1, v_1) \pm \sqrt{-g_6(u_1, v_1)}} \right) .\end{aligned} \tag{15}$$

The integrations on dt and dx are trivial because the metric does not depend on the coordinate time t or brane variable x. According to Eq. (9), the Lorentz signature is maintained as long as $g_{tx}^2 > g_{tt}\,(-g_{xx})$. We have shown that the world line for x_+ lies below the x-axis when $g_{xx} < 0$, and so we require $g_{xx}(u_1, v_1) < 0$ in order to gain negative time $\Delta(T_1)_+$ during travel on the (u_1, v_1) hyperslice. From here on, we will simply use the label ΔT_1 for this negative $\Delta(T_1)_+$ solution on the (u_1, v_1) hyperslice:

$$\Delta T_1 \equiv \Delta(T_1)_+ = \Delta X \left(\frac{g_{xx}(u_1, v_1)}{g_{tx}(u_1, v_1) + \sqrt{-g_6(u_1, v_1)}} \right) . \tag{16}$$

To close the worldline, the lightlike signal must return from positive ΔX to the origin $x = 0$ in a time ΔT_2 less than or equal to $|\Delta T_1|$. If this were to occur in a negative time, then we would have $g_{xx} < 0$ and $\dot{x} > 0$. Table **??** shows that there is no solution of this type available. So the return path must take place in positive time, with $\dot{x} < 0$. Reference again to Table **??** reveals that the return solution is $\dot{x}_-$. In principle, the $\dot{x}_-$ solution on the (u_1, v_1) hyperslice provides a return path. However, it is easy to show that the return time ΔT_2 for this solution exceeds $|\Delta T_1|$ and so fails to close the world line. Thus, we must go to a second hyperslice at (u_2, v_2). We have

$$\Delta T_2 = \left[\int_{\Delta X}^{0} dx = -\Delta X \right] \left(\frac{g_{xx}(u_2, v_2)}{g_{tx}(u_2, v_2) - \sqrt{-g_6(u_2, v_2)}} \right) , \tag{17}$$

with $g_{xx}(u_2, v_2)$ of either sign.

The necessary condition relating the outgoing and return paths of a CTC is that the sum $\Delta T_2 + \Delta T_1$ be less than zero. Equivalently, the CTC conditions are that

$$\frac{-g_{xx}(u_2, v_2)}{g_{tx}(u_2, v_2) - \sqrt{-g_6(u_2, v_2)}} + \frac{g_{xx}(u_1, v_1)}{g_{tx}(u_1, v_1) + \sqrt{-g_6(u_1, v_1)}} < 0 \,, \tag{18a}$$

and that

$$g_{xx}(u_1, v_1) < 0 \,, \tag{18b}$$

(recall our sign convention (8) for g_{xx}). Here $\Delta T_1 < 0$, $\Delta T_2 > 0$ has been used. Note that (18a) is both necessary and sufficient, while (18b) is implied by (18a), assuming that the negative time ΔT_1 is accumulated during the travel on the (u_1, v_1) hyperslice. Thus (18b) is a necessary but not a sufficient condition. It is nevertheless a useful guide for our analysis of candidates for CTC spacetimes. The two transit times ΔT_1 and ΔT_2 can be made arbitrarily long, and so the short-time paths from the brane to the (u_1, v_1) hyperslice, from (u_1, v_1) to the (u_2, v_2) hyperslice, and from (u_2, v_2) back to the brane, can be neglected; if we can show the existence of metric elements on the (u_2, v_2) and (u_1, v_1) hyperslices satisfying the constraints of Eqs. (18), we will have demonstrated the existence of a closed worldline for lightlike quanta. Since ΔT_1 and ΔT_2 can be made arbitrarily long, finite mass effects of order $1/\gamma^2$ may be neglected, and so the same conditions enable CTCs for extremely relativistic timelike quanta.

It is worth remarking that besides the necessity of the inequality $g_{xx}(u_1, v_1) < 0$ to generate a negative time path, it is also necessary that $g_{tx}(u_1, v_1) \neq 0$. Without this latter condition, the Lorentz signature could not be maintained when $g_{xx} < 0$, and indeed, the square root in the second term in (18a) would become imaginary.

Noting that ΔT_1 is negative and ΔT_2 positive, we have $\Delta T_1/\Delta X = -1/|\dot{x}_1|$ and $\Delta T_2/\Delta X = +1/|\dot{x}_2|$. Thus, we may also interpret Eqs. (18) to say that

$$\frac{1}{|\dot{x}_2|} + \frac{-1}{|\dot{x}_1|} < 0\,, \quad \text{i.e., } |\dot{x}_2| > |\dot{x}_1|\,. \tag{19}$$

In words, the quantum must return to the brane at a speed even more superluminal than that with which it exited. This requirement of a superluminal return speed tells us that the return path cannot be on our brane.

Since Eqs. (18) are the necessary and sufficient condition for a CTC, any metric failing to satisfy the inequalities in (18) has no CTC. On the other hand, we have seen that the GTvS model contains a CTC in the 2+1 dimensional space $(r, \phi; t)$. Thus, we expect that CTCs will populate some metrics in $N+1$ spaces, for any $N > 2$, as well. Indeed, Eq. (18) summarizes the straightforward recipe for constructing metrics with CTCs in spaces equal to or larger than 2+1. One simply needs (i) a g_{xx} that passes through zero as a function of another spatial coordinate "u", (ii) a nonzero g_{tx} in the region of u where $g_{xx} < 0$, and (iii) a return path suitably arranged with nonzero values g_{xx} and g_{tx} in another coordinate region of u. The mathematical construction of such metrics is not in question. What may be debated is the physics motivation for such metrics. In the following sections we will develop a metric with CTCs, motivated by a popular concept in particle physics and gravitation, extra-dimensional "warped" spaces.

Causality with one warped extra dimension

We first consider the five-dimensional asymmetrically-warped line element with a single extra dimension which we label as "u":

$$ds^2 = dt^2 - \sum_i \alpha^2(u)\,(dx^i)^2 - du^2, \tag{20}$$

$i = 1, 2, 3$, with our brane located at the $u = 0$ submanifold. With no loss of generality, we may take $\alpha(u)$ to be positive.

Variants of this warped spacetime (20) have been proposed as solutions to the cosmological horizon problem [**?**], and as a possible source for a small cosmological constant [**?**]. They also have been discussed in the context of the gravitational generation of cosmic acceleration [**?**], and infrared modification of gravity [**?**]. Very recently it has been shown that sterile neutrinos propagating in such a spacetime can account for the LSND neutrino oscillation evidence, without the problems faced by conventional four-dimensional four-neutrino scenarios [**?**].

The warped spacetime of (20) allows shortcut geodesics connecting spacelike-separated events on the brane if $|\alpha(u)| < |\alpha(0)|$ for any $u \neq 0$. However, the metric (20) exhibits a global time function t. Thus, taken by itself this spacetime is causally stable and does not allow for CTCs. The failure of (20) to support a CTC can also be seen in our CTC equations (18). Since $g_{xx} = \alpha^2$ in (20) cannot be negative without violating the assumed Lorentzian signature, the CTC condition (18b) cannot be satisfied.

Given that the metric (20) does allow spacelike geodesics (as viewed from the brane), a boosted observer may see a negative time for the outgoing path. It is of pedagogical value to investigate (20) in the coordinates of this boosted observer. This effort will serve as a precursor for a successful construction of a metric with CTCs in six dimensions in the next section.

The metric in (20) is in Gaussian normal form with respect to u (i.e., $g_{tu} = g_{x_i u} = 0$), so the induced metric on each hypersurface with constant u is simply given by the extra-dimensional metric evaluated on the hypersurface. These induced metrics are purely Minkowskian, albeit with a different constant limiting velocity $c(u) = \alpha^{-1}(u)$ on each hypersurface. This means that a Lorentz symmetry can be defined for each hypersurface, but each hypersurface's Lorentz symmetry will not hold on any other hypersurface. Lorentz invariance is not a global symmetry.

It is natural to choose $c(u=0)=1$ such that the induced metric on the brane is given by $ds^2_{\text{brane}} = dt^2 - dx^2$. There then follows the usual Lorentz symmetry under the familiar transformations along our brane direction:

$$x' = \gamma\,(x - \beta t)\,, \qquad t' = \gamma\,(t - \beta x)\,, \qquad u' = u = 0, \tag{21}$$

or equivalently, the inverse transformation

$$x = \gamma\,(x' + \beta t')\,, \qquad t = \gamma\,(t' + \beta x')\,, \tag{22}$$

with the usual definition $\gamma = (1-\beta^2)^{-1/2}$. The complete metric in the brane-boosted system is given by the tensor transformation law

$$g'_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta}\, g_{\mu\nu}, \tag{23}$$

where $g_{\mu\nu} = \text{diag}(1, -\alpha^2, -1)$ is the Gauss-normal metric of Eq. (20). Using Eq. (22), the resulting boosted metric is

$$g'_{\mu\nu} = \begin{pmatrix} \gamma^2(1-\beta^2\alpha^2) & \gamma^2\beta(1-\alpha^2) & 0 \\ \gamma^2\beta(1-\alpha^2) & -\gamma^2(\alpha^2-\beta^2) & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{24}$$

Notice that only for $\alpha^2 = 1$ is the metric Lorentz invariant. Such is the case on our brane, but generally not the case on other hypersurfaces. On other hypersurfaces, Eq. (20) gives the limiting velocity seen by local inhabitants in the rest frame as $\alpha^{-1}(u_j) \equiv \alpha_j^{-1}$. However, this value is not invariant under Lorentz boosts defined on our brane.

At first glance the metric (24) seems to belong to the to the broad class of metrics (1), which includes the Gödel- and Tipler-van-Stockum (GTvS) spacetimes. After all, $g_{tx} \neq 0$ where $\alpha \neq 1$, i.e., off the brane. Moreover, $\text{sign}(g_{xx})$ is adjustable. However, a significant difference from the GTvS metric is that, in the case here the variable x is not periodic (unless our universe has the topology of a flat torus). It is thus required to construct an explicit return path to the spacetime point of origin. It is not possible to construct a return path that arrives sufficiently quickly to close the curve.

A rigorous way to show the absence of a CTC is to inject the boosted metric elements of (24) into the CTC conditions (18). A boost does not alter the determinant of the metric, so $\sqrt{g_5} = |\alpha|$. With $g_{tx} = \gamma^2\beta(1-\alpha^2)$, and $g_{xx} = \gamma^2(\alpha^2 - \beta^2)$, one finds that the CTC condition in Eq. (18a) is satisfied only if $|\alpha_1| < -|\alpha_2|$, which is a contradiction. So there is no CTC.

Thus, the return path cannot be superluminal and thus cannot close the timelike curve, and there is no CTC.

So we conclude that in an asymmetrically warped space with only one extra dimension (20), as well as its boosted equivalent (24), no CTCs exist if space dimensions are non-periodic. 5-dimensional brane universes with one asymmetrically warped extra dimension are causally stable. (The exception is the topology of a flat torus, which maps spacetime into the class of GTvS spacetimes of Eq. (1).) We are thus led to consider next a spacetime with two asymmetrically warped extra dimensions. There we will find that CTCs do exist. The lesson learned from the attempt to formulate a 5D metric having CTCs will provide intuitive input into the construction of the 6D metric.

CTCs in two warped extra dimensions

We now proceed by constructing a 6D metric exhibiting CTCs, which is a natural generalization of the metric (20). Let "u" and "v" label the two extra space dimensions. We assume that these dimensions have warp factors $\alpha(u)$ and $\eta(v)$, respectively.

In our attempt to construct an asymmetrically warped metric exhibiting CTCs in 5D, we found that the metric in Eq. (20) allowed a quantum to travel superluminally into the bulk. Being outside our light-cone, the worldline of this quantum could be boosted to negative time by a Lorentz transformation on the brane. However, we showed that a superluminal return path to the brane was required to close the worldline and there was no such path. This failure can be traced to the fact that the Lorentz transformation was just a coordinate change, and so provided a change of view, but no new physics. What is needed is a nonzero g_{tx} that cannot be removed by a linear transformation among brane coordinates. Introducing the 6th dimension provides a solution, first because it allows a superluminal return path along the additional 6th dimension, and second because it allows $g_{tx}(u, v)$ to be "hard-wired" into the metric so that it is not removable by a linear coordinate transformation on the brane. (Recall that we learned in Section (**??**), via Eqs. (18) and the discussion just below these, that a nonzero $g_{tx}(u, v)$ is a necessary ingredient for the existence of CTCs.)

A natural 6D generalization of (20) can be realized by assuming that the metric for the u- and v dimensions exhibits the simple form in (20), but in different Lorentz frames. This assumption seems natural for any spacetime with two or more extra dimensions, since there is no preferred Lorentz frame for the bulk, from the viewpoint of the brane. In analogy to (20), the 6D generalization could be realized by assuming two AdS-Schwarzschild or AdS-Reissner-Nordström black holes being located in the u and v dimension and moving with a relative velocity. This choice also ensures superluminal travel to as well as from the brane, and a Minkowskian metric on the brane. To construct this 6-dimensional metric explicitly, let us denote by β_{uv} the "relative velocity" between the two Lorentz frames in which the u and v dimensions assume the simple form (20), respectively. We incorporate the "u-frame" slice at $v = 0$ by retaining the warp factor $\alpha(u)$ on the brane coordinate dx, and we incorporate the "v-frame" slice at $u = 0$ by writing the boosted metric in Eq. (**??**) with the warp $\alpha(u)$ now replaced by $\eta(v)$. The resulting full 6-dimensional metric then has

Figure 2: A spacetime map of a closed timelike curve in an asymmetrically warped universe: (i) A signal leaves the brane and arrives at the $v > 0$ with $u = 0$ hyperslice. The transit time is negligible compared to the next step.
(ii) As viewed from the brane, the signal takes a spacelike shortcut at constant $v > 0$, $u = 0$, from the origin O to point $P1$ with $t_1 < 0$. The dt/dx slope $s1$ is outside of the light-cone.
(iii) The signal travels from $(v, u) = (> 0, 0)$ to $(v, u) = (0, > 0)$ in negligible time compared to step (ii).
(iv) As viewed from the brane, the signal then travels back along a path of constant $u > 0$, with $v = 0$, from $P1$ to $P2 = (t, 0)$. The dt/dx slope $s2$ is more outside of the light-cone, such that elapsed time t remains negative.
(v) The signal returns to the brane at $(v, u) = (0, 0)$ in negligible time compared to (ii).
A Lorentz boost transforms $P1$ and $P2$ along their respective brane-space hyperbolas to $P1'$, and to $P2' = (t', x')$. It can be shown that time t' is again negative, and $|x'| < |t'|$ so that a world-line inside the light-cone closes the curve. The Lorentz-transformed paths are related to shortcuts via $u > 0$ with $v = 0$ first, and $v > 0$ with $u = 0$ second.

the form

$$ds^2 = \gamma_{uv}^2 \left\{ [1 - \beta_{uv}^2 \eta^2(v)]\, dt^2 + 2\beta_{uv}\alpha(u)[1 - \eta^2(v)]\, dxdt - \alpha^2(u)[\eta^2(v) - \beta_{uv}^2]\, dx^2 \right\} - du^2 - dv^2. \tag{25}$$

One easily finds that $-Det \equiv -g_6 = \alpha^2(u)\, \eta^2(v)$. That this determinant is independent of β_{uv} is consistent with the interpretation of β_{uv} as a kind of boost parameter. Of special importance for the existence of the CTC is the off-diagonal metric element g_{tx}, which is nonzero for $\eta(v) \neq 1$ (i.e., off the brane), and the metric element g_{xx} which is of indeterminate sign. As a consistency check on the metric, we note that for $u = v = 0$, i.e., on the brane, Eq. (25) reduces to 4-dimensional Minkowski spacetime.

The existence of a CTC is guaranteed by showing that the metric elements in (25) can be chosen to satisfy the two CTC conditions of Eq. (18). Inputting the metric elements into (18), one finds that the conditions reduce to

$$\frac{\alpha_2\,(\beta_{uv} + \eta_2)}{1 + \beta_{uv}\eta_2} < \frac{\alpha_1\,(\beta_{uv} - \eta_1)}{1 - \beta_{uv}\eta_1}\,, \tag{26}$$

and

$$\eta_1 < \beta_{uv}. \tag{27}$$

The new feature here, as opposed to the 5D metric, is the freedom to choose α_1 and α_2 to ensure that the CTC conditions are satisfied. We see that any pair (α_1, α_2) will do, as long as they satisfy

$$\frac{\alpha_2}{\alpha_1} < \left(\frac{\beta_{uv} - \eta_1}{1 - \beta_{uv}\eta_1}\right) \left(\frac{1 + \beta_{uv}\eta_2}{\beta_{uv} + \eta_2}\right) . \tag{28}$$

This inequality can always be satisfied by an arbitrarily small choice for α_2.

One simple and successful choice is to set $\alpha_1 = 1$ and $\eta_2 = 1$, i.e., to take the outgoing path on the $u = 0$ hyperslice and the return path on the $v = 0$ hyperslice. With these choices, (28) reduces to $\alpha_2 < (\beta - \eta_1)/(1 - \beta\eta_1)$. This is guaranteed to be satisfiable by (27). The choices $u_1 = 0$ and $v_2 = 0$ will lead to an explicit CTC. With $u = 0$, Eq. (25) reduces to (**??**) with $\eta^2(v)$ replacing $\alpha^2(u)$:

$$ds^2|_{u=0} = \gamma_{uv}^2 \left\{ [1 - \beta_{uv}^2 \eta^2(v)]dt^2 + 2\beta_{uv}[1 - \eta^2(v)]dxdt - [\eta^2(v) - \beta_{uv}^2]dx^2 \right\} - dv^2. \tag{29}$$

Thus we see explicitly that choosing $\eta_1 < \beta_{uv}$ on the $u = 0$ hyperslice sets $g_{xx} < 0$, so that our outgoing path necessarily accumulates negative time (original frame in Table **??**). On the return path, we set $v = 0$. Then the 6D metric of Eq. (25) reduces to (20), repeated here:

$$ds^2|_{v=0} = dt^2 - \alpha^2(u)dx^2 - du^2 ; \tag{30}$$

It is clear that this return path can be made arbitrarily brief by choosing α_2 arbitrarily small. The CTC is revealed.

We note that when the metric (25) is transformed into the v-frame by a Lorentz transformation on the brane with $\beta = -\beta_{uv}$, then the metric along the v-dimension assumes the simple form of (30) (with obvious replacements) and the metric along the u-dimension becomes non-diagonal.

To summarize this section, we have identified a CTC beginning and ending on our brane and superluminally transiting two paths parallel to our brane but in the asymmetrically warped u- and v-dimensions. The physics that enables the CTC is the breaking of global Lorentz invariance away from the brane.

Figure 3: Nonzero elements of the Einstein tensor $G^{\mu}{}_{\nu}$ (in arbitrary units): $G^{\delta}{}_{\delta} \equiv G^{0}{}_{0} = G^{y}{}_{y} = G^{z}{}_{z} = G^{v}{}_{v}$, on the $v = 0$ slice, as a function of u. Assumed are warp factors $\alpha(u) = 1/(u^2 + c^2)$ and $\eta(v) = 1/(v^4 + c^2)$, with $c = 1$. We find that the weak and dominant energy conditions are violated in the bulk, while all energy conditions with the exception of the SEC are satisfied on the brane.

Stress-energy tensor and energy conditions

As a check on the consistency of the picture, we should diagnose the stress-energy tensor which sources the extra-dimensional metric, for any pathologies. In particular, we will be interested in the resulting matter distributions on and off the brane. Thus, our task is to calculate the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{31}$$

from the spacetime metric of Eq. (25), and then to obtain the stress-energy tensor $T_{\mu\nu}$ via the Einstein equation

$$T_{\mu\nu} = \frac{1}{8\,\pi\,\mathrm{G_N}}\, G_{\mu\nu}\,. \tag{32}$$

Consequently, we proceed to evaluate $T_{\mu\nu} = (8\pi\,G_{\mathrm{N}})^{-1}\,G_{\mu\nu}$ with no preconceptions as to its form. We note that in general, $T_{\mu\nu}$ contains contributions from matter, fields, and cosmological constant on and off the brane, and from brane tension on the brane.

Instead of complicated analytic expressions for $T_{\mu\nu}$, we present some visual output [**?**] of the Einstein tensor versus u, on the $v = 0$ slice. We do so for warp factors $\alpha(u)$ and $\eta(v)$ chosen to satisfy energy conditions discussed below. An analogous figure is the Einstein tensor versus v, on the $u = 0$ slice. However, this Einstein tensor has off-diagonal elements, which increases the number of figures. Furthermore, it offers us no additional enlightenment, so we do not show this Einstein tensor.

There is considerable theoretical prejudice that stable Einstein tensors should satisfy certain "energy conditions" relating energy density ρ and directional pressures p^j. The null, weak, strong and dominant energy conditions state that

$$\text{NEC} \;:\; \rho + p^j \geq 0, \quad \forall j\,. \tag{33}$$

$$\text{WEC} \;:\; \rho \geq 0\,; \quad \text{and} \quad \rho + p^j \geq 0, \quad \forall j\,. \tag{34}$$

$$\text{SEC} \;:\; \rho + p^j \geq 0, \quad \forall j; \quad \text{and} \quad \rho + \sum_j p^j \geq 0\,. \tag{35}$$

$$\text{DEC} \;:\; \rho \geq 0\,; \quad \text{and} \quad p^j \in [\rho, -\rho], \quad \forall j\,. \tag{36}$$

For the purpose of definiteness in the identification of ρ and p^j, we assume the anisotropic fluid relations

$$T^{\mu\nu} = -p\, g^{\mu\nu} + (\rho + p)\, U^\mu U^\nu\,, \tag{37}$$

with $u^\mu = (1, \vec{0})$ being the net four velocity of the fluid. Then, with a diagonal metric with $g_{tt} = 1$ (Gaussian-normal coordinates), one obtains for the nonzero elements of $T^\mu_{\ \mu}$,

$$\rho = T^0_{\ 0} \quad \text{and} \quad p^j = -T^j_{\ j}\,. \tag{38}$$

These are the relations appropriate for the $v = 0$ slice of our metric, since one sees in Eq. (30) that the $v = 0$ metric is manifestly diagonal with $g_{tt} = 1$.

It is not difficult to find a functional form for the warp factors α and η which conserves some of the energy conditions, at least on the brane. One such example is given by $\alpha(u) = 1/(u^2 + c^2)$ and $\eta(v) = 1/(v^4 + c^2)$. For this case the elements of the Einstein tensor on the $v = 0$ slice are shown as a function of u in Fig. 3. The null, weak and dominant energy conditions are conserved on the brane, while the strong energy condition is violated both on the brane and in the bulk.

The negative energy density that afflicts many wormhole and CTC solutions in four dimensions is avoided on the brane in the example for an extra-dimensional CTC presented here. However, ρ becomes negative as one moves away from the brane into the bulk, so that the WEC and DEC are violated off the brane, while the NEC remains conserved. We have successfully constructed a metric exhibiting CTCs in an extra-dimensional spacetime by "moving" the negative energy density from the brane to the bulk. One might even speculate that the negative energy density in the bulk is related to the compactification of the extra dimensions, or possibly to the repulsion of Standard matter from the bulk.

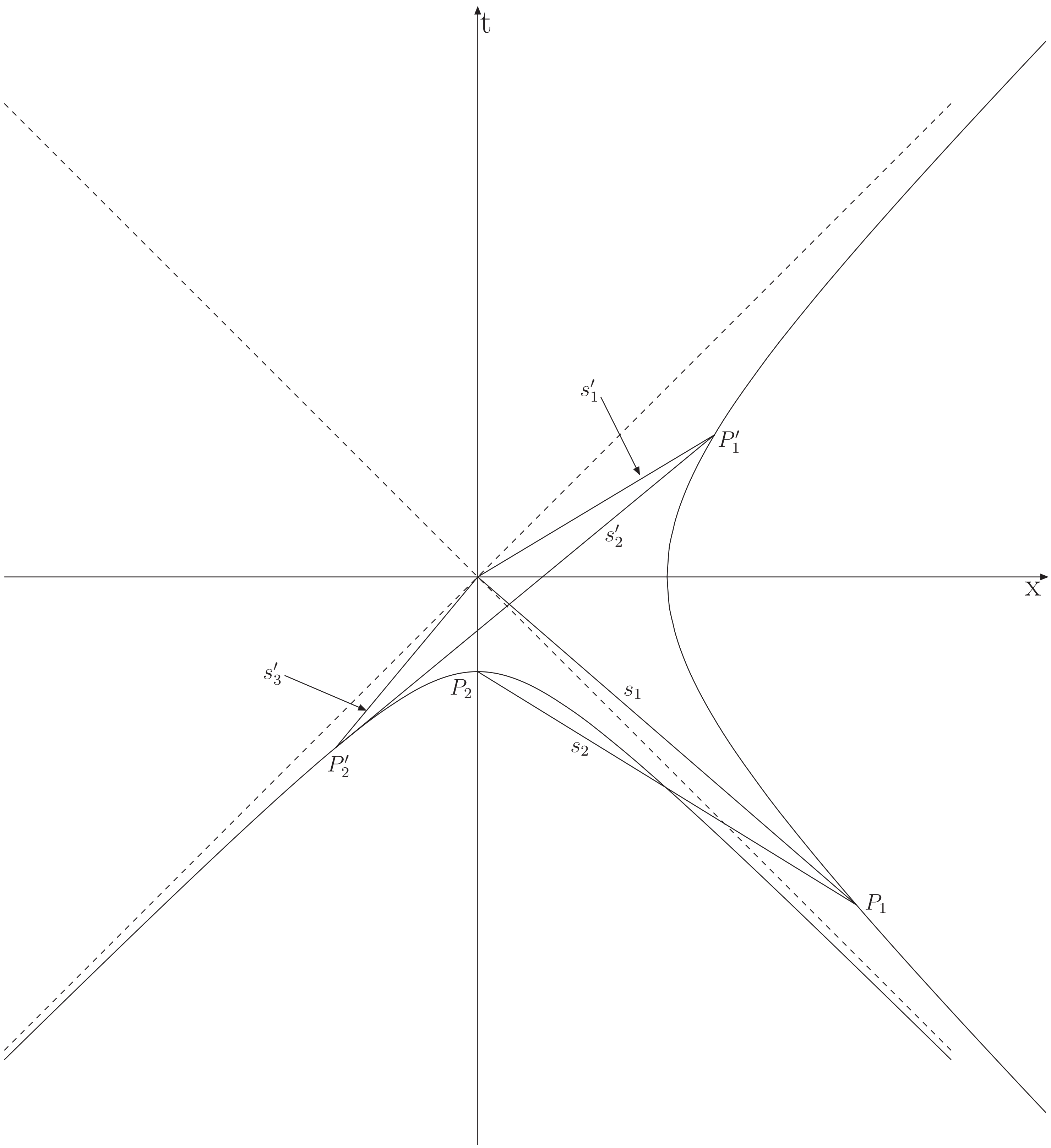
One also sees in Fig. (3) that $G^y_{\ y} = G^z_{\ z} = G^v_{\ v}$ are equal to $G^0_{\ 0}$ on the $v = 0$ slice. This equality amounts to a dark energy or cosmological constant equation of state for the y-, t-, and v-directed pressures, namely, $w^j \equiv p^j/\rho = -1$. There may be some intriguing physics underlying this result.

Discussion and Conclusion

We have derived the general conditions on metric elements which allow spacetimes to contain closed timelike curves (CTCs). Then, we have demonstrated the existence of CTCs for a rather generic spacetime with two asymmetrically warped extra dimensions. In addition, we have found particular warp factors for the metric which yield positive energy density on the brane. However, negative energy density is not completely banished, as it does appear in the bulk. Since one cannot observe the bulk energy density, we may at least say that negative energy density is banished from sight. It is also possible that an anthropic argument applies here: Life may evolve only where energy density is positive. Then lifeless bulk regions of negative energy density can communicate their existence to living beings only via geometry, perhaps mediated by the exchange of gravitons or appropriately named, "sterile" neutrinos.

It should be stressed that realistic graviton or bulk fermion signals, rather than following restricted bulk trajectories with constant u or v as constructed here, will instead propagate on the path of least action to minimize the travel time. Since the effectively superluminal velocities in our constructed example produced a CTC, we expect that a truly geodesic signal will also generate a CTC. In this case the causal structure of extra dimensions may be studied with sterile neutrino beams by utilizing resonant conversion of active neutrinos via matter effects into sterile neutrinos and back. We note that the model presented herein is complete in that the geodesic equations of motion are derivable from the metric in Eq. (25). We have not investigated the geodesic equations in this work.

A thorough discussion of whether CTCs in the observable universe are hidden behind chronology horizons where the stress-energy tensor diverges (one may consult the discussion in [**?**, **?**]), is beyond the scope of this work. We have confined ourselves to the pragmatic attitude that even if chronology were protected by some mechanism operative near the chronology horizon, it remains a highly rewarding effort to study the physics near this horizon. The CTC we have constructed is particularly interesting in this respect, since it could be available to gauge-singlet particles which have previously been hypothesized to propagate in the extra-dimensional bulk.



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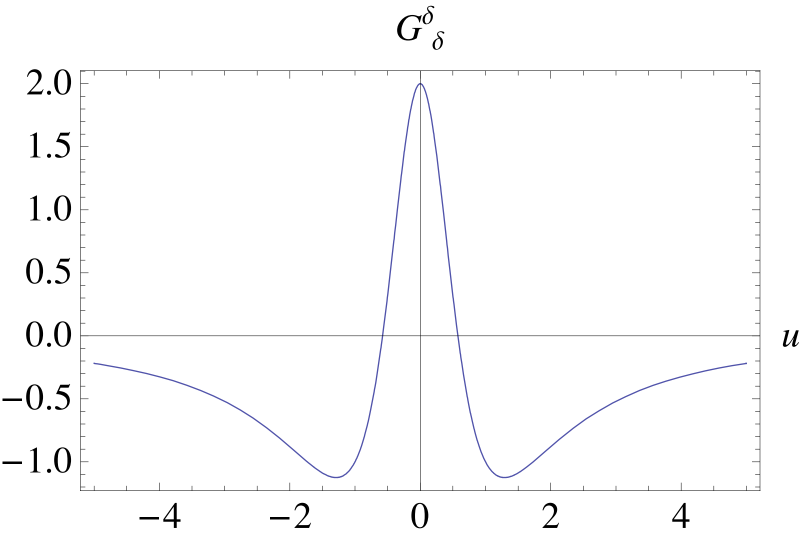

G^δ_δ
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
u
−4
−2
0
2
4